THREE ZUTOT

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Abstract. Three topics in dynamical systems are discussed. In the first two sections we solve some open problems concerning, respectively, Furstenberg entropy of stationary dynamical systems, and uniformly rigid actions admitting a weakly mixing fully supported invariant probability measure. In the third section we provide a new example that displays some unexpected properties of strictly ergodic actions of non-amenable groups.

Introduction

We collect in this paper three short notes [1]. They are independent of each other and are collected here just because they occurred to us in recent discussions. The first two actually solve some open problems concerning, respectively, Furstenberg entropy of stationary dynamical systems, and uniformly rigid actions admitting a weakly mixing fully supported invariant probability measure. The third, which we discovered long ago, provides a new interesting example that displays some unexpected properties of strictly ergodic actions of non-amenable groups.

1. Furstenberg entropy

Let G be a locally compact second countable topological group, m an admissible probability measure on G, i.e., m has the following two properties: (i) For some $k \geq 1$ the convolution power m^{*k} is absolutely continuous with respect to Haar measure. (ii) the smallest closed subgroup containing the support of m is all of G. Let $(X, \mathcal{B})$ be a standard Borel space and let G act on it in a measurable way. A probability measure μ on X is called m-stationary, or just stationary when m is understood, if $m * \mu = \mu$. As was shown by Nevo and Zimmer, every m-stationary probability measure μ on a G-space X is quasi-invariant; i.e. for every $g \in G$, μ and $g\mu$ have the same null sets.

The Furstenberg (or m) entropy of an m-stationary system (X, μ) is given by

$$h_m(X,\mu) = -\int_G \int_X \log\big(\frac{dg\mu}{d\mu}(x)\big) d\mu(x) dm(g),$$

or

$$h_m(X,\mu) = -\sum m(g) \int_X \log\big(\frac{dg\mu}{d\mu}(x)\big) d\mu(x),$$

when G is discrete. We refer to [2], [3], [11], [13], [14], [15] and [4] for more details.

Quoting [1], the work which motivated our note, we consider the:

Date: February 1, 2011.

2000 *Mathematics Subject Classification.* Primary 22D40, 37B05, 37A30, 37A40, 37A50.

Key words and phrases. Stationary systems, Furstenberg entropy, uniform rigidity, measure weak mixing, strict ergodicity, amenability.

[1]Zuta is minutia (or miniature) in Hebrew; zutot is the plural, minutiae

Furstenberg entropy realization problem: Given (G, m) what are all possible values of the m-entropy $h_m(X, \mu)$ as (X, μ) varies over all ergodic m-stationary actions of G?

In [14], Theorem 2.7 and [15], Theorem 3, a remarkable fact is proven: if G is a connected semisimple Lie group with finite center and real rank at least 2 then there is a finite set $F(m)$ such that if a m-stationary action satisfies a certain technical mixing condition then its m-entropy is contained in $F(m)$. These finite values correspond with the actions of G on homogeneous spaces $(G/Q, \mu_Q)$ where $Q < G$ is a parabolic subgroup. Indeed, it is shown that any such (G, m)-space is a relatively measure-preserving extension of one of these actions.

In [14], page 323, the authors remark that they do not know the full set of possible values of the Furstenberg entropy for a given (G, m) or even whether this set of values contains an interval (for any non-amenable group G). However, they prove that if G is $PSL_2(\mathbb{R})$ or a semisimple group of real rank ≥ 2 containing a parabolic subgroup that maps onto $PSL_2(\mathbb{R})$ then infinitely many different values are achieved ([14], Theorem 3.4). It is also proven that if G has property T then there is an open interval $(0, \epsilon(m))$ containing no values of $h_m(X, \nu)$ for ergodic m-stationary G-systems $h(X, \nu)$ [12].

Now, it turns out that some simple skew-product constructions similar to those described in [4] yield an easy solution of this problem for the free group on two generators. We follow the notation and terminology of [4].

Let Z be the space of right infinite reduced words on the letters $\{a, a^{-1}, b, b^{-1}\}$. The free group F_2 on the generators a, b acts on Z by concatenation on the left and reduction. Let η be the probability measure on Z given by

$$\eta(C(u_1, \dots, u_n)) = \frac{1}{4 \cdot 3^{n-1}},$$

where for $u_j \in \{a, a^{-1}, b, b^{-1}\}$, $C(u_1, \dots, u_n) = \{z \in Z : z_j = u_j, \ j = 1, \dots, n\}$. The measure η is m-stationary for the standard measure $m = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$, and the m-system $\mathcal{Z} = (Z, \eta)$ is m-proximal (or a boundary). In fact $\mathcal{Z}$ is the Poisson boundary $\Pi(F_2, m)$.

