A certain generalization of SPFA to higher cardinals.

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Abstract

Itay Neeman found in [6] a new way of iterating proper forcing notions and extended it in [7] to $\aleph_2$. In [4] his construction for $\aleph_1$ ([6]) was generalized to semi-proper forcing notions. We continue here [4] and extend the construction to higher cardinals still using finite conditions. In the final model a weak form of SPFA will hold.

1 Basic definitions and main results

The following two definitions are due to S. Shelah [8].

Definition 1.1 A forcing notion Q is called a $\{\eta\}$–*proper* iff for every $M \prec \langle H(\chi), \in, < \rangle$ of a size η with $Q \in M$ the following holds:
for every $q \in M$ there is $p \geq q$ which is (M, Q)-generic, i.e. $p \Vdash ((M[\underset{\sim}{G}])^V = M)$.
If Q is $\{\eta'\}$–proper for every regular cardinal $\eta' \leq \eta$, then we call Q a $\{\leq \eta\}$–*proper*.

Definition 1.2 A forcing notion Q is called a $\{\eta\}$–*semi-proper* iff for every $M \prec \langle H(\chi), \in, < \rangle$ of a size η with $Q \in M$ the following holds:
for every $q \in M$ there is $p \geq q$ which is (M, Q)–semi-generic, i.e. $p \Vdash (M[\underset{\sim}{G}] \cap \eta^+ = M \cap \eta^+)$.
If Q is $\{\eta'\}$–proper for every regular cardinal $\eta' \leq \eta$, then we call Q a $\{\leq \eta\}$–*semi-proper*.

Remark 1.3 Further we will use a bit weaker notions. Instead of arbitrary M's in Definitions 1.1,1.2 we restrict ourself to models closed under $< \eta$–sequences in GCH situations and once GCH breaks down - to models which are generic extensions of closed under $< \eta$–sequences models from the ground model which satisfies GCH.

Dealing with finite $\in$-increasing sequences closed under intersections, as it was done in [5], [6] and worked fine at $\aleph_1$, seems to be problematic here in context of larger cardinals. The problems appear already at $\aleph_2$, i.e. once models of cardinalities $\aleph_0$ and $\aleph_1$ are around.

The basic problem is with $\{\aleph_0\}$–properness. The proof of it requires kind of nice restrictions of conditions to a countable submodel which may not exist now.
We will follow the intuition of [3] and use instead of $\in$-increasing sequences closed under intersections – finite structures with pistes from [3].

Definition 1.4 Let $\mathcal{A}$ be a finite structure with pistes (see [3]) and Q a forcing notion such that $Q \in \bigcap \mathcal{A}$. We call a condition $p \in Q$ $(\mathcal{A}, Q)$*-generic* iff p is (A, Q)- generic for every $A \in \mathcal{A}$.

Definition 1.5 A forcing notion Q is called $\leq \eta$*–piste structures proper (or just piste proper)* iff for every finite structure with pistes $\mathcal{A}$ which consists of models of cardinalities $\leq \eta$ with $Q \in \bigcap \mathcal{A}$ the following holds:
for every $q \in Q \cap \bigcap \mathcal{A}$ there is $p \geq q$ which is $(\mathcal{A}, Q)$-generic.

Definition 1.6 Let Q be a $\leq \eta$–piste structures proper forcing, $\mathcal{A}$ a finite structure with pistes which consists of models of cardinalities $\leq \eta$ with $Q \in \bigcap \mathcal{A}$. Let p be $(\mathcal{A}, Q)$- generic. We call p *a minimal generic for* $\mathcal{A}$ if for every $\mathcal{B}$ which extends $\mathcal{A}$ (in sense of [3]) there is $q \geq p$ which is $(\mathcal{B}, Q)$-generic.

Remark 1.7 It is possible to weaken a bit the requirement to an existence of $q \geq p$ which is $(\mathcal{B}, Q)$-generic, for every $\mathcal{B}$ which extends $\mathcal{A}$ which has the same countable models as $\mathcal{A}$ has.

Definition 1.8 A forcing notion Q is called $\leq \eta$*–strongly piste structures proper (or just strongly piste proper)* iff it is $\leq \eta$–piste structures proper and for every a finite structure with pistes $\mathcal{A}$ which consists of models of cardinalities $\leq \eta$ with $Q \in \bigcap \mathcal{A}$, for every $(\mathcal{A}, Q)$-generic condition q there are $p \geq q$ and a finite structure with pistes $\mathcal{B}$ which extends $\mathcal{A}$ such that p is a minimal generic for $\mathcal{B}$.

Remark 1.9 1. Note that in the original Neeman setting at $\aleph_1$ [6] or at those of [4] there was no problem to find a common generic or semi-generic condition for $\in$-increasing sequences of models, since only countable models or models of inaccessible cardinalities were involved. In present situation starting with $\aleph_2$, there are also models of size $\aleph_1$. This complicates the matter. Thus let for example Q be the Levy collapse of ω_3 to ω_2. Define a sequence A_0, A_1, A_2, A_3 of elementary submodels such that

(a) A_3 is countable,

(b) $A_i, i \leq 2$ are of size $\aleph_1$,

(c) $A_0 \in A_1 \in A_2 \in A_3$,

(d) $A_3 \cap A_2 \subseteq A_0$ and $\sup(A_3 \cap A_2 \cap \omega_2) = \sup(A_0 \cap \omega_2)$,

(e) there is no $p \in Q$ which is a generic simultaneously over each of A_i's.

Assume CH. Pick any $A_2 \preceq H(\chi)$, for a regular χ large enough, which is a limit of increasing continuous sequence of the length $\aleph_1$ of elementary submodels of $H(\chi)$ each of size $\aleph_1$. Let $A_3 \prec H(\chi)$ with $A_2 \in A_3$ and the sequence in A_3 as well. Then $A_2 \cap A_3 \in A_2$ and there is a model A of the sequence such that $A_3 \cap A_2 \subseteq A$ and $\sup(A_3 \cap A_2 \cap \omega_2) = \sup(A \cap \omega_2)$. Let $A_0 = A$.
Now let choose A_1.
Pick a sequence $\langle A^i \mid i < \omega_1 \rangle \in A_2$ such that

- for every $i < \omega_1$, $|A^i| = \aleph_1$,
- for every $i < \omega_1$, $A_0 \in A^i$,
- for every $i, j < \omega_1$, $A_i \cap \omega_2 = A_j \cap \omega_2$
- for every $i, j < \omega_1$, $i \neq j \Rightarrow A_i \cap \omega_3 \neq A_j \cap \omega_3$.

