Some constructions of ultrafilters over a measurable cardinal.

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Abstract

Some non-normal $\kappa$–complete ultrafilters over a measurable $\kappa$ with special properties are constructed. Questions by A. Kanamori [4] about infinite Rudin-Frolik sequences, discreteness and products are answered.

1 Introduction.

We present here several constructions of $\kappa$–complete ultrafilters over a measurable cardinal $\kappa$ and examine their consistency strength. Some questions of Aki Kanamori from [4] are answered.

Section 2 deals with Rudin-Frolik ordering and answers Question 5.11 from [4] about infinite increasing Rudin-Frolik sequences. In Section 3, an example of non-discrete family of ultrafilters is constructed, answering Question 5.12 from [4]. Also the strength of existence of such family is examined. Section 4 deals with products of ultrafilters. A negative answer to Question 5.8 from [4] given.

2 On Rudin-Frolik increasing sequences.

In [4], Aki Kanamori asked if there exists a $\kappa$–ultrafilter with an infinite number of Rudin-Frolik predecessors.

We show that starting with $o(\kappa) = 2$ it is possible.

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Assume GCH. Let

$$
\bar{U} = \langle U(\alpha, \beta) \mid (\alpha, \beta) \in \text{dom}(\bar{U}), \alpha \leq \kappa, \beta < \sigma^\bar{U}(\alpha) \rangle
$$

be a coherent sequence such that $\sigma^\bar{U}(\kappa) = 2$ and for every $\alpha < \kappa$, $\sigma^\bar{U}(\alpha) \leq 1$. Let

$$
A = \{ \alpha \mid \exists \beta(\alpha, \beta) \in \text{dom}(\bar{U}) \}.
$$

Then for every $\alpha \in A$, $\sigma^\bar{U}(\alpha) = 1$ and $U(\alpha, 0)$ is a normal ultrafilter over $\alpha$.

We force with Easton support iteration of the Prikry forcings with $U(\alpha, 0)$'s (and their extensions), $\alpha \in A$, as in [1] (a better presentation appears in [2]). Let $G$ be a generic. Then for every increasing sequence $t$ of ordinals less than $\kappa$, the normal ultrafilter $U(\kappa, 1)$ of $V$ extends to a $\kappa$-complete ultrafilter $U(\kappa, 1, t)$ in $V[G]$, see [1], p.291.

Denote by $b_\alpha$ the Prikry sequence from $G$ added to $\alpha$, for every $\alpha \in A$. Then $U(\kappa, 1, t)$ concentrates on $\alpha \in A$ for which $b_\alpha$ starts from $t$, i.e. $b_\alpha \upharpoonright |t| = t$.

Let $\bar{U}(\kappa, 0)$ be the canonical extension of $U(\kappa, 0)$ to a normal ultrafilter in $V[G]$ defined as in [2] on page 290.

Denote $U(\kappa, 1, (\cdot))$ by $\bar{U}(\kappa, 1)$.

**Lemma 2.1** For every $n, 0 < n < \omega$, $\bar{U}(\kappa, 1) = \bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$.

**Proof.** Recall the definition of $U(\kappa, 1, t)$, $t \in [\kappa]^m, m < \omega$:

$x \in U(\kappa, 1, t)$ iff for some $r \in G, \gamma < \kappa^+, B \in \bar{U}(\kappa, 0), \in M_{U(\kappa, 1)}$ the following holds:

$$
\Gamma \cup \{ (t, b) \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, 1)}(X),
$$

where $p_\gamma$ is the $\gamma$-th element of the canonical master sequence.

In particular, $x \in \bar{U}(\kappa, 1)$ iff for some $r \in G, \gamma < \kappa^+, B \in \bar{U}(\kappa, 0), \in M_{U(\kappa, 1)}$ the following holds:

$$
\Gamma \cup \{ (\cdot, B) \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, 1)}(X).
$$

Then, for every $t \in [B]^n$, we will have

$$
\Gamma \cup \{ (t, B \setminus \text{max}(t) + 1) \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, 1)}(X).
$$

So, $x \in U(\kappa, 1, t)$. But $[B]^n \in \bar{U}(\kappa, 0)^n$, hence $x \in \bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$.

Hence we showed that $\bar{U}(\kappa, 1) \subseteq \bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$. But this already implies the equality, since both $\bar{U}(\kappa, 1)$ and $\bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$ are ultrafilters.

$\square$
Lemma 2.2 The family \( \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle \) is a discrete family of ultrafilters.

Proof. For each \( t \in [\kappa]^n \) set
\[
A_t := \{ \alpha \in A \mid b_\alpha \upharpoonright n = t \}.
\]
Let \( t, t' \in [\kappa]^n \) be two different sequences, then, clearly, \( A_t \cap A_{t'} = \emptyset \).
\( \square \)

Recall the following definition:

Definition 2.3 (Frolik and M.E. Rudin) Let \( U, D \) be ultrafilters over \( I \). \( U \geq_{R-F} D \) iff there is a discrete family \( \{ E_i \mid i \in I \} \) of ultrafilters over some \( J \) such that \( U = D - \lim \{ E_i \mid i \in I \} \).

So we obtain the following:

Theorem 2.4 \( \bar{U}(\kappa, 1) \) has infinitely many predecessors in the Rudin-Frolik ordering.

Proof. For every \( n, 0 < n < \omega \), use a bijection between \( [\kappa]^n \) and \( \kappa \) and transfer \( \bar{U}(\kappa, 0)^n \) to \( \kappa \). The rest follows by Lemmas 2.1, 2.2.
\( \square \)

Note that for \( \kappa \)-complete ultrafilters \( U \) and \( D \) over \( \kappa \), \( U \geq_{R-F} D \) implies \( U \geq_{R-K} D \). So, by [5], the existence of a \( \kappa \)-complete ultrafilter over \( \kappa \) with infinitely many predecessors in the Rudin-Frolik ordering implies by Kanamori [4], that \( 0^\dagger \) exists. Let us improve this in order to give the exact strength.

Theorem 2.5 The existence of a \( \kappa \)-complete ultrafilter over \( \kappa \) with infinitely many predecessors in the Rudin-Frolik ordering implies that \( o(\kappa) \geq 2 \) in the core model.

Proof. Note first that for \( \kappa \)-complete ultrafilters \( U \) and \( D \) over \( \kappa \), \( U \geq_{R-F} D \) implies \( U \geq_{R-K} D \). So, by [5], the existence of a \( \kappa \)-complete ultrafilter over \( \kappa \) with infinitely many predecessors in the Rudin-Frolik ordering implies that \( \exists \lambda o(\lambda) \geq 2 \). Let us argue that actually \( o(\kappa) \geq 2 \) in the core model.

Suppose otherwise. So, \( o(\kappa) = 1 \). Let \( U(\kappa, 0) \) be the unique normal measure over \( \kappa \) in the core model \( K \).

Suppose that, in \( V \), we have a \( \kappa \)-complete ultrafilter \( E \) over \( \kappa \) with infinitely many predecessors in the Rudin-Frolik ordering. Let \( \langle E_n \mid n < \omega \rangle \) be a Rudin-Frolik increasing sequence of predecessors of \( E \). Recall that by M.E. Rudin (see [4], 5.5) the predecessors of \( E \) are linearly ordered.
Consider $i := i_E \upharpoonright \mathcal{K}$. Then, by [5], it is an iterated ultrapower of $\mathcal{K}$ by its measures. The critical point of $i_E$ is $\kappa$, hence $U(\kappa, 0)$ is applied first. Note that $U(\kappa, 0)$ (and its images) can be applied only finitely many times, since $M_E$ is closed under countable (and even $\kappa$) sequences of its elements. Denote by $k^*$ the number of such applications.

Let $n \leq \omega$. Similar, consider $i_n := i_{E_n} \upharpoonright \mathcal{K}$. Again, the critical point of $i_{E_n}$ is $\kappa$, hence $U(\kappa, 0)$ is applied first. The number of applications of $U(\kappa, 0)$ (and its images) is finite. Denote by $k_n$ the number of such applications.

Now let $n < m < \omega$. We have $E_n <_{R-F} E_m$. Hence, there is a discrete sequence $\langle E_{n^{\alpha}_m} \mid \alpha < \kappa \rangle$ of ultrafilters over $\kappa$ such that

$$E_m = E_n - \lim (E_{n^{\alpha}_m} \mid \alpha < \kappa).$$

Then the ultrapower $M_{E_m}$ of $V$ by $E_m$ is $\text{Ult}(M_{E_n}, E'_{nm} \upharpoonright \text{id}_{E_n})$, where $E'_{nm} \upharpoonright \text{id}_{E_n} = i_{E_n}((E_{n^{\alpha}_m} \mid \alpha < \kappa))(\text{id}_{E_n})$ is an ultrafilter over $i_{E_n}(\kappa)$.

Now, in $i_n(\mathcal{K})$, the only normal ultrafilter over $i_{E_n}(\kappa) = i_n(U(\kappa, 0))$. But this means that $i_{E_m}$ is obtained by more applications of $U(\kappa, 0)$ than $i_{E_n}$, i.e. $k_n < k_m$.

Similar, $k^* > k_n$, for every $n < \omega$. This means, in particular, that $k^* \geq \omega$, which is impossible. Contradiction.

Remark 2.6 Note that the situation with Rudin-Keisler order is different in this respect. Thus, by [3], starting with a measurable $\kappa$ with $\{o(\kappa) \mid \alpha < \kappa\}$ unbounded in it, it is possible to construct a model with an increasing Rudin-Keisler sequence of the length $\kappa^+$. A similar arguments can be used to produce long increasing Rudin-Frolik sequences.