Next let (Y, σ) be the topological Bernoulli system with $Y = \{0, 1\}^{\mathbb{Z}}$ and σ the shift homeomorphism. For $t \in [0, 1]$ let μ_t be the coin tossing measure with a $1 - t, t$ coin. Let $X = Y \times Z$, and define the action of F_2 on X by:

$$a(\omega, z) = (\sigma\omega, a_\omega z), \qquad a^{-1}(\omega, z) = (\sigma^{-1}\omega, a^{-1}_{\sigma^{-1}\omega} z),$$
$$b(\omega, z) = (\sigma\omega, b_\omega z), \qquad b^{-1}(\omega, z) = (\sigma^{-1}\omega, b^{-1}_{\sigma^{-1}\omega} z),$$

where $a_\omega = e$ when $\omega_0 = 0$, $a_\omega = a$ when $\omega_0 = 1$, and b_ω is defined similarly.

The action of F_2 on X is continuous and $\nu_t = \mu_t \times \eta$ is m-stationary and ergodic. Finally, the Furstenberg entropy of ν_t is t times the maximal entropy. [2]

Denoting $h_{max} = h_m(\Pi(F_2, m), \eta)$ we have thus shown:

1.1. **Theorem.** *Let $G = F_2$ be the free group on two generators a and b, and m the probability measure on G given by $m = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$. Let $(\Pi(F_2, m), \eta)$ be the Poisson (or universal) boundary corresponding to the pair (G, m). Then for*

[2]A slight modification (choosing Y to be an appropriate minimal subshift) will produce minimal examples.

every value t in the interval $[0, h_{max}]$ there is an m-stationary system (X, μ_t) with $h_m(X, \mu_t) = t$.

2. Uniform rigidity

Recall that a topological dynamical system (X, T), where X is a compact Hausdorff space and $T : X \to X$ a self homeomorphism, is called *uniformly rigid* if there is a sequence $n_k \nearrow \infty$ such that the sequence of homeomorphisms T^{n_k} tends uniformly to the identity. This notion was formally introduced in [7] where such systems were analyzed. Building on examples constructed in [8] it was shown in [7] that there exist strictly ergodic (hence minimal) systems which are both topologically weakly mixing and uniformly rigid.

In a recent paper [10] the authors posed the question whether there are uniformly rigid systems which also admit a weakly mixing measure of full support (question 3.1).

Now in the paper [9] we show the existence, in a certain class $\mathcal{S}$ of self homeomorphisms of the infinite torus $\mathbb{T}^\infty$ — which has the form

$$\mathcal{S} = \mathrm{cls}\,\{g\sigma g^{-1} : g \in G\}$$

— of a residual subset $\mathcal{R}$ of $\mathcal{S}$, such that each T in $\mathcal{R}$ is (i) strictly ergodic and (ii) measure weakly mixing with respect to the Haar measure on $\mathbb{T}^\infty$. The group G here is a certain subgroup of homeomorphisms of $\mathbb{T}^\infty$, see [9] for more details. Now this result, together with the observation in the paper [7] (on page 319) that, automatically, a residual subset of $\mathcal{S}$ consists of uniformly rigid homeomorphisms, immediately yield a residual set of uniformly rigid, measure weakly mixing, strictly ergodic homeomorphisms.

A similar construction can be carried out also in the setup of [8] thus producing such examples on the 2-torus $\mathbb{T}^2$.

2.1. **Theorem.** *There exists a compact metric, strictly ergodic (hence minimal), uniformly rigid dynamical system (X, μ, T) such that the corresponding measure dynamical system is weakly mixing.*

3. Strict ergodicity

Let G be a topological group. A topological G-system is a pair (X, G) where X is a compact Hausdorff space on which G acts by homeomorphisms in such a way that the map $G \times X \to X$, $(g, x) \mapsto gx$ is continuous. The system is minimal if every orbit is dense and it is uniquely ergodic if there is on X a unique G-invariant probability measure. Finally the system is strictly ergodic if it is uniquely ergodic and the unique G-invariant measure, say μ, has full support, i.e. $\mathrm{supp}\,(\mu) = X$.

