# Errata to "Ergodic theory via joinings"

January, 2011

Page 4, line 6:
(see [97], [9] and [10])

---

Page 5, line -8:
In the second section Choquet's theorem is used to prove the ...

---

Page 16, line 5:
(e) $(X, \Gamma)$ is point transitive and for every $x \in X$ and neighborhood $U$ of $x$, the set $N(x, U) = \{\gamma \in \Gamma : \gamma x \in U\}$ is syndetic in $\Gamma$.

---

Page 26, line 5:
Show that the system $(X, \Gamma)$ is minimal, admits no nontrivial equicontinuous factor, but the relation $Q$ is not an equivalence relation ...

---

Page 37, line -5:
For $m \in \mathbb{N}$ set $V_{1/m} = B_{\delta_{1/m}}(x_{1/m})$, $U_m = \Gamma V_{1/m}$ and let $R = \bigcap_{m \in \mathbb{N}} U_m$.

---

Page 47, line -18:
General references to topological dynamics, structure theory and Ellis' algebraic theory of minimal systems are [106], [58], [85], [36], [11] and [256].

---

Page 52, line 13:
By part 2 ...

---

Page 57, line 3:
$\hat{d}(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{1}{2} \big(\mu(\phi A_n \triangle \psi A_n) + \mu(\phi^{-1} A_n \triangle \psi^{-1} A_n)\big)$.

---

Page 58, line 3:

$$\alpha(n, y) = \begin{cases} \alpha(T^{n-1}y) \cdots \alpha(Ty)\alpha(y) & \text{for } n \geq 1 \\ \text{id} & \text{for } n = 0 \\ \alpha(T^n y)^{-1} \cdots \alpha(T^{-1}y)^{-1} & \text{for } n < 0. \end{cases}$$

---

Page 65, line -13:
Conversely, assume that $\pi$ admits no non-zero invariant vectors and suppose that $\mathfrak{m}(f) \neq 0$ for some function $0 \neq f \in B_\pi$. By Theorem 1.51.2(c) we can assume that $f$ is in $AP(\Gamma)$ and, therefore, that it arises as $f(\gamma) = \langle \pi(\gamma)x, x \rangle$ for a finite dimensional irreducible sub-representation of $\pi$. This however contradicts the previous lemma and we conclude that $\mathfrak{m}(f) = 0$ for every function $f \in B_\pi$.

---

Page 68, line -3:

Denote

$$\text{weak-cls}\,\pi(X) = Y \subset L^2(\mu), \quad \text{and} \quad \nu = \pi_*(\mu),$$

then, since the action of $\Gamma$ on the compact space $Y$ is WAP and topologically transitive, we can apply Lemma 1.50 to deduce that $(Y, \nu, \Gamma)$ is a nontrivial isometric factor.

---

Page 79, line 6:
... the Tits alternative ...

---

Page 81, line 11:
This is wrong; condition 2 does not imply mixing.

---

Page 82, line 7:
... iff no eigenvalue of $A$ is a root of 1.

---

Page 98, line 5:
In the first two questions in Exercise 4.8 one needs to assume that the acting group is amenable. Here is a counterexample for the acting group $G = SL(2, \mathbb{R})$.

**Example**: Consider the topological dynamical system $(Y, G)$ where $Y = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $G$ acts by automorphisms. It is well known that the only ergodic $G$-invariant probability measures on $Y$ are the Lebesgue measure $\lambda$ and finitely supported measures on periodic orbits.
Now, by a well known procedure, one can "blow-up" a periodic point into a projective line $\mathbb{P}^1$, consisting of all the lines through the origin in $\mathbb{R}^2$. Thus, e.g. the point $(0,0) \in \mathbb{T}^2$ is replaced by $(0,0) \times \mathbb{P}^1$, in such a way that a sequence $(x_n, y_n)$ in $\mathbb{T}^2$ approaches $((0,0), \ell)$ iff $\lim_{n\to\infty}(x_n, y_n) = (0,0)$ and the sequence of lines $\ell_n$, where $\ell_n$ is the unique line through the origin and $(x_n, y_n)$, tends to the line $\ell \in \mathbb{P}^1$. The $G$-action on the larger space is clear.
We enumerate the periodic orbits and attach a projective line with diameter $\epsilon_n$ at each point of the $n$-th orbit. An appropriate choice of a sequence of positive numbers $\epsilon_n$ tending to zero will ensure that the resulting space $X$ is compact and metrizable. Again the action of $G$ on $X$ is naturally defined and we obtain the system $(X, G)$. Finally by collapsing each $\mathbb{P}^1$ back to the point it is attached to we get a natural homomorphism $\pi : X \to Y$.
It is easy to check now that $X$ carries a unique invariant measure (the natural lift of the Lebesgue measure on $Y$) which is full. Thus the system $(X, G)$ is strictly ergodic, but of course it is not minimal. Also, the factor $(Y, G)$ is not uniquely ergodic.