Let us show how to get a sequence of the length $\kappa + 1$

Assume GCH. Let

$$\vec{U} = \langle U(\alpha, \beta) \mid (\alpha, \beta) \in \text{dom}(\vec{U}), \alpha \leq \kappa, \beta < o^\beta(\alpha) \rangle$$

be a coherent sequence such that $o^\beta(\kappa) = \kappa + 1$ and for every $\alpha < \kappa$, $o^\beta(\alpha) \leq \kappa$. Let

$$A = \{ \alpha \mid \exists \beta(\alpha, \beta) \in \text{dom}(\vec{U}) \}.$$

Then for every $\alpha \in A$, $o^\vec{U}(\alpha) \leq \kappa$.

\footnote{Theorem 5.10 of [4] states that this is impossible, however we think that there is a problem in the argument. Namely, on page 346, line 7 - sets depend on $\beta$’s; this effects the further definition of a function $f$ (line 16). Its unclear how to insure $f(\xi) > f(\xi')$ for most $\xi$’s, and, so $f$ may be constant mod $D_0$.}
We force with Easton support iteration of the Prikry type forcings with extensions of \( U(\alpha, \beta) \mid \beta < \omega^\alpha(\alpha) \)'s, \( \alpha \in A \), as in [1]. Let \( G \) be a generic. Then, for every \( \alpha \in A \) with \( \omega^\alpha(\alpha) = 1 \) or being a regular uncountable cardinal, Prikry sequence or Magidor sequence of order type \( \omega^\alpha(\alpha) \) is added by \( G \) (more sequences are added, see [1] for detailed descriptions, but do not need them here). Denote such sequences by \( b_\alpha \).

Let \( \bar{U}(\kappa, 0) \) be the canonical extension of \( U(\kappa, 0) \) to a normal ultrafilter in \( V[G] \) defined as in [2].

Denote by \( A' \) the subset of \( A \) which consists of \( \alpha \)'s with \( \omega^\alpha(\alpha) = 1 \) or being a regular uncountable cardinal.

For every \( \delta, \alpha \in A' \cup \{ \kappa \}, \delta < \alpha \) we will use an extensions \( U(\kappa, \alpha, \langle \rangle) \) and \( U(\kappa, \alpha, \langle \delta \rangle) \) of \( U(\kappa, \alpha) \). They were defined in [1] as follows:

\[
X \in U(\kappa, \alpha, \langle \rangle) \text{ iff for some } r \in G, \gamma < \kappa^+ \text{ and a tree } T, \text{ in } M_{U(\kappa, 1)} \text{ the following holds:}
\]

\[
r \cup \{ \langle \langle \rangle, T \rangle \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, 1)}(\bar{X}),
\]

where \( p_\gamma \) is the \( \gamma \)-th element of the canonical master sequence.

\[
X \in U(\kappa, \alpha, \langle \delta \rangle) \text{ iff for some } r \in G, \gamma < \kappa^+ \text{ and a tree } T, \text{ in } M_{U(\kappa, 1)} \text{ the following holds:}
\]

\[
r \cup \{ \langle \langle \delta \rangle, T \rangle \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, 1)}(\bar{X}),
\]

where \( p_\gamma \) is the \( \gamma \)-th element of the canonical master sequence.

Denote further \( U(\kappa, \alpha, \langle \rangle) \) by \( \bar{U}(\kappa, \alpha) \).

Notice that \( U(\kappa, \alpha, \langle \delta \rangle) \) concentrates on \( \nu \)'s with \( \omega^\delta(\nu) = \alpha, \delta \in b_\nu \) and \( b_\nu \cap \delta = b_\delta \).

We have now the following analog of 2.1:

**Lemma 2.7** For every \( \alpha \in A' \), \( \bar{U}(\kappa, \kappa) = \bar{U}(\kappa, \alpha) - \lim \langle U(\kappa, \kappa, \langle \nu \rangle) \mid \omega^\delta(\nu) = \alpha \rangle \).

**Proof.** \( X \in \bar{U}(\kappa, \kappa) \) iff for some \( r \in G, \gamma < \kappa^+, T, \) in \( M_{U(\kappa, \kappa)} \) the following holds:

\[
r \cup \{ \langle \langle \rangle, T \rangle \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, \kappa)}(\bar{X}).
\]

Recall that \( T \) is a tree consisting of coherent sequences and \( Suc_T(\langle \rangle) \in \bar{U}(\kappa, \alpha) \). Then, for every \( \nu \in Suc_T(\langle \rangle) \) with \( \omega^\delta(\nu) = \alpha \), we will have

\[
r \cup \{ \langle \langle \nu \rangle, T_{(\nu)} \rangle \} \cup p_\gamma \vDash \kappa \in i_{U(\kappa, \kappa)}(\bar{X}).
\]

So, \( X \in U(\kappa, \kappa, \langle \nu \rangle) \). But this holds for \( \bar{U}(\kappa, \alpha) \)-measure one many \( \nu \)'s, hence \( X \in \bar{U}(\kappa, \alpha) - \lim \langle U(\kappa, \kappa, \langle \nu \rangle) \mid \omega^\delta(\nu) = \alpha \rangle \).
Hence we showed that \( \bar{U}(\kappa, \kappa) \subseteq \bar{U}(\kappa, \alpha) - \lim \langle U(\kappa, \kappa, \langle \nu \rangle) | o^U(\nu) = \alpha \rangle \). But this already implies the equality, since both \( \bar{U}(\kappa, \kappa) \) and \( \bar{U}(\kappa, \alpha) - \lim \langle U(\kappa, \kappa, \langle \nu \rangle) | o^U(\nu) = \alpha \rangle \) are ultrafilters.

□

The same argument shows the following:

**Lemma 2.8** For every \( \gamma, \alpha \in A', \alpha < \gamma \), \( \bar{U}(\kappa, \gamma) = \bar{U}(\kappa, \alpha) - \lim \langle U(\kappa, \gamma, \langle \nu \rangle) | o^U(\nu) = \alpha \rangle \).

**Lemma 2.9** The family \( \langle U(\kappa, \gamma, \langle \nu \rangle) | o^U(\nu) = \alpha \rangle \) is a discrete family of ultrafilters, for every \( \gamma, \alpha \in A' \cup \{ \kappa \}, \alpha < \gamma \).

**Proof.** Fix \( \gamma, \alpha \in A' \cup \{ \kappa \}, \alpha < \gamma \). For each \( \nu \) with \( o^U(\nu) = \alpha \) set

\[ A_{\nu} := \{ \xi \in A' | o^\bar{U}(\xi) = \gamma, \nu \in b_\gamma \text{ and } b_\gamma \cap \nu = b_\nu \}. \]

Let \( \nu, \nu' \in A' \) be two different elements with \( o^U(\nu) = o^U(\nu') = \alpha \), then, clearly, \( A_{\nu} \cap A_{\nu'} = \emptyset \).

□

So, again as above, we obtain the following:

**Theorem 2.10** \( \bar{U}(\kappa, \kappa) \) has \( \kappa \)-many predecessors in the Rudin-Frolik ordering.

**Proof.** By Lemmas 2.7, 2.8, the sequence \( \langle \bar{U}(\kappa, \gamma) | \gamma \in A' \cup \{ \kappa \} \rangle \) is R-F-increasing.

□

It follows now that:

**Corollary 2.11** The consistency strength of existence of a \( \kappa \)-complete ultrafilter over \( \kappa \) with \( \kappa \)-many predecessors in the Rudin-Frolik ordering is is at least \( \{ o(\alpha) | \alpha < \kappa \} \) is unbounded in \( \kappa \) and at most \( o(\kappa) = \kappa + 1 \).

### 3 Discrete families of ultrafilters.

Aki Kanamori asked in [4] the following natural question:

If \( \{ U_\tau | \tau < \kappa \} \) is a family of distinct \( \kappa \)-complete ultrafilters over \( \kappa \) and \( E \) is any \( \kappa \)-complete ultrafilter over \( \kappa \), is there an \( X \in E \) so that \( \{ U_\tau | \tau \in X \} \) is a discrete family?

We will give a negative answer to this question below.

Let us use the previous construction. We preserve all the notation made there.

Consider the family

\[ \{ U(\kappa, \kappa, \langle \delta \rangle) | \delta, \alpha \in A', \delta < \alpha \}. \]
Lemma 3.1 The family \( \{ U(\kappa, \kappa, \langle \delta \rangle) \mid \delta \in A', \delta < \kappa \} \) consists of different ultrafilters.

Proof. Let \( U(\kappa, \kappa, \langle \delta \rangle), U(\kappa, \kappa, \langle \delta' \rangle) \) be two different members of the family. If \( o^U(\delta) = o^U(\delta') \), then they are different by Lemma 2.9. Suppose that \( o^U(\delta) < o^U(\delta') \). Then the set

\[
\{ \nu < \kappa \mid o^U(\nu) = \nu, \delta \in b_\nu, b_\nu \cap \delta = b_\delta \text{ and } \delta' \not\in b_\nu \} \in U(\kappa, \kappa, \langle \delta \rangle) \setminus U(\kappa, \kappa, \langle \delta' \rangle).
\]

So we are done.

\[ \square \]

Pick now a \( \kappa \)-complete (non-principal) ultrafilter \( D \) such that the set

\[ Z := \{ \alpha < \kappa \mid \alpha \text{ is a regular uncountable cardinal } \} \in D. \]

Define now a \( \kappa \)-complete ultrafilter \( E \) over \( [\kappa]^2 \) as follows:

\[ X \in E \text{ iff } \{ \alpha < \kappa \mid \{ \delta < \kappa \mid (\alpha, \delta) \in X \} \in U(\kappa, \alpha, \langle \rangle) \} \in D. \]

I.e. \( E = D - \Sigma \alpha U(\kappa, \alpha, \langle \rangle) \). We can assume that if \( (\alpha, \delta) \in X \), for a set \( X \in E \), then \( o^U(\delta) = \alpha \), since \( U(\kappa, \alpha) \) concentrates on such \( \delta \)'s.

Now, for every pair \( (\alpha, \delta) \) with \( o^U(\delta) = \alpha \), define \( U(\alpha, \delta) = U(\kappa, \kappa, \langle \rangle) \).