Suppose now that G is an amenable group and that the system (X, G) is strictly ergodic. Then, if $Y \subsetneq X$ is a nonempty, closed, G-invariant, proper subset, it follows by the amenability of G, that there is a G-invariant probability measure on Y, say ν. The measure ν can not coincide with μ because $\mathrm{supp}\,(\nu) \subseteq Y$. This however contradicts the unique ergodicity of X and we conclude that X admits no nonempty, closed, G-invariant, proper subsets; i.e., (X, G) is minimal. Thus, *when G is amenable, a strictly ergodic system is necessarily minimal.* A similar argument shows that *a factor of a strictly ergodic G-system is strictly ergodic.* If $\pi : X \to Y$ is a factor map from the strictly ergodic system (X, μ, G) onto Y, then one needs to show that for an invariant measure ν on Y there always is an invariant measure on X whose push forward in Y is ν. For this one uses the amenability of G in the disguise of the fixed point property.

In [6] (Page 98, line 5) in the first two questions in Exercise 4.8 one is asked to prove the above statements, but the assumption that G be amenable is missing. Now it turns out that these statements need not be true for a general group action. Here is a counterexample for the acting group $G = SL(2, \mathbb{Z})$.

Example: Consider the topological dynamical system (Y, G) where $Y = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and G acts by automorphisms. It is well known that the only ergodic G-invariant probability measures on Y are the Lebesgue measure λ and finitely supported measures on periodic orbits.

Now, by a well known procedure, one can "blow-up" a periodic point into a projective line $\mathbb{P}^1$, consisting of all the lines through the origin in $\mathbb{R}^2$. Thus, e.g. the point $(0,0) \in \mathbb{T}^2$ is replaced by $(0,0) \times \mathbb{P}^1$, in such a way that a sequence (x_n, y_n) in $\mathbb{T}^2$ approaches $((0,0), \ell)$ iff $\lim_{n\to\infty}(x_n, y_n) = (0,0)$ and the sequence of lines ℓ_n, where ℓ_n is the unique line through the origin and (x_n, y_n), tends to the line $\ell \in \mathbb{P}^1$. The G-action on the larger space is clear.

We enumerate the periodic orbits and attach a projective line with diameter ϵ_n at each point of the n-th orbit. An appropriate choice of a sequence of positive numbers ϵ_n tending to zero will ensure that the resulting space X is compact and metrizable. Again the action of G on X is naturally defined and we obtain the system (X, G). Finally by collapsing each $\mathbb{P}^1$ back to the point it is attached to we get a natural homomorphism $\pi : X \to Y$.

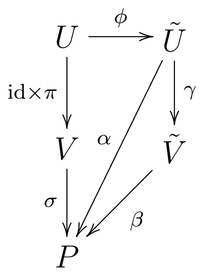
It is easy to check now that X carries a unique invariant measure (the natural lift of the Lebesgue measure on Y) which is full. Thus the system (X, G) is strictly ergodic, but of course it is not minimal. Also, the factor (Y, G) is not uniquely ergodic.

3.1. **Theorem.** *There exist a compact metric minimal dynamical system $(\tilde{U}, G)$, with $G = SL(2, \mathbb{Z})$, and a continuous homomorphism of topological dynamical systems $\gamma : (\tilde{U}, G) \to (\tilde{V}, G)$ such that the system $(\tilde{U}, G)$ is strictly ergodic but its factor $(\tilde{V}, G)$ is not.*

Proof. We apply the Furstenberg-Weiss construction [5] to the above example, as follows. Start with the profinite system (P, G) which is of course minimal and strictly ergodic. Consider the product system $(P \times X, G) := (U, G)$, where (X, G) is the example constructed above. Clearly this product system is topologically transitive and it admits $(P \times Y, G) := (V, G)$ as a factor $\mathrm{id} \times \pi : U \to V$. Let $\sigma : V = P \times Y \to P$ be the projection map. Now apply (a slight strengthening of [3]) the Furstenberg-Weiss

[3] See [16], Remark 3. I

theorem to obtain a commutative diagram:



where, $(\tilde{U}, G)$ and $(\tilde{V}, G)$ are minimal systems, the homomorphism α (and hence also β and γ) is an almost one to one extension, and the map $\phi : U \to \tilde{U}$ is a Borel isomorphism which induces an affine isomorphism of the simplex of invariant measures on U to that on $\tilde{U}$. The map $\gamma : \tilde{U} \to \tilde{V}$ is then the required map.

□

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