Some constructions of ultrafilters over a measurable cardinal.

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Abstract

Some non-normal $\kappa$–complete ultrafilters over a measurable $\kappa$ with special properties are constructed. Questions by A. Kanamori [4] about infinite Rudin-Frolik sequences, discreteness and products are answered.

1 Introduction.

We present here several constructions of $\kappa$–complete ultrafilters over a measurable cardinal $\kappa$ and examine their consistency strength. Some questions of Aki Kanamori from [4] are answered.

Let us state some basic definitions, we refer to a basic paper by A. Kanamori [4] for a comprehensive account.

Definition 1.1 The Rudin-Keisler ordering ($\leq_{R-K}$) on ultrafilters is defined as follows: If $U$ is an ultrafilter on a set $I$ and $V$ an ultrafilter on a set $J$, $V \leq_{R-K} U$ iff there is a function $f : I \to J$ so that $V = f_*(U)$, where

$$f_*(U) = \{X \subseteq J | f^{-1}(X) \in U\}.$$  

Let $U =_{R-K} V$ iff both $V \leq_{R-K} U$ and $U \leq_{R-K} V$; in this case, $U$ is said to be isomorphic to $V$. Finally, let $V <_{R-K} U$ iff $V \leq_{R-K} U$ and $V \neq U$.

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If $U = R - K \mathcal{V}$, then a function $f$ above can be picked one to one on a set in $U$.

The order $\leq_{R-K}$ is well-founded on $\kappa-$complete ultrafilters and isomorphic ultrafilters have the same ultrapower.

**Definition 1.2** Let $D$ be an ultrafilter over a set $I$ and $E_i$ be an ultrafilter over a set $J$, for every $i \in I$.

(i) The $D-$limit of $\langle E_i \mid i \in I \rangle$ is the ultrafilter $D - \lim \langle E_i \mid i \in I \rangle$ over $J$ defined by

$$X \in D - \lim \langle E_i \mid i \in I \rangle \text{ iff } \{i \in I \mid X \in E_i\} \in D.$$ 

(ii) The $D-$sum of $\langle E_i \mid i \in I \rangle$ is the ultrafilter $D - \Sigma \langle E_i \mid i \in I \rangle$ over $I \times J$ defined by

$$X \in D - \Sigma \langle E_i \mid i \in I \rangle \text{ iff } \{i \in I \mid \{j \in J \mid \langle i,j \rangle \in X\} \in E_i\} \in D.$$ 

**Definition 1.3** A family of ultrafilters $\langle E_i \mid i \in I \rangle$ is called a discrete family iff there is a disjoint family $\langle X_i \mid i \in I \rangle$ such that $X_i \in E_i$, for every $i \in I$.

**Definition 1.4** The Rudin-Frolik ordering ($\leq_{R-F}$) on ultrafilters is defined as follows:

Let $D$ be an ultrafilters over a set $I$. $U \geq_{R-F} D$ iff for some $J$ and a discrete family $\{E_i \mid i \in I\}$ of ultrafilters over $J$, $U = D - \lim \{E_i \mid i \in I\}$.

$\leq_{R-F}$ is a sub-ordering of the Rudin-Keisler ordering.

**Definition 1.5** The Mitchell ordering ($\ll$) on $\kappa-$complete ultrafilters is defined as follows:

Let $\mathcal{U}, \mathcal{V}$ be two $\kappa-$complete ultrafilters over a measurable cardinal $\kappa$. $\mathcal{U} \ll \mathcal{V}$ iff $\mathcal{U}$ belongs to the ultrapower by $\mathcal{V}$.

The Mitchell order on $\kappa-$complete ultrafilters over $\kappa$ is well-founded and inside the Mitchell Core Model $\mathcal{K}$ ([5], [6]) it is a well-order.

In [1], [3] forcing notions which allow to turn the Mitchell order into the Rudin-Keisler order were introduced.

We assume some familiarity with basics of $\mathcal{K}$ and this forcing notions.

The structure of the paper is as follows:

Section 2 deals with Rudin-Frolik ordering and answers Question 5.11 from [4] about infinite increasing Rudin-Frolik sequences. In Section 3, an example of non-discrete family of ultrafilters is constructed, answering Question 5.12 from [4]. Also the strength of existence of such family is examined. Section 4 deals with products of ultrafilters. A negative answer to Question 5.8 from [4] given.
2 On Rudin-Frolik increasing sequences.

In [4], Aki Kanamori asked if there exists a $\kappa$–ultrafilter with an infinite number of Rudin-Frolik predecessors.

We show that starting with $o(\kappa) = 2$ it is possible.

Assume GCH. Let

$$\vec{U} = \langle U(\alpha, \beta) \mid (\alpha, \beta) \in \text{dom}(\vec{U}), \alpha \leq \kappa, \beta < \omega(\alpha) \rangle$$

be a coherent sequence such that $o(\vec{U}) = 2$ and for every $\alpha < \kappa$, $o(\vec{U}) \leq 1$. Let

$$A = \{ \alpha \mid \exists \beta(\alpha, \beta) \in \text{dom}(\vec{U}) \}.$$ 

Then for every $\alpha \in A$, $o(\vec{U}) = 1$ and $U(\alpha, 0)$ is a normal ultrafilter over $\alpha$.

We force with Easton support iteration of the Prikry forcings with $U(\alpha, 0)$'s (and their extensions), $\alpha \in A$, as in [1] (a better presentation appears in [2]). Let $G$ be a generic.

Then for every increasing sequence $t$ of ordinals less than $\kappa$, the normal ultrafilter $U(\kappa, 1)$ of $V$ extends to a $\kappa$–complete ultrafilter $U(\kappa, 1, t)$ in $V[G]$, see [1], p.291.

Denote by $b_\alpha$ the Prikry sequence from $G$ added to $\alpha$, for every $\alpha \in A$. Then $U(\kappa, 1, t)$ concentrates on $\alpha \in A$ for which $b_\alpha$ starts from $t$, i.e. $b_\alpha \upharpoonright |t| = t$.

Let $\vec{U}(\kappa, 0)$ be the canonical extension of $U(\kappa, 0)$ to a normal ultrafilter in $V[G]$ defined as in [2] on page 290.

Denote $U(\kappa, 1, \langle \rangle)$ by $\vec{U}(\kappa, 1)$.

**Lemma 2.1** For every $n, 0 < n < \omega$, $\vec{U}(\kappa, 1) = \vec{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$.

**Proof.** Recall the definition of $U(\kappa, 1, t)$, $t \in [\kappa]^m, m < \omega$:

$X \in U(\kappa, 1, t)$ iff for some $r \in G, \gamma < \kappa^+, B \in \vec{U}(\kappa, 0)$, in $M_{U(\kappa, 1)}$ the following holds:

$$r \cup \{ \langle t, B \rangle \} \cup p_\gamma \forces \kappa \in i_{U(\kappa, 1)}(X),$$

where $p_\gamma$ is the $\gamma$–th element of the canonical master sequence.

In particular, $X \in \vec{U}(\kappa, 1)$ iff for some $r \in G, \gamma < \kappa^+, B \in \vec{U}(\kappa, 0)$, in $M_{U(\kappa, 1)}$ the following holds:

$$r \cup \{ \langle \langle \rangle, B \rangle \} \cup p_\gamma \forces \kappa \in i_{U(\kappa, 1)}(X).$$

Then, for every $t \in [B]^n$, we will have

$$r \cup \{ \langle t, B \setminus \text{max}(t) + 1 \} \cup p_\gamma \forces \kappa \in i_{U(\kappa, 1)}(X).$$
So, $X \in U(\kappa, 1, t)$. But $[B]^n \in \bar{U}(\kappa, 0)^n$, hence $X \in \bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$.

Hence we showed that $\bar{U}(\kappa, 1) \subseteq \bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$. But this already implies the equality, since both $\bar{U}(\kappa, 1)$ and $\bar{U}(\kappa, 0)^n - \lim \langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$ are ultrafilters.

□

**Lemma 2.2** The family $\langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$ is a discrete family of ultrafilters.

**Proof.** For each $t \in [\kappa]^n$ set

$$A_t := \{ \alpha \in A \mid b_\alpha \upharpoonright n = t \}.$$

Let $t, t' \in [\kappa]^n$ be two different sequences, then, clearly, $A_t \cap A_{t'} = \emptyset$.

□

Recall the following definition:

**Definition 2.3** The Rudin-Frolik ordering (≤\text{RF}) on ultrafilters is defined as follows:
Let $D$ be an ultrafilters over a set $I$. $U \geq \text{RF} D$ iff for some $J$ and a discrete family $\{ E_i \mid i \in I \}$ of ultrafilters over $J$, $U = D - \lim \{ E_i \mid i \in I \}$.

So we obtain the following:

**Theorem 2.4** $\bar{U}(\kappa, 1)$ has infinitely many predecessors in the Rudin-Frolik ordering.

**Proof.** For every $n, 0 < n < \omega$, use a bijection between $[\kappa]^n$ and $\kappa$ and transfer $\bar{U}(\kappa, 0)^n$ to $\kappa$. The rest follows by Lemmas 2.1, 2.2.

□

Note that for $\kappa$–complete ultrafilters $U$ and $D$ over $\kappa$, $U \geq \text{RF} D$ implies $U \geq \text{K} D$.

So, by [5], the existence of a $\kappa$–complete ultrafilter over $\kappa$ with infinitely many predecessors in the Rudin-Frolik ordering implies by Kanamori [4], that $0^+$ exists. Let us improve this in order to give the exact strength.

**Theorem 2.5** The existence of a $\kappa$–complete ultrafilter over $\kappa$ with infinitely many predecessors in the Rudin-Frolik ordering implies that $\mathcal{O}(\kappa) \geq 2$ in the core model$^1$.

**Proof.** Note first that for $\kappa$–complete ultrafilters $U$ and $D$ over $\kappa$, $U \geq \text{RF} D$ implies $U \geq \text{K} D$. So, by [5], the existence of a $\kappa$–complete ultrafilter over $\kappa$ with infinitely many predecessors in the Rudin-Frolik ordering implies that $\exists \lambda \mathcal{O}(\lambda) \geq 2$. Let us argue that actually $\mathcal{O}(\kappa) \geq 2$ in the core model.

$^1$By the core model we mean the Mitchell core model for sequences of measures [5].
Suppose otherwise. So, $o(\kappa) = 1$. Let $U(\kappa, 0)$ be the unique normal measure over $\kappa$ in the core model $\mathcal{K}$.

Suppose that, in $V$, we have a $\kappa$–complete ultrafilter $E$ over $\kappa$ with infinitely many predecessors in the Rudin-Frolik ordering. Let $\langle E_n \mid n < \omega \rangle$ be a Rudin-Frolik increasing sequence of predecessors of $E$. Recall that by M.E. Rudin (see [4], 5.5) the predecessors of $E$ are linearly ordered by $\leq_{R-F}$ and also, $\leq_{R-F}$ is well founded on $\kappa$–complete ultrafilters.