Lemma 3.2 For every \( X \in E \), the family \( \{ U_\tau \mid \tau \in X \} \) is not discrete.

Proof. Let \( X \in E \). Suppose that there is a separating sequence \( \langle Y_{\alpha, \delta} \mid (\alpha, \delta) \in X \rangle \) for \( \langle U_{\alpha, \delta} \mid (\alpha, \delta) \in X \rangle \). Pick some \( \alpha, \alpha' \in \text{dom}(X), \alpha < \alpha' \). Let

\[ A_\alpha = \{ \delta < \kappa \mid (\alpha, \delta) \in X \} \]

and

\[ A_{\alpha'} = \{ \delta < \kappa \mid (\alpha', \delta) \in X \}. \]

Then \( A_\alpha \in U(\kappa, \alpha, \langle \rangle) \) and \( A_{\alpha'} \in U(\kappa, \alpha', \langle \rangle) \). By shrinking \( X \) if necessary, assume that \( \delta \in A_\alpha \) implies \( o^U(\delta) = \alpha \) and \( \delta' \in A_{\alpha'} \) implies \( o^U(\delta') = \alpha' \).

Consider the following set

\[ B = \{ \nu < \kappa \mid o^U(\nu) = \nu \text{ and (there are } \delta \in A_\alpha, \delta' \in A_{\alpha'} \text{ such that } \delta < \delta' \text{ and } \delta, \delta' \in b_\nu \}. \]

Then \( B \in U(\kappa, \kappa, \langle \rangle) \). Just take the witnessing tree \( T_B \) (as in the definition of \( U(\kappa, \kappa, \langle \rangle) \)) with the first level

\[ A_\alpha \cup A_{\alpha'} \cup (\kappa \setminus (A_\alpha \cup A_{\alpha'})). \]
Then for every $\delta \in A_\alpha$, $B \in U(\kappa, \kappa, \langle \delta \rangle)$. So, $B' := B \cap Y_{(a, \delta)}$ is a subset of $B$ in $U(\kappa, \kappa, \langle \delta \rangle)$.

But then an extension of $T_B$ will witness this. In particular there will be $\delta' \in A_{\alpha'}$ such that $B' \in U(\kappa, \kappa, \langle \delta' \rangle)$. This implies that both $Y_{(a, \delta)}$ and $Y_{(a', \delta')}$ are in $U(\kappa, \kappa, \langle \delta' \rangle) = U_{(a, \delta')}$. Hence, $Y_{(a, \delta)} \cap Y_{(a', \delta')} \neq \emptyset$. Contradiction.

Now combining Lemmas 3.1, 3.2 we obtain the following:

**Theorem 3.3** In $V[G]$ there are a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa$–complete ultrafilters over $\kappa$ and a $\kappa$–complete ultrafilter $E$ over $\kappa$, so that $\{U_\tau \mid \tau \in X\}$ is a not discrete family for any $X \in E$.

**Corollary 3.4** The consistency strength of existence a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa$–complete ultrafilters over $\kappa$ and a $\kappa$–complete ultrafilter $E$ over $\kappa$, so that $\{U_\tau \mid \tau \in X\}$ is a not discrete family for any $X \in E$, is at most $o(\kappa) = \kappa + 1$.

Let us argue now that that $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in $\kappa$ is necessary for this.

**Theorem 3.5** Suppose that there are a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa$–complete ultrafilters over $\kappa$ and a $\kappa$–complete ultrafilter $E$ over $\kappa$, so that $\{U_\tau \mid \tau \in X\}$ is not a discrete family for any $X \in E$. Then $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in $\kappa$ in the Mitchell core model.

**Proof.** Suppose otherwise. Let $\{U_\tau \mid \tau < \kappa\}$ be a family of distinct $\kappa$–complete ultrafilters over $\kappa$ and $E$ be a $\kappa$–complete ultrafilter over $\kappa$, so that $\{U_\tau \mid \tau \in X\}$ is a discrete family for any $X \in E$.

Let $\mathcal{K}$ be the Mitchell core model and $o(\kappa) = \eta < \kappa$.

For every $\tau < \kappa$, let $j_\tau$ be $i_{U_\tau} \upharpoonright \mathcal{K}$. Then, by [5], $j_\tau$ is an iterated ultrapower of $\mathcal{K}$. By [3], the are less than $\kappa$ possibilities for $j_\tau(\kappa)$. By $\kappa$–completeness of $E$, we can assume that for every $\tau < \kappa$, $j_\tau(\kappa)$ has a fixed value $\theta$. Denote by $Gen_\tau$ the set of generators of $j_\tau$, i.e. the set of ordinals $\nu, \kappa \leq \nu < \theta$ such that for every $n < \omega$, $f : [\kappa]^n \rightarrow \kappa, f \in \mathcal{K}$ and $a \in [\nu]^n$, $\nu \neq j_\tau(f)(a)$. Let $Gen^*_\tau$ be the subset of $Gen_\tau$ consisting of all principle generators of $j_\tau$, i.e. of all $\nu \in Gen_\tau$ such that for every $n < \omega$, $f : [\kappa]^n \rightarrow \kappa, f \in \mathcal{K}$ and $a \in [\nu]^n$, $\nu > j_\tau(f)(a)$.

Again by [3], the are less than $\kappa$ possibilities for $Gen^*_\tau$’s. So, by $\kappa$–completeness of $E_\tau$, we can assume that for every $\tau < \kappa$, $Gen^*_\tau = Gen^*$.

Suppose that $\nu \in Gen_\tau$ and $\nu$ is not a principle generator. Then there are finite set of generators $b \subseteq \nu$ and $f : [\kappa]^{|b|} \rightarrow \kappa, f \in \mathcal{K}$ such that $\nu < j_\tau(f)(b)$. Set, following W. Mitchell,

$$\alpha(\nu) = \min\{j_\tau(f)(b) \mid b \subseteq \nu \text{ is a finite set of generators,}$$
\[ f : [\kappa]^{|b|} \rightarrow \kappa, \ f \in \mathcal{K} \text{ and } \nu < j_\tau(f)(b). \]

Let \( b_\nu \subseteq \nu \) be the smallest finite set of generators such that for some \( f : [\kappa]^{|b_\nu|} \rightarrow \kappa, \ f \in \mathcal{K}, \ \alpha(\nu) = j_\tau(f)(b_\nu). \)

Let us call a finite set of generators \( a \subseteq \text{Gen}_\tau \) nice iff for each \( \nu \in a \) either \( \nu \) is a principle generator or it is not and then \( b_\nu \subseteq a. \)

Consider now \( [id]_{U_\tau}. \) Find the smallest finite nice set of generators \( a_\tau \) in \( \text{Gen}_\tau \) such that for some \( h_\tau : [\kappa]^{|a_\tau|} \rightarrow \kappa, \ h_\tau \in \mathcal{K} \) we have \( [id]_{U_\tau} = j_\tau(h_\tau)(a_\tau). \) We may assume, using \( \kappa \)-completeness of \( E, \) that \( a_\tau \cap \text{Gen}_\tau \) has a constant value. Denote it by \( a^*. \)

Let us deal first with simpler particular cases.

Suppose first that \( a^* = a_\tau \) and it consists only of \( \kappa \) itself, for every \( \tau < \kappa \) (or on an \( E \)-measure one set). Then, for some \( \theta < o(\kappa), \) each \( j_\tau \) is just the ultrapower embedding \( i_{U(\kappa, \theta)} \) by a normal measure \( U(\kappa, \theta) \) from the sequence of \( \mathcal{K}. \)

Now the functions \( h_\tau, \ \tau < \kappa \) represent ordinals between \( \kappa \) and \( i_{U(\kappa, \theta)}(\kappa) \) in this ultrapower. Hence, they are one to one mod \( U(\kappa, \theta). \) This means that each \( U_\tau \) is equivalent to its normal measure as witnessed by \( h_\tau. \) But such ultrafilters can be easily separated.

Suppose next that \( a_\tau = a^* = \{ \kappa, \kappa_1 \}, \) for every \( \tau < \kappa \) (or on an \( E \)-measure one set). Assume that each \( j_\tau \) is the second ultrapower embedding by a normal measure \( U(\kappa, \theta) \) over \( \kappa \) in \( \mathcal{K}, \) where \( \kappa_1 \) is the image of \( \kappa \) under \( i_{U(\kappa, \theta)}(\kappa) \).

Denote \( i_{U(\kappa, \theta)} \) by \( i_1 : \mathcal{K} \rightarrow \mathcal{K}_1, \) the ultrapower embedding of \( \mathcal{K}_1 \) by \( i_1(U(\kappa, \theta)) \) by \( i_{1,2} = i_{i_1(U(\kappa, \theta))} : \mathcal{K}_1 \rightarrow \mathcal{K}_2 \) and the second ultrapower embedding (the one equal to \( j_\tau \)'s) by \( i_2 = i_{1,2} \circ i_1 : \mathcal{K} \rightarrow \mathcal{K}_2. \) Let \( \kappa_2 = i_2(\kappa). \) Then we have \( [id]_{U_\tau} = i_2(h_\tau)(\kappa_1, \kappa_1) \in [\kappa_1, \kappa_2], \) for every \( \tau < \kappa. \)

Let us deal first with different mod \( U(\kappa, \theta)^2 \) functions among \( h_\tau \)'s. So, let \( Z \subseteq \kappa \) be a set of such functions, i.e. for every \( \tau \neq \tau' \) in \( Z, \ h_\tau \neq h_{\tau'} \) mod \( U(\kappa, \theta)^2. \)

Our prime interest will be in \( \langle \text{rng}(h_\tau) \mid \tau \in Z \rangle. \) We will argue that there is a set \( C \in U(\kappa, \theta)^2 \) such that \( \langle h_\tau''|C \setminus \tau + 1]^2 \mid \tau \in Z \rangle \) is a disjoint family, which in turn will witness that the family \( \langle U_\tau \mid \tau \in Z \rangle \) is discrete.