Set $\delta = A^0 \cap \omega_2$. Consider the set $S = \{f''\delta \mid f \in Q \cap A_3\}$. Then S a countable set of subsets of ω_3. Pick $i < \omega_1$ such that $A^i \cap \omega_3 \notin S$. Set $A_1 = A^i$.
Now suppose that there is $p \in Q$ which is Q-generic over each $A_i, i \leq 3$. Then $p \restriction A_2 \cap \omega_2 \in A_3$, since $A_2 \in A_3$. Hence $p"\delta \in S$, and so $p"\delta \neq A_1 \cap \omega_3$. This prevents p from being generic over A_1.

2. It is possible to weaken a little applying restrictions of 1.3.

A combination of Neeman's ideas from [6] with a models produced in [3] allows to show the following:

Theorem 1.10 *Let κ be a supercompact cardinal and $\eta < \kappa$ be a regular cardinal. Then in a forcing extension which preserves all the cardinals $\leq \eta^+$ and turns κ into η^{++} the $\leq \eta$–strongly piste structures PFA holds, i.e. for every $\leq \eta$–strongly piste structures proper forcing notion Q and for every collection $\mathcal{D}$ of $\leq \eta^+$ dense subsets of Q there is a filter on Q that meets all of them.*

We will proceed here by replacing properness by a certain variation of semi-properness.

Definition 1.11 A forcing notion Q is called $\leq \eta$*–piste structures semi-proper (or just piste semi-proper)* iff for every finite structure with pistes $\mathcal{A}$ which consists of models of cardinalities $\leq \eta$ with $Q \in \bigcap \mathcal{A}$ the following holds:
for every $q \in Q \cap \bigcap \mathcal{A}$ there is $p \geq q$ which is is $(\mathcal{A}, Q)$-semi-generic, i.e. for every $A \in \mathcal{A}$, p is (A, Q)-semi-generic. We call such p further $(\mathcal{A}, Q)$*-semi-generic.*

Definition 1.12 Let Q be a $\leq \eta$–piste structures almost proper forcing, $\mathcal{A}$ a finite structure with pistes which consists of models of cardinalities $\leq \eta$ with $Q \in \bigcap \mathcal{A}$. Let p be $(\mathcal{A}, Q)$-semi-generic. We call p *a minimal semi-generic for* $\mathcal{A}$ if for every $\mathcal{B}$ which extends $\mathcal{A}$ (in sense of [3]) there is $q \geq p$ which is $(\mathcal{B}, Q)$-semi-generic.

Definition 1.13 A forcing notion Q is called $\leq \eta$*–strongly piste structures semi-proper (or just strongly piste semi-proper)* iff it is $\leq \eta$–piste structures semi-proper and for every a finite structure with pistes $\mathcal{A}$ which consists of models of cardinalities $\leq \eta$ with $Q \in \bigcap \mathcal{A}$, for every $(\mathcal{A}, Q)$- semi-generic condition q there are $p \geq q$ and a finite structure with pistes $\mathcal{B}$ which extends $\mathcal{A}$ such that p is a minimal semi-generic for $\mathcal{B}$.

Our purpose will be to show the following:

Theorem 1.14 *Let κ be a supercompact cardinal and $\eta < \kappa$ be a regular cardinal. Then in a forcing extension which preserves all the cardinals $\leq \eta^+$ and turns κ into η^{++} the $\{\leq \eta\}$–strongly piste structures semi-proper SPFA holds, i.e. for every $\{\leq \eta\}$–strongly piste structures semi-proper forcing notion Q and for every collection $\mathcal{D}$ of $\leq \eta^+$ dense subsets of Q there is a filter on Q that meets all of them.*

Our prime interest was to try to apply this to certain variations of the forcing of [1] for sealing antichains. We had in mind the following variation of the sealing antichains forcing for a regular $\eta > \omega$. First we restrict ourself only to stationary subsets of η^+ consisting of ordinals of cofinality η. Second instead of using the Levy collapse and forcing a club into diagonal union by approximations of length η, we force both by finite conditions. Unfortunately the combination of this two forcings is not $\{\leq \eta\}$–strongly piste structures semi-proper.

2 Forcing conditions

We deal here with $\eta = \aleph_1$–case. The treatment of the general case is similar.

Let κ be a Mahlo cardinal. Fix an increasing continuous chain $\langle \mathfrak{M}_\alpha \mid \alpha < \kappa \rangle$ of $V_{\kappa+1}$ such that

1. $|\mathfrak{M}_\alpha| < \kappa$,

2. $\mathfrak{M}_\alpha \cap V_\kappa = V_{\kappa_\alpha}$, for some $\kappa_\alpha < \kappa$,

3. if κ_α is a regular cardinal, then it is an inaccessible,

4. κ_0 and each $\kappa_{\alpha+1}$ are inaccessible cardinals.

The following notion was introduced in [4]:

Definition 2.1 Suppose that $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is an iteration with initial segments in V_κ. Let $A \preceq V_\kappa$ and $X \in V_\kappa$.
We say α *is reachable from* A *in* 0*–steps* iff $\alpha \in A$. Define α is reachable from A in 1–step iff there are $\nu \in A \cap \alpha$ with $P_\nu \in A$ and $p \in P_\nu$, $p \Vdash_{P_\nu} \alpha \in A[\underset{\sim}{G}(P_\nu)]$. Call such p and ν a 1*–step reachability witnesses for* α. Continue by induction. α is reachable from A in $n+1$–steps iff there are n-steps reachability witnesses $\langle\langle p_k, \nu_k\rangle \mid k \leq n\rangle$, $\nu < \alpha$ and $p \in P_\nu$ such that

1. $p_i \in P_{\nu_i}$, for every $i \leq n$,

2. $p_i \leq p_j \restriction \nu_i$, for all $i \leq j \leq n$,

3. $\nu_i < \nu_j < \nu$, for all $i < j \leq n$,

4. p_i, ν_i are a 1–step reachability witnesses for ν_{i+1} with model $A[\underset{\sim}{G}(P_{\nu_{i-1}})]$, if $i > 0$ or with A, if $i = 0$, for every $i < n$,

5. $p_n \Vdash_{P_{\nu_n}} \nu, P_\nu \in A[\underset{\sim}{G}(P_{\nu_n})]$,

6. $p \restriction \nu_n \geq p_n$,

7. $p \Vdash_{P_\nu} \alpha \in A[\underset{\sim}{G}(P_\nu)]$.

Let us call α *is reachable from* A iff for some $n < \omega$, α is reachable from A in n–steps.

Given some $q \in P_\kappa$, let us define reachability from A relatively to q similar only requiring that witnesses extend q.

If $X \in V_\kappa$, then X *is reachable from* A iff $\text{rank}(X)$ and $f_{\text{rank}(X)}(X)$ are reachable from A, where $f_\xi : V_\xi \leftrightarrow |V_\xi|$ is some fixed in advance well ordering.

Remark 2.2 The only reachable ordinals from A will be members of A, once forcing notions under the consideration are proper instead of semi-proper.

Let us define now the forcing. The definition is similar to those of [4] with some modifications made in order to deal with the present situation.