Consider $i := i_E \upharpoonright \mathcal{K}$. Then, by [5], it is an iterated ultrapower of $\mathcal{K}$ by its measures. The critical point of $i_E$ is $\kappa$, hence $U(\kappa, 0)$ is applied first. Note that $U(\kappa, 0)$ (and its images) can be applied only finitely many times, since $\mathcal{M}$ is closed under countable (and even $\kappa$) sequences of its elements. Denote by $k^*$ the number of such applications.

Let $n \leq \omega$. Similar, consider $i_n := i_{E_n} \upharpoonright \mathcal{K}$. Again, the critical point of $i_{E_n}$ is $\kappa$, hence $U(\kappa, 0)$ is applied first. The number of applications of $U(\kappa, 0)$ (and its images) is finite. Denote by $k_n$ the number of such applications.

Now let $n < m < \omega$. We have $E_n <_{R-F} E_m$. Hence, there is a discrete sequence $\langle E_{n\alpha} \mid \alpha < \kappa \rangle$ of ultrafilters over $\kappa$ such that

$$E_m = E_n - \lim \langle E_{n\alpha} \mid \alpha < \kappa \rangle.$$

Then the ultrapower $M_{E_m}$ of $V$ by $E_m$ is $\text{Ult}(M_{E_n}, E'_{nm}[id]_{E_n})$, where $E'_{nm}[id]_{E_n} = i_{E_n}(\langle E_{n\alpha} \mid \alpha < \kappa \rangle)([id]_{E_n})$ is an ultrafilter over $i_{E_n}(\kappa)$.

Now, in $i_n(\mathcal{K})$, the only normal ultrafilter over $i_{E_n}(\kappa) = i_n(\kappa)$ is $i_n(U(\kappa, 0))$. But this means that $i_{E_m}$ is obtained by more applications of $U(\kappa, 0)$ than $i_{E_n}$, i.e. $k_n < k_m$.

Similar, $k^* > k_n$, for every $n < \omega$. This means, in particular, that $k^* \geq \omega$, which is impossible. Contradiction.

□

Remark 2.6 Note that the situation with Rudin-Keisler order is different in this respect. Thus, by [3], starting with a measurable $\kappa$ with $\{o(\alpha) \mid \alpha < \kappa\}$ unbounded in it, it is possible to construct a model with an increasing Rudin-Keisler sequence of the length $\kappa^+$.

A similar arguments can be used to produce long increasing Rudin-Frolik sequences.

Let us show how to get a sequence of the length $\kappa + 1^2$.

Assume GCH. Let

$$\bar{U} = \langle U(\alpha, \beta) \mid (\alpha, \beta) \in \text{dom}(\bar{U}), \alpha \leq \kappa, \beta < o(\alpha) \rangle.$$

\footnote{Theorem 5.10 of [4] states that this is impossible, however we think that there is a problem in the argument. Namely, on page 346, line 7 - sets depend on $\beta$'s; this effects the further definition of a function $f$ (line 16). Its unclear how to insure $f(\xi) > f(\xi')$ for most $\xi$'s, and, so $f$ may be constant mod $D_0$.}
be a coherent sequence such that $o^U(\kappa) = \kappa + 1$ and for every $\alpha < \kappa$, $o^U(\alpha) \leq \kappa$. Let

$$A = \{ \alpha \mid \exists \beta(\alpha, \beta) \in \text{dom}(U) \}.$$ 

Then for every $\alpha \in A$, $o^U(\alpha) \leq \kappa$.

We force with Easton support iteration of the Prikry type forcings with extensions of $\langle U(\alpha, \beta) \mid \beta < o^U(\alpha) \rangle$’s, $\alpha \in A$, as in [1]. Let $G$ be a generic. Then, for every $\alpha \in A$ with $o^U(\alpha) = 1$ or being a regular uncountable cardinal, Prikry sequence or Magidor sequence of order type $o^U(\alpha)$ is added by $G$ (more sequences are added, see [1] for detailed descriptions, but be do not need them here). Denote such sequences by $b_\alpha$.

Let $\bar{U}(\kappa, 0)$ be the canonical extension of $U(\kappa, 0)$ to a normal ultrafilter in $V[G]$ defined as in [2].

Denote by $A'$ the subset of $A$ which consists of $\alpha$’s with $o^U(\alpha) = 1$ or being a regular uncountable cardinal.

For every $\delta, \alpha \in A' \cup \{ \kappa \}$, $\delta < \alpha$ we will use an extensions $U(\kappa, \alpha, \langle \rangle)$ and $U(\kappa, \alpha, \langle \delta \rangle)$ of $U(\kappa, \alpha)$. They were defined in [1] as follows:

$$X \in U(\kappa, \alpha, \langle \rangle) \iff \text{for some } r \in G, \gamma < \kappa^+ \text{ and a tree } T, \text{ in } M_{U(\kappa, \alpha)} \text{ the following holds:}$$

$$r \cup \{ \langle \langle \rangle, T \rangle \} \cup p_\gamma \models \kappa \in i_{U(\kappa, \alpha)}(X),$$

where $p_\gamma$ is the $\gamma$–th element of the canonical master sequence.

$$X \in U(\kappa, \alpha, \langle \delta \rangle) \iff \text{for some } r \in G, \gamma < \kappa^+ \text{ and a tree } T, \text{ in } M_{U(\kappa, \alpha)} \text{ the following holds:}$$

$$r \cup \{ \langle \langle \delta \rangle, T \rangle \} \cup p_\gamma \models \kappa \in i_{U(\kappa, \alpha)}(X),$$

where $p_\gamma$ is the $\gamma$–th element of the canonical master sequence.

Denote further $U(\kappa, \alpha, \langle \rangle)$ by $\bar{U}(\kappa, \alpha)$.

Notice that $U(\kappa, \alpha, \langle \delta \rangle)$ concentrates on $\nu$’s with $o^U(\nu) = \alpha$, $\delta \in b_\nu$ and $b_\nu \cap \delta = b_\delta$.

We have now the following analog of 2.1:

**Lemma 2.7** For every $\alpha \in A'$, $\bar{U}(\kappa, \kappa) = \bar{U}(\kappa, \alpha) - \lim \langle U(\kappa, \kappa, \langle \nu \rangle) \mid o^U(\nu) = \alpha \rangle$.

**Proof.** $X \in \bar{U}(\kappa, \kappa)$ iff for some $r \in G, \gamma < \kappa^+, T$, in $M_{U(\kappa, \kappa)}$ the following holds:

$$r \cup \{ \langle \langle \nu \rangle, T(\nu) \} \cup p_\gamma \models \kappa \in i_{U(\kappa, \kappa)}(X).$$

Recall that $T$ is a tree consisting of coherent sequences, as defined in [1], and $\text{Suc}_T(\langle \rangle) \in \bar{U}(\kappa, \alpha)$. Then, for every $\nu \in \text{Suc}_T(\langle \rangle)$ with $o^U(\nu) = \alpha$, we will have

$$r \cup \{ \langle \langle \nu \rangle, T(\nu) \} \cup p_\gamma \models \kappa \in i_{U(\kappa, \kappa)}(X).$$
So, \( X \in U(\kappa, \kappa, \langle \nu \rangle) \). But this holds for \( \dot{U}(\kappa, \alpha) \)–measure one many \( \nu \)'s, hence \( X \in \dot{U}(\kappa, \alpha) \)–\lim \langle U(\kappa, \kappa, \langle \nu \rangle) | o^\dot{U}(\nu) = \alpha \rangle.

Hence we showed that \( \dot{U}(\kappa, \kappa) \subseteq \dot{U}(\kappa, \alpha) \)–\lim \langle U(\kappa, \kappa, \langle \nu \rangle) | o^\dot{U}(\nu) = \alpha \rangle \). But this already implies the equality, since both \( \dot{U}(\kappa, \kappa) \) and \( \dot{U}(\kappa, \alpha) \)–\lim \langle U(\kappa, \kappa, \langle \nu \rangle) | o^\dot{U}(\nu) = \alpha \rangle \) are ultrafilters.

□

The same argument shows the following:

**Lemma 2.8** For every \( \gamma, \alpha \in A', \alpha < \gamma \), \( \dot{U}(\kappa, \gamma) = \dot{U}(\kappa, \alpha) \)–\lim \langle U(\kappa, \gamma, \langle \nu \rangle) | o^\dot{U}(\nu) = \alpha \rangle.

**Lemma 2.9** The family \( \langle U(\kappa, \gamma, \langle \nu \rangle) | o^\dot{U}(\nu) = \alpha \rangle \) is a discrete family of ultrafilters, for every \( \gamma, \alpha \in A' \cup \{ \kappa \}, \alpha < \gamma \).

**Proof.** Fix \( \gamma, \alpha \in A' \cup \{ \kappa \}, \alpha < \gamma \). For each \( \nu \) with \( o^\dot{U}(\nu) = \alpha \) set

\[
A_\nu := \{ \xi \in A' | o^\dot{U}(\xi) = \gamma, \nu \in b_\gamma \text{ and } b_\gamma \cap \nu = b_\nu \}.
\]

Let \( \nu, \nu' \in A' \) be two different elements with \( o^\dot{U}(\nu) = o^\dot{U}(\nu') = \alpha \), then, clearly, \( A_\nu \cap A_{\nu'} = \emptyset \).

□

So, again as above, we obtain the following:

**Theorem 2.10** \( \dot{U}(\kappa, \kappa) \) has \( \kappa \)–many predecessors in the Rudin-Frolik ordering.

**Proof.** By Lemmas 2.7, 2.8, the sequence \( \langle \dot{U}(\kappa, \gamma) | \gamma \in A' \cup \{ \kappa \} \rangle \) is R-F-increasing.

□

It follows now that:

**Corollary 2.11** The consistency strength of existence of a \( \kappa \)–complete ultrafilter over \( \kappa \) with \( \kappa \)–many predecessors in the Rudin-Frolik ordering is at least \( \{ o(\alpha) | \alpha < \kappa \} \) is unbounded in \( \kappa \) and at most \( o(\kappa) = \kappa + 1 \).

3 Discrete families of ultrafilters.

Aki Kanamori asked in [4] the following natural question (Question 5.12):

If \( \{ U_\tau | \tau < \kappa \} \) is a family of distinct \( \kappa \)–complete ultrafilters over \( \kappa \) and \( E \) is any \( \kappa \)–complete ultrafilter over \( \kappa \), is there an \( X \in E \) so that \( \{ U_\tau | \tau \in X \} \) is a discrete family?

We will give a negative answer to this question below.
Let us use the previous construction. We preserve all the notation made there.

Consider the family
\[ \{ U(\kappa, \kappa, \langle \delta \rangle) \mid \delta \in A', \delta < \kappa \}. \]

**Lemma 3.1** The family \( \{ U(\kappa, \kappa, \langle \delta \rangle) \mid \delta \in A', \delta < \kappa \} \) consists of different ultrafilters.