Let \( \tau \in Z \) and \( \beta < \kappa. \) Define \( h_\beta^\tau : \beta \rightarrow \kappa \setminus \beta \) by setting \( h_\beta^\tau(\alpha) = h_\tau(\alpha, \beta). \)

Consider \( i_1((h_\beta^\tau \mid \beta < \kappa))(\kappa) : \kappa \rightarrow \kappa_1 \setminus \kappa. \) Denote it by \( h'_\beta. \)

Suppose for a moment that for some \( \tau, \tau' \in Z, \ \tau \neq \tau', \ h_\tau = h_{\tau'} \) mod \( U(\kappa, \theta). \) Then there is a set \( H \in U(\kappa, \theta) \) such that

\[ \{ \beta < \kappa \mid h_\beta^\tau \upharpoonright H \cap \beta = h_{\tau'}^\beta \upharpoonright H \cap \beta \} \subseteq U(\kappa, \theta). \]

But then

\[ H \subseteq \{ \alpha < \kappa \mid \{ \beta < \kappa \mid h_\tau(\alpha, \beta) = h_{\tau'}(\alpha, \beta) \} \subseteq U(\kappa, \theta) \} \]
we have
\[ \{ \alpha < \kappa \mid \{ \beta < \kappa \mid h_\tau(\alpha, \beta) = h_{\tau'}(\alpha, \beta) \} \in U(\kappa, \theta) \}. \]

Which is impossible.

Hence, \( \tau, \tau' \in Z, \tau \neq \tau' \) implies \( h_\tau' \neq h_{\tau'} \) mod \( U(\kappa, \theta) \).

Now, using normality of \( U(\kappa, \theta) \) and covering by a set in \( K \) of cardinality \( \kappa \), it is easy to find \( A \in U(\kappa, \theta) \) such that \( \tau, \tau' \in Z, \tau < \tau' \) implies
\[
\text{rng}(h_\tau' \upharpoonright A \setminus \tau') \cap \text{rng}(h_{\tau'} \upharpoonright A \setminus \tau') = \emptyset.
\]

This statement is true in \( K_1 \), hence by elementarity,
\[
\{ \beta < \kappa \mid \text{rng}(h_\tau \upharpoonright (A \cap \beta) \setminus \tau') \cap \text{rng}(h_{\tau'} \upharpoonright (A \cap \beta) \setminus \tau') = \emptyset \} \in U(\kappa, \theta).
\]

Fix \( \tau \in Z \). Let \( \tau' \in Z \) be different from \( \tau \). Set
\[
B_\tau'' = \{ \beta < \kappa \mid \text{rng}(h_\tau \upharpoonright (A \cap \beta) \setminus \tau') \cap \text{rng}(h_{\tau'} \upharpoonright (A \cap \beta) \setminus \tau') = \emptyset \},
\]
if \( \tau < \tau' \) and
\[
B_\tau'' = \{ \beta < \kappa \mid \text{rng}(h_\tau \upharpoonright (A \cap \beta) \setminus \tau) \cap \text{rng}(h_{\tau'} \upharpoonright (A \cap \beta) \setminus \tau) = \emptyset \},
\]
if \( \tau' < \tau \). Then \( B_\tau'' \in U(\kappa, \theta) \). The set
\[
E_\tau = \{ \beta < \kappa \mid \forall \alpha < \beta' < \beta \langle h_\tau(\alpha, \beta') < \beta \rangle \} \in U(\kappa, \theta).
\]

Set \( C_\tau = (A \setminus \tau) \cap E_\tau \cap \Delta_{\tau' \in Z, \tau' \neq \tau} B_\tau'' \). Then for every \( \alpha, \alpha', \beta \in C_\tau \) with \( \alpha, \alpha' < \beta \), \( \alpha \neq \alpha' \) we have
\[
\langle \ast \rangle h_\tau(\alpha, \beta) \neq h_{\tau'}(\alpha', \beta),
\]
once \( \tau' \in Z, \tau' \neq \tau \) and \( \tau' < \beta \).

Suppose now \( \tau, \tau' \in Z, \tau \neq \tau' \), \( (\alpha, \beta), (\alpha', \beta') \in [C_\tau] \cap [C_\tau] \). Assume for a moment that
\[
h_\tau(\alpha, \beta) = h_{\tau'}(\alpha', \beta').
\]
Note first that \( \beta = \beta' \), since \( h_\tau(\alpha, \beta) \geq \beta \), \( h_{\tau'}(\alpha', \beta') \geq \beta' \) and \( \beta, \beta' \in E_\tau \cap E_{\tau'} \). But then
\[
h_\tau(\alpha, \beta) \neq h_{\tau'}(\alpha', \beta),
\]
by the previous paragraph.

Finally let \( C = \Delta_{\tau \in Z} C_\tau \). The sequence \( \langle h_\tau''[C \setminus \tau + 1] \mid \tau \in Z \rangle \) will be as desired.

Thus let \( \tau < \tau', \tau, \tau' \in Z \) and \( (\alpha, \beta) \in [C \setminus \tau + 1], (\alpha', \beta') \in [C \setminus \tau' + 1] \). If \( \beta \leq \tau' \), then
\[ h_\tau(\alpha, \beta) < \beta' \leq h_\tau(\alpha', \beta') , \text{ since } \beta' \in C' \setminus \tau' + 1 , \text{ and so, } \beta' \in C_\tau \subseteq E_\tau . \] If \( \beta > \tau' \), then \( \beta \in C_{\tau'} \). So, \( \beta \neq \beta' \), say \( \beta > \beta' \) will imply
\[ \beta' \leq h_\tau(\alpha', \beta') < \beta \leq h_\tau(\alpha, \beta) . \]

Suppose that \( \beta = \beta' \). But \( \beta > \tau' \), hence by (*) above \( h_\tau(\alpha, \beta) \neq h_\tau(\alpha', \beta) \).

Let us deal now with ultrafilters from the sequence \( \langle U_\tau \mid \tau < \kappa \rangle \) such that the ordinals \([id]_{U_\tau}'s\) are the same and of the form \( i_2(h)(\kappa, \kappa_1) \), for some \( h : [\kappa]^2 \to \kappa, h \in \mathcal{K} \). Assume for simplicity that every \( \tau < \kappa \) is like this.

Denote \( h_\alpha U(\kappa, \theta)^2 \) by \( \mathcal{V} \). We have then that for every \( X \subseteq \kappa, X \in \mathcal{K} \),
\[ X \in \mathcal{V} \iff i_2(h)(\kappa, \kappa_1) \in i_2(X) \iff [id]_{U_\tau} \in i_2(X) \iff [id]_{U_\tau} \in i_{U_\tau}(X) \iff X \in U_\tau. \]

So, \( U_\tau \supseteq \mathcal{V} \), for every \( \tau < \kappa \).

Let \( \pi : \kappa \to \kappa, \pi \in \mathcal{K} \) be a projection of \( \mathcal{V} \) to the normal ultrafilter Rudin–Keisler below \( \mathcal{V} \), i.e. to \( U(\kappa, \theta) \). Assume that \( \mathcal{V} \) is Rudin-Keisler equivalent to \( U(\kappa, \theta)^2 \). The case \( \mathcal{V} =_{R–K} U(\kappa, \theta) \) is similar and no other possibility can occur here. So,
\[ \kappa = [\pi]_\mathcal{V} = i_2(\pi)([id]_\mathcal{V}) = i_2(\pi)(h(\kappa, \kappa_1)) = i_2(\pi)([id]_{U_\tau}), \]
for every \( \tau < \kappa \). Which means that for every \( \tau < \kappa \), \( \pi \) is a projection of \( U_\tau \) to its normal measure.

Now the conclusion follows by the following likely known lemma.

\textbf{Lemma 3.6} Let \( \langle E_\alpha \mid \alpha < \kappa \rangle \) be a family of pairwise different \( \kappa \)-complete ultrafilters over \( \kappa \) which have the same projection to their least normal measures. Then the family is discrete.

\textit{Proof.} Denote by \( \pi \) this common projection.

Let \( \alpha < \kappa \). For every \( \beta < \kappa, \beta \neq \alpha \), pick \( A^\beta_\alpha \in E_\alpha \setminus E_\beta \). Let
\[ B_\alpha = \{ \nu < \kappa \mid \pi(\nu) > \alpha \} . \]

Then \( B_\alpha \in E_\alpha \), since \( \pi_* E_\alpha \) is not principal ultrafilter. Set
\[ A_\alpha = \Delta^*_\beta < \kappa, \beta \neq \alpha A^\beta_\alpha = \{ \nu < \kappa \mid \forall \beta < \pi(\nu)(\beta \neq \alpha \to \nu \in A^\beta_\alpha) \} . \]

Then \( A_\alpha \in E_\alpha \). Let
\[ A^*_\alpha = A_\alpha \cap B_\alpha \cap \bigcap_{\beta < \alpha} (\kappa \setminus A^0_\beta) . \]
Clearly, $A_\alpha^* \in E_\alpha$.

Let us argue that the sets $\langle A_\alpha^* | \alpha < \kappa \rangle$ are pairwise disjoint. So, let $\alpha < \alpha' < \kappa$. Suppose that $\nu \in A_\alpha^* \cap A_{\alpha'}^*$. Then $\nu \in B_{\alpha'}$, and hence, $\pi(\nu) > \alpha' > \alpha$. But then, $\nu \in A_\alpha$ implies that $\nu \in A_{\alpha'}^*$, which is impossible since $\nu \in A_{\alpha'}^* \subseteq \kappa \setminus A_\alpha^*$.

\[ \Box \]

Let us turn now to the general case. So, we have for each $\tau < \kappa$, the smallest finite nice set of generators $a_\tau$ in $Gen_\tau$ and $h_\tau : [\kappa]^{\aleph_\tau} \rightarrow \kappa$, $h_\tau \in K$ such that $[id]_{U_\tau} = j_\tau(h_\tau)(a_\tau)$. Also, $i_U \upharpoonright K = j_\tau$ is an iterated ultrapower of $K$ by its measures.

