Errata to “Ergodic theory via joinings”
January, 2011

Page 4, line 6:
(see [97], [9] and [10])

Page 5, line -8:
In the second section Choquet’s theorem is used to prove the ...

Page 16, line 5:
(e) \((X, \Gamma)\) is point transitive and for every \(x \in X\) and neighborhood \(U\) of \(x\), the set \(N(x, U) = \{\gamma \in \Gamma : \gamma x \in U\}\) is syndetic in \(\Gamma\).

Page 26, line 5:
Show that the system \((X, \Gamma)\) is minimal, admits no nontrivial equicontinuous factor, but the relation \(Q\) is not an equivalence relation ...

Page 37, line -5:
For \(m \in \mathbb{N}\) set \(V_1/m = B_{\delta_{1/m}}(x_{1/m}), U_m = \Gamma V_1/m\) and let \(R = \bigcap_{m \in \mathbb{N}} U_m\).

Page 47, line -18:
General references to topological dynamics, structure theory and Ellis’ algebraic theory of minimal systems are [106], [58], [85], [36], [11] and [256].

Page 52, line 13:
By part 2 ...

Page 57, line 3:
\[ d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n \frac{1}{2}} (\mu(\phi A_n \triangle \psi A_n) + \mu(\phi^{-1} A_n \triangle \psi^{-1} A_n)) \]

Page 58, line 3:
\[ \alpha(n, y) = \begin{cases} \alpha(T^{-1} y) \cdots \alpha(T y) \alpha(y) & \text{for } n \geq 1 \\ \text{id} & \text{for } n = 0 \\ \alpha(T^n y)^{-1} \cdots \alpha(T^{-1} y)^{-1} & \text{for } n < 0. \end{cases} \]

Page 65, line -13:
Conversely, assume that \(\pi\) admits no non-zero invariant vectors and suppose that \(m(f) \neq 0\) for some function \(0 \neq f \in B_\pi\). By Theorem 1.51.2(c) we can assume that \(f\) is in \(\mathcal{AP}(\Gamma)\) and, therefore, that it arises as \(f(\gamma) = \langle \pi(\gamma) x, x \rangle\) for a finite dimensional irreducible sub-representation of \(\pi\). This however contradicts the previous lemma and we conclude that \(m(f) = 0\) for every function \(f \in B_\pi\).

Page 68, line -3:
Denote weak-cls \( \pi(X) = Y \subset L^2(\mu) \), and \( \nu = \pi_\ast(\mu) \), then, since the action of \( \Gamma \) on the compact space \( Y \) is WAP and topologically transitive, we can apply Lemma 1.50 to deduce that \( (Y, \nu, \Gamma) \) is a nontrivial isometric factor.

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Page 79, line 6:
... the Tits alternative ...

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Page 81, line 11:
This is wrong; condition 2 does not imply mixing.

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Page 82, line 7:
... iff no eigenvalue of \( A \) is a root of 1.

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Page 98, line 5:
In the first two questions in Exercise 4.8 one needs to assume that the acting group is amenable. Here is a counterexample for the acting group \( G = \text{SL}(2, \mathbb{R}) \).

**Example:** Consider the topological dynamical system \((Y, G)\) where
\( Y = T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) and \( G \) acts by automorphisms. It is well known that the only ergodic \( G \)-invariant probability measures on \( Y \) are the Lebesgue measure \( \lambda \) and finitely supported measures on periodic orbits.

Now, by a well known procedure, one can “blow-up” a periodic point into a projective line \( \mathbb{P}^1 \), consisting of all the lines through the origin in \( \mathbb{R}^2 \). Thus, e.g. the point \( (0, 0) \in T^2 \) is replaced by \( (0, 0) \times \mathbb{P}^1 \), in such a way that a sequence \((x_n, y_n) \) in \( T^2 \) approaches \((0, 0), \ell \) iff \( \lim_{n \to \infty} (x_n, y_n) = (0, 0) \) and the sequence of lines \( \ell_n \), where \( \ell_n \) is the unique line through the origin and \((x_n, y_n) \), tends to the line \( \ell \in \mathbb{P}^1 \). The \( G \)-action on the larger space is clear.

We enumerate the periodic orbits and attach a projective line with diameter \( \epsilon_n \) at each point of the \( n \)-th orbit. An appropriate choice of a sequence of positive numbers \( \epsilon_n \) tending to zero will ensure that the resulting space \( X \) is compact and metrizable. Again the action of \( G \) on \( X \) is naturally defined and we obtain the system \((X, G)\). Finally by collapsing each \( \mathbb{P}^1 \) back to the point it is attached to we get a natural homomorphism \( \pi : X \to Y \).

It is easy to check now that \( X \) carries a unique invariant measure (the natural lift of the Lebesgue measure on \( Y \)) which is full. Thus the system \((X, G)\) is strictly ergodic, but of course it is not minimal. Also, the factor \((Y, G)\) is not uniquely ergodic.

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Page 112, line 5:
... with \( T^i z \in U \). Now \( T^i z \in U \) implies ...

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Page 116, line -10:
In particular \( |\sigma_{x,y}| \ll \sigma_x \).

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Page 119, line 11:
If \((X, X, \mu, T)\) is not totally ergodic ...
Group extensions and Veech’s theorem: My main source here is [138].

A joining characterization of homogeneous skew-products: Theorem 6.19, an extension of Veech’s characterization of group extensions to isometric extensions, seems to be new (see also Lemańczyk, Thouvenot and Weiss [168] for related results).

In Chapter 3, Section 9 we have defined ...

\[ \text{gr} (\mu, \gamma_n)(A_0 \times A_1 \times \cdots \times A_k) = \mu(A_0 \cap (\gamma_n)^{-1} A_1 \cap \cdots \cap (\gamma_n)^{-1} A_k). \]

... we must have \( \hat{\pi}_{k+1}(\hat{\lambda}) = \nu_{k+1} \). The assumption that \( \Gamma \) is abelian implies that \( \hat{\lambda} \) is \( \Gamma \)-invariant. Set \( \lambda = \sigma^{k+1}(\hat{\lambda}); \)

... the measure \( \lambda \) is ergodic and symmetric ...

Thus every ergodic symmetric ...

... for \( M_n \), and write ...

We show that \( c(n_1, n_2, n_3) = \hat{\rho}_1(n_1, n_2, n_3) \) is the Fourier transform of a measure \( \rho_1 \) supported on the subgroup \( H \) of \( \mathbb{T}^3 \).

Replace \( C(X) \) by \( \text{Aut} (X) \) (4 times in the statement and proof of Theorem 12.7)

Replace \( n \) by \( r \).

Replace 3 by \( r \).
Page 228, lines -11 and -17:
... Theorem 12.14 ...

Page 237, line 20:
Replace $H$ by $g$ throughout the proof.

Page 254, line -10:
$$I(\alpha)(x) = -\sum_{j \in J} 1_{A_j}(x) \log \mu(A_j),$$

Page 292, line 1:
A subset $A \subset X$ is called \textit{uniform} if
$$\lim_{N \to \infty} \text{ess sup}_x \left| \frac{1}{N} \sum_{n=0}^{N-1} 1_A(T^n x) - \mu(A) \right| = 0.$$ 
A partition $\alpha$ is \textit{uniform} if every set in $\bigcup_{n=0}^{\infty} \alpha_n$ is uniform.

Page 303, line -10:
The rigidity resides in the $b_1$ part of the tower $b = t_{n+1}$.

Page 338, line -13:
By (iv) ...

Page 383, line -4 right:
syndetically transitive, 24, 111