**Proof.** Let \( U(\kappa, \kappa, \langle \delta \rangle), U(\kappa, \kappa, \langle \delta' \rangle) \) be two different members of the family. If \( o^U(\delta) = o^U(\delta') \), then they are different by Lemma 2.9. Suppose that \( o^U(\delta) < o^U(\delta') \). Then the set
\[ \{ \nu < \kappa \mid o^U(\nu) = \nu, \delta \in b_\nu, b_\nu \cap \delta = b_\delta \text{ and } \delta' \notin b_\nu \} \in U(\kappa, \kappa, \langle \delta \rangle) \setminus U(\kappa, \kappa, \langle \delta' \rangle). \]

So we are done.

\( \square \)

Pick now a \( \kappa \)-complete (non-principal) ultrafilter \( D \) such that the set
\[ Z := \{ \alpha < \kappa \mid \alpha \text{ is a regular uncountable cardinal } \} \in D. \]

Define now a \( \kappa \)-complete ultrafilter \( E \) over \([\kappa]^2\) as follows:
\[ X \in E \text{ iff } \{ \alpha \in Z \mid \{ \delta < \kappa \mid (\alpha, \delta) \in X \} \in U(\kappa, \alpha, \langle \rangle) \} \in D. \]

I.e. \( E = D - \lim_{\alpha} U(\kappa, \alpha, \langle \rangle) \). We can assume that if \( (\alpha, \delta) \in X \), for a set \( X \in E \), then \( o^U(\delta) = \alpha \), since \( U(\kappa, \alpha) \) concentrates on such \( \delta \)'s.

Now, for every pair \( (\alpha, \delta) \) with \( o^U(\delta) = \alpha \), define \( U_{(\alpha, \delta)} = U(\kappa, \kappa, \langle \delta \rangle) \).

**Lemma 3.2** For every \( X \in E \), the family \( \{ U_\tau \mid \tau \in X \} \) is not discrete.

**Proof.** Let \( X \in E \). Suppose that there is a separating sequence \( \langle Y_{(\alpha, \delta)} \mid (\alpha, \delta) \in X \rangle \) for \( \langle U_{(\alpha, \delta)} \mid (\alpha, \delta) \in X \rangle \). Pick some \( \alpha, \alpha' \in \text{dom}(X), \alpha < \alpha' \). Let
\[ A_\alpha = \{ \delta < \kappa \mid (\alpha, \delta) \in X \} \]
and
\[ A_{\alpha'} = \{ \delta < \kappa \mid (\alpha', \delta) \in X \}. \]

Then \( A_\alpha \in U(\kappa, \alpha, \langle \rangle) \) and \( A_{\alpha'} \in U(\kappa, \alpha', \langle \rangle) \). By shrinking \( X \) if necessary, assume that \( \delta \in A_\alpha \) implies \( o^U(\delta) = \alpha \) and \( \delta' \in A_{\alpha'} \) implies \( o^U(\delta') = \alpha' \).

Consider the following set
\[ B = \{ \nu < \kappa \mid o^U(\nu) = \nu \text{ and (there are } \delta \in A_\alpha, \delta' \in A_{\alpha'} \text{ such that } \delta < \delta' \text{ and } \delta, \delta' \in b_\nu) \}. \]
Then \(B \in U(\kappa, \kappa, \varnothing)\). Just take the witnessing tree \(T_B\) (as in the definition of \(U(\kappa, \kappa, \varnothing)\)) with the first level
\[
A_\alpha \cup A_{\alpha'} \cup (\kappa \setminus (A_\alpha \cup A_{\alpha'})).
\]
Then for every \(\delta \in A_\alpha, B \in U(\kappa, \kappa, \{\delta\})\). So, \(B' := B \cap Y_{(\alpha, \delta)}\) is a subset of \(B\) in \(U(\kappa, \kappa, \{\delta\})\). But then an extension of \(T_B\) will witness this. In particular there will be \(\delta' \in A_{\alpha'}\) such that \(B' \in U(\kappa, \kappa, \{\delta'\})\). This implies that both \(Y_{(\alpha, \delta)}\) and \(Y_{(\alpha', \delta')}\) are in \(U(\kappa, \kappa, \{\delta'\}) = U_{(\alpha, \delta')}.\)
Hence, \(Y_{(\alpha, \delta)} \cap Y_{(\alpha', \delta')} \neq \emptyset\). Contradiction.
\[\square\]

Now combining Lemmas 3.1, 3.2 we obtain the following:

**Theorem 3.3** In \(V[G]\) there are a family \(\{U_\tau \mid \tau < \kappa\}\) of distinct \(\kappa\)-complete ultrafilters over \(\kappa\) and a \(\kappa\)-complete ultrafilter \(E\) over \(\kappa\), so that \(\{U_\tau \mid \tau \in X\}\) is a not discrete family for any \(X \in E\).

**Corollary 3.4** The consistency strength of existence a family \(\{U_\tau \mid \tau < \kappa\}\) of distinct \(\kappa\)-complete ultrafilters over \(\kappa\) and a \(\kappa\)-complete ultrafilter \(E\) over \(\kappa\), so that \(\{U_\tau \mid \tau \in X\}\) is a not discrete family for any \(X \in E\), is at most \(o(\kappa) = \kappa + 1\).

Let us argue now that that \(\{o(\alpha) \mid \alpha < \kappa\}\) is unbounded in \(\kappa\) is necessary for this.

**Theorem 3.5** Suppose that there are a family \(\{U_\tau \mid \tau < \kappa\}\) of distinct \(\kappa\)-complete ultrafilters over \(\kappa\) and a \(\kappa\)-complete ultrafilter \(E\) over \(\kappa\), so that \(\{U_\tau \mid \tau \in X\}\) is not a discrete family for any \(X \in E\). Then \(\{o(\alpha) \mid \alpha < \kappa\}\) is unbounded in \(\kappa\) in the Mitchell core model.

**Proof.** Suppose that for some large enough \(\eta < \kappa\), there is no \(\alpha \leq \kappa\) with \(o(\alpha) \geq \eta\) in the Mitchell core model \(\mathcal{K}\).

Let \(\{U_\tau \mid \tau < \kappa\}\) be a family of distinct \(\kappa\)-complete ultrafilters over \(\kappa\) and \(E\) be a \(\kappa\)-complete ultrafilter over \(\kappa\).

For every \(\tau < \kappa\), let \(j_\tau\) be \(i_{U_\tau} \restriction \mathcal{K}\). Then, by [5], \(j_\tau\) is an iterated ultrapower of \(\mathcal{K}\). By [3], the are less than \(\kappa\) possibilities for \(j_\tau(\kappa)\). By \(\kappa\)-completeness of \(E\), we can assume that for every \(\tau < \kappa\), \(j_\tau(\kappa)\) has a fixed value \(\theta\). Denote by \(Gen_\tau\) the set of generators of \(j_\tau\), i.e. the set of ordinals \(\nu, \kappa \leq \nu < \theta\) such that for every \(n < \omega, f : [\kappa]^n \to \kappa, f \in \mathcal{K}\) and \(a \in [\nu]^n, \nu \neq j_\tau(f)(a)\).
Let \(\text{Gen}_\tau^*\) be the subset of \(\text{Gen}_\tau\) consisting of all principal generators of \(j_\tau\), i.e. of all \(\nu \in \text{Gen}_\tau\) such that for every \(n < \omega, f : [\kappa]^n \to \kappa, f \in \mathcal{K}\) and \(a \in [\nu]^n, \nu > j_\tau(f)(a)\). Again by [3], there are less than \(\kappa\) possibilities for \(\text{Gen}_\tau^*\)'s. So, by \(\kappa\)-completeness of \(E\), we can assume that for every \(\tau < \kappa\), \(\text{Gen}_\tau^* = \text{Gen}^*\).
Suppose that \( \nu \in \text{Gen}_\tau \) and \( \nu \) is not a principal generator. Then there are finite set of generators \( b \subseteq \nu \) and \( f : [\kappa]^{[b]} \to \kappa, f \in \mathcal{K} \) such that \( \nu < j_\tau(f)(b) \).

Set, following W. Mitchell [6],
\[
\alpha(\nu) = \min \{ j_\tau(f)(b) \mid b \subseteq \nu \text{ is a finite set of generators, } f : [\kappa]^{[b]} \to \kappa, f \in \mathcal{K} \text{ and } \nu < j_\tau(f)(b) \}.
\]

Let \( b_\nu \subseteq \nu \) be the smallest finite set of generators such that for some \( f : [\kappa]^{[b_\nu]} \to \kappa, f \in \mathcal{K}, \alpha(\nu) = j_\tau(f)(b_\nu). \)

Let us call a finite set of generators \( a \subseteq \text{Gen}_\tau \) nice iff for each \( \nu \in a \) either \( \nu \) is a principal generator or it is not and then \( b_\nu \subseteq a \).

Consider now \( [id]_{U_\tau} \). Find the smallest finite nice set of generators \( a_\tau \) in \( \text{Gen}_\tau \) such that for some \( h_\tau : [\kappa]^{[a_\tau]} \to \kappa, h_\tau \in \mathcal{K} \) we have \( [id]_{U_\tau} = j_\tau(h_\tau)(a_\tau) \). We may assume, using \( \kappa \)-completeness of \( E \), that \( a_\tau \cap \text{Gen}^* \) has a constant value. Denote it by \( a^* \).

We would like to stabilize much of the things involved here, using \( \kappa \)-completeness of \( E \). In order to state this precise, let us deal with substructures of \( H_\chi \) for some large enough cardinal \( \chi \).

For every \( \tau < \kappa \) pick \( \mathcal{A}_\tau \preceq H_\chi \) of cardinality \( \eta \) which includes \( \{ \tau, U_\tau, a^* \} \cup \eta + 1 \).

The number of non-isomorphic structures among \( \mathcal{A}_\tau \) is less than \( \kappa \).

Now, using \( \kappa \)-completeness of \( E \), we can find \( X \in E \) such that for every \( \tau, \tau' \in X, \mathcal{A}_\tau \) and \( \mathcal{A}_{\tau'} \) are isomorphic.

Assume for simplicity that \( X = \kappa \).

Let \( \tau < \kappa \). Then \( j_\tau \) can be decomposed into a composition of two embeddings \( j^* : \mathcal{K} \to \mathcal{K}^* \) followed by \( k_\tau : \mathcal{K}^* \to (\mathcal{K})^{M_{U_\tau}}, \) where \( j^* \) is generated by the principal generators \( \text{Gen}^* \), and so, it is an iterated ultrapower which critical points are \( \kappa \) and its images, and \( k_\tau \) is the rest of the iteration generated by the non-principal generators \( \text{Gen}_\tau \setminus \text{Gen}^* \).