If $a_\tau = a^*$ or just $a_\tau$'s are the same, for most (mod $E$) $\tau$'s, then the previous arguments apply without much changes. Suppose that this does not happen, i.e. for an $E$—measure one set of $\tau$, $a_\tau \neq a^*$. Assume that this is true for every $\tau < \kappa$ and also that $|a_\tau| = |a_{\tau'}|$, for every $\tau, \tau' < \kappa$.

Then for every $\tau < \kappa$, let $\langle \mu_{\tau,k} | k < m \rangle$ be an increasing enumeration of $Gen_\tau \cap (a_\tau \setminus a^*)$. Then $\alpha(\mu_{\tau,0}) > \mu_{\tau,0}$. By the definition of $\alpha(\mu_{\tau,0})$, we have $b_{\mu_{\tau,0}} \subseteq a^* \cap a_{\tau,0}$ and $f_{\mu_{\tau,0}} \in K$ such that

$$j_\tau(f_{\mu_{\tau,0}})(b_{\mu_{\tau,0}}) = \alpha(\mu_{\tau,0}).$$

Similar, for each $k, 0 < k < m$, $\alpha(\mu_{\tau,k}) > \mu_{\tau,k}$ and there are $b_{\mu_{\tau,k}} \subseteq a_\tau \cap \mu_{\tau,k}$ and $f_{\mu_{\tau,k}} \in K$ such that

$$j_\tau(f_{\mu_{\tau,k}})(b_{\mu_{\tau,k}}) = \alpha(\mu_{\tau,k}).$$

Note if $\mu_{\tau,k} < \mu_{\tau,k'}$ and no generator of $j_\tau$ seats in between, then $\alpha(\mu_{\tau,k}) \geq \alpha(\mu_{\tau,k'})$.

Also note that if $\delta$ is of a form $\alpha(\mu_{\tau,k})$, for some $\tau < \kappa$, then the number of generators with this $\delta$ bounded in $\kappa$, since the set $\{\delta_\eta | \eta < \kappa\}$ is bounded in $\kappa$.

Using the $\kappa$—completeness of $E$, we can assume that all $a_\tau$'s are generated in the same fashion over $a^*$ with respect to the order and number and order of applications of the $\alpha(\cdot), b_-$. Stating this more precisely the structures

$$A_\tau = \langle a_\tau, <, a^*, \alpha(-), b_-, ... \rangle$$

are isomorphic over $a^*$.

Let us deal with the following partial case, in the general one mainly the notation are more complicated.

Assume that there is a set $Z \subseteq \kappa$ of cardinality $\kappa$ such that for some $a^{**} \subseteq a^*$, for every $\tau \in Z$ there is $\mu_{\tau} \in a_\tau \setminus \max(a^{**})$ such that

1. $\alpha(\mu_{\tau}) = j_\tau(f_{\mu_{\tau}})(a^{**})$,
2. \( \mu_r \leq [id]_{U_r} < \alpha(\mu_r) \),

3. if \( \tau \neq \tau' \) are in \( Z \), then \( \alpha(\mu_r) \neq \alpha(\mu_{r'}) \).

Note that once \( \alpha(\mu_r) \) is fixed, the number of possible \( \mu_r \)'s with \( \alpha(\mu_r) = \alpha(\mu_{r'}) \) is below \( \kappa \), since \( \{ o(\xi) \mid \xi < \kappa \} \) is bounded in \( \kappa \). So the condition 3 above is not really very restrictive.

Note also that if \( \tau \neq \tau' \) are in \( Z \), then \( \mu_r < \mu_{r'} \) implies \( \alpha(\mu_r) < \alpha(\mu_{r'}) \) and \( \mu_r > \mu_{r'} \) implies \( \alpha(\mu_r) < \alpha(\mu_{r'}) \). Since \( \mu_r, \mu_{r'} \) are generators (indiscernibles) corresponding to different measurables \( \alpha(\mu_r), \alpha(\mu_{r'}) \) and this measurables depend (were generated by ) on \( a^{**} \) only.

Now we would like to use the arguments similar to the previous considered case and split not only \( \alpha(\mu_r) \)'s but rather the intervals they generate.

First note that the set 
\[
\{ \alpha(\mu_r) \mid \tau' \in Z \text{ and } \mu_r < \mu_{r'} \}
\]
is bounded below \( \mu_r \), due to the cofinality considerations. So we can pick some \( \alpha^- (\mu_r) \) of a form \( j_r(f_{\mu_r}^-(\alpha)) \) in the interval (sup\( (\{ \alpha(\mu_r) \mid \tau' \in Z \text{ and } \mu_r < \mu_{r'} \}, \alpha) \)).

Let 
\[
U = \{ X \subseteq [\kappa]^{\text{a**}} \mid X \in \mathcal{K}, a^{**} \in j_r(X) \}.
\]
Then it is a \( \kappa^- \)-complete ultrafilter over \([\kappa]^{a^{**}}\) in \( \mathcal{K} \) which is a product of finitely many normal measures over \( \kappa \).

Our aim will be to find a set \( C \subseteq [\kappa]^{a^{**}} \) in \( \mathcal{K} \) such that

1. \( a^{**} \in j_r(C) \), for all \( \tau \in Z \),

2. the intervals \( [f_{\mu_r}^-(\bar{\nu}), f_{\mu_r}(\bar{\nu})], [f_{\mu_{r'}}^-(\bar{\nu'}), f_{\mu_{r'}}(\bar{\nu'})] \) are disjoint whenever \( \tau \neq \tau' \) are in \( Z \) and \( \bar{\nu} \in C, \min(\bar{\nu}) > \tau, \bar{\nu'} \in C, \min(\bar{\nu'}) > \tau' \).

Denote max\( (a^{**}) \) by \( \beta \) and \( a^{**} \setminus \{ \beta \} \) by \( \bar{\alpha} \).

Let \( U(\kappa, \theta) \) be the last measure of \( U \), i.e. \( U = (U \mid [\kappa]^{a^{**}} - 1) \times U(\kappa, \theta) \).

Let \( \tau \in Z \) and \( \beta < \kappa \). Define \( \gamma_{\beta} : \beta \to \kappa \setminus \beta \) by setting \( \gamma_{\beta} (\bar{\alpha}) = f_{\mu_r}(\bar{\alpha}, \beta) \) and \( \gamma_{\beta}^- : \beta \to \kappa \setminus \beta \) by setting \( \gamma_{\beta}^- (\bar{\alpha}) = f_{\mu_r}^-(\bar{\alpha}, \beta) \).

Consider 
\[
i_{U(\kappa, \theta)}(\gamma_{\beta}^\beta(\beta < \kappa))(\kappa) : [\kappa]^{a^{**}} - 1 \to i_{U(\kappa, \theta)}(\kappa) \setminus \kappa.
\]
Denote it by \( g_{\tau}^\beta \). Similar let 
\[
i_{U(\kappa, \theta)}(\gamma_{\beta}^-\beta(\beta < \kappa))(\kappa) : [\kappa]^{a^{**}} - 1 \to i_{U(\kappa, \theta)}(\kappa) \setminus \kappa.
\]
Denote it by $g_{\tau}'$.

Suppose for a moment that for some $\tau, \tau' \in Z, \tau \neq \tau'$, $g_{\tau}' < g_{\tau'}' \leq g_\tau'$ mod $U \upharpoonright [\kappa]^{\alpha*+1}$.

Then there is a set $H \in U \upharpoonright [\kappa]^{\alpha*+1}$ such that for every $\alpha \in H$, the set

$$\{ \beta < \kappa \mid g_{\tau}'(\alpha) < g_{\tau'}'(\alpha) \leq g_{\tau}'(\alpha) \} \in U(\kappa, \theta).$$

But then

$$H \subseteq \{ \alpha \in [\kappa]^{\alpha*+1} \mid \{ \beta < \kappa \mid g_{\tau}'(\alpha, \beta) < g_{\tau'}'(\alpha, \beta) \leq g_{\tau}'(\alpha, \beta) \} \in U(\kappa, \theta) \}.$$

Hence,

$$\{ \alpha \in [\kappa]^{\alpha*+1} \mid \{ \beta < \kappa \mid g_{\tau}'(\alpha, \beta) < g_{\tau'}'(\alpha, \beta) \leq g_{\tau}'(\alpha, \beta) \} \in U(\kappa, \theta) \} \in U \upharpoonright [\kappa]^{\alpha*+1}.$$

Which is impossible.

Hence, $\tau, \tau' \in Z, \tau \neq \tau'$ implies $\neg(g_{\tau}' < g_{\tau'}' \leq g_\tau')$ mod $U \upharpoonright [\kappa]^{\alpha*+1}$. Which means, by switching between $\tau$ and $\tau'$ is necessary, that $g_{\tau}' < g_{\tau'}'$ mod $U \upharpoonright [\kappa]^{\alpha*+1}$ or $g_{\tau}' < g_{\tau'}'$ mod $U \upharpoonright [\kappa]^{\alpha*+1}$.

Now, using induction, normality of components of $U \upharpoonright [\kappa]^{\alpha*+1}$ and covering the set

$$\{ \{g_{\tau}', g_{\tau}'\} \mid \tau \in Z \}$$

by a set in $K$ of cardinality $\kappa$, if necessary, we can find $A \in U \upharpoonright [\kappa]^{\alpha*+1}$ such that $\tau, \tau' \in Z, \tau \neq \tau'$ implies that for every $\vec{\nu}, \vec{\nu}' \in A$ with $\min(\vec{\nu}) > \tau, \min(\vec{\nu}') > \tau'$ the intervals

$$[g_{\tau}'(\vec{\nu}), g_{\tau'}'(\vec{\nu})], [g_{\tau}'(\vec{\nu}'), g_{\tau'}'(\vec{\nu}')]$$

are disjoint.

Thus, we can assume that the functions $g_{\tau}', g_{\tau}'$ are not constant, just otherwise the set of relevant generators can be reduced to a smaller one.