Definition 2.3 Define by induction on $\tau \leq \kappa$ an iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \tau, \beta < \tau \rangle$.

1. If $\tau = \tau' + 1$ and $\kappa_{\tau'}$ is a singular, then let $P_\tau = P_{\tau'} * Cohen(\omega)$ (or just let $Q_{\tau'}$ to be trivial).

2. If τ is a limit ordinal or if $\tau = \tau' + 1$ and $\kappa_{\tau'}$ is a regular, then set $p = \langle \underset{\sim}{p}_\beta \mid \beta < \tau \rangle \in P_\tau$ iff

   (a) for each $\gamma < \tau$, with κ_γ regular, we have

      i. $p \restriction \gamma = \langle \underset{\sim}{p}_\beta \mid \beta < \gamma \rangle \in P_\gamma$,

      ii. $p \restriction \gamma \Vdash \underset{\sim}{Q}_\gamma$ is a semi-proper forcing notion and $|\underset{\sim}{Q}_\gamma| \leq \kappa_{\gamma+1}$.

   (b) There are three finite sets $s(p), j(p)$ and $m(p)$ (this sets can be read from p except when τ is a singular) such that

      i. $s(p) \subseteq \tau$ called the *support* of p is such that for each $\gamma \in s(p)$
      $p \restriction \gamma \Vdash \underset{\sim}{p}_\gamma \in \underset{\sim}{Q}_\gamma$,

      ii. $j(p) \subseteq \tau$ called the *jumps* of p.
      This will be places in p where change of sequences of models will be allowed.

      iii. $m(p) = \{A_0, ..., A_{k(p)-1}\}$, $k(p) < \omega$, is a finite set called the *models* of p.
      Let us state the requirements on $m(p)$. For every $i < k(p)$ the following hold:

      A. $A_i \in V$,

      B. $|A_i| = \aleph_0$ or $|A_i| = \aleph_1$ or $A_i = V_\delta$ for some inaccessible $\delta < \kappa_\tau$,

      C. for every $i < k(p)$,$s, 1 \leq s \leq n_i$,
      $A_i \prec V_{\kappa_\tau}$, if κ_τ is an inaccessible,
      and $A_i \prec V_{\kappa_{\tau+1}}$, otherwise, i.e. whenever τ is a limit ordinal and κ_τ is a singular cardinal,

   (c) For each $\gamma < \tau$ let us specify a sequence of models based on a set of models from $m(p)$ which will stand over the coordinate γ.
   Let $\gamma < \tau$. Take all models $A \in m(p)$ such that
   $p \restriction \gamma \Vdash_{P_\gamma} \kappa_\gamma, P_\gamma, \underset{\sim}{Q}_\gamma$ are reachable from A.
   Denote the set of all such models by $\tilde{A}''_\gamma$.
   Let $G(P_\gamma) \subseteq P_\gamma$ be a generic with $p \restriction \gamma \in G(P_\gamma)$. Set $\tilde{A}'_\gamma = \{A[G(P_\gamma)] \cap$

$V_{\kappa_{\gamma+1}}[G(P_\gamma)] \mid A \in \tilde{A}''_\gamma, \kappa_{\gamma+1} \in A[G(P_\gamma)]\} \cup \{A[G(P_\gamma)] \mid A \in \tilde{A}''_\gamma, A[G(P_\gamma)] \subseteq V_{\kappa_{\gamma+1}}[G(P_\gamma)]$. Let $\tilde{A}_\gamma$ be the set obtained from $\tilde{A}'_\gamma$ by adding to it all intersections of countable members of $\tilde{A}'_\gamma$ with uncountable ones.
Back in V, let $\underset{\sim}{\tilde{A}}_\gamma$ be a name of such $\tilde{A}_\gamma$.
We require the following:

i. $p \restriction \gamma \Vdash_{P_\gamma}$ the set $\underset{\sim}{\tilde{A}}_\gamma$ is a structure with pistes in sense of [3].

ii. If $\gamma \in s(p)$, then $p \restriction \gamma \Vdash_{P_\gamma} \underset{\sim}{p}_\gamma$ is a minimal semi-generic for $(\underset{\sim}{Q}_\gamma, \underset{\sim}{\tilde{A}}_\gamma[\underset{\sim}{G}(P_\gamma)])$.

(d) (Jumps) Suppose that for some $\gamma < \tau, k, l < \omega$
we have $p \restriction \gamma \Vdash_{P_\gamma} \underset{\sim}{A}_{\gamma 0}[\underset{\sim}{G}(P_\gamma)], \underset{\sim}{A}_{\gamma 1}[\underset{\sim}{G}(P_\gamma)] \in \underset{\sim}{\tilde{A}}_\gamma[\underset{\sim}{G}(P_\gamma)]), \underset{\sim}{A}_{\gamma 0}[\underset{\sim}{G}(P_\gamma)] \in \underset{\sim}{A}_{\gamma 1}[\underset{\sim}{G}(P_\gamma)]$ and $|\underset{\sim}{A}_{\gamma 1}[\underset{\sim}{G}(P_\gamma)]| < |\underset{\sim}{A}_{\gamma 0}[\underset{\sim}{G}(P_\gamma)]|$.
Let $\gamma^* \leq \gamma$ be the maximal element of $j(p)$ below γ, if exists or 0 otherwise. Let $A \in m(p)$ be any model such that $\underset{\sim}{A}_{\gamma 1}$ is obtained from A, as in (2c) above(i.e. as a member of $\tilde{A}'_\gamma$ or $\tilde{A}_\gamma$, forced by $p \restriction \gamma$). Pick the least element $\gamma^{**} \geq \gamma^*$ which is reachable from A. We require that there is $B \in m(p)$ such that

i. $p \restriction \gamma \Vdash_{P_\gamma} \underset{\sim}{A}_{\gamma 0}$ can be obtained from B, as in (2c) above.

ii. $p \restriction \gamma \Vdash_{P_{\gamma^{**}}} B \in A[\underset{\sim}{G}(P_{\gamma^{**}})]$.
In particular, if $j(p) \cap \gamma + 1 = \emptyset$, then just $B \in A$.

Definition 2.4 Let $\tau \leq \kappa$ and let $p = \langle \underset{\sim}{p}_\beta \mid \beta < \tau \rangle, p' = \langle \underset{\sim}{p}'_\beta \mid \beta < \tau \rangle \in P_\kappa$. Set $p \geq p'$ (p is stronger than p') iff

1. $m(p) \supseteq m(p')$,
2. $s(p) \supseteq s(p')$,
3. $j(p) \supseteq j(p')$,
4. for every $\beta \in s(p')$ we have $p \restriction \beta \Vdash_{P_\beta} \underset{\sim}{p}_\beta \geq_{\underset{\sim}{Q}_\beta} \underset{\sim}{p}'_\beta$,

Let us start with a following lemma.