In order to construct a separating family, we will not use all of \( j_\tau = k_\tau \circ j^* \), but rather their finite part which \( a_\tau \) generates. Let \( j'_{\tau}, j'_*, k'_\tau \) be this parts, i.e., \( j' \) is generated by \( a^*, k'_\tau \) by \( a_\tau \setminus a^* \) and \( j'_* = k'_\tau \circ j' \).

Note that every embedding \( j'_\tau, j'_*, k'_\tau \) is in \( \mathcal{K} \), since it is finite.

Pick embeddings \( \sigma_\tau, \sigma' \) such that \( j_\tau = \sigma_\tau \circ j'_\tau, j^* = \sigma' \circ j' \).

Denote by \( a'_\tau \) and \( a^* \) the corresponding pre-images of \( a_\tau \) and \( a^* \).

Define in \( \mathcal{K} \) an ultrafilter over \( \kappa \):
\[
\mathcal{V}_\tau = \{ X \subseteq \kappa \mid j'_{\tau}(h_\tau)(a'_\tau) \in j'_\tau(X) \}.
\]
Then, clearly, $\mathcal{V}_\tau \subseteq U_\tau$.

Let $\pi_\tau : \kappa \to \kappa$, $\pi_\tau \in K$ be a projection of $\mathcal{V}_\tau$ to the normal ultrafilter Rudin–Keisler below $\mathcal{V}_\tau$.

The next lemma is likely well known:

**Lemma 3.6** Let $\langle E_\alpha \mid \alpha < \kappa \rangle$ be a family of pairwise different $\kappa$–complete ultrafilters over $\kappa$ which have the same projection to their least normal measures. Then the family is discrete.

**Proof.** Denote by $\pi$ this common projection.

Let $\alpha < \kappa$. For every $\beta < \kappa$, $\beta \neq \alpha$, pick $A_\alpha^\beta \in E_\alpha \setminus E_\beta$. Let

$$B_\alpha = \{ \nu < \kappa \mid \pi(\nu) > \alpha \}.$$ 

Then $B_\alpha \in E_\alpha$, since $\pi_*, E_\alpha$ is not principal ultrafilter. Set

$$A_\alpha = \Delta_{\beta < \kappa, \beta \neq \alpha} A_\alpha^\beta = \{ \nu < \kappa \mid \forall \beta < \pi(\nu) (\beta \neq \alpha \to \nu \in A_\alpha^\beta) \}.$$ 

Then $A_\alpha \in E_\alpha$. Let

$$A_\alpha^* = A_\alpha \cap B_\alpha \cap \bigcap_{\beta < \alpha} (\kappa \setminus A_\alpha^\beta).$$

Clearly, $A_\alpha^* \in E_\alpha$.

Let us argue that the sets $\langle A_\alpha^* \mid \alpha < \kappa \rangle$ are pairwise disjoint. So, let $\alpha < \alpha' < \kappa$. Suppose that $\nu \in A_\alpha^* \cap A_{\alpha'}^*$. Then $\nu \in B_{\alpha'}$, and hence, $\pi(\nu) > \alpha' > \alpha$. But then, $\nu \in A_\alpha$ implies that $\nu \in A_{\alpha'}^*$, which is impossible since $\nu \in A_{\alpha'}^* \subseteq \kappa \setminus A_\alpha^\alpha$.

□

**Lemma 3.7** Let $Z \subseteq \kappa$ be such that for every $\tau, \tau' \in Z$, $\pi_\tau = \pi_{\tau'}$. Then the family $\langle \mathcal{V}_\tau \mid \tau \in Z \rangle$, and hence also $\langle U_\tau \mid \tau \in Z \rangle$, is discrete.

**Proof.** Note that $\mathcal{V}_\tau$’s are ultrafilters only inside $K$ and $Z$ is not required to be in $K$.

However, our initial assumption allows to use the Mitchell Covering Lemma [6] in order to find $Z^* \subseteq Z$, $Z^* \in K$, $|Z^*| \leq \kappa$ and a sequence in $K$ of $\kappa$–complete ultrafilters $\langle \mathcal{V}_\tau \mid \tau \in Z^* \rangle$ which agrees with the original on $\tau$’s in $Z$ and has the same projections to the normal measure.

Now we can work in $K$ and to apply Lemma 3.6 to the sequence $\langle \mathcal{V}_\tau \mid \tau \in Z^* \rangle$.

□

The following is a particular case of the previous lemma:
Lemma 3.8 Let $\mathcal{Z} \subseteq \kappa$ be such that for every $\tau, \tau' \in \mathcal{Z}$, $a'_\tau = a'_\tau$. Then the family $\langle \mathcal{V}_\tau \mid \tau \in \mathcal{Z} \rangle$, and hence, also $\langle U_\tau \mid \tau \in \mathcal{Z} \rangle$, is a discrete family.

Note that in general (by forcing over $\mathcal{K}$), it is possible to have many ultrafilters $U_\tau$ with the same $j_\tau$, and so with the same $\mathcal{V}_\tau$. For example, forcing a Cohen subset to $\kappa$ (with appropriate preparation below) will give a situation where a ground model embedding extends in different ways.

We would like to argue that $\langle \mathcal{V}_\tau \mid \tau < \kappa \rangle$ is a discrete family, i.e., can be separated. It follows then that $\langle U_\tau \mid \tau < \kappa \rangle$ is a discrete family as well, since $\mathcal{V}_\tau \subseteq U_\tau$, for every $\tau < \kappa$.

Without loss of generality, we can assume that $\langle \mathcal{V}_\tau \mid \tau < \kappa \rangle$ is in $\mathcal{K}$. Just otherwise use the Mitchell Covering Lemma [6] and cover this family by a family in $\mathcal{K}$ of the same cardinality $\kappa$.

Work in $\mathcal{K}$. We separate $\langle \mathcal{V}_\tau \mid \tau < \kappa \rangle$ there.

Suppose first that $a' = \{\kappa\}$.

Then $j'$ is just the ultrapower embedding of $\mathcal{K}$ into $\mathcal{K}_1 = j'(\mathcal{K})$ by a normal measure $\mathcal{F}$ over $\kappa$ in $\mathcal{K}$.

Let $\tau < \kappa$. Then $k'_\tau$ is an ultrapower embedding of $\mathcal{K}_1$ by the ultrafilter generated by $a'_\tau \setminus \{\kappa\}$.

Let $\delta_\tau$ be the least $\delta$ such that $k'_\tau(\delta) > \max(a'_\tau)$.

For example, if $\alpha(\max(a'_\tau))$ depend only on $\kappa$, then $\delta_\tau = \alpha(\max(a'_\tau))$.

Define in $\mathcal{K}_1$ a $\min(a_\tau \setminus \{\kappa\})$-complete ultrafilter $W_\tau$ over $[\delta_\tau]^{|a'_\tau|} - 1$:

$$X \in W_\tau \text{ iff } a_\tau \setminus \{\kappa\} \in k'_\tau(X).$$

$W_\tau$ is a uniform ultrafilter, i.e., every $X \in W_\tau$ has cardinality $\delta_\tau$. Note that $\delta_\tau$ can be a singular cardinal.

Consider in $\mathcal{K}_1$ a function $h'_\tau$ on $[\delta_\tau]^{|a'_\tau|} - 1$ defined as follows:

$$h'_\tau(\nu_1, ..., \nu_{|a'_\tau|} - 1) = j'(h_\tau)(\kappa, \nu_1, ..., \nu_{|a'_\tau|} - 1).$$

Then $k'_\tau(h'_\tau)(a_\tau \setminus \{\kappa\}) = j'_\tau(h_\tau)(a'_\tau)$.

Consider $h'_\tau * W_\tau$.

Due to the minimality of $a_\tau$, the ultrafilter $h'_\tau * W_\tau$ is Rudin-Keisler equivalent to $W_\tau$.

Let us shrink it a bit. Proceed as follows.

Let $\tau < \kappa$. Set $\mu_\tau = j'_\tau(h_\tau)(a'_\tau)$.

Let $\alpha_\tau = \min(\alpha \mid k'_\tau(\alpha) > \mu_\tau)$.

Set in $\mathcal{K}_1$,

$$W_\tau = \{X \subseteq \alpha_\tau \mid \mu_\tau \in k'_\tau(X)\}.$$
Clearly, $\mathcal{W}_\tau$ is a min$(a_\tau \setminus \{\kappa\})$—complete ultrafilter (in $\mathcal{K}_1$). Also, by minimality of $\alpha_\tau$, every co-bounded subset of $\alpha_\tau$ is in $\mathcal{W}_\tau$. In particular, $\text{cof}(\alpha_\tau) > \kappa$.

**Suppose now that for every $\tau_0, \tau_1 < \kappa$, if $\tau_0 \neq \tau_1$, then $\alpha_{\tau_0} \neq \alpha_{\tau_1}$**.

Set $\beta_\tau = \sup\{\alpha_\rho \mid \rho < \kappa \text{ and } \alpha_\rho < \alpha_\tau\} + \kappa$. Then $\beta_\tau < \alpha_\tau$, and so, $\alpha_\tau \setminus \beta_\tau \in \mathcal{W}_\tau$.

Then the family $\langle \alpha_\tau \setminus \beta_\tau \mid \tau < \kappa \rangle$ separates $\langle \mathcal{W}_\tau \mid \tau < \kappa \rangle$.

Let us move this below $\kappa$.

Just pick one to one functions $s_\tau, r_\tau : \kappa \to \kappa, \tau < \kappa$ such that

1. $j'(s_\tau)(\kappa) = \alpha_\tau$,
2. $j'(r_\tau)(\kappa) = \beta_\tau$,
3. for every $\nu < \kappa$, $\nu < r_\tau(\nu) < s_\tau(\nu)$.

Let

$$C = \{\nu < \kappa \mid \forall \rho < \nu \forall \tau < \nu (s_\tau(\rho) < \nu)\}.$$ 

Clearly, it is a club in $\kappa$.

Consider

$$Z = \{\nu < \kappa \mid \forall \tau, \xi < \nu (\text{if } \xi \neq \tau \text{ then } (\alpha_\xi < \alpha_\tau \rightarrow s_\xi(\nu) < r_\tau(\nu))) \text{ and } (\alpha_\xi > \alpha_\tau \rightarrow s_\tau(\nu) < r_\xi(\nu))\}.$$ 

Clearly, $Z$ and $C$ are in $\mathcal{F}$.

For every $\tau < \kappa$, let

$$A_\tau = \bigcup\{[r_\tau(\nu), s_\tau(\nu)) \mid \nu \in Z \cap C \setminus \tau + 1\}.$$ 

Note that if $\tau, \rho < \kappa$ and $\tau \neq \rho$, then $A_\tau \cap A_\rho = \emptyset$. Thus, say $\rho < \tau$ and $\zeta \in A_\tau \cap A_\rho$, then there are $\nu_\tau > \tau, \nu_\rho > \rho$ such that $\zeta \in [r_\tau(\nu_\tau), s_\tau(\nu_\tau))$ and $\zeta \in [r_\tau(\nu_\rho), s_\tau(\nu_\rho))$. But this intervals should be disjoint, since $\nu_\rho, \nu_\tau \in Z \cap C$, and so, $r_\tau(\nu_\tau) > \nu_\tau > s_\rho(\nu_\rho)$.