Split into two cases according to the supremums of the ranges.

**Case 1. Same supremum.**

So assume for simplification of notation that for every $\tau \in Z$ the ranges of the functions $g_{\tau}', g_{\tau}'$ have the same supremum $\chi$. Then $\chi$ has cofinality $\kappa$, and let $\langle \chi, \gamma \mid \gamma < \kappa \rangle$ be a cofinal sequence.

Now we proceed similar to what was done in the beginning with $h_{\tau}$, only an induction on size of $a^{**}$ should be used.

**Case 1. Different supremums.**

Then we deal with this different supremums and split them. This will provide the desired conclusion also for $g_{\tau}', g_{\tau}'$'s.

Now, the statement that for every $\vec{\nu}, \vec{\nu}' \in A$ with $\min(\vec{\nu}) > \tau, \min(\vec{\nu}') > \tau'$ the intervals

$$[g_{\tau}'(\vec{\nu}), g_{\tau'}'(\vec{\nu})], [g_{\tau}'(\vec{\nu}'), g_{\tau'}'(\vec{\nu}')]$$

are disjoint,
is true in $K$, hence by elementarity,

$$\{ \beta < \kappa \mid \forall \bar{\nu}, \bar{\nu}' \in A \cap [\beta]^{\alpha+1}_\tau \min(\bar{\nu}) > \tau \land \min(\bar{\nu}') > \tau' \rightarrow [g^-_{\tau}(\bar{\nu}), g^\beta_\tau(\bar{\nu})] \cap [g^-_{\tau'}(\bar{\nu}'), g^\beta_{\tau'}(\bar{\nu}')] = \emptyset \} \in U(\kappa, \theta).$$

Fix $\tau \in Z$. Let $\tau' \in Z$ be different from $\tau$. Set

$$B'_\tau = \{ \beta < \kappa \mid \forall \bar{\nu}, \bar{\nu}' \in A \cap [\beta \setminus \tau']^{\alpha+1}_\tau ([g^-_{\tau}(\bar{\nu}), g^\beta_\tau(\bar{\nu})] \cap [g^-_{\tau'}(\bar{\nu}'), g^\beta_{\tau'}(\bar{\nu}')] = \emptyset) \},$$

if $\tau < \tau'$ and

$$B''_\tau = \{ \beta < \kappa \mid \forall \bar{\nu}, \bar{\nu}' \in A \cap [\beta \setminus \tau]^{\alpha+1}_\tau ([g^-_{\tau}(\bar{\nu}), g^\beta_\tau(\bar{\nu})] \cap [g^-_{\tau'}(\bar{\nu}'), g^\beta_{\tau'}(\bar{\nu}')] = \emptyset) \},$$

if $\tau' < \tau$. Then $B'_\tau \subseteq U(\kappa, \theta)$. The set

$$E_\tau = \{ \beta < \kappa \mid \forall \bar{\alpha} < \beta < \beta'(g_{\tau}(\bar{\alpha}, \beta') < \beta) \} \in U(\kappa, \theta).$$

Set $C_\tau = E_\tau \cap \Delta_{\tau' \in Z, \tau' \neq \tau} B''_\tau$. Then for every $\bar{\alpha}, \bar{\alpha}' \in (A \setminus \tau), \beta \in C_\tau$ with $\alpha, \alpha' < \beta, \alpha \neq \alpha'$ we have

$$(**)[g^-_{\tau}(\bar{\alpha}, \beta), g_{\tau}(\bar{\alpha}, \beta)] \cap [g^-_{\tau'}(\bar{\alpha}', \beta'), g_{\tau'}(\bar{\alpha}', \beta')] = \emptyset$$

once $\tau' \in Z, \tau' \neq \tau$ and $\tau' < \beta$.

Suppose now $\tau, \tau' \in Z, \tau \neq \tau', \bar{\alpha}, \bar{\alpha}' \in (A \setminus \tau) \cap (A \setminus \tau'), \beta \in C_\tau, \beta' \in C_{\tau'}$. Assume for a moment that

$$[g^-_{\tau}(\bar{\alpha}, \beta), g_{\tau}(\bar{\alpha}, \beta)] \cap [g^-_{\tau'}(\bar{\alpha}', \beta'), g_{\tau'}(\bar{\alpha}', \beta')] \neq \emptyset$$

Note first that $\beta = \beta'$, since $\beta \leq g^-_{\tau}(\bar{\alpha}, \beta) \leq g_{\tau}(\alpha, \beta), \beta' \leq g^-_{\tau'}(\bar{\alpha}', \beta') \leq g_{\tau'}(\alpha', \beta')$ and $\beta, \beta' \in E_\tau \cap E_{\tau'}$. But then

$$[g^-_{\tau}(\bar{\alpha}, \beta), g_{\tau}(\bar{\alpha}, \beta)] \cap [g^-_{\tau'}(\bar{\alpha}', \beta'), g_{\tau'}(\bar{\alpha}', \beta')] \neq \emptyset,$$

by the previous paragraph.

Finally let $\tilde{C} = \Delta_{\tau \in Z} C_\tau$ and

$$C = \{(\bar{\alpha}, \beta) \mid \bar{\alpha} \in A, \beta \in \tilde{C} \text{ and } \beta > \max(\bar{\alpha}) \}.$$

Such $C$ will be as desired. Thus let $\tau < \tau', \tau' \in Z$ and $(\bar{\alpha}, \beta) \in C \setminus \tau + 1, (\bar{\alpha}', \beta') \in C \setminus \tau' + 1.$

If $\beta \leq \tau'$, then $g_{\tau}(\alpha, \beta) < \beta' \leq g^-_{\tau'}(\alpha', \beta')$, since $(\bar{\alpha}', \beta') \in C \setminus \tau' + 1$, and so, $\beta' \in C_\tau \subseteq E_\tau$.

If $\beta > \tau'$, then $\beta \in C_{\tau'}$. So, $\beta \neq \beta'$, say $\beta > \beta'$ will imply

$$\beta' \leq g_{\tau'}(\bar{\alpha'}, \beta') < \beta \leq g^-_{\tau}(\bar{\alpha}, \beta).$$
Suppose that $\beta = \beta'$. But $\beta > \tau'$, hence by (***) above

$$[g_\tau(\vec{\alpha}, \beta), g_\tau(\vec{\alpha}, \beta)] \cap [g'_\tau(\vec{\alpha}', \beta), g'_\tau(\vec{\alpha}', \beta)] = \emptyset.$$  

□

4 Products of ultrafilters.

In [4], Aki Kanamori asked the following question (Question 5.8 there):

If $\mathcal{U}$ and $\mathcal{V}$ are $\kappa$-complete ultrafilters over $\kappa$ such that $\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{V} \times \mathcal{U}$, is there a $\mathcal{W}$ and integers $n$ and $m$ so that $\mathcal{U} \cong \mathcal{W}^n$ and $\mathcal{V} \cong \mathcal{W}^m$?

Solovay gave an affirmative answer once "$\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{V} \times \mathcal{U}$" is replaced by "$\mathcal{U} \times \mathcal{V} \cong \mathcal{V} \times \mathcal{U}$", and Kanamori once $\mathcal{U}$ is a $p$-point, see [4] 5.7, 5.9.

We would like to show that the negative answer is consistent assuming $o(\kappa) = \kappa$. Two examples will be produced. The following will be shown:

**Theorem 4.1** Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa$-complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\kappa$ such that

1. $\mathcal{V} \not\leq_{R-K} \mathcal{U}$,
2. $\mathcal{V} \times \mathcal{U} \not\leq_{R-K} \mathcal{U} \times \mathcal{V}$.

**Theorem 4.2** Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa$-complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\kappa$ such that

1. $\mathcal{V}$ is a normal measure,
2. $\mathcal{V}$ is the projection of $\mathcal{U}$ to its least normal measure,
3. $\mathcal{V} \times \mathcal{U} \not\leq_{R-K} \mathcal{U} \times \mathcal{V}$.

**Proof of the first theorem.**

Let us keep the notation of the previous section.

So, we have $\kappa$-complete ultrafilters $U(\kappa, \alpha, t), \alpha < \kappa, t \in [\kappa]^{<\omega}$ which extend $U(\kappa, \alpha)$'s. Denote $U(\kappa, \alpha, \langle \rangle)$ by $\bar{U}(\kappa, \alpha)$.

Let $f : \kappa \to \kappa$. Define

$$U_f = \{ X \subseteq \kappa \mid \{ \alpha < \kappa \mid X \in \bar{U}(\kappa, f(\alpha)) \} \in \bar{U}(\kappa, 0) \}.$$
i.e.

\[ U_f = \bar{U}(\kappa, 0) - \lim_{\alpha \prec \kappa} \bar{U}(\kappa, f(\alpha)). \]

Then \( U_f \) is a \( \kappa \)-complete ultrafilter over \( \kappa \).

It is noted in [3], that if \( f \leq g \mod \bar{U}(\kappa, 0) \), then \( U_f \leq R - K U_g \).

Our prime interest will be in \( f = id \) and \( g = id + 1 \).

Set \( U = U_{id} \) and \( V = U_{id + 1} \).

We would like to argue that \( U \times V <_{R - K} V \times U \).

Note that neither \( U \) nor \( V \) are of the form \( W^n \), for \( n > 1 \), since the only ultrafilters Rudin-Keisler below \( U \) are \( \bar{U}(\kappa, \alpha) \), \( \alpha < \kappa \) and their finite powers, those below \( V \) are \( \bar{U}(\kappa, \alpha) \), \( \alpha < \kappa \), \( U \) and their finite powers. Just examine the ultrapowers by \( U \) nor \( V \).

In particular, \( V \neq U^n, n < \omega \).