Lemma 2.5 *Let Q be an $\{\omega, \omega_1\}$–semi proper forcing. Let A, B be elementary submodels of cardinalities $\aleph_0$ and $\aleph_1$ respectively such that ${}^\omega B \subseteq B$, $Q \in A, B$ and $B \in A$. Then for every $q \in Q \cap A \cap B$ and every $p \geq q$ which is semi-generic for both A and B, we have*

$$p \Vdash A[\underset{\sim}{G}(Q)] \cap B[\underset{\sim}{G}(Q)] = (A \cap B)[\underset{\sim}{G}(Q)].$$

Remark 2.6 1. A special kind of reflection of A into B was needed in [4] in the argument that κ remains a cardinal, and a similar argument will be used here with ω replaced by ω_1. In present situation - once A is countable and B has cardinality $\aleph_1$ it looks impossible make such reflections once $Q \not\subseteq B$. The lemma allows us to use a simple reflection over $A \cap B$ of A to some $A' \in B$ instead. Then $A \cap B = A' \cap B'$, where B' is the image of B under the isomorphism between A and A'. So

$$A[G(Q)] \cap B[G(Q)] = (A \cap B)[G(Q)] = (A' \cap B')[G(Q)],$$

and the intersection is handled from inside of B.
The splitting property of Q will be used in order to generate A, A'-generic conditions.

2. Let $G(Q)$ be a generic subset of Q with $p \in G(Q)$. Note that $(A \cap B)[G(Q)] \cap \omega_1 \subseteq A[G(Q)] \cap \omega_1 = A \cap \omega_1$ and $(A \cap B)[G(Q)] \cap \omega_1 \subseteq B[G(Q)] \cap \omega_1 = B \cap \omega_1$. Hence $(A \cap B)[G(Q)] \cap \omega_1 \subseteq A \cap B \cap \omega_1$. So $(A \cap B)[G(Q)] \cap \omega_1 = A \cap B \cap \omega_1$. Which means that p is $A \cap B$–semi-generic.

3. It follows from the lemma that $A[G(Q)] \cap B[G(Q)] \in B[G(Q)]$, since $A \cap B \in B$ (just B is closed under ω–sequences which are in V) and so $(A \cap B)[G(Q)] \in B[G(Q)]$.
Note that $B[G(Q)]$ need not be closed under ω-sequences. Just, for example take Q to be the Cohen forcing adding ω_2–Cohen reals.

Proof. Let $G(Q) \subseteq Q$ generic with $p \in G(Q)$.
Clearly $A[G(Q)] \cap B[G(Q)] \supseteq (A \cap B)[G(Q)]$, for every generic $G(Q) \subseteq Q$.
Let us prove the opposite direction. Let $\alpha \in A[G(Q)] \cap B[G(Q)]$. So there is a Q-name $\underset{\sim}{\alpha} \in B$ such that $\alpha = \underset{\sim}{\alpha}_{G(Q)}$. Then, by elementarity,

$$A[G(Q)] \models (\exists \underset{\sim}{\alpha} \in B \quad \underset{\sim}{\alpha}_G(Q) = \alpha).$$

Pick such a name $\underset{\sim}{\alpha}$ of α in $(A[G(Q)])^V$. Recall that $|B| = \aleph_1$. So there is $\langle x_\tau \mid \tau < \omega_1 \rangle$ an enumeration of B in A. By ω–semi-properness we have $A[G(Q)] \cap \omega_1 = A \cap \omega_1$. Hence no elements of B can appear in $A[G(Q)] \setminus A$. In particular $\underset{\sim}{\alpha} \in A$. Hence $\alpha = \underset{\sim}{\alpha}_{G(Q)} \in (A \cap B)[G(Q)]$.
□

In a general case (i.e. $\eta \geq \omega_1$) we use the following observation:

Lemma 2.7 *Suppose that Q be an $\leq \eta$–piste structures semi proper forcing. Let A be a countable elementary submodel, $Q \in A$. Then for every $q \in Q \cap A$ there is $p \geq q$ which is $\leq \eta$–semi-generic for A, i.e.*

$$p \Vdash A[\underset{\sim}{G}(Q)] \cap \eta^+ = A \cap \eta^+.$$

Proof. Let $n^* < \omega$. Suppose that Q is $\leq \omega_{n^*}$–semi-proper. We show by induction that for every $n \leq n^*$, there is $p_n \geq q$ such that

$$p_n \Vdash A[\underset{\sim}{G}(Q)] \cap \omega_{n+1} = A \cap \omega_{n+1}.$$

Start with $n = 1$. Use the fact that for each term (Skolem function) t, $\sup(t''\omega_1) \in A$. Hence, if $A_1 = Hull(A \cup \omega_1)$ then $A_1 \cap \omega_2 = \sup(A \cap \omega_2)$. Now if $p_1 \geq q$ is both $\{\omega\}$–semi-generic for A and $\{\omega_1\}$–semi-generic for A_1, then

$$p_1 \Vdash A[\underset{\sim}{G}(Q)] \cap \omega_2 = A \cap \omega_2.$$

Piste structures semi properness provides such p_1, since $\{A, A_1\}$ is (or easily generates) a structure with pistes.
At stage n the same argument works only we will need $p_n \geq q$ which is now simultaneously semi-generic for models $A \subseteq A_1 \subseteq ... \subseteq A_n$, where $A_k = Hull(A \cup \omega_k)$, $1 \leq k \leq n$. Again $A_k \cap \omega_k = \sup(A \cap \omega_k)$.

The case $\eta \geq \omega_\omega$ is treated similar.

□

Next three lemmas deal with semi properness and their proofs are similar to those of [4].

Lemma 2.8 *The forcing P_α is $\{\omega\}$–semi-proper (i.e. semi-proper) for every $\alpha \leq \kappa$.*

Proof. Let $M \prec H(\chi)$ be a countable elementary submodel, $r \in P_\kappa$ and $r, P_\kappa \in M$, for some $\chi > \kappa$ large enough. Extend r to a condition r^* by adding M to every sequence in $m(r)$ (as the largest model under $\in$). We claim that $r^* \restriction \alpha$ is P_α–semi-generic over M, for every $\alpha \leq \kappa, \alpha \in M$. Let us prove this by induction on α. Let $p \geq r^* \restriction \alpha, p \in P_\alpha$ and $\underset{\sim}{\mu} \in M$ a P_α–name of a countable ordinal.

It is enough to find some $\tilde{p}$ which is compatible with p and forces $\underset{\sim}{\mu} \in M \cap \omega_1$.

Case 1. *α is a successor ordinal or α is a limit ordinal and there are ordinals in $M \cap \alpha$ above $s(p) \cup j(p)$.*

Let $\eta = \alpha'$, if $\alpha = \alpha' + 1$ and if α is a limit ordinal, then let η be the first element $M \cap \alpha$ of above members of $s(p) \cup j(p)$.
Force with P_η. Let G_η be generic with $p \restriction \eta \in G_\eta$. By induction, $M[G_\eta] \cap \omega_1 = M \cap \omega_1$.
Work in $(M[G_\eta])^V$(i.e. the ground model of $M[G_\eta]$; note that it may be bigger than M, need not be in V, but it is equal to $M[G_\eta] \cap V$) pick an extension r' of $r \restriction \alpha$ such that $m(r')$ includes the restrictions to $(M[G_\eta])^V$ of sequences of $m(p)$. Recall that there are only finitely many models that are involved in this sequences and all relevant ones are in $(M[G_\eta])^V$ by the choice of η.
Consider the set $D_\eta = \{t \in P_\eta \mid \exists t' \in P_\alpha$ such that $t' \restriction \eta = t \in P_\alpha, t' \geq r'$ and it decides $\underset{\sim}{\mu}\}$. D_η is in $(M[G_\eta])^V$ and is dense in P_η. Pick some $t \in G_\eta \cap D_\eta$ in $M[G_\eta]$. Let $t' \in (M[G_\eta])^V$ be a witness forcing $\underset{\sim}{\mu}$ to be some $\nu < \omega_1$. Then $\nu \in M[G_\eta] \cap \omega_1 = M \cap \omega_1$.
Let us argue that t' is compatible with p. The only problem that may lead to incompatibility is that for some $\beta \in \alpha \setminus s(p)$ we have $\beta \in s(t')$ and t'_β is not semi-generic for some countable model of p_β. Consider such β. Then β must be one of the coordinates of t', since t was in G_η which is generic for P_η. Remember that $t' \in (M[G_\eta])^V$ and hence $\underset{\sim}{t}'_\beta \in (M[G_\eta])^V$ as well. By the definition of order the model M (and actually, $M[G_\eta]$) appears among models of p_β. The part of p_β which consists of elements of M^{P_β} is in fact included into r'_β. But t'_β is Q_β–semi-generic for every countable model of r'_β, and hence of the part of p_β below M^{P_β}. Also $t'_\beta \in (M[G_\eta])^V$, since there are only finitely many places where $s(r')$ is increased and so all of them are inside $(M[G_\eta])^V$. Now everything follows, since p_β is $\in$–increasing, closed under intersections sequence with M inside and Q_β is a semi-proper forcing. Note that there may be a need to add η to the set of jumps, once some of models of t' are in $(M[G_\eta])^V \setminus M$. Now everything follows, since p_β is a structure with pistes with M inside and Q_β is a $\{\leq \aleph_1\}$–strongly piste structures semi-proper forcing.

Case 2. *α is a limit ordinal and the ordinals of $M \cap \alpha$ are bounded below* $\max(s(p) \cup j(p))$.

By extending p if necessary we can assume that $\max(s(p))$ is above $j(p)$.
A new point here relatively to the iteration of proper forcing notions is that a generic extension $M[G_\alpha]$ can have new ordinals (i.e. ordinals not in M) even if $M[G_\alpha] \cap \omega_1 = M \cap \omega_1$. If $\alpha < \omega_2$, then the treatment of semi-proper and proper cases is identical, but if $\alpha \geq \omega_2$, then in a semi-proper case $M[G_\alpha] \setminus M$ may have ordinals in $s(p)$.
If no ordinal $\zeta, \min(s(p) \setminus M) \leq \zeta < \alpha$ is reachable from M (relatively to p), then take η to be an element of $M \cap \alpha$ above $M \cap s(p)$ and repeat the argument of the previous case.
Otherwise find some $n < \omega$, $\xi < \alpha$ and $q \in P_\xi$ compatible with $p \restriction P_\xi$ which witness reachability from M of some $\zeta < \alpha$ in n–steps and so that $s(p) \subseteq \zeta$ or no element of

$\geq \min(s(p) \setminus \zeta + 1)$ is reachable relatively to q. Suppose for simplicity that $n = 1$. The treatment of the general case is similar.
Then $\xi \in M$, by the definition of reachability witnesses. Force with P_ξ. Let G_ξ be a generic with $p \restriction \xi, q \in G_\xi$. By induction, $M[G_\xi] \cap \omega_1 = M \cap \omega_1$. We have $\zeta \in M[G_\xi]$.
Work in $V[G_\xi]$. Take $\eta = \zeta$ and repeat the argument of Case 1. The requirement (2d) of 2.3 applies in order to deal with elements of $j(p)$ which are below $\sup(M \cap \alpha)$, but not in $M[G_\eta]$ (if any).
Let us clarify the point that does not occur in Case 1.
There are ordinals ξ in $s(p)$ above η. Now we extend inside M in order to decide $\underset{\sim}{\mu}$. As a result models which belong to M may be added. There is no problem with countable models and those of inaccessible size, since the former are contained in M, and so ξ and then $\underset{\sim}{Q}_\xi$ do not belong to them which implies that there are no requirements about genericity. There are no genericity requirements for models of inaccessible size. But once a model B of size $\aleph_1$ is added, then ξ may belong to B. Thus let, for simplicity $\alpha = \omega_1$. Suppose $\xi \in M \cap \omega_1$. Then $\xi \in B$, for every B of size $\aleph_1$. So why $\underset{\sim}{q}_\xi$ and B are not contradictory?
The reason is that $\underset{\sim}{q}_\xi$ is minimal semi-generic. Here is the only place where the minimality is used. We apply 2(c)ii of Definition 2.3.
$\square$

Lemma 2.9 *The forcing P_α is $\{\omega_1\}$–semi proper for every $\alpha \leq \kappa$.*

Proof. Let $M \prec H(\chi)$ be an elementary submodel of cardinality $\aleph_1$, $r \in P_\alpha$ and $r, P_\alpha \in M$, for some $\chi > \kappa$ large enough. Extend r to a condition r^* by adding M to every sequence in $m(r)$ (as the largest model under $\in$). We claim that r^* is P_α–semi-generic over M, for every $\alpha \leq \kappa, \alpha \in M$. Let us prove this by induction on α. Let $p \geq r^*$ and $\underset{\sim}{\mu} \in M$ a P_α–name of an ordinal less than ω_2. It is enough to find some $\tilde{p}$ which is compatible with p and forces $\underset{\sim}{\mu} \in M \cap \omega_2$. We can assume without loss of generality that p decides $\underset{\sim}{\mu}$. Just extend it to such condition if necessary. Also assume that there is a countable model A which maximal $(\in)$ in $m(p)$, i.e. for every $B \in m(p)$, either $B \in A$ or $B = A$.

Case 1. *α is a successor ordinal or α is a limit ordinal and there are ordinals in $M \cap \alpha$ above $s(p)$ and $j(p)$.*

Let $\eta = \alpha'$, if $\alpha = \alpha' + 1$ and if α is a limit ordinal, then let η be the first element $M \cap \alpha$ of above members of $s(p) \cup j(p)$.
Force with P_η. Let G_η be generic with $p \restriction \eta \in G_\eta$. By induction, $M[G_\eta] \cap \omega_2 = M \cap \omega_2$.
Reflect A to some $A' \in (M[G_\eta])^V$ over $A \cap M$. Recall that by Lemma 2.8 we have $A[G_\eta] \cap$

$M[G_\eta] = (A \cap M)[G_\eta]$. Use the isomorphism $\pi_{AA'}$ between A and A' to move the part of p above η down into A'. Let p' be the result. The reflection is strong enough to insure that p' decides $\underset{\sim}{\mu}_{G_\eta}$. Also, $p' \restriction \eta \in G_\eta$. Now $p' \in (M[G_\eta])^V$, hence the decided value is in $M[G_\eta] \cap \omega_2 = M \cap \omega_2$.
Set $\tilde{p} = p'$. Then it will be as desired.

Case 2. *α is a limit ordinal and the ordinals of $M \cap \alpha$ are bounded below* $\max(s(p) \cup j(p))$.

By extending p if necessary we can assume that $\max(s(p))$ is above $j(p)$.

If no ordinal $\zeta, \min(s(p) \setminus M) \leq \zeta < \alpha$ is reachable from M (relatively to p), then take η to be an element of $M \cap \alpha$ above $M \cap s(p)$ and repeat the argument of the previous case. Reflect A to some $A' \in (M[G_\eta])^V$ over $A \cap M$. Recall that by Lemma 2.8 we have $A[G_\eta] \cap M[G_\eta] = (A \cap M)[G_\eta]$. Use the isomorphism $\pi_{AA'}$ between A and A' to move the part of p above η down into A'. Let p' be the result.
The reflection here strong enough to insure that p' decides $\underset{\sim}{\mu}_{G_\eta}$. $s(p)$ contains elements outside and even above those reachable from M, but note that once we have $\xi \in s(p) \setminus M$, then $\underset{\sim}{Q}_\xi \notin M$ and hence there is no need to worry about $\underset{\sim}{q}_\beta$ being semi-generic over M (and models that belong to it) by Definition 2.4(4). Recall also that M has size $\aleph_1$, hence it contains every $B \in M$ of cardinality $\aleph_0, \aleph_1$. So, once $\underset{\sim}{Q}_\xi \notin M$, then it will not be a member of any such B.

Otherwise find some $n < \omega$, $\xi < \alpha$ and $q \in P_\xi$ compatible with $p \restriction P_\xi$ which witness reachability from M of some $\zeta < \alpha$ in n–steps and so that $s(p) \subseteq \zeta$ or no element of $\geq \min(s(p) \setminus \zeta + 1)$ is reachable relatively to q. Suppose for simplicity that $n = 1$. The treatment of the general case is similar.
Then $\xi \in M$, by the definition of reachability witnesses. Force with P_ξ. Let G_ξ be a generic with $p \restriction \xi, q \in G_\xi$. By induction, $M[G_\xi] \cap \omega_2 = M \cap \omega_2$. We have $\zeta \in M[G_\xi]$.
Work in $V[G_\xi]$. Take $\eta = \zeta$ and proceed again as in Case 1.
□

Lemma 2.10 *The forcing P_κ preserves κ.*

Proof. It repeats the proof of Lemma 1.8 of [4] only replace a countable model A there by a model of cardinality $\aleph_1$.
□

Now, if κ is a supercompact and a Laver function $F : \kappa \to V_\kappa$ supplies semi-proper forcings, then $\{\leq \aleph_1\}$–strongly piste structure semi-proper SPFA will hold in $V[G(P_\kappa)]$.

3 Some examples of $\{\leq \omega_1\}$–strongly piste structures semi-proper and proper forcings.

Unfortunately, we do not have many examples. Let us state few that we do have.

3.1 Prikry type forcings.

Prikry, Magidor, Radin and similar forcings are $\{\leq \omega_1\}$–strongly piste structures semi-proper forcings. The point is that we can just take the intersection of all sets of measures one which are in a given finite piste structure $\mathcal{A}$. This will produce a minimal semi-generic condition.

Consider the forcing of [2] which changes a cofinality of a measurable cardinal λ to ω_1 without adding new countable sequences of ordinals. This forcing, denote it by Q, is $\leq \aleph_1$–semi-proper, but there is a stationary set $S \subseteq [H(\chi)]^\omega$ such that for each $A \in S$ we have the following:

1. $A \preceq H(\chi)$,
2. $Q \in A$,
3. for every $q \in A \cap Q$ there is $p \geq q$ such that p is (A, Q)-generic.

This forcing fails to be ω–strongly piste proper, since there are countable models without generic conditions over them. But one can easily redefine the notion and require existence of generic conditions only for models in a stationary set S. Let us call such the result $\leq \aleph_1$–strongly almost piste proper with respect to S. The proofs of the corresponding iteration lemmas remains the same only in the reflection argument a model from S should be reflected into a model of size $\aleph_1$.

3.2 Examples of $\{\leq \omega_1\}$–strongly piste structures proper forcings

Trivially, c.c.c. forcings are $\{\leq \omega_1\}$–strongly piste structures proper forcings. But also Cohen forcings for adding subsets to $\aleph_1$ are such as well.

The next example will be adding of a collapsing function will be added by finite conditions. A variation of the forcing of [3] will be used for this purpose. Let us describe the forcing conditions.

Definition 3.1 Let Q be the set of all finite structures with pistes of length 2 over ω_3, i.e. the set of all pairs $\langle\langle A^{0\omega}, A^{1\omega}, C^\omega\rangle, \langle A^{0\omega_1}, A^{1\omega_1}, C^{\omega_1}\rangle\rangle$ such that

1. $A^{0\omega_1} \preceq \langle H(\omega_3), \in, \leq \rangle$,

2. $|A^{0\omega_1}| = \aleph_1$,

3. $A^{0\omega_1} \in A^{1\omega_1}$,

4. $A^{1\omega_1}$ is a finite $\in$-increasing sequence of elementary submodels of $A^{0\omega_1}$ of cardinality $\aleph_1$, i.e. basically ordinals,

5. If $B \in A^{1\omega_1}$ and $\mathrm{cof}(B \cap \omega_2) = \omega_1$, then ${}^{\omega}B \subseteq B$ or, in not CH situation, as stated in 3.3, $B = B'[G(P_\alpha)]$ with $B' \in V$ and $({}^{\omega}B') \cap V \subseteq B'$.

6. If $X \in A^{1\omega_1}$, $\mathrm{cof}(X \cap \omega_2) = \delta$, then there is an increasing continuous chain $\langle X_i \mid i < \delta \rangle$ of elementary submodels of X such that

   (a) $\bigcup_{i<\delta} X_i = X$,

   (b) $X_i \in X$, for every $i < \delta$,

   (c) $|X_i| = \aleph_1$, for every $i < \delta$.