Now, for every $\tau < \kappa$,

$$j'(A_\tau) \supseteq [j'(r_\tau)(\kappa), j'(s_\tau)(\kappa)) = [\beta_\tau, \alpha_\tau) \in \mathcal{W}_\tau.$$ 

So, applying $k'_\tau$,

$$[id]_{\mathcal{V}_\tau} = j'_\tau(h_\tau)(a'_\tau) \in j'_\tau(A_\tau).$$

Hence, $A_\tau \in \mathcal{V}_\tau$, and we are done.

Note that such defined $A_\tau$’s depend on $\alpha_\tau$’s, rather then on $\mathcal{W}_\tau$’s.
Next stage will be to deal with ultrafilters having same $\alpha_\tau$.

We will split their common set of measure one $A_\tau$, and hence, the split from the rest of ultrafilters will remain.

So fix some $\alpha^*$ and consider a set

$$Z(\alpha^*) = \{ \tau < \kappa \mid \alpha_\tau = \alpha^* \}.$$ 

Let $\tau, \tau' \in Z(\alpha^*)$. Both $W_\tau$ and $W_{\tau'}$ are obtained in the same fashion. If $a'_\tau \setminus a'$ (and so $a'_{\tau'} \setminus a'$) consists of a single element, then $\alpha_\tau = \alpha_{\tau'}$ will imply $W_\tau = W_{\tau'}$ and $\nu_\tau = \nu_{\tau'}$.

Hence, $a'_\tau \setminus a'$ (and so $a'_{\tau'} \setminus a'$) consists of at least two elements. Then $W_\tau$ (and so $W_{\tau'}$) is either a product of the projection of $W_\tau$ to the first coordinate with the rest or a sum (see Definition 1.2(ii)) according to the projection of $W_\tau$ to the first coordinate with the rest.

The treatment of both cases is similar. Let us deal with the later possibility.

Denote by $W^0_\tau$ the projection of $W_\tau$ to the first coordinate. It is a normal ultrafilter over $\alpha(\min(a'_\tau \setminus a'))$. For every $\zeta < \alpha(\min(a'_\tau \setminus a'))$, there will be an ultrafilter $W_{\tau\zeta}$ such that

$$X \in W_\tau \text{ iff } \{ \zeta \mid \{ \bar{\rho} \mid \zeta \bar{\rho} \in X \} \in W_{\tau\zeta} \} \in W^0_\tau.$$ 

Assume for simplicity that each $W_{\tau\zeta}$ is just a normal ultrafilter over some $\chi(\tau, \zeta)$.

The sequence $(\chi(\tau, \zeta) \mid \zeta < \alpha(\min(a'_\tau \setminus a')))$ is increasing mod $W^0_\tau$. Otherwise, using normality of $W^0_\tau$, we can stabilize it and the sum will be replaced by a product.

Assume, by shrinking if necessary, that the sequence $(\chi(\tau, \zeta) \mid \zeta < \alpha(\min(a'_\tau \setminus a')))$ is increasing. Set $\chi(\tau) = \bigcup_{\zeta < \alpha(\min(a'_\tau \setminus a'))} \chi(\tau, \zeta)$. Then $\chi(\tau)$ is a cardinal of cofinality $\alpha(\min(a'_\tau \setminus a'))$. So, $W_\tau$ concentrates on a set of cardinality $\chi(\tau)$ with cofinality $\alpha(\min(a'_\tau \setminus a'))$.

**Claim.** $\text{cof}(\alpha_\tau) = \alpha(\min(a'_\tau \setminus a'))$.

**Proof.** Set $\varepsilon = \text{cof}(\alpha_\tau)$. Fix $(\alpha_\tau(\gamma) \mid \gamma < \varepsilon)$ a witnessing cofinal sequence in $\alpha_\tau$.

Suppose that $\varepsilon \neq \alpha(\min(a'_\tau \setminus a'))$.

**Case 1.** $\varepsilon < \alpha(\min(a'_\tau \setminus a'))$.

Then the ultrapower embedding $j_{W^0_\tau}$ of $W^0_\tau$ moves the sequence $(\alpha_\tau(\gamma) \mid \gamma < \varepsilon)$ to a sequence $(j_{W^0_\tau}(\alpha_\tau(\gamma)) \mid \gamma < \varepsilon)$ of the same length.

Then the further ultrapower by $W'_{\tau\alpha(\min(a'_\tau \setminus a'))}$ is taken, where $W'_{\tau\alpha(\min(a'_\tau \setminus a'))}$ denotes the $\alpha(\min(a'_\tau \setminus a'))$-th member of the sequence $j_{W^0_\tau}((W_{\tau\zeta} \mid \zeta < \alpha(\min(a'_\tau \setminus a'))))$.

The critical point of such embedding is much above $\varepsilon$, so the sequence $(j_{W^0_\tau}(\alpha_\tau(\gamma)) \mid \gamma < \varepsilon)$ is moved to $(j_{W'_{\tau\alpha(\min(a'_\tau \setminus a'))}}(j_{W^0_\tau}(\alpha_\tau(\gamma)))) \mid \gamma < \varepsilon)$. Now recall that $j_{W'_{\tau\alpha(\min(a'_\tau \setminus a'))}} \circ j_{W^0_\tau}$ is just $k_\varepsilon'$ and $\alpha_\tau = \min(\alpha \mid k_\varepsilon'(\alpha) > \mu_\tau)$,
where $\mu_\tau = j_\tau^*(h_\tau)(a'_\tau)$.

This is impossible since $k'_\tau(\alpha_\tau) = \bigcup_{\gamma < \epsilon} k'_\tau(\alpha_\tau(\gamma))$.

**Case 2.** $\epsilon > \alpha(\min(a'_\tau \setminus a'))$.

Then $j_{W^0}^\prime \epsilon$ is unbounded in $j_{W^0}(\epsilon)$, and so, the sequence $\langle j_{W^0}(\alpha_\tau(\gamma)) \mid \gamma < \epsilon \rangle$ is unbounded in $j_{W^0}(\alpha_\tau)$.

Now turn to the further ultrapower by $W^\prime_{\alpha(\min(a'_\tau \setminus a'))}$.

Clearly, still $j_{W^\prime_{\alpha(\min(a'_\tau \setminus a'))}}^\prime (j_{W^0}(\epsilon))$ is unbounded in $j_{W^\prime_{\alpha(\min(a'_\tau \setminus a'))}}(j_{W^0}(\epsilon))$.

Hence, again, $k'_\tau(\alpha_\tau) = \bigcup_{\gamma < \epsilon} k'_\tau(\alpha_\tau(\gamma))$. Contradiction.

$\square$ of the claim.

Hence, $\text{cof}(\alpha_\tau) = \alpha(\min(a'_\tau \setminus a'))$.

Now, in order to have $\alpha_\tau = \alpha_{\cdot \tau}$ in this situation, we must have $\alpha(\min(a'_\tau \setminus a')) = \alpha(\min(a'_\tau \setminus a'))$.

Then the choice of the structures $A_{\cdot \tau}, A_{\cdot \tau}$ above implies that $W^0_\tau = W^0_\tau$, and so, $\min(a'_\tau \setminus a') = \min(a'_\tau \setminus a')$.

We continue the iteration $j'$ by taking ultrapower with $W^0_\tau$ and then proceed as before, but with less generators.

Suppose now that $a'$ contains more elements than just $\kappa$.

Suppose for simplicity that $a' = \{\kappa, \kappa_1\}$. The general case treated similar only the notations are more complicated.

Then $j'$ is the iterated ultrapower using two normal measures over $\kappa$, or just the ultrapower embedding by their product.

Denote the first normal ultrafilter used by $\mathcal{F}$ and the second by $\mathcal{F}_1$.

Let $j : \mathcal{K} \rightarrow \mathcal{K}_1 \simeq \text{Ult}(\mathcal{K}, \mathcal{F})$ and $j_1 : \mathcal{K}_1 \rightarrow \mathcal{K}_2 \simeq \text{Ult}(\mathcal{K}_1, j(\mathcal{F}_1))$.

The idea will be to apply the previous argument inside $\mathcal{K}_1$ and then to move down to $\mathcal{K}$.

We preserve the notation of the previous case with obvious adoptions to $\mathcal{K}_1$.

So, we will have disjoint unions of intervals $\langle A^1_\tau \mid \tau < \kappa \rangle$

subsets of $\kappa_1 \setminus \kappa$ such that for every $\tau < \kappa$,

$$A^1_\tau = \bigcup \{ [r^1_\tau(\nu), s^1_\tau(\nu)) \mid \nu \in Z^1 \cap C^1 \},$$

where $C^1$ and $Z^1$ are defined as $C$ and $Z$ above, but with $\kappa_1$ replacing $\kappa$.

Namely,

$$C^1 = \{ \nu < \kappa_1 \mid \forall \rho < \nu \forall \tau < \kappa (s^1_\tau(\rho) < \nu) \},$$

$$Z^1 = \{ \nu < \kappa_1 \mid \forall \tau < \kappa (\nu < r^1_\tau(\nu)) \land \forall \tau, \xi < \kappa (\text{if } \xi \neq \tau \text{ then } (\alpha_\xi < \alpha_\tau \rightarrow s^1_\xi(\nu) < r^1_\tau(\nu)) \land (\alpha_\xi > \alpha_\tau \rightarrow s^1_\xi(\nu) < r^1_\xi(\nu))) \}. $$

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Also, $C^1 \cap Z^1 \in j(F_1)$ now. By shrinking if necessary, assume that $\min(C^1)$ and $\min(Z^1)$ are above $\kappa$.

We have $\alpha_\tau = j'(s_\tau)(\kappa, \kappa_1)$, $\beta_\tau = j'(r_\tau)(\kappa, \kappa_1)$ and $s^1_\tau, r^1_\tau : \kappa_1 \to \kappa_1$ represent them mod $j(F_1)$.

So, $s^1_\tau(\nu) = j(s_\tau)(\kappa, \nu)$ and $r^1_\tau(\nu) = j(r_\tau)(\kappa, \nu)$.

For $\alpha < \kappa$ set

$C^{1\alpha} = \{ \nu < \kappa \mid \forall \rho < \nu \forall \tau < \alpha \exists! s_\tau(\alpha, \rho) < \nu \}$

and

$Z^{1\alpha} = \{ \nu < \kappa \mid \forall \tau < \alpha(\nu < r_\tau(\alpha, \nu)) \wedge \exists! \xi < \alpha(\text{if } \xi \neq \tau \text{ then} \alpha_\xi < \alpha_\tau \to s_\xi(\alpha, \nu) < r_\tau(\alpha, \nu) \text{ and } (\alpha_\xi > \alpha_\tau \to s_\xi(\alpha, \nu) < r_\xi(\alpha, \nu)) \}.$

Then the functions $\alpha \mapsto C^{1\alpha}$ and $\alpha \mapsto Z^{1\alpha}$ represent $C^1$ and $Z^1$ mod $F$.