Suppose that \( B \in U \times V \). Then

\[ \{ \mu < \kappa \mid \{ \xi < \kappa \mid (\mu, \xi) \in B \} \in V \} \in U. \]

Denote

\[ A = \{ \mu < \kappa \mid \{ \xi < \kappa \mid (\mu, \xi) \in B \} \in V \} \]

and for each \( \mu < \kappa \), let

\[ A_\mu = \{ \xi < \kappa \mid (\mu, \xi) \in B \}. \]

Recall that

\[ U = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha) \mid \alpha < \kappa \rangle. \]

Hence, there is \( Z \in \bar{U}(\kappa, 0) \) such that for every \( \alpha \in Z \), \( A \in \bar{U}(\kappa, \alpha) \).

Similar,

\[ V = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha + 1) \mid \alpha < \kappa \rangle. \]

Hence, for every \( \mu \in A \), there is \( Y_\mu \in \bar{U}(\kappa, 0) \) such that for every \( \alpha \in Y_\mu \), \( A_\mu \in \bar{U}(\kappa, \alpha + 1) \).

Set

\[ X = Z \cap \Delta_{\mu \in A} Y_\mu. \]

Then \( X \in \bar{U}(\kappa, 0) \) and for every \( \alpha \in X \) we have

\[ A \in \bar{U}(\kappa, \alpha) \text{ and } \forall \mu \in A \cap \alpha (A_\mu \in \bar{U}(\kappa, \alpha + 1)). \]

Then, by elementarity, in \( M_\nu \), for every \( \alpha \in i_\nu(X) \),

\[ i_\nu(A) \in \bar{U}(i_\nu(\kappa), \alpha) \text{ and } \forall \mu \in i_\nu(A) \cap \alpha (A'_\mu \in \bar{U}(i_\nu(\kappa), \alpha + 1)), \]

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where $i_V(\langle A_\mu | \mu < \kappa \rangle) = \langle A'_\mu | \mu < i_V(\kappa) \rangle$.

Let $\rho^U$ denotes $[id]_U$. Then $\rho^U \in i_U(A)$. We have a natural embedding $\sigma : M_U \to M_V$ and it does not move $\rho^U$, since its critical point is $i_U(\kappa)$.

Then,

$$\rho^U = \sigma(\rho^U) \in \sigma(i_U(A)) = i_V(A).$$

Note that generators of $\bar{U}(\kappa, 0)$ appear unboundedly many times below $\rho_V > \rho_U$. Let $\alpha^*$ be, say, the least generator such generator above $\rho^U$.

Then $\alpha^* \in i_V(X) \setminus \rho^U + 1$. So,

$$\forall \mu \in i_V(A) \cap \alpha^*(A'_\mu \in \bar{U}(i_V(\kappa), \alpha^* + 1)).$$

Now, $\bar{U}(i_V(\kappa), \alpha^* + 1) <_{R-K} \bar{U}(i_V(\kappa), id) = i_V(U)$. Let $\eta$ represents a corresponding projection function in the ultrapower of $M_V$ by $i_V(U)$.

Then for all $\mu \in i_V(A) \cap \alpha^*$, $\eta \in i_V(U)(A'_\mu)$.

Hence,

$$\eta \in i_V(U)(A'_\mu).$$

So,

$$(\rho^U, \eta) \in i_V(U)(B).$$

We are done, since then

$$\{E \subseteq [\kappa]^2 | (\rho^U, \eta) \in i_V(U)(E)\} \supseteq \mathcal{U} \times \mathcal{V},$$

but $\mathcal{U} \times \mathcal{V}$ is an ultrafilter, so

$$\{E \subseteq [\kappa]^2 | (\rho^U, \eta) \in i_V(U)(E)\} = \mathcal{U} \times \mathcal{V},$$

which means that

$$\mathcal{U} \times \mathcal{V} <_{R-K} \mathcal{V} \times \mathcal{U}.$$
We have
\[ U = U(\kappa, 0) - \lim (\bar{U}(\kappa, \alpha) \mid \alpha < \kappa). \]
So, the ultrapower with \( U \) is obtained as follows. First \( \bar{U}(\kappa, 0) \) is applied. We have
\[ i_{\bar{U}(\kappa, 0)} : V \to M_{\bar{U}(\kappa, 0)}. \]
Next \( \bar{U}(i_{\bar{U}(\kappa, 0)})(\kappa), \kappa \) is applied over \( M_{\bar{U}(\kappa, 0)} \). We have
\[ i_{U(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)} : M_{\bar{U}(\kappa, 0)} \to M_{\bar{U}(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)}. \]
The composition is the ultrapower embedding by \( U \), i.e.
\[ i_U = i_{U(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)} \circ i_{\bar{U}(\kappa, 0)} : V \to M_U = M_{U(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)}. \]
Consider \( \bar{U}(\kappa, 0) \times U \).
So, we have \( i_{U(\kappa, 0)} : V \to M_{U(\kappa, 0)} \) followed by \( i_{U(\kappa, 0)}(U) = U(i_{U(\kappa, 0)}(\kappa), id) \). The application of \( U(i_{U(\kappa, 0)}(\kappa), id) \) to \( M_{U(\kappa, 0)} \) has the similar description to the one above.
Namely, \( i_{U(\kappa, 0)}(U(\kappa, 0)) \) is used first followed by
\[ \bar{U}(i_{i_{U(\kappa, 0)}(U(\kappa, 0))}(i_{U(\kappa, 0)}(\kappa)), i_{U(\kappa, 0)}(\kappa)). \]
In order to simplify the notation, let us denote \( i_{U(\kappa, 0)} \) by \( i_1 \), \( M_{U(\kappa, 0)} \) by \( M_1 \), \( i_{U(\kappa, 0)}(\kappa) \) by \( \kappa_1 \), the second ultrapower of \( U(\kappa, 0) \) by \( M_2 \) and the image of \( \kappa_1 \) there by \( \kappa_2 \).
Then \( i_{U(\kappa, 0) \times U} : V \to M_{U(\kappa, 0) \times U} \) is \( i_1 : V \to M_1 \) followed by \( i_{U(\kappa_1, 0)} : M_1 \to M_2 \) and then by \( i_{\bar{U}(\kappa_2, \kappa_1)} : M_2 \to M_{\bar{U}(\kappa, 0) \times U} \).
Note that in \( M_2 \), we have \( \bar{U}(\kappa_2, \kappa_1) >_{R-K} \bar{U}(\kappa_2, \kappa) \) and even \( \bar{U}(\kappa_2, \kappa_1) >_{R-K} \bar{U}(\kappa_2, \kappa) \times \bar{U}(\kappa_2, 0) \).
Pick \( (\eta, \rho) \) which represents a corresponding projection function in the ultrapower of \( M_2 \) by \( \bar{U}(\kappa_2, \kappa_1) \).
Let us argue that
\[ \{ E \subseteq [\kappa]^2 \mid (\eta, \rho) \in i_{U(\kappa, 0) \times U}(E) \} \supseteq U \times \bar{U}(\kappa, 0). \]
Let \( A \in U \), then
\[ [id]_{\bar{U}(\kappa_1, \kappa)} \in i_U(A) = i_{\bar{U}(\kappa_1, \kappa)}(i_1(A)). \]
Then, in \( M_1 \),
\[ i_1(A) \in \bar{U}(\kappa_1, \kappa). \]
Apply the second ultrapower embedding $i\bar{U}(\kappa_1,0)$ to it. Note that its critical point is $\kappa_1 > \kappa$. Then,

$$i_2(A) = i\bar{U}(\kappa_1,0)(i_1(A)) \in i\bar{U}(\kappa_1,0)(\bar{U}(\kappa_1,\kappa)) = \bar{U}(\kappa_2,\kappa).$$

Next apply $i\bar{U}(\kappa_2,\kappa_1) : M_2 \to M\bar{U}(\kappa,0)$. So, by the choice of $\eta$,

$$\eta \in i\bar{U}(\kappa,0)x\bar{U}(A) = i\bar{U}(\kappa_2,\kappa_1)(i_2(A)).$$

Suppose now that $B \in \mathcal{U} \times \bar{U}(\kappa,0)$. Set

$$A := \{ \mu < \kappa \mid \{ \xi < \kappa \mid (\mu, \xi) \in B \} \in \bar{U}(\kappa,0) \}.$$ 

Then $A \in \mathcal{U}$ and for every $\mu \in A$ the set

$$A_\mu := \{ \xi < \kappa \mid (\mu, \xi) \in B \} \in \bar{U}(\kappa,0).$$

Apply $i_2$. Then, in $M_2$,

$$\forall \mu \in i_2(A)(A_\mu \in \bar{U}(\kappa_2,0)).$$

But, by above, we have

$$i_2(A) \in \bar{U}(\kappa_2,\kappa),$$

hence,

$$i_2(B) \in \bar{U}(\kappa_2,\kappa) \times \bar{U}(\kappa_2,0).$$

So,

$$(\eta, \rho) \in i\bar{U}(\kappa,0)x\bar{U}(B),$$

and we are done.

□

Let us address now the strength issue.

**Theorem 4.3** Suppose that there is no inner model in which $\kappa$ is a measurable with

$\{ o(\alpha) \mid \alpha < \kappa \}$ unbounded in it. Then for any two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\kappa$, if $\mathcal{V} \times \mathcal{U} \geq_{R-K} \mathcal{U} \times \mathcal{V}$, then there is an integer $n$ such that $\mathcal{V} =_{R-K} \mathcal{U}^n$.

**Proof.** Suppose that there is no inner model in which $\kappa$ is a measurable with

$\{ o(\alpha) \mid \alpha < \kappa \}$ unbounded in it. Then the separation holds and there are no $\kappa$ non-Rudin-Keisler equivalent ultrafilters which are Rudin-Keisler below some $\kappa-$complete ultrafilter.
Let $\mathcal{U}$ and $\mathcal{V}$ be two $\kappa$–complete ultrafilters over $\kappa$ and $\mathcal{V} \times \mathcal{U} \geq _{R-K} \mathcal{U} \times \mathcal{V}$.

