CHAPTER 15

Symbolic Representations

2. Kakutani, Rohlin and K-R towers

Let $\mathbf{X}$ be a dynamical system and $B \subset X$; an array $\mathfrak{c} = \{B, TB, \dots, T^{N-1}B\}$ with $T^jB,\ 0 \le j < N$ pairwise disjoint is called a *Rohlin tower* or a *column over B of height N*. The set B is called the *base* of the tower and $T^{N-1}B$ is its *roof*. Let $|\mathfrak{c}| = \cup_{j=0}^{N-1} T^jB$, the *carrier* of the tower $\mathfrak{t}$. A collection (finite or countable) $\mathfrak{t}$ of disjoint columns $\mathfrak{c}_k$ (with bases B_k and heights N_k) is called a *tower* and we let $|\mathfrak{t}| = \cup_{k=0}^{\infty}|\mathfrak{c}_k|$. The union of the bases $B = \cup_k B_k$ is the *base* of $\mathfrak{t}$, and the union of the roofs is the *roof* of $\mathfrak{t}$. We sometimes write (a bit imprecisely) $\mathfrak{t} = \{\mathfrak{c}_k : k = 1, 2, \dots\}$. The sets $\{T^ix :\ 0 \le i < N_k(x)\}$ for $x \in B$ are called the *fibers* of $\mathfrak{t}$.

Given a tower $\mathfrak{t}$ with columns $\{\mathfrak{c}_k : k = 1, 2, \dots\}$, base $B = \cup_k B_k$, and a finite (or countable) partition $\alpha = \{A_1, \dots, A_t\}$, we define an equivalence relation on B as follows: $x \sim y$ iff x and y are in the same B_k and for every $0 \le j < N_k$, T^jx and T^jy are in the same element of α; i.e. x and y have the same (α, N_k)-*name*. We now consider each equivalence class $B_{k,\mathbf{a}}$, with $\mathbf{a}$ a name in $\alpha_0^{N_k-1}$, as a basis of a column $\mathfrak{c}_{k,\mathbf{a}} = \{B_{k,\mathbf{a}}, TB_{k,\mathbf{a}}, \dots, T^{N_k-1}B_{k,\mathbf{a}}\}$ and say that the resulting tower $\mathfrak{t}_\alpha = \{\mathfrak{c}_{k,\mathbf{a}} : \mathbf{a} \in \alpha_0^{N_k-1},\ k = 1, \dots\}$ is the tower $\mathfrak{t}$ *refined* according to α. When $\alpha = \{C, X \setminus C\}$, C a subset of X, we write $\mathfrak{t}_\alpha = \mathfrak{t}_C$ and say that $\mathfrak{t}$ is refined according to C.

A subset B of X is called a *sweeping set* if $\cup_{n\ge0}T^nB = X$. For an ergodic system every set of positive measure is sweeping. Given any set B of positive measure, define the *return time function* $r_B : B \to \mathbb{N} \cup \{\infty\}$ by

$$r_B(x) = \min\{n \ge 1 : T^nx \in B\},$$

when this minimum is finite and $r_B(x) = \infty$ otherwise. Let $B_k = \{x \in B : r_B(x) = k\}$ and note that by Poincaré's recurrence theorem B_∞ is a null set. We let $\mathfrak{c}_k$ be the column $\{B_k, TB_k, \dots, T^{k-1}B_k\}$ and we call the tower $\mathfrak{t} = \mathfrak{t}(B) = \{\mathfrak{c}_k : k = 1, 2, \dots\}$, the *Kakutani tower over B*. Thus the Kakutani tower $\mathfrak{t}$ is a partition of the set $|\mathfrak{t}|$. For a sweeping set B, $|\mathfrak{t}| = X$ and the Kakutani tower over B is then a partition of the whole space. If the Kakutani tower over B has finitely many columns (i.e. the function r_B is bounded) we say that B has a *finite height* and we call the Kakutani tower over B a *K-R tower*. The number $\max r_B$ is called the *height* of B or the *height* of the K-R tower.

Note that for an ergodic $\mathbb{Z}$-system $\mathbf{X}$, either the space X consists of a finite set of points on which μ is equidistributed, or the measure μ is atom-less. In the first case the system is called *periodic*, and it is called *non-periodic* in the latter. Next we present the famous "Rohlin lemma". We shall prove it in the ergodic case but in fact it holds for every aperiodic $\mathbb{Z}$-system (see Halmos' book [113]).

Theorem 2.1 (Rohlin's lemma). *Let $\mathbf{X}$ be a non-periodic ergodic system, N a positive integer and $\epsilon > 0$, then there exists a subset B such that the sets $B, TB, \dots, T^{N-1}B$ are pairwise disjoint and $\mu(\cup_{j=0}^{N-1}T^jB) > 1 - \epsilon$.*

Proof. Let $C \subset X$ be a set with measure $0 < \mu(C) < \epsilon/N$. Consider the Kakutani tower $\mathfrak{t}(C)$ over C. For every $k \ge N$ divide the column $\mathfrak{c}_k = \{T^iC_k : 0 \le i < k\}$, starting from its base $C_k = \{x \in C : r_C(x) = k\}$, into blocks of size N. Mark the first level of each of these blocks as belonging to B. Taking the union of these marked levels over the columns $\mathfrak{c}_k, k = N, N+1, \dots$, gives us a set B with $r_B \ge N$. Clearly $\mathfrak{q} = \{B, TB, \dots, T^{N-1}B\}$ is a Rohlin tower; i.e. the sets T^jB for $0 \le j < N$ are disjoint. Now $|\mathfrak{t}| \setminus |\mathfrak{q}| = X \setminus |\mathfrak{q}|$ is composed of the first N columns $\mathfrak{c}_k$ of height $k < N$ and some top levels from the other columns, with a contribution of at most $N-1$ levels from each. A simple calculation shows therefore that

$$\mu(X \setminus |\mathfrak{q}|) < N\mu(C) < \epsilon.$$

□

Theorem 2.2. *Let $\mathbf{X}$ be a non-periodic ergodic system.*

(1) *For any positive integer N there exists a set C of finite height such that the K-R tower $\mathfrak{t}(C)$ satisfies* range $r_C \subset \{N, N+1\}$.
(2) *Given a K-R tower with base C and height N, for any sufficiently large n there is a bounded K-R tower with base D contained in C whose column heights are all at least n and at most $n + 4N$.*

Proof. 1. Let $n > 10N^2$ and use Rohlin's lemma to construct a Rohlin tower $\mathfrak{q} = \{B, TB, \ldots, T^{n-1}B\}$. Thus the return time function $r_B(x)$ is greater than $10 \cdot N^2$ on B. Let

$$B_k = \{x \in B : r_B(x) = k\}.$$

When B_k is non-empty one can therefore write k as a positive combination of N and $N+1$, say

$$k = Nu_k + (N+1)v_k.$$

Now consider the Kakutani tower over B and divide the column $\mathfrak{c}_k = \{T^iB_k : 0 \leq i < k\}$ into u_k blocks of size N and v_k blocks of size $N+1$. The set C is now defined as the union over the various columns, of the first levels of these blocks. Clearly the function r_C takes only two values, either N or $N+1$ as required.

2. Start with a Rohlin tower $\mathfrak{q} = \{B, TB, \ldots, T^{M-1}B\}$ with $M = 10(n+2N)^2$, and look at the unbounded (in general) Kakutani tower $\mathfrak{t}$ over B. As can be seen by the proof of Rohlin's lemma we can choose $B \subset C$. Refine this tower according to C and call the refined tower $\mathfrak{t}_C$. For each $m \geq 10\ (n+2N)^2$, the column $\mathfrak{q}_m$ over $B_m = \{x \in B : r_B = m\}$ in $\mathfrak{t}$, is split in $\mathfrak{t}_C$ into a finite number of columns so that each level is either a subset of C or of C^c.

As in the first part of the proof we partition each column of the tower $\mathfrak{t}_C$ into blocks of sizes $n + 2N$ and $n + 2N + 1$. Since we want the base D of the K-R tower we construct to belong to C we move the base level of each block (up or down) to the nearest level that belongs to C. Since the height of C is N we do not move these levels more than $N-1$ steps. The new blocks, with bases in C, are of size between n and $n+4N$, and we let D be the union of these bases. □