Fix a set $T \in F$ such that for each $\alpha \in T$, $C^{1\alpha}$ is a club in $\kappa$ and each $Z^{1\alpha}$ is in $F_1$.

Set

$C^* = \Delta_{\alpha < \kappa} C^{1\alpha}$ and $Z^* = \Delta_{\alpha < \kappa} Z^{1\alpha}$.

Then $C^*, Z^* \in F_1$, due to normality. Also, $j(C^*) \setminus \kappa + 1 \subseteq C^1$ and $j(Z^*) \setminus \kappa + 1 \subseteq Z^1$.

Consider a functions $s^0_\tau : \kappa \to \kappa_1$ and $r^0_\tau : \kappa \to \kappa_1$ defined as follows:

$s^0_\tau(\alpha) = j_{F_1}(s_\tau)(\alpha, \kappa), r^0_\tau(\alpha) = j_{F_1}(r_\tau)(\alpha, \kappa)$.

Then,

$j(s^0_\tau)(\kappa) = j'(s_\tau)(\kappa, \kappa_1), j(r^0_\tau)(\kappa) = j'(r_\tau)(\kappa, \kappa_1)$.

Note that any function $j : \kappa \to On$ is either constant or increasing mod $F$ due to normality. $s^0_\tau$ cannot be constant, since then we can remove $\kappa$ from the set of generators of $\alpha_\tau$.

The function $r^0_\tau$ can be constant.

If $\alpha$ is not a constant function, let us compare $\sup(\text{rng}(r^0_\tau))$ with $\sup(\text{rng}(s^0_\tau))$.

Clearly, $\sup(\text{rng}(r^0_\tau)) \leq \sup(\text{rng}(s^0_\tau))$.

If $\sup(\text{rng}(r^0_\tau)) < \sup(\text{rng}(s^0_\tau))$, then we can replace $r^0_\tau$ by the constant function $\sup(\text{rng}(r^0_\tau))$.

Note that due to isomorphism of models $A_\xi, \xi < \kappa$, the situation is the same in this respect, for all $\xi < \kappa$.

Suppose that $\sup(\text{rng}(r^0_\tau)) = \sup(\text{rng}(s^0_\tau))$. The treatment of the case of constant function is similar and simpler.

Let $\theta_\tau = \sup(\text{rng}(r^0_\tau)) = \sup(\text{rng}(s^0_\tau))$ and let $\langle \theta_\alpha \mid \alpha < \kappa \rangle$ be a closed cofinal in $\theta_\tau$ sequence.
Pick a set $T_\tau \subseteq T$ in $\mathcal{F}$ such that for every $\alpha \in T_\tau$ the following hold:

1. $\alpha > \tau$,
2. $\theta_{\alpha \tau} \leq r_\tau^0(\alpha) < s_\xi^0(\alpha)$, for every $\alpha < \kappa$,
3. $r_\tau^0(\alpha') < r_\tau^0(\alpha)$, for every $\alpha' < \alpha < \kappa$,
4. $\forall_\tau, \xi < \alpha < \kappa$ (if $\xi \neq \tau$ then $(\alpha_\xi < \alpha_\tau \rightarrow s_\xi^0(\alpha) < r_\xi^0(\alpha))$ and $(\alpha_\xi > \alpha_\tau \rightarrow s_\xi^0(\alpha) < r_\xi^0(\alpha))$),
5. for every $\xi < \tau$,
   (a) if $\theta_\xi < \theta_\tau$, then $\theta_\xi < \theta_{\alpha \tau}$,
   (b) if $\theta_\xi > \theta_\tau$, then $\theta_{\alpha \xi} > \theta_{\alpha' \xi}$ and $s_\xi^0(\alpha')$, whenever $\alpha' < \kappa$ and $\theta_{\alpha' \xi}, s_\xi^0(\alpha') < \theta_\tau$.

Note that $\text{cof}(\theta_{\alpha \xi}) = \kappa$, and so, the sequence $\theta_{\alpha \xi} (\alpha' < \kappa)$ should be bounded below $\theta_\tau$. Also, $\tau < \kappa$, hence the total number of such $\xi$'s is less than $\kappa$, and so, there is a common bound.

Now we reflect this down using $\mathcal{F}_1$. Find a set $Y \in \mathcal{F}_1$ such that for every $\beta \in Y$ the following hold:

1. $\theta_\beta^ \beta > \beta$ has cofinality $\beta$ and $\{\theta_\alpha^\beta : \alpha < \beta\}$ be a closed cofinal in $\theta_\beta^ \beta$ sequence.
2. $\theta_\alpha^\beta = r_\tau(\alpha, \beta) < s_\tau(\alpha, \beta)$, for every $\alpha < \beta$,
3. for every $\xi < \alpha$, if $\theta_\xi^\beta = \theta_\tau^\beta$, then $s_\xi^0(\alpha') < \theta_{\alpha \tau}^\beta$, for every $\alpha' < \alpha$,
4. for every $\xi < \alpha$, if $\theta_\xi^\beta < \theta_\alpha^\beta$, then $\theta_\xi^\beta < \theta_{\alpha \tau}^\beta$,
5. $r_\tau(\alpha', \beta) < r_\tau^0(\alpha, \beta)$, for every $\alpha' < \alpha < \beta$,
6. $\forall_\tau, \xi < \alpha < \beta$ (if $\xi \neq \tau$ then $(\alpha_\xi < \alpha_\tau \rightarrow s_\xi(\alpha, \beta) < r_\xi(\alpha, \beta))$ and $(\alpha_\xi > \alpha_\tau \rightarrow s_\xi(\alpha, \beta) < r_\xi(\alpha, \beta))$).
7. for every $\xi < \tau$,
   (a) if $\theta_\xi < \theta_\tau$, then $\theta_\xi^\beta < \theta_{\alpha \tau}^\beta$,
   (b) if $\theta_\xi > \theta_\tau$, then $\theta_\xi^\beta > \theta_\tau^\beta$, $\theta_{\alpha' \xi}^\beta > \theta_{\alpha' \tau}^\beta$ and $\theta_{\alpha' \tau}^\beta > s_\xi(\alpha', \beta)$, whenever $\alpha' < \beta$ and $\theta_{\alpha' \xi}^\beta, s_\xi(\alpha', \beta) < \theta_\tau^\beta$. 

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Set

\[ A_\tau = \bigcup \{ [r_\tau(\alpha, \beta), s_\tau(\alpha, \beta)] \mid \alpha \in T_\tau, \tau < \alpha < \beta \in C^* \cap Z^* \cap Y \}. \]

Let us show that if \( \xi, \tau < \kappa, \xi \neq \tau \), then \( A_\xi \cap A_\tau = \emptyset \).

Suppose otherwise. Let \( \xi, \tau < \kappa, \xi \neq \tau \) and \( A_\xi \cap A_\tau \neq \emptyset \).

Let \( \zeta \in A_\xi \cap A_\tau \). Then there are \( \alpha < \beta, \alpha' < \beta', \tau < \alpha, \xi < \alpha', \beta, \beta' \in C^* \cap Z^* \cap Y \) such that \( \zeta \in [r_\tau(\alpha, \beta), s_\tau(\alpha, \beta) \cap [r_\xi(\alpha', \beta'), s_\xi(\alpha', \beta')]) \).

Let us argue that the intersection of such intervals must be empty.

We assume that \( \xi \neq \tau \) implies \( \alpha_\xi \neq \alpha_\tau \). Due to a symmetry of the situation it is enough to consider the case \( \xi < \tau \).

First note that if \( \beta' < \beta \) (or if \( \beta < \beta' \)), then for any \( \gamma < \beta, \gamma' < \beta' \),

\[ s_\xi(\gamma', \beta') < \beta < r_\tau(\gamma, \beta)( \text{ or } s_\tau(\gamma, \beta) < \beta' < r_\xi(\gamma', \beta')) \]

Which is impossible.

So, suppose that \( \beta = \beta' \). Split into three cases.

**Case 1.** \( \theta_\tau^\beta > \theta_\xi^\beta \).

Then

\[ s_\xi(\alpha', \beta') < \theta_\xi^\beta < \theta_\tau^\beta \leq r_\tau(\alpha, \beta), \]

by 7(a) above, which is impossible.

**Case 2.** \( \theta_\tau^\beta < \theta_\xi^\beta \).

Then

\[ s_\xi(\alpha', \beta') < \theta_\tau^\beta \leq r_\tau(\alpha, \beta), \]

by 7(b) above, which is impossible.

**Case 3.** \( \theta_\tau^\beta = \theta_\xi^\beta \).

If \( \alpha' < \alpha \), then

\[ s_\xi(\alpha') < \theta_\tau^\beta \leq r_\tau(\alpha, \beta), \]

by the items 2,3 above, which is again impossible.

If \( \alpha < \alpha' \), then \( \tau < \alpha \) implies that \( \tau < \alpha' \). So, again,

\[ s_\tau(\alpha) < \theta_\alpha^\beta \leq r_\xi(\alpha', \beta), \]

by the items 2,3 above, which is impossible.

Finally, if \( \alpha = \alpha' \), then we apply the item 6.

\( \square \)
4 Products of ultrafilters.

In [4], Aki Kanamori asked the following question (Question 5.8 there):

If $U$ and $V$ are $\kappa$-complete ultrafilters over $\kappa$ such that $U \times V \leq R^{\kappa} V \times U$, is there a $W$ and integers $n$ and $m$ so that $U = R^{\kappa} W^n$ and $V = R^{\kappa} W^m$?

Solovay gave an affirmative answer once "$U \times V \leq R^{\kappa} V \times U$" is replaced by "$U \times V = R^{\kappa} V \times U$", and Kanamori if $U$ is a $p$-point, see [4] 5.7, 5.9.

We would like to show that the negative answer is consistent assuming $o(\kappa) = \kappa$. Two examples will be produced. The following will be shown:

**Theorem 4.1** Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa$-complete ultrafilters $U$ and $V$ over $\kappa$ such that

1. $V >_{R^{\kappa}} U$,

2. $V \times U >_{R^{\kappa}} U \times V$.

**Remark 4.2** Note that if there are $W, n, m < \omega$ such that $U = R^{\kappa} W^n$ and $V = R^{\kappa} W^m$, then $V \times U = R^{\kappa} U \times V$.

**Theorem 4.3** Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa$-complete ultrafilters $U$ and $V$ over $\kappa$ such that

1. $V$ is a normal measure,

2. $V$ is the projection of $U$ to its least normal measure,

3. $V \times U >_{R^{\kappa}} U \times V$.