Let $(\rho, \eta) \in [i_{\mathcal{U} \times \mathcal{U}}(\kappa)]^2$ generates $\mathcal{U} \times \mathcal{V}$, i.e.

$$\mathcal{U} \times \mathcal{V} = \{ X \subseteq [\kappa]^2 \mid (\rho, \eta) \in i_{\mathcal{U} \times \mathcal{U}}(X) \}.$$  

Clearly, then $\eta > i_{\mathcal{V}}(\kappa)$. Consider in $M_\mathcal{V}$ an ultrafilter $\mathcal{W}$ defined by $\eta$, i.e.

$$\mathcal{W} := \{ Z \subseteq i_{\mathcal{V}}(\kappa) \mid \eta \in i_{\mathcal{V}}(Z) \}.$$  

Clearly, $\mathcal{W} \leq _{R-K} i_{\mathcal{V}}(\mathcal{U})$. Find a sequence of ultrafilters $\langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle$ which represents $\mathcal{W}$ in the ultrapower by $\mathcal{V}$, i.e.

$$i_{\mathcal{V}}(\langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle)([id]_\mathcal{V}) = \mathcal{W}.$$  

So, for most (mod $\mathcal{V}$) $\alpha$'s, $\mathcal{W}_\alpha \leq _{R-K} \mathcal{U}$.

Note that $\mathcal{V} = \mathcal{V} - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle$.

Namely,

$$X \in \mathcal{V} \Leftrightarrow \eta \in i_{\mathcal{U} \times \mathcal{U}}(X) \Leftrightarrow i_{\mathcal{V}}(X) \in \mathcal{W}$$

$$\Leftrightarrow \{ \alpha < \kappa \mid X \in \mathcal{W}_\alpha \} \in \mathcal{V} \Leftrightarrow X \in \mathcal{V} - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle.$$  

The sequence $\langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle$ may contain same ultrafilters, but among them must be $\kappa$ different. Just otherwise, mod $\mathcal{V}$ they will be the same. Let $\mathcal{W}'$ be this ultrafilter. Then, $\mathcal{V} = \mathcal{V} - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle$, implies $\mathcal{V} = \mathcal{W}'$. So, $\mathcal{V} \leq _{R-K} \mathcal{U}$.

Now, if $\rho < i_{\mathcal{V}}(\kappa)$, then $\mathcal{U} \leq _{R-K} \mathcal{V}$. Hence, $\mathcal{U} = _{R-K} \mathcal{V}$, which is impossible.

Assume for a while that $\rho < i_{\mathcal{V}}(\kappa)$.

Still among this different $\mathcal{W}_\alpha$'s may be many which are Rudin-Keisler equivalent.

If the number of the equivalence classes has cardinality $\kappa$ then we are done. Suppose otherwise. Then there is $\mathcal{W}'$ such that $\mathcal{W}_\alpha = _{R-K} \mathcal{W}'$, for almost every $\alpha$ mod $\mathcal{V}$.

Set $\alpha \sim \beta$ iff $\mathcal{W}_\alpha = \mathcal{W}_\beta$. Let $t : \kappa \rightarrow \kappa$ be a function which picks exactly one ultrafilter in such equivalence classes.

Set $\mathcal{V}' = t_\ast \mathcal{V}$. Then

$$\mathcal{V} = \mathcal{V}' - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa \rangle.$$  

Now, using the separation property, the ultrapower by $\mathcal{V}$ is the ultrapower by $\mathcal{V}'$ followed by $\mathcal{W}_{[id]_{\mathcal{V}'}}$.

But $\mathcal{W}_{[id]_{\mathcal{V}'}} = _{R-K} i_{\mathcal{V}'}(\mathcal{W}')$, so its ultrapower is the same as those by $i_{\mathcal{V}'}(\mathcal{W}')$. This means
that the iterated ultrapower is just $V' \times W'$.

So, $V' \times W' =_{R-K} V$.

Then

$$V \leq_{R-K} V' \times U \text{ and } V' <_{R-K} V.$$  

Following Kanamori [4],5.9, we would like to argue that $U \times V' \leq_{R-K} V'$ and then to apply induction to

$$U \times V' \leq_{R-K} V' \times U.$$  

I.e. there will be $n < \omega$ such that $V' =_{R-K} U^n$, and then

$$U \times V' \leq_{R-K} V \leq_{R-K} V' \times U$$

will imply that $V =_{R-K} U^{n+1}$. Denote $[t]_V$ by $\rho'$. By Kanamori [4],5.4, it is enough to show that for any not constant mod $V$ function $g : \kappa \to \kappa$,

$$\rho < i_{V \times U}(g)(\rho').$$

Also, Kanamori [4],5.4, we know that for any not constant mod $V$ function $g : \kappa \to \kappa$,

$$\rho < i_{V \times U}(g)(\eta).$$

So it will be enough to show that there is $s : \kappa \to \kappa$ such that

$$\rho' = i_{V \times U}(s)(\eta).$$

Define such $s$ by using the separation property $W_\alpha$’s relatively to $V'$.

Thus let

$$\langle A_\alpha \mid \alpha \in B \rangle$$

be a disjoint family of sets, $B \in V'$ such that each $A_\alpha \in W_\alpha$. Consider

$$\langle A'_\alpha \mid \alpha \in i_{V \times U}(B) \rangle = i_{V \times U}(\langle A_\alpha \mid \alpha \in B \rangle).$$

Then $\eta \in A'_\rho$, since $\eta$ generates $W_{\rho'}$ in $M_V$.

So, define $s : \kappa \to \kappa$ by setting

$$s(\mu) = \min(\{\alpha \mid \mu \in A_\alpha\}).$$

Suppose now that $\rho \geq i_V(\kappa)$. Then, as above, replacing $\eta$ by $(\rho, \eta)$, we will have in $M_V$ an ultrafilter $W$ defined by $(\rho, \eta)$, i.e.

$$W := \{Z \subseteq [i_V(\kappa)]^2 \mid (\rho, \eta) \in i_{V \times U}(Z)\}.$$
Clearly, $W \leq_{R-K} i_V(U)$. Find a sequence of ultrafilters $\langle W_\alpha \mid \alpha < \kappa \rangle$ which represents $W$ in the ultrapower by $V$, i.e.

$$i_V(\langle W_\alpha \mid \alpha < \kappa \rangle)([id]_V) = W.$$ 

So, for most (mod $V$) $\alpha$’s, $W_\alpha \leq_{R-K} U$.

Note that

$$U \times V = V - \lim (W_\alpha \mid \alpha < \kappa).$$

Namely,

$$X \in U \times V \iff (\rho, \eta) \in i_V \times U(X) \iff \exists \alpha < \kappa \ X \in W_\alpha \iff X \in V - \lim (W_\alpha \mid \alpha < \kappa).$$

The sequence $\langle W_\alpha \mid \alpha < \kappa \rangle$ may contain same ultrafilters, but among them must be $\kappa$ different. Just otherwise, mod $V$ they will be the same. Let $W'$ be this ultrafilter. Then, $U \times V = V - \lim (W_\alpha \mid \alpha < \kappa)$, implies $U \times V = W'$. So, $U \times V \leq_{R-K} U$, which is impossible.

Still among this different $W_\alpha$’s may be many which are Rudin-Keisler equivalent.

If the number of the equivalence classes has cardinality $\kappa$ then we are done. Suppose otherwise. Then there is $W'$ such that $W_\alpha =_{R-K} W'$, for almost every $\alpha$ mod $V$.

Set $\alpha \sim \beta$ iff $W_\alpha = W_\beta$. Let $t : \kappa \to \kappa$ be a function which picks exactly one ultrafilter in such equivalence classes.

Set $V' = t_* V$. Then

$$U \times V = V' - \lim (W_\alpha \mid \alpha < \kappa).$$

Now, using the separation property, the ultrapower by $U \times V$ is the ultrapower by $V'$ followed by $W'_{[id]_{V'}}$. But $W'_{[id]_{V'}} =_{R-K} i_V(W')$, so its ultrapower is the same as those by $i_V(W')$. This means that the iterated ultrapower is just $V' \times W'$.

So, $V' \times W' =_{R-K} U \times V$. Then by Kanamori [4] (5.6), at least one of the following three possibilities must holds:

1. $W' =_{R-K} V$ and $V' =_{R-K} U$;

2. there is a $\kappa$--complete ultrafilter $F$, such that $V' =_{R-K} U \times F$ and $V =_{R-K} F \times W'$;

3. there is a $\kappa$--complete ultrafilter $G$ such that $U =_{R-K} V' \times G$ and $W' =_{R-K} G \times V$.

Suppose for a moment that the first possibility occurs. Then

$$U \geq_{R-K} W' =_{R-K} V \geq_{R-K} V' =_{R-K} U.$$
So, $\mathcal{U} =_{R-K} \mathcal{V}$, and then $\mathcal{U} \times \mathcal{V} =_{R-K} \mathcal{V} \times \mathcal{U}$, which is impossible.

Suppose now that the second possibility occurs. Then $\mathcal{V} \geq_{R-K} \mathcal{V}'$ and $W' \leq_{R-K} U$ imply

$$\mathcal{U} \times F \leq_{R-K} F \times W' \leq_{R-K} F \times \mathcal{U}.$$ 

But, also (2) implies that $\mathcal{V} >_{R-K} F$. So, we can apply the induction to

$$\mathcal{U} \times F \leq_{R-K} F \times \mathcal{U}.$$ 

Consider now the third possibility. Then $\mathcal{U} \geq_{R-K} W'$ and $\mathcal{V} \geq_{R-K} \mathcal{V}'$ imply

$$\mathcal{V} \times G \geq_{R-K} \mathcal{V}' \times G \geq_{R-K} G \times \mathcal{V}.$$ 

But, also (3) implies that $\mathcal{U} >_{R-K} G$. So, we can apply the induction to

$$\mathcal{V} \times G \geq_{R-K} G \times \mathcal{V}.$$ 

□
References


[5] W. Mitchell, Core model for sequences of measures,