BANACH REPRESENTATIONS AND AFFINE COMPACTIFICATIONS OF DYNAMICAL SYSTEMS

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ABSTRACT. To every Banach space V we associate a compact right topological affine semigroup $\mathcal{E}(V)$. We show that a separable Banach space V is Asplund if and only if $\mathcal{E}(V)$ is metrizable, and it is Rosenthal (i.e. it does not contain an isomorphic copy of l_1) if and only if $\mathcal{E}(V)$ is a Rosenthal compactum. We study representations of compact right topological semigroups in $\mathcal{E}(V)$. In particular, representations of tame and HNS-semigroups arise naturally as enveloping semigroups of tame and HNS (hereditarily non-sensitive) dynamical systems, respectively. As an application we obtain a generalization of a theorem of R. Ellis. A main theme of our investigation is the relationship between the enveloping semigroup of a dynamical system X and the enveloping semigroup of its various affine compactifications $Q(X)$. When the two coincide we say that the affine compactification $Q(X)$ is E-compatible. This is a refinement of the notion of injectivity. We show that distal non-equicontinuous systems do not admit any E-compatible compactification. We present several new examples of non-injective dynamical systems and examine the relationship between injectivity and E-compatibility.

CONTENTS

INTRODUCTION

In this work we pursue our ongoing investigation of representations of dynamical systems on Banach spaces (see [47, 49, 25, 26, 51, 30, 32, 27]).

Recall that a representation of a dynamical system (G, X) on a Banach space V is given by a pair (h, α), where $h : G \to \mathrm{Iso}(V)$ is a *co-homomorphism* (i.e., $h(g_1 g_2) = h(g_2)h(g_1)$ for all $g_1, g_2 \in G$) of the group G into the group $\mathrm{Iso}(V)$ of linear isometries of V, and $\alpha : X \to V^*$ is a weak* continuous bounded G-map with respect to the dual action of $h(G)$ on V^*. For semigroup actions (S, X) we consider the co-homomorphisms $h : S \to \Theta(V)$, where $\Theta(V)$ is the semigroup of all contractive operators. For every representation (h, α), taking $Q = \overline{co}^{w^*}(\alpha(X))$, we get natural affine S-compactifications $\alpha : X \to Q$. This way of obtaining affine compactifications establishes a direct link to our earlier works which were mainly concerned with representations on reflexive, Asplund and Rosenthal Banach spaces.

In Section 2 we discuss semigroup compactifications which arise from certain linear representations, the so-called *operator compactifications*. These were studied by Witz [75] and Junghenn

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[40]. In the weakly almost periodic case this approach retrieves the classical work of de Leeuw and Glicksberg [16].

To every Banach space V we associate a compact right topological affine semigroup $\mathcal{E}(V)$. This is actually the enveloping semigroup of the natural dynamical system $(\Theta(V)^{op}, B^*)$, where $B^* \subset V^*$ is the weak* compact unit ball and $\Theta(V)^{op}$ is the adjoint semigroup of $\Theta(V)$. We show that a separable Banach space V is Asplund if and only if $\mathcal{E}(V)$ is metrizable, and it is Rosenthal (i.e. it does not contain an isomorphic copy of l_1) if and only if $\mathcal{E}(V)$ is a Rosenthal compactum, Theorems 6.11 and 6.22 respectively. We note that the first assertion, about Asplund spaces, can in essence be already found in [30].

Among the representations of compact right topological semigroups in $\mathcal{E}(V)$ we are especially interested in tame and HNS-semigroups. These arise naturally in the study of tame and HNS (= hereditarily non-sensitive) dynamical systems.

Tame dynamical metric systems appeared first in the work of Köhler [42] under the name of *regular* systems. In [25] we formulated a dynamical version of the Bourgain-Fremlin-Talagrand (in short: BFT) dichotomy (Fact 6.21 below). According to this an enveloping semigroup is either *tame*: has cardinality $\leq 2^{\aleph_0}$ and consists of Baire class 1 maps, or it is topologically wild and contains a copy of $\beta\mathbb{N}$, the Čech-Stone compactificaltion of a discrete countable set. This dichotomy combined with a characterization of Rosenthal Banach spaces, Theorem 6.18, lead to a dichotomy theorem for Banach spaces (Theorem 6.22).