7. $\mathrm{dom}(C^{\omega_1}) = A^{1\omega_1}$,

8. for every $B \in \mathrm{dom}(C^{\omega_1})$,$C^{\omega_1}(B) = (A^{1\omega_1} \cap B) \cup \{B\}$,

9. If $X \in A^{1\omega_1}$, $\mathrm{cof}(X \cap \omega_2) = \delta$, $A \in A^{1\omega}$ and $X \in A$, then there is an increasing continuous chain $\langle X_i \mid i < \delta \rangle$ of elementary submodels of X such that

   (a) $\langle X_i \mid i < \delta \rangle \in A$,

   (b) $\bigcup_{i<\delta} X_i = X$,

   (c) $X_i \in X$, for every $i < \delta$,

   (d) $X_A := \bigcup_{i \in A} X_i \in A^{1\omega_1}$.

   Note that if $\delta = \omega$, then $X_A = X$.

10. Let $A \in A^{1\omega}, X \in A^{1\omega_1}$ and $X \in A$. If $Z \in C^{\omega_1}(X_A)$ (i.e. $Z \in X_A \cap A^{1\omega_1}$), then there is $Z' \in C^{\omega_1}(X_A) \cap A$ such that $Z' \supseteq Z$.

11. Suppose that Y is not the minimal element of $A^{1\omega_1}$. Let Y_0 be its immediate predecessor in $A^{1\omega_1}$. If $X \in A^{1\omega} \cap Y$, then

- $Y_0 \in X$

  or
- $X \in Y_0$

  or
- $X \subset Y_0, X \notin Y_0$ and then for every $Z \in C^{\omega_1}(Y_0)$ there is $Z' \in C^{\omega_1}(Y_0) \cap X$ such that $Z' \supseteq Z$.

12. If $A \in A^{1\omega}$ and $A \not\subseteq A^{0\omega_1}$, then $A^{0\omega_1} \in A$.

    Let us state requirements on $A^{1\omega}$.

13. $A^{0\omega} \preceq \langle H(\omega_3), \in, \leq \rangle$,
14. $|A^{0\omega}| = \aleph_0$,
15. $A^{0\omega} \in A^{1\omega}$,
16. in not CH situation we require that each member A of $A^{1\omega}$ is of the form $A = A'[G(P_\alpha)]$ with $A' \in V$,
17. if $X, Y \in A^{1\omega}$, then $X \in Y$ iff $X \subsetneq Y$,
18. if $X \in A^{1\omega}$, then X has at most two immediate predecessor in $A^{1\omega}$,
19. $\mathrm{dom}(C^{\omega}) = A^{1\omega}$,
20. for every $B \in A^{1\omega}$ the following holds:
    (a) $B \in C^{\omega}(B)$,
    (b) $C^{\omega}(B)$ is an $\in$–chain of models in $A^{1\omega} \cap (B \cup \{B\})$,
    (c) if $X \in C^{\omega}(B)$, then $C^{\omega}(X) = \{Y \in C^{\omega}(B) \mid Y \in X \cup \{X\}\}$,
    (d) if B has immediate predecessors in $A^{1\omega}$, then one of them is in $C^{\omega}(B)$.
21. Let $X \in A^{1\omega}$. Then either
    (a) X is minimal under $\in$ (or equivalently under $\supseteq$) in $A^{1\omega}$

    or

    (b) X has a unique immediate predecessors in $A^{1\omega}$,

    or

(c) X has exactly two immediate predecessors X_0, X_1 in $A^{1\omega}$ and then

- X, X_0, X_1 form a Δ–*system triple* relatively to some $F_0, F_1 \in A^{1\omega_1}$ which means the following:

  i. $F_0 \subsetneq F_1$(or $F_1 \subsetneq F_0$,

  ii. $F_0 \in X_0$ and $F_1 \in X_1$,

  iii. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

  iv. the structures
  $\langle X_0, \in X_0 \cap A^{1\omega}, X_0 \cap A^{1\omega_1}, (C^{\omega} \upharpoonright X_0 \cap A^{1\omega}) \cap X_0, (C^{\omega_1} \upharpoonright X_0 \cap A^{1\omega_1}) \cap X_0\rangle$
  and
  $\langle X_1, \in X_1 \cap A^{1\omega}, X_1 \cap A^{1\omega_1}, (C^{\omega} \upharpoonright X_1 \cap A^{1\omega}) \cap X_1, (C^{\omega_1} \upharpoonright X_1 \cap A^{1\omega_1}) \cap X_1\rangle$
  are isomorphic over $X_0 \cap X_1$

Further we will refer to such X as a *splitting point.*

22. If X is a splitting point, X_0, X_1 are its immediate predecessors and $F_0, F_1 \in A^{1\omega_1}$ witness this, then $F_0 \in F_1$ implies that there are no elements of $A^{1\omega_1}$ between X_1 and X. $F_1 \in F_0$ implies that there are no elements of $A^{1\omega_1}$ between X_0 and X.

23. If $X \in A^{1\omega}$, $Y \in A^{1\omega} \cup A^{1\omega_1}$ and $Y \in X$, then Y is a *piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i < n\rangle$ of elements of $A^{1\omega}$ which we call a piste leading to Y such that

    (a) $X = X(0)$,

    (b) for every $i, 0 < i \leq n$, $X(i) \in C^{\omega}(X(i-1))$ or $X(i-1)$ has two immediate predecessors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^{\omega}(X(i-1)), X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,

    (c) $Y = X(n)$, if $Y \in A^{1\omega}$ and if $Y \in A^{1\omega_1}$, then $Y \in X(n)$, $X(n)$ and Y is not a member of any element of $X(n) \cap A^{1\omega}$.

24. Let $A, B \in A^{1\omega} \cup A^{1\omega_1}$ and $|A| \neq |B|$. If A is a least model in $A^{1|A|}$ with $B \in A$ and B is a potentially limit point, then A is such as well.

25. Either $A^{0\omega} \in A^{0\omega_1}$ and then $A^{1\omega} \subseteq A^{0\omega_1}$
    or $A^{0\omega_1} \in A^{0\omega}$ and then $A^{1\omega_1} \setminus \{A^{0\omega_1}_{A^{0\omega}}\} \subseteq A^{0\omega}$.
    Or $A^{0\omega} \subseteq A^{0\omega_1}$ and $\sup(A^{0\omega} \cap \omega_2) = \sup(A^{0\omega_1} \cap \omega_2)$.

26. It is allowed that one or both of $A^{1\omega}, A^{1\omega_1}$ are empty.

Define the order on Q.

Definition 3.2 Let $p_0 = \langle\langle A_0^{0\omega}, A_0^{1\omega}, C_0^{\omega}\rangle, \langle A_0^{0\omega_1}, A_0^{1\omega_1}, C_0^{\omega_1}\rangle\rangle$,
$p_1 = \langle\langle A_1^{0\omega}, A_1^{1\omega}, C_1^{\omega}\rangle, \langle A_1^{0\omega_1}, A_1^{1\omega_1}, C_1^{\omega_1}\rangle\rangle$ be in Q.
Then $p_0 \leq p_1$ iff

1. $A_0^{1\omega_1} \subseteq A_1^{1\omega_1}$,

2. $A_0^{1\omega} \subseteq A_1^{1\omega}$,

3. for every $A \in A_0^{1\omega}, C_0^{\omega}(A) \subseteq C_1^{\omega}(A)$.

Remark 3.3 Below (Lemma 3.6) the following property will be used:
for models M of cardinality ω_1 and A countable with $M \in A$, $A \cap M \in M$.
It is automatic once M is closed under ω–sequences. But note that the iteration P_κ adds κ-many reals, so models M in V^{P_α}'s cannot be closed. However we can restrict ourself only to models of the form $B[G(P_\alpha)]$ with $B \in V$ and $({}^{\omega}B) \cap V \subseteq B$ once $|B| = \aleph_1$. Then Lemma 2.5, 2.6(3) can be applied.

Let us state the intersection properties of models involved in Q. They are just simplified versions of more general intersection properties of [3]. We refer to [3] for the proofs.

Lemma 3.4 *Let* $\langle\langle A^{0\omega}, A^{1\omega}, C^{\omega}\rangle, \langle A^{0\omega_1}, A^{1\omega_1}, C^{\omega_1}\rangle\rangle \in Q$ *and* $A \in A^{1\omega}, B \in A^{1\omega_1}$. *Then either* $A \in B$ *or there is* $B' \in A^{1\omega_1} \cap A$ *such that* $A \cap B = A \cap A'$.

Lemma 3.5 *Let* $\langle\langle A^{0\omega}, A^{1\omega}, C^{\omega}\rangle, \langle A^{0\omega_1}, A^{1\omega_1}, C^{\omega_1}\rangle\rangle \in \mathcal{P}(S)$ *and* $A, B \in A^{1\omega}$. *Then either* $A = B$ *or* $A \in B$ *or* $B \in A$ *or there are* $X \in A^{1\omega_1} \cap A, A' \in A^{1\omega} \cap (A \cup \{A\})$ *such that* $A \cap B = A' \cap X$.

Lemma 3.6 Q *is* $\{\omega_1\}$*–proper forcing.*

Proof. Let $M \preceq H(\chi)$ of cardinality $\aleph_1$, ${}^{\omega}M \subseteq M$ (or $M = B[G(P_\alpha)]$ with $B \in V$ and $({}^{\omega}B) \cap V \subseteq B$, as explained in 3.3), $q \in Q \cap M$. Extend q by adding to it $M \cap H(\omega_3)$ as the largest model. Denote the result by $q^\frown M \cap H(\omega_3)$. We claim that $q^\frown M \cap H(\omega_3)$ is (M, Q)–generic.
Let $p \geq q^\frown M \cap H(\omega_3)$, $D \subseteq Q$ is dense and $D \in M$. Find a condition in $D \cap M$ which is compatible with p.

Extending p if necessary, we can assume that there is countable A in p such that all the rest of p is in A.
Reflect now A to some $A' \in M$ which realizes the same type as A does in a rich enough language over $A \cap M$. Then p will reflect down to some $p' \in M \cap D$. Such p' is compatible with p. Just combine sequences of p' and p into one sequence.
□

Lemma 3.7 *Q is $\{\omega\}$–proper forcing.*

Proof. Let $M \preceq H(\chi)$ of be countable, $q \in Q \cap M$. Extend q by adding to it $M \cap H(\omega_3)$ as the largest model. Denote the result by $q^\frown M \cap H(\omega_3)$. We claim that $q^\frown M \cap H(\omega_3)$ is (M, Q)–generic.
Let $p \geq q^\frown M \cap H(\omega_3)$, $D \subseteq Q$ is dense and $D \in M$. Find a condition in $D \cap M$ which is compatible with p.
Let $p \restriction M$ be a subsequence of p which consists of the models of p which belong to M. Extend $p \restriction M$ to some $p' \in D \cap M$. Such p' is compatible with p. Just combine sequences of p' and p into one sequence. Note that there was no need to reflect here since if $X \in Y$ implies $X \subset Y$ for models X, Y of a same size. So if for some countable B in p we have $M \cap H(\omega_3) \in B$, then $M \cap H(\omega_3) \subseteq B$ and all the editions made in p' will in B automatically.
□

Lemma 3.8 *The forcing with Q collapses $\aleph_3$ to $\aleph_2$.*

Proof. Let $G \subseteq Q$ generic. Consider the set $X := \{B \mid |B| = \aleph_1, \exists p \in G, B \in p\}$. Let F be a partial function from ω_2 to ω_3^V defined by setting $F(B \cap \omega_2) = \sup(B \cap \omega_3^V)$. It is well defined and has an unbounded range in ω_3^V.
Alternatively, it is possible and probably more convenient to use $H(B \cap \omega_2) = B \cap \omega_3^V$. By genericity $\bigcup \text{ran}(H) = \omega_3^V$.
□

Lemma 3.9 *The forcing Q is $\{\leq \aleph_1\}$–strongly piste structures proper forcing.*

Proof. Let $\langle A_0, ..., A_n \rangle$ be a structure with pistes which consists of elementary submodels of $H(\chi)$ of sizes $\aleph_0, \aleph_1$ with Q inside. Let us show that for every $q \in Q \cap \bigcap_{i \leq n} A_i$ there is $p \geq q$ which is (A_i, Q)–generic for every $i \leq n$.

Extend q by adding to it $A_i \cap H(\omega_3)$, for every $i \leq n$. Let p be the result. By Lemmas 3.6,3.7, p will be (A_i, Q)–generic for each $i \leq n$.
$\square$

Recall that in not CH situation we need to weaken the notion of strongly piste structures proper forcing (Definition 1.5) in order to include Q. It is possible to restrict ourself to this situation using only submodels that are generic extensions of submodels in V.

Remark 3.10 Note that if $B_0, B_1 \prec H(\chi)$ (with χ large enough) of cardinality $\aleph_0, \aleph_1$ etc. with $Q \in B_0, B_1$ are of a Δ–system type witnessed by F_0, F_1 of inaccessible sizes, then then $B_0 \cap H(2^{|Q|}) = B_1 \cap H(2^{|Q|})$, since $H(2^{|Q|}) \in B_0 \cap B_1$, $H(2^{|Q|}) \subseteq F_0, F_1$ and $B_0 \cap B_1 = B_0 \cap F_0 = B_1 \cap F_1$.
So below we do not need to worry about Δ–system type pairs witnessed by big models. Just they agree far enough to have a common generic for Q.

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