---

Page 112, line 5:
... with $T^l z \in U$. Now $T^l z \in U$ implies ...

---

Page 116, line -10:
In particular $|\sigma_{x,y}| \ll \sigma_x$.

---

Page 119, line 11:
If $(X, \mathcal{X}, \mu, T)$ is not totally ergodic ...

Page 144, line -4:

**Group extensions and Veech's theorem:** My main source here is [138].

**A joining characterization of homogeneous skew-products:** Theorem 6.19, an extension of Veech's characterization of group extensions to isometric extensions, seems to be new (see also Lemańczyk, Thouvenot and Weiss [168] for related results).

---

Page 148, line -17:
In Chapter 3, Section 9 we have defined ...

---

Page 148, line -1:

$$\mathrm{gr}\,(\mu, \boldsymbol{\gamma}_n)(A_0 \times A_1 \times \cdots \times A_k) = \mu(A_0 \cap (\gamma_n^{(1)})^{-1} A_1 \cap \cdots \cap (\gamma_n^{(k)})^{-1} A_k).$$

---

Page 149, line -6:
... we must have $\hat{\pi}^{k+1}(\hat{\lambda}) = \nu^{k+1}$. The assumption that $\Gamma$ is abelian implies that $\hat{\lambda}$ is $\Gamma$-invariant.
Set $\lambda = \sigma^{k+1}(\hat{\lambda})$; ...

---

Page 152, line 18:
... $h_{l+1}$ ... (not $h_{1+1}$)

---

Page 174, line 10:
... the measure $\lambda$ is ergodic and symmetric ...

---

Page 174, line 11:
Thus every ergodic symmetric ...

---

Page 191, line -3:
... for $M_n$, and write ...

---

Page 206, line 1:
We show that $c(n_1, n_2, n_3) = \hat{\rho}_1(n_1, n_2, n_3)$ is the Fourier transform of a measure $\rho_1$ supported on the subgroup $H$ of $\mathbb{T}^3$.

---

Page 218, line -10:
Replace $C(X)$ by $\mathrm{Aut}\,(\mathbf{X})$ (4 times in the statement and proof of Theorem 12.7)

---

Page 220, line 6:
Replace $C(X)$ by $\mathrm{Aut}\,(\mathbf{X})$

---

Page 223, lines 6, 7 and 8:
Replace $n$ by $r$.

---

Page 223, lines 17 and 19:
Replace 3 by $r$.

---

Page 228, lines -11 and -17:
... Theorem 12.14 ...

---

Page 237, line 20:
Replace $H$ by $\mathfrak{H}$ throughout the proof.

---

Page 254, line -10:

$$I(\alpha)(x) = -\sum_{j\in J} \mathbf{1}_{A_j}(x) \log \mu(A_j),$$

---

Page 292, line 1:
A subset $A \subset X$ is called *uniform* if
$\lim_{N\to\infty}$ ess $\sup_x \left| \frac{1}{N} \sum_0^{N-1} \mathbf{1}_A(T^i x) - \mu(A) \right| = 0.$
A partition $\alpha$ is *uniform* if every set in $\cup_{n=0}^{\infty} \alpha_{-n}^{n}$ is uniform.

---

page 303, line -10:
The rigidity resides in the $\mathbf{b}_1$ part of the tower $\mathbf{b} = \mathbf{t}_{n+1}$.

---

Page 338, line -13:
By (iv) ...

---

Page 383, line -4 right:
syndetically transitive, **24**, 111