The enveloping semigroup characterization of (metric) tame systems in [30] led us in [27] to a general, more flexible definition of tame systems. A (not necessarily metrizable) compact dynamical system X is *tame* if every member of its enveloping semigroup is a fragmented (Baire 1, for metrizable X) self-map on X.

In the papers [27, 25, 49] we have shown that a metric system is HNS (tame, WAP) if and only if it admits a faithful representation on an Asplund (respectively, Rosenthal, reflexive) Banach space. The algebra of all Asplund (tame) functions on a semigroup S is defined as the collection of all functions on S which come from HNS (respectively, tame) S-compactifications $S \to X$. These algebras are denoted by $\mathrm{Asp}(S)$ and $\mathrm{Tame}(S)$ respectively. Tame and HNS dynamical systems were investigated in several recent publications. See for example the papers by Huang [35] and Kerr-Li [41].

In Section 7 we strengthen some of our earlier results regarding representations on Banach spaces. We show in Theorem 7.8 that the Polish group $G = H_+[0,1]$, which admits only trivial Asplund representations, is however Rosenthal representable.

One of the main topics treated in this work is a refinement of the notion of "injectivity". The latter was introduced by Köhler [42] (who, in turn, was motivated by a problem of Pym [61]) and examined systematically in [22, 23]. A compact dynamical G-system X is called *injective* if the canonical (restriction) homomorphism $r : E(P(X)) \to E(X)$ — where $E(X)$ denotes the enveloping semigroup of the system (G, X) and $P(X)$ is the compact space of probability measures on X — is an injection, hence an isomorphism. The refinement we investigate in the present work is the following one (Section 4). Instead of considering just the space $P(X)$ we consider any embedding $(G, X) \hookrightarrow (G, Q)$ into an affine G-system (G, Q) with $Q = \overline{co}^{w^*}(X)$ and we say that this embedding is *E-compatible* if the homomorphism $r : E(Q) \to E(X)$ is injective (hence an isomorphism).

Distal affine dynamical systems have quite rigid properties. See for example the work of Namioka [55]. It was shown in [20] that a minimally generated metric distal affine G-flow is equicontinuous. Using a version of this result we show that for a minimal distal dynamical system E-compatibility in any faithful affine compactification implies equicontinuity. Thus such embedding is never E-compatible when the system is distal but not equicontinuous (Proposition 5.4). In particular this way we obtain in Theorem 5.6 a concrete example of a semigroup compactification which is not an *operator compactification*. More precisely, for the algebra $D(\mathbb{Z})$ of all distal functions on $\mathbb{Z}$ the corresponding semigroup compactification $\alpha : \mathbb{Z} \to \mathbb{Z}^{D(\mathbb{Z})}$ is not an operator compactification.

Non-injectivity is not restricted to distal systems. We construct examples of Toeplitz systems which are not injective, Theorem 5.7. We don't have such examples for a weakly mixing system. We also describe an example of a $\mathbb{Z}^2$-system which admits an E-compatible embedding yet is not injective (Example 5.12). We don't have such an example for $\mathbb{Z}$-systems.

The notion of a *left introverted* (we say shortly: *introverted*) linear subspace of $C(S)$ was introduced by M.M. Day in 1957. It is an important tool in the study of semigroups of means and affine semigroup compactifications. It also plays a major role in the theory of Banach semigroup algebras and their second duals, see for example, [8, 62, 14]. A weaker property of subalgebras of $C(S)$ is being m-Introverted. It turns out that a subalgebra of $\mathrm{RUC}(G)$ is m-introverted iff the corresponding dynamical system is point-universal iff it is isomorphic (as a dynamical system) to its own enveloping semigroup. It is well known that the algebras $\mathrm{RUC}(G)$ and $\mathrm{WAP}(G)$ are introverted. In general there is a large room between the algebras $\mathrm{RUC}(G)$ and $\mathrm{WAP}(G)$ for topological groups G. Indeed, by [52] ([2] for monothetic G), $\mathrm{RUC}(G) = \mathrm{WAP}(G)$ iff G is precompact.

We provide new non-trivial examples of introverted spaces. We show that $\mathrm{Tame}(S)$ is always introverted. Moreover, all of its m-introverted S-subalgebras (like, $\mathrm{Asp}(S)$ and $\mathrm{WAP}(S)$) are introverted. As a particular case (Theorem 8.4) it follows that every m-introverted *separable S-subalgebra* of $C(S)$ is introverted. Note also that, by [28], the algebra $\mathrm{Asp}(G)$ (which contains the algebra $\mathrm{WAP}(G)$) is (left) amenable for every topological group G. This is in contrast to the fact that the larger algebra $\mathrm{Tame}(G)$ is, in general, non-amenable.