**Proof of the first theorem.**

Let us keep the notation of the previous section.

So, we have $\kappa$-complete ultrafilters $U(\kappa, \alpha, t), \alpha < \kappa, t \in [\kappa]^{<\omega}$ which extend $U(\kappa, \alpha)$’s. Denote $U(\kappa, \alpha, \langle \rangle)$ by $\bar{U}(\kappa, \alpha)$.

Let $f : \kappa \to \kappa$. Define

$$U_f = \{X \subseteq \kappa \mid \alpha < \kappa \mid X \in \bar{U}(\kappa, f(\alpha)) \in \bar{U}(\kappa, 0)\},$$

i.e.

$$U_f = \bar{U}(\kappa, 0) - \lim_{\alpha < \kappa} \bar{U}(\kappa, f(\alpha)).$$
Then $U_f$ is a $\kappa$–complete ultrafilter over $\kappa$.

It is noted in [3], that if $f \leq g \mod \bar{U}(\kappa, 0)$, then $U_f \leq \kappa \setminus U_g$.

Our prime interest will be in $f = id$ and $g = id + 1$.

Set $\mathcal{U} = U_{id}$ and $\mathcal{V} = U_{id+1}$.

We would like to argue that $\mathcal{U} \times \mathcal{V} < \mathcal{R} - K \mathcal{V} \times \mathcal{U}$.

Note that neither $\mathcal{U}$ nor $\mathcal{V}$ are of the form $\mathcal{W}^n$, for $n > 1$, since the only ultrafilters Rudin-Keisler below $\mathcal{U}$ are $\bar{U}(\kappa, \alpha)$, $\alpha < \kappa$ and their finite powers, those below $\mathcal{V}$ are $\bar{U}(\kappa, \alpha)$, $\alpha < \kappa$, $\mathcal{U}$ and their finite powers. Just examine the ultrapowers by $\mathcal{U}$ and $\mathcal{V}$, we refer to [1],[3] for this type analyzes.

In particular, $\mathcal{V} \not\cong \mathcal{U}^n$, $n < \omega$.

Suppose that $B \in \mathcal{U} \times \mathcal{V}$. Then

$$\{\mu < \kappa \mid \{\xi < \kappa \mid (\mu, \xi) \in B\} \in \mathcal{V}\} \in \mathcal{U}.$$  

Denote

$$A = \{\mu < \kappa \mid \{\xi < \kappa \mid (\mu, \xi) \in B\} \in \mathcal{V}\}$$

and for each $\mu < \kappa$, let

$$A_\mu = \{\xi < \kappa \mid (\mu, \xi) \in B\}.$$  

Recall that

$$\mathcal{U} = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha) \mid \alpha < \kappa \rangle.$$  

Hence, there is $Z \in \bar{U}(\kappa, 0)$ such that for every $\alpha \in Z$, $A \in \bar{U}(\kappa, \alpha)$.

Similar,

$$\mathcal{V} = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha + 1) \mid \alpha < \kappa \rangle.$$  

Hence, for every $\mu \in A$, there is $Y_\mu \in \bar{U}(\kappa, 0)$ such that for every $\alpha \in Y_\mu$, $A_\mu \in \bar{U}(\kappa, \alpha + 1)$.

Set

$$X = Z \cap \Delta_{\mu \in A} Y_\mu.$$  

Then $X \in \bar{U}(\kappa, 0)$ and for every $\alpha \in X$ we have

$$A \in \bar{U}(\kappa, \alpha) \text{ and } \forall \mu \in A \cap \alpha (A_\mu \in \bar{U}(\kappa, \alpha + 1)).$$  

Then, by elementarity, in $M_{\mathcal{V}}$ (the ultrapower by $\mathcal{V}$), for every $\alpha \in i_{\mathcal{V}}(X)$,

$$i_{\mathcal{V}}(A) \in \bar{U}(i_{\mathcal{V}}(\kappa), \alpha) \text{ and } \forall \mu \in i_{\mathcal{V}}(A) \cap \alpha (A'_\mu \in \bar{U}(i_{\mathcal{V}}(\kappa), \alpha + 1)),$$

where $i_{\mathcal{V}}(\langle A_\mu \mid \mu < \kappa \rangle) = \langle A'_\mu \mid \mu < i_{\mathcal{V}}(\kappa) \rangle$.  

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Let $\rho^\U$ denote $[id]_\U$. Then $\rho^\U \in i_\U(A)$. We have a natural embedding $\sigma : M_\U \to M_\V$ and it does not move $\rho^\U$, since its critical point is $i_\U(\kappa)$. Then,

$$\rho^\U = \sigma(\rho^\U) \in \sigma(i_\U(A)) = i_\V(A).$$

Note that generators of $\bar{U}(\kappa, 0)$ appear unboundedly many times below $\rho_\V > \rho_\U$. Let $\alpha^*$ be, say, the least such generator above $\rho^\U$. Then $\alpha^* \in i_\V(X) \setminus \rho^\U + 1$. So,

$$\forall \mu \in i_\V(A) \cap \alpha^*(A'_\mu \in \bar{U}(i_\V(\kappa), \alpha^* + 1)).$$

Now, $\bar{U}(i_\V(\kappa), \alpha^* + 1)) <_{R-K} U(i_\V(\kappa), id) = i_\V(\U)$. Let $\eta$ represents a corresponding projection function in the ultrapower of $M_\V$ by $i_\V(\U)$. Then for all $\mu \in i_\V(A) \cap \alpha^*$, $\eta \in i_\V(\U)(A'_\mu)$. Hence,

$$\eta \in i_\V(\U)(A'_\mu).$$

So,

$$(\rho^\U, \eta) \in i_\V(\U)(B).$$

We are done, since then

$$\{ E \subseteq [\kappa]^2 \mid (\rho^\U, \eta) \in i_\V(\U)(E) \} \supseteq \U \times \V,$$

but $\U \times \V$ is an ultrafilter, so

$$\{ E \subseteq [\kappa]^2 \mid (\rho^\U, \eta) \in i_\V(\U)(E) \} = \U \times \V,$$

which means that

$$\U \times \V <_{R-K} \V \times \U.$$

□

The second theorem can be deduced from the first, but let us give a direct argument.

Proof of the second theorem.

Let us show now that $\bar{U}(\kappa, 0) \times \U >_{R-K} \U \times \bar{U}(\kappa, 0)$.

Note that $\bar{U}(\kappa, 0)$ is normal. By Kanamori [4], it is impossible to have $\V \times \U >_{R-K} \U \times \V$ once $\U$ is normal or even a $P$–point.

We have

$$\U = \bar{U}(\kappa, 0) - \lim (\bar{U}(\kappa, \alpha) \mid \alpha < \kappa).$$
So, the ultrapower with $\mathcal{U}$ is obtained as follows. First $\bar{U}(\kappa, 0)$ is applied. We have

$$i_{\bar{U}(\kappa, 0)} : V \to M_{\bar{U}(\kappa, 0)}.$$  

Next $\bar{U}(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)$ is applied over $M_{\bar{U}(\kappa, 0)}$. We have

$$i_{\bar{U}(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)} : M_{\bar{U}(\kappa, 0)} \to M_{\bar{U}(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)}.$$  

The composition is the ultrapower embedding by $\mathcal{U}$, i.e.

$$i_{\mathcal{U}} = i_{\bar{U}(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)} \circ i_{\bar{U}(\kappa, 0)} : V \to M_{\mathcal{U}} = M_{\bar{U}(i_{\bar{U}(\kappa, 0)}(\kappa), \kappa)}.$$  

Consider $\bar{U}(\kappa, 0) \times \mathcal{U}$. So, we have $i_{\bar{U}(\kappa, 0)} : V \to M_{\bar{U}(\kappa, 0)}$ followed by $i_{\bar{U}(\kappa, 0)}(\mathcal{U}) = U(i_{\bar{U}(\kappa, 0)}(\kappa), id)$. The application of $U(i_{\bar{U}(\kappa, 0)}(\kappa), id)$ to $M_{\bar{U}(\kappa, 0)}$ has the similar description to the one above. Namely, $i_{\bar{U}(\kappa, 0)}(\bar{U}(\kappa, 0))$ is used first followed by

$$\bar{U}(i_{\bar{U}(\kappa, 0)}(\bar{U}(\kappa, 0))(i_{\bar{U}(\kappa, 0)}(\kappa)), i_{\bar{U}(\kappa, 0)}(\kappa)).$$  

In order to simplify the notation, let us denote $i_{\bar{U}(\kappa, 0)}$ by $i_1$, $M_{\bar{U}(\kappa, 0)}$ by $M_1$, $i_{\bar{U}(\kappa, 0)}(\kappa)$ by $\kappa_1$, the second ultrapower of $\bar{U}(\kappa, 0)$ by $M_2$ and the image of $\kappa_1$ there by $\kappa_2$.

Then $i_{\bar{U}(\kappa, 0) \times \mathcal{U}} : V \to M_{\bar{U}(\kappa, 0) \times \mathcal{U}}$ is $i_1 : V \to M_1$ followed by $i_{\bar{U}(\kappa_1, 0)} : M_1 \to M_2$ and then by $i_{\bar{U}(\kappa_2, \kappa_1)} : M_2 \to M_{\bar{U}(\kappa, 0) \times \mathcal{U}}$.

Note that in $M_2$, we have $\bar{U}(\kappa_2, \kappa_1) >_{R-K} \bar{U}(\kappa_2, \kappa)$ and even $\bar{U}(\kappa_2, \kappa_1) >_{R-K} \bar{U}(\kappa_2, \kappa) \times \bar{U}(\kappa_2, 0)$.

Pick $(\eta, \rho)$ which represents a corresponding projection function in the ultrapower of $M_2$ by $\bar{U}(\kappa_2, \kappa_1)$.

Let us argue that

$$\{E \subseteq [\kappa]^2 \mid (\eta, \rho) \in i_{\bar{U}(\kappa, 0) \times \mathcal{U}}(E)\} \supseteq \mathcal{U} \times \bar{U}(\kappa, 0).$$  

Let $A \in \mathcal{U}$, then

$$[id]_{\bar{U}(\kappa_1, \kappa)} \in i_{\mathcal{U}}(A) = i_{\bar{U}(\kappa_1, \kappa)}(i_1(A)).$$  

Then, in $M_1$,

$$i_1(A) \in \bar{U}(\kappa_1, \kappa).$$

---

$^3$V denotes here the generic extension, where all the ultrafilters $\bar{U}(\kappa, \alpha)$ are defined.
Apply the second ultrapower embedding \( i(U_{\kappa_1,0}) \) to it. Note that its critical point is \( \kappa_1 > \kappa \). Then,
\[
i_2(A) = i(U_{\kappa_1,0})(i_1(A)) \in i(U_{\kappa_1,0})(\bar{U}(\kappa_1, \kappa)) = \bar{U}(\kappa_2, \kappa).
\]
Next apply \( i(U_{\kappa_2,\kappa_1}) : M_2 \to M_{U(\kappa,0)\times U} \). So, by the choice of \( \eta \),
\[
\eta \in i(U_{\kappa,0}\times U)(A) = i(U_{\kappa_2,\kappa_1})(i_2(A)).
\]
Suppose now that \( B \in U \times \bar{U}(\kappa, 0) \). Set
\[
A := \{ \mu < \kappa \mid \{ \xi < \kappa \mid (\mu, \xi) \in B \} \in \bar{U}(\kappa, 0) \}.
\]
Then \( A \in U \) and for every \( \mu \in A \) the set
\[
A_\mu := \{ \xi < \kappa \mid (\mu, \xi) \in B \} \in \bar{U}(\kappa, 0).
\]
Apply \( i_2 \). Then, in \( M_2 \),
\[
\forall \mu \in i_2(A)(A_\mu \in \bar{U}(\kappa_2, 0)).
\]
But, by above, we have
\[
i_2(A) \in \bar{U}(\kappa_2, \kappa),
\]
hence,
\[
i_2(B) \in \bar{U}(\kappa_2, \kappa) \times \bar{U}(\kappa_2, 0).
\]
So,
\[
(\eta, \rho) \in i(U(\kappa,0)\times U)(B),
\]
and we are done.
\[\square\]

Let us address now the strength issue.

**Theorem 4.4** Suppose that there is no inner model in which \( \kappa \) is a measurable with \( \{ o(\alpha) \mid \alpha < \kappa \} \) unbounded in it. Then for any two \( \kappa \)-complete ultrafilters \( U \) and \( V \) over \( \kappa \), if \( V \times U \geq_{R\neg K} U \times V \), then there is a \( \kappa \)-complete ultrafilter \( W \) over \( \kappa \) and there are integers \( n, m \) such that \( V =_{R\neg K} W^n \) and \( U =_{R\neg K} W^m \).

**Proof.** Suppose that there is no inner model in which \( \kappa \) is a measurable with \( \{ o(\alpha) \mid \alpha < \kappa \} \) unbounded in it. Then there are no \( \kappa \)-non-Rudin-Keisler equivalent ultrafilters which are Rudin-Keisler below some \( \kappa \)-complete ultrafilter. Also, the separation property holds, i.e., if \( \{ U_\tau \mid \tau < \kappa \} \) is a family of distinct \( \kappa \)-complete ultrafilters over \( \kappa \)
and $E$ is a $\kappa$–complete ultrafilter over $\kappa$, then there is $X \in E$ such that $\{U_\tau \mid \tau \in X\}$ is a discrete family.

Suppose now that the theorem fails. Using the well-foundedness of $\leq_{R-K}$, pick then $\mathcal{U}$ and $\mathcal{V}$ to be two $\kappa$–complete ultrafilters over $\kappa$, $\mathcal{V} \times \mathcal{U} \supseteq_{R-K} \mathcal{U} \times \mathcal{V}$ which are a counterexample to the theorem and assume that the pair $(\mathcal{U}, \mathcal{V})$ is the least in the following sense:

whenever

1. $\mathcal{U}' \leq_{R-K} \mathcal{U}$,
2. $\mathcal{V}' \leq_{R-K} \mathcal{V}$,
3. $\mathcal{U}' <_{R-K} \mathcal{U}$ or $\mathcal{V}' <_{R-K} \mathcal{V}$,
4. $\mathcal{V}' \times \mathcal{U}' \supseteq_{R-K} \mathcal{U}' \times \mathcal{V}'$

hold, then there is a $\kappa$–complete ultrafilter $W'$ over $\kappa$ and there are integers $n, m$ such that $\mathcal{V}' =_{R-K} W'^n$ and $\mathcal{U}' =_{R-K} W'^m$.

Let $(\rho, \eta) \in [i_{V \times U}(\kappa)]^2$ generates $\mathcal{U} \times \mathcal{V}$, i.e.

$$\mathcal{U} \times \mathcal{V} = \{X \subseteq [\kappa]^2 \mid (\rho, \eta) \in i_{V \times U}(X)\}.$$ 

Clearly, then $\eta > i_{V}(\kappa)$.

Consider in $M_V$ an ultrafilter $W$ defined by $(\rho, \eta)$, i.e.

$$W := \{Z \subseteq [i_{\mathcal{V}}(\kappa)]^2 \mid (\rho, \eta) \in i_{\mathcal{V}}(U)(Z)\}.$$ 

Clearly, $W \leq_{R-K} i_{\mathcal{V}}(U)$. Find a sequence of ultrafilters $(W_\alpha \mid \alpha < \kappa)$ which represents $W$ in the ultrapower by $\mathcal{V}$, i.e.

$$i_{\mathcal{V}}((W_\alpha \mid \alpha < \kappa))(\text{id}_\mathcal{V}) = W.$$ 

So, for most $(\text{mod } \mathcal{V}) \alpha$’s, $W_\alpha \leq_{R-K} \mathcal{U}$.

Note that

$$\mathcal{U} \times \mathcal{V} = \mathcal{V} - \lim (W_\alpha \mid \alpha < \kappa).$$ 

Namely,

$$X \in \mathcal{U} \times \mathcal{V} \iff (\rho, \eta) \in i_{\mathcal{V} \times U}(X) \iff i_{\mathcal{V}}(X) \in W$$

$$\iff \{\alpha < \kappa \mid X \in W_\alpha\} \in \mathcal{V} \iff X \in \mathcal{V} - \lim (W_\alpha \mid \alpha < \kappa).$$

The sequence $(W_\alpha \mid \alpha < \kappa)$ may contain same ultrafilters, but among them must be $\kappa$ different. Just otherwise, mod $\mathcal{V}$ they will be the same. Let $W'$ be this ultrafilter. Then,
\( \mathcal{U} \times \mathcal{V} = \mathcal{V} - \lim (W_\alpha \mid \alpha < \kappa) \), implies \( \mathcal{U} \times \mathcal{V} = W' \). So, \( \mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{U} \), which is impossible. Let \( t : \kappa \to \kappa \) be a function such that \( t(\alpha) = t(\beta) \) iff \( W_\alpha = W_\beta \). Then \( t \) is not constant mod \( \mathcal{V} \). Set \( \mathcal{V}' = t_* \mathcal{V} \). Then

\[
\mathcal{U} \times \mathcal{V} = \mathcal{V}' - \lim (W_\alpha \mid \alpha < \kappa),
\]

implies \( \mathcal{U} \times \mathcal{V} = W' \). So, \( \mathcal{U} \times \mathcal{V} \leq \mathcal{R} - K \mathcal{U} \), which is impossible.


Let \( t: \kappa \to \kappa \) be a function such that \( t(\alpha) = t(\beta) \) iff \( W_\alpha = W_\beta \). Then \( t \) is not constant mod \( \mathcal{V} \). Set \( \mathcal{V}' = t_* \mathcal{V} \). Then

\[
\mathcal{U} \times \mathcal{V} = \mathcal{V}' - \lim (W_\alpha \mid \alpha < \kappa),
\]

Now, using the separation property, the ultrapower by \( \mathcal{U} \times \mathcal{V} \) is the ultrapower by \( \mathcal{V}' \) followed by \( W[\text{id}_{\mathcal{V}'}] \).

Still, among this different \( W_\alpha \)'s may be many which are Rudin-Keisler equivalent. It is impossible that the number of the equivalence classes has cardinality \( \kappa \), since then we will have \( \kappa \)—many non-equivalent (R-K) ultrafilters below \( \mathcal{U} \).

Suppose that the number of the equivalence classes has cardinality \( < \kappa \). Then there is \( W' \) such that \( W_\alpha = R-K W' \), for almost every \( \alpha \) mod \( \mathcal{V}' \).

Hence, \( W[\text{id}_{\mathcal{V}'}] = R-K i_{\mathcal{V}'}(W') \), so its ultrapower is the same as those by \( i_{\mathcal{V}'}(W') \). This means that the iterated ultrapower is just \( \mathcal{V}' \times W' \).

So, \( \mathcal{V}' \times W' = R-K \mathcal{U} \times \mathcal{V} \). Then by Kanamori [4] (5.6), at least one of the following three possibilities must holds:

1. \( W' = R-K \mathcal{V} \) and \( \mathcal{V}' = R-K \mathcal{U} \);
2. there is a \( \kappa \)—complete ultrafilter \( F \), such that \( \mathcal{V}' = R-K \mathcal{U} \times F \) and \( \mathcal{V} = R-K F \times W' \);
3. there is a \( \kappa \)—complete ultrafilter \( G \) such that \( \mathcal{U} = R-K \mathcal{V}' \times G \) and \( W' = R-K G \times \mathcal{V} \).

Suppose for a moment that the first possibility occurs. Then

\[
\mathcal{U} \supseteq_{R-K} \mathcal{W}' = R-K \mathcal{V} \supseteq_{R-K} \mathcal{V}' = R-K \mathcal{U}.
\]

So, \( \mathcal{U} = R-K \mathcal{V} \), and then \( \mathcal{U} \times \mathcal{V} = R-K \mathcal{V} \times \mathcal{U} \).

Suppose now that the second possibility occurs. Then \( \mathcal{V}' \leq_{R-K} \mathcal{V} \) and \( W' \leq_{R-K} \mathcal{U} \) imply

\[
\mathcal{U} \times F = R-K \mathcal{V}' \leq_{R-K} \mathcal{V} = R-K F \times W' \leq_{R-K} F \times \mathcal{U}.
\]

But, also (2) implies that \( \mathcal{V} >_{R-K} F \). Hence, we can apply the minimality assumption to the pair \( (\mathcal{U}, F) \).

Then there will be a \( \kappa \)—complete ultrafilter \( Z \) and \( k, \ell < \omega \) such that

\[
\mathcal{U} = Z^k \text{ and } F = Z^\ell.
\]
This implies, in particular, 
\[ U \times F =_{R-K} F \times U. \]

But then also
\[ U \times F =_{R-K} F \times W'(=_{R-K} V) =_{R-K} F \times U. \]

Hence \( V = Z^{k+\ell} \), and we are done.

Consider now the third possibility.
Then \( U \geq_{R-K} W' \) and \( V \geq_{R-K} V' \) imply
\[ V \times G \geq_{R-K} V' \times G =_{R-K} U \geq_{R-K} W' =_{R-K} G \times V. \]

But, also (3) implies that \( U >_{R-K} G \). Hence, we can apply the minimality assumption to the pair \((V, G)\).

Then there will be a \( \kappa \)-complete ultrafilter \( Y \) and \( n, m < \omega \) such that
\[ V = Y^n \text{ and } G = Y^m. \]

This implies, in particular,
\[ V \times G =_{R-K} G \times V. \]

But then also
\[ V \times G =_{R-K} V' \times G =_{R-K} U =_{R-K} G \times V. \]

Hence \( U = Y^{n+m} \), and we are done.
\[ \square \]
References