We show that a semigroup compactification $\nu : S \to P$ is an operator compactification iff the corresponding algebra of this compactification $\mathcal{A}_\nu$ is *intro-generated*. The latter means that $\mathcal{A}_\nu$ contains an introverted subspace $F \subset \mathcal{A}_\nu$ such that the minimal closed subalgebra of $C(S)$ containing F is $\mathcal{A}_\nu$. (This phenomenon reflects the existence of an E-compatible system which is not injective.) The space $D(\mathbb{Z})$ of all distal functions on $\mathbb{Z}$ is not intro-generated (Theorem 5.6). The $\mathbb{Z}^2$-flow from Example 5.12 mentioned above provides an intro-generated subspace of $l_\infty(\mathbb{Z}^2)$ which is not introverted.

In Section 8 we first show, in Theorem 8.1, that affine compactifications coming from representations on Rosenthal spaces are E-compatible. The core of the proof is Haydon's characterization of Rosenthal spaces in terms of the w^*-Krein-Milman property. Using results of Section 7 about representations of tame systems on Rosenthal spaces we show in Theorem 8.2 that every tame S-space X is injective. This result was proved by Köhler [42] for metrizable systems. In [22] there is a simple proof of this which uses the fact that for a tame metrizable system X its enveloping semigroup is a Fréchet space.

Next we prove a representation theorem (Theorem 8.5) according to which the enveloping semigroup of a tame (respectively, HNS) system, admits an admissible embedding into $\mathcal{E}(V)$, where V runs over the class of Rosenthal (respectively, Asplund) Banach spaces. These results extend the following well known theorem: the class of reflexively representable compact right topological semigroups coincides with the class of compact semitopological semigroups (proved in [68, 47]). As an applications of Theorem 8.5, using Theorem 5.5, we obtain a *generalized Ellis theorem*: a tame compact right topological group is a topological group (Theorem 8.7).

Finally, a representation theorem for S-affine compactifications (Theorem 8.18), shows that for tame (HNS, WAP) compact metrizable S-systems, their S-affine compactifications can be *affinely S-represented* on Rosenthal (Asplund, reflexive) separable Banach spaces.

1. Preliminaries

Topological spaces are always assumed to be Hausdorff and completely regular. The closure of a subset $A \subset X$ is denoted by $\overline{A}$ or $cl(A)$. Banach spaces and locally convex vector spaces are over the field $\mathbb{R}$ of real numbers. For a subset A of a Banach space we denote by $sp(A)$ and $\overline{sp}^{norm}(A)$ the linear span and the norm-closed linear span of K respectively. We denote by $co(A)$ and $\overline{co}(A)$ the convex hull and the closed convex hull of a set A, respectively. If $A \subset V^*$ is a subset of the dual space V^* of V we mostly mean the weak* topology on A and $\overline{co}(A)$ or $\overline{co}^{w^*}(A)$ will denote the w^*-closure of $co(A)$ in V^*. For a topological space X we denote by $C(X)$ the Banach algebra of real valued continuous and bounded functions equipped with the supremum norm. For a subset $A \subset C(X)$ we denote by $\langle A \rangle$ the smallest unital (i.e., containing the constants) closed subalgebra of $C(X)$ containing A.

1.1. **Semigroups and actions.** Let S be a semigroup which is also a topological space. By $\lambda_a :$ $S \to S, x \mapsto ax$ and $\rho_a : S \to S, x \mapsto xa$ we denote the left and right a-transitions. The subset $\Lambda(S) := \{a \in S : \lambda_a$ is continuous$\}$ is called the *topological center* of S.

Definition 1.1. A semigroup S as above is said to be:
  (1) a *right topological semigroup* if every ρ_a is continuous.
  (2) *semitopological* if the multiplication $S \times S \to S$ is separately continuous.
  (3) [54] *admissible* if S is right topological and $\Lambda(S)$ is dense in S.

Let A be a subsemigroup of a right topological semigroup S. If $A \subset \Lambda(S)$ then the closure $cl(A)$ is a right topological semigroup. In general, $cl(A)$ is not necessarily a subsemigroup of S (even if S is compact right topological and A is a left ideal). Also $\Lambda(S)$ may be empty for general compact right topological semigroup S. See [8, p. 29].

Definition 1.2. Let S be a semitopological semigroup with a neutral element e. Let $\pi : S \times X \to X$ be a *left action* of S on a topological space X. This means that $ex = x$ and $s_1(s_2 x) = (s_1 s_2)x$ for all $s_1, s_2 \in S$ and $x \in X$, where as usual, we write sx instead of $\pi(s,x) = \lambda_s(x) = \rho_x(s)$. Let $S \times X \to X$ and $S \times Y \to Y$ be two actions. A map $f : X \to Y$ between S-spaces is an *S-map* if $f(sx) = sf(x)$ for every $(s,x) \in S \times X$.

We say that X is a *dynamical S-system* (or an *S-space* or an *S-flow*) if the action π is separately continuous (that is, if all orbit maps $\rho_x : S \to X$ and all translations $\lambda_s : X \to X$ are continuous). We sometimes write it as a pair (S, X).

A *right system* (X, S) can be defined analogously. If S^{op} is the *opposite semigroup* of S with the same topology then (X, S) can be treated as a left system (S^{op}, X) (and vice versa).

Fact 1.3. [44] *Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then every separately continuous action of G on a compact space X is continuous.*

Notation: *All semigroups S are assumed to be monoids, i.e, semigroups with a neutral element which will be denoted by e. Also actions are monoidal (meaning $ex = x, \forall x \in X$) and separately continuous. We reserve the symbol G for the case when S is a group. All right topological semigroups below are assumed to be admissible.*

Given $x \in X$, its *orbit* is the set $Sx = \{sx : s \in S\}$ and the closure of this set, $cl(Sx)$, is the *orbit closure* of x. A point x with $cl(Sx) = X$ is called a *transitive point*, and the set of transitive points is denoted by X_{tr}. We say that the system is *point-transitive* when $X_{tr} \neq \emptyset$. The system is called *minimal* if $X_{tr} = X$.

1.2. **Representations of dynamical systems.** A *representation* of a semigroup S on a normed space V is a co-homomorphism $h : S \to \Theta(V)$, where $\Theta(V) := \{T \in L(V) : ||T|| \leq 1\}$ and $h(e) = id_V$. Here $L(V)$ is the space of continuous linear operators $V \to V$ and id_V is the identity operator. This is equivalent to the requirement that $h : S \to \Theta(V)^{op}$ be a monoid homomorphism, where $\Theta(V)^{op}$ is the opposite semigroup of $\Theta(V)$. If $S = G$, is a group then $h(G) \subset \text{Iso}(V)$, where $\text{Iso}(V)$ is the group of all linear isometries from V onto V. The adjoint operator $adj : L(V) \to L(V^*)$ induces an injective co-homomorphism $adj : \Theta(V) \to \Theta(V^*), adj(s) = s^*$. We will identify $adj(L(V))$ and the opposite semigroup $L(V)^{op}$; as well as $adj(\Theta(V)) \subset L(V^*)$ and its opposite semigroup $\Theta(V)^{op}$. Mostly we use the same symbol s instead of s^*. Since $\Theta(V)^{op}$ acts from the right on V and from the left on V^* we sometimes write vs for $h(s)(v)$ and $s\psi$ for $h(s)^*(\psi)$.

A pair of vectors $(v, \psi) \in V \times V^*$ defines a function (called a *matrix coefficient* of h)

$$m(v, \psi) : S \to \mathbb{R}, \quad s \mapsto \psi(vs) = \langle vs, \psi \rangle = \langle v, s\psi \rangle.$$

The weak operator topology on $\Theta(V)$ (similarly, on $\Theta(V)^{op}$) is the weak topology generated by all matrix coefficients. So $h : S \to \Theta(V)^{op}$ is *weakly continuous* iff $m(v, \psi) \in C(S)$ for every $(v, \psi) \in V \times V^*$. The strong operator topology on $\Theta(V)$ (and on $\Theta(V)^{op}$) is the pointwise topology with respect to its left (respectively, right) action on the Banach space V.

Lemma 1.4. *Let $h : S \to \Theta(V)$ be a weakly continuous co-homomorphism. Then for every $v \in V$ the following map*

$$T_v : V^* \to C(S), \quad T_v(\psi) = m(v, \psi)$$

is a well defined linear bounded weak*-pointwise continuous S-map between left S-actions.

Definition 1.5. (See [49, 25]) Let X be a dynamical S-system.

(1) A *representation* of (S, X) on a normed space V is a pair

$$(h, \alpha) : S \times X \rightrightarrows \Theta(V) \times V^*$$

where $h : S \to \Theta(V)$ is a co-homomorphism of semigroups and $\alpha : X \to V^*$ is a weak* continuous bounded S-mapping with respect to the dual action

$$S \times V^* \to V^*, \ (s\varphi)(v) := \varphi(h(s)(v)).$$

We say that the representation is weakly (strongly) continuous if h is weakly (strongly) continuous. A representation (h, α) is said to be *faithful* if α is a topological embedding.

(2) If $\mathcal{K}$ is a subclass of the class of Banach spaces, we say that a dynamical system (S, X) is weakly (respectively, strongly) $\mathcal{K}$-*representable* if there exists a weakly (respectively, strongly) continuous faithful representation of (S, X) on a Banach space $V \in \mathcal{K}$.

(3) A subdirect product, i.e. an S-subspace of a direct product, of weakly (strongly) $\mathcal{K}$-representable S-spaces is said to be weakly (strongly) $\mathcal{K}$-*approximable*.

We consider in particular the following classes of Banach spaces: Reflexive, Asplund, and Rosenthal spaces. A reflexively (Asplund) representable compact dynamical system is a dynamical version of the purely topological notion of an *Eberlein* (respectively, a *Radon-Nikodym*) compactum, in the sense of Amir and Lindenstrauss (respectively, in the sense of Namioka).

1.3. **Background on Banach spaces and fragmentability.**

Definition 1.6. Let (X, τ) be a topological space and (Y, μ) a uniform space.

(1) [38] X is (τ, μ)-*fragmented* by a (typically, not continuous) function $f : X \to Y$ if for every nonempty subset A of X and every $\varepsilon \in \mu$ there exists an open subset O of X such that $O \cap A$ is nonempty and the set $f(O \cap A)$ is ε-small in Y. We also say in that case that the function f is *fragmented*. Notation: $f \in \mathcal{F}(X, Y)$, whenever the uniformity μ is understood. If $Y = \mathbb{R}$ then we write simply $\mathcal{F}(X)$.

(2) [25] We say that a *family of functions* $F = \{f : (X, \tau) \to (Y, \mu)\}$ is *fragmented* if condition (1) holds simultaneously for all $f \in F$. That is, $f(O \cap A)$ is ε-small for every $f \in F$.

(3) [29] We say that F is an *eventually fragmented family* if every infinite subfamily $C \subset F$ contains an infinite fragmented subfamily $K \subset C$.

In Definition 1.6.1 when $Y = X, f = id_X$ and μ is a metric uniform structure, we get the usual definition of fragmentability (more precisely, (τ, μ)-fragmentability) in the sense of Jayne and Rogers [39]. Implicitly it already appears in a paper of Namioka and Phelps [57].

Remark 1.7. [25, 27]

(1) It is enough to check the condition of Definition 1.6 only for closed subsets $A \subset X$ and for $\varepsilon \in \mu$ from a *subbase* γ of μ (that is, the finite intersections of the elements of γ form a base of the uniform structure μ).

(2) When X and Y are Polish spaces, $f : X \to Y$ is fragmented iff f is a Baire class 1 function.

(3) When X is compact and (Y, ρ) metrizable uniform space then $f : X \to Y$ is fragmented iff f has a *point of continuity property* (i.e., for every closed nonempty $A \subset X$ the restriction $f_{|A} : A \to Y$ has a continuity point).

(4) When Y is compact with its unique compatible uniformity μ then $p : X \to Y$ is fragmented if and only if $f \circ p : X \to \mathbb{R}$ has a point of continuity property for every $f \in C(Y)$.

Lemma 1.8. (1) *Suppose F is a compact space, X is Čech-complete, Y is a uniform space and we are given a separately continuous map $w : F \times X \to Y$. Then the naturally associated family $\tilde{F} := \{\tilde{f} : X \to Y\}_{f \in F}$ is fragmented, where $\tilde{f}(x) = w(f, x)$.*

(2) *Suppose F is a compact metrizable space, X is hereditarily Baire and M is separable and metrizable. Assume we are given a map $w : F \times X \to M$ such that every $\tilde{x} : F \to M, f \mapsto w(f, x)$ is continuous and $y : X \to M$ is continuous at every $\tilde{y} \in Y$ for some dense subset Y of F. Then the family $\tilde{F}$ is fragmented.*

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Proof. (1): There exists a collection of uniform maps $\{\varphi_i : Y \to M_i\}_{i \in I}$ into metrizable uniform spaces M_i which generates the uniformity on Y. Now for every closed subset $A \subset X$ apply Namioka's joint continuity theorem to the separately continuous map $\varphi_i \circ w : F \times A \to M_i$ and take into account Remark 1.7.1.

(2): Since every $\tilde{x} : F \to M$ is continuous, the natural map $j : X \to C(F, M)$, $j(x) = \tilde{x}$ is well defined. Every closed nonempty subset $A \subset X$ is Baire. By [30, Proposition 2.4], $j|_A : A \to C(F, M)$ has a point of continuity, where $C(F, M)$ carries the sup-metric. Hence, $\tilde{F}_A = \{\tilde{f}\restriction_A : A \to M\}_{f \in F}$ is equicontinuous at some point $a \in A$. This implies that the family $\tilde{F}$ is fragmented. $\square$

For other properties of fragmented maps and fragmented families refer to [49, 25, 27].

Recall that a Banach space V is an *Asplund* space if the dual of every separable Banach subspace is separable. In the following result the equivalence (1) $\Leftrightarrow$ (2) is a well known criterion [56], and (3) is a reformulation of (2) in terms of fragmented families. When V is a Banach space we denote by B, or B_V, the closed unit ball of V. $B^* = B_{V^*}$ and $B^{**} := B_{V^{**}}$ will denote the weak* compact unit balls in the dual V^* and second dual V^{**} of V respectively.

Fact 1.9. [57, 56] *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is an Asplund space.*
(2) *Every bounded subset A of the dual V^* is (weak*,norm)-fragmented.*
(3) *B is a fragmented family of real valued maps on the compactum B^*.*

Assertion (3) is a reformulation of (2). Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. For more details cf. [18, 56].

We say that a Banach space V is *Rosenthal* if it does not contain an isomorphic copy of l_1. Clearly, every Asplund space is Rosenthal.

Definition 1.10. [27] Let X be a topological space. We say that a subset $F \subset C(X)$ is a *Rosenthal family* (for X) if F is norm bounded and the pointwise closure $\mathrm{cl}_p(F)$ of F in $\mathbb{R}^X$ consists of fragmented maps, that is, $\mathrm{cl}_p(F) \subset \mathcal{F}(X)$.

Let $f_n : X \to \mathbb{R}$ be a uniformly bounded sequence of functions on a *set* X. Following Rosenthal we say that this sequence is an *l_1-sequence* on X if there exists a real constant $a > 0$ such that for all $n \in \mathbb{N}$ and all choices of real scalars $c_1, \cdots, c_n$ we have

$$a \cdot \sum_{i=1}^n |c_i| \leq \|\sum_{i=1}^n c_i f_i\|.$$

This is the same as requiring that the closed linear span in $l_\infty(X)$ of the sequence f_n be linearly homeomorphic to the Banach space l_1. In fact, in this case the map

$$l_1 \to l_\infty(X), \quad (c_n) \to \sum_{n \in \mathbb{N}} c_n f_n$$

is a linear homeomorphic embedding.

The equivalence of (1) and (2) in the following fact is well known. See for example, [70].

Fact 1.11. [70, 27] *Let X be a compact space and $F \subset C(X)$ a bounded subset. The following conditions are equivalent:*

(1) *F does not contain a subsequence equivalent to the unit basis of l_1.*
(2) *F is a Rosenthal family for X.*
(3) *F is an eventually fragmented family.*

We need some known characterizations of Rosenthal spaces.

Fact 1.12. *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is a Rosenthal Banach space.*
(2) *(Rosenthal [63]) Every bounded sequence in V has a weak-Cauchy subsequence.*
(3) *(E. Saab and P. Saab [66]) Each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B^*. Equivalently, if $B^{**} \subset \mathcal{F}(B^*)$.*

(4) (Haydon [34, Theorem 3.3]) *For every weak* compact subset $Y \subset V^*$ the weak* and norm closures of the convex hull $co(Y)$ in V^* coincide: $\overline{co}^{w^*}(Y) = \overline{co}^{norm}(Y)$.*

(5) B *is a* Rosenthal family *for the weak* compact unit ball B^*.*

Condition (3) is a reformulation (in terms of fragmented maps) of a criterion from [66] which was originally stated in terms of the point of continuity property. (5) can be derived from (3).

Fact 1.13. (Banach-Grothendieck theorem) [4, Cor. 2.6] *If V is a Banach space then for every continuous linear functional $u : V^* \to \mathbb{R}$ on the dual space V^* the following are equivalent:*

(1) u *is w^*-continuous.*

(2) *The restriction $u|_{B^*}$ is w^*-continuous.*

(3) u *is the evaluation at some point of V. That is, $u \in i(V)$, where $i : V \hookrightarrow V^{**}$ is the canonical embedding.*

Let $\{V_i\}_{i \in I}$ be a family of Banach spaces. The l_2-sum of this family, denoted by $V := (\Sigma_{i \in I} V_i)_{l_2}$, is defined as the space of all functions $(x_i)_{i \in I}$ on I such that $x_i \in V_i$ and

$$||x|| := \left(\sum_{i \in I} ||x_i||^2\right)^{\frac{1}{2}} < \infty.$$

Lemma 1.14. (1) $V^* = (\Sigma_{i \in I} V_i)^*_{l_2} = (\Sigma_{i \in I} V_i^*)_{l_2}$ *and the pairing $V \times V^* \to \mathbb{R}$ is defined by $\langle v, f \rangle = \sum_{i \in I} f_i(v_i)$.*

(2) *If every V_i is reflexive (Asplund, Rosenthal) then V is reflexive (respectively: Asplund, Rosenthal).*

(3) *For every semitopological semigroup S the classes of reflexively (Asplund, Rosenthal) representable compact S-spaces are closed under countable products.*

Proof. (1) This is well known (see, for example, [57]).

(2) The reflexive case follows easily from (1). For the Asplund case see [57] (or [18] for a simpler proof). Now suppose that each V_i is Rosenthal and $l_1 \subset V = (\Sigma_{i \in I} V_i)_{l_2}$. Since l_1 is separable one may easily reduce the question to the case of countably many Rosenthal spaces V_i. So we can suppose that $V := (\Sigma_{n \in \mathbb{N}} V_n)_{l_2}$. In view of Fact 1.12 it suffices to show that every element $u = (u_n)_{\in V^{**}}$ is a fragmented map on the weak* compact unit ball B_{V^*}. That is, we need to check that $u \in \mathcal{F}(B_{V^*})$. The set $\mathcal{F}(X) \cap l_\infty(X)$ is a Banach subspace of $l_\infty(X)$ for every topological space X. So the proof can be reduced to the case of coordinate functionals u_{n_0}. Also, $\langle u_{n_0}, (f_n)_{n \in \mathbb{N}} \rangle = f_{n_0}(u_{n_0})$. Now use the fact that u_{n_0} is a fragmented map on $B_{V^*_{n_0}}$ because V_{n_0} is Rosenthal (Fact 1.12).

(3) Similar to [51, Lemma 3.3] (or [49, Lemma 4.9]) using (2) and the l_2-sum of representations (h_n, α_n) of (S, X_n) on V_n where $||\alpha_n(x)|| \leq 2^{-n}$ for every $x \in X_n$ and $n \in \mathbb{N}$. $\square$

1.4. S-Compactifications and functions. A *compactification* of X is a pair (ν, Y) where Y is a compact (Hausdorff, by our assumptions) space and ν is a continuous map with a dense range.

The Gelfand-Kolmogoroff theory [19] establishes an order preserving bijective correspondence (up to equivalence of compactifications) between Banach *unital* subalgebras $\mathcal{A} \subset C(X)$ and compactifications $\nu : X \to Y$ of X. Every Banach unital S-subalgebra $\mathcal{A}$ induces the *canonical $\mathcal{A}$-compactification* $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$, where $X^{\mathcal{A}}$ is the spectrum (or the Gelfand space — the collection of continuous multiplicative functionals on $\mathcal{A}$). The map $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}} \subset \mathcal{A}^*$ is defined by the *Gelfand transform*, the evaluation at x functional, $\alpha_{\mathcal{A}}(x)(f) := f(x)$. Conversely, every compactification $\nu : X \to Y$ is equivalent to the *canonical $\mathcal{A}_\nu$-compactification* $\alpha_{\mathcal{A}_\nu} : X \to X^{\mathcal{A}_\nu}$, where the algebra $\mathcal{A}_\nu$ is defined as the image $j_\nu(C(Y))$ of the embedding $j_\nu : C(Y) \hookrightarrow C(X)$, $\phi \mapsto \phi \circ \nu$.

Definition 1.15. Let X be an S-system. An S-compactification of X is a continuous S-map $\alpha : X \to Y$, with a dense range, into a compact S-system Y. An S-compactification is said to be *jointly continuous* (respectively, *separately continuous*) if the action $S \times Y \to Y$ is jointly continuous (respectively, separately continuous).

By S_d we denote the discrete copy of S.

Remark 1.16. If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two compactifications, then ν_2 *dominates* ν_1, that is, $\nu_1 = q \circ \nu_2$ for some (uniquely defined) continuous map $q : Y_2 \to Y_1$ iff $\mathcal{A}_{\nu_1} \subset \mathcal{A}_{\nu_2}$. If in addition,

X, Y_1 and Y_2 are S_d-systems (i.e., all the s-translations on X, Y_1 and Y_2 are continuous) and if ν_1 and ν_2 are S-maps, then q is also an S-map. Furthermore, if the action on Y_1 is (separately) continuous then the action on Y_2 is (respectively, separately) continuous. If ν_1 and ν_2 are homomorphisms of semigroups then q is also a homomorphism. See [73, App. D].

1.5. **From representations to compactifications.** Representations of dynamical systems (S, X) lead to S-compactifications of X. Let V be a normed space and let

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, V^*)$$

be a representation of (S, X), where α is a weak* continuous map. Consider the induced compactification $\alpha : X \to Y := \overline{\alpha(X)}$, the weak* closure of $\alpha(X)$. Clearly, the induced natural action $S \times Y \to Y$ is well defined and every left translation is continuous. So, Y is an S_d-system.

Remark 1.17.   (1) The induced action $S \times Y \to Y$ is separately continuous iff the matrix coefficient $m(v, y) : S \to \mathbb{R}$ is continuous $\forall\, v \in V$, $y \in Y$.
  (2) If h is strongly (weakly) continuous then the induced dual action of S on the weak* compact unit ball B^* and on Y is jointly (respectively, separately) continuous.

To every S-space X we associate the *regular representation* on the Banach space $V := C(X)$ defined by the pair (h, α) where $h : S \to \Theta(V), s \mapsto L_s$ (with $L_s f(x) = f(sx)$) is the natural co-homomorphism and $\alpha : X \to V^*, x \mapsto \delta_x$ is the evaluation map $\delta_x(f) = f(x)$. Denote by $(\mathrm{WRUC}(X))$ $\mathrm{RUC}(X)$ the set of all (weakly) right uniformly continuous functions. That is functions $f \in C(X)$ such that the orbit map $\tilde{f} : S \to C(X), s \mapsto fs = L_s(f)$ is (weakly) norm continuous. Then $\mathrm{RUC}(X)$ and $\mathrm{WRUC}(X)$ are norm closed S-invariant unital linear subspaces of $C(X)$ and the restriction of the regular representation is continuous on $\mathrm{RUC}(X)$ and weakly continuous on $\mathrm{WRUC}(X)$. Furthermore, $\mathrm{RUC}(X)$ is a Banach subalgebra of $C(X)$. If $S \times X \to X$ is continuous and X is compact then $C(X) = \mathrm{RUC}(X)$. In particular, for the left action of S on itself $X := S$ we write simply $\mathrm{RUC}(S)$ and $\mathrm{WRUC}(S)$. If $X := G$ is a topological group with the left action on itself then $\mathrm{RUC}(G)$ is the usual algebra of right uniformly continuous functions on G. Note that $W\mathrm{RUC}(S)$ plays a major role in the theory of semigroups being the largest left introverted linear subspace of $C(S)$ (Rao's theorem; see for example, [7]).

We say that a function $f \in C(X)$ on an S-space X *comes from* an S-compactification $\nu : X \to Y$ (recall that we require only that the actions on X, Y are separately continuous) if there exists $\tilde{f} \in C(Y)$ such that $f = \tilde{f} \circ \nu$. Denote by $\mathrm{RMC}(X)$ the set (in fact a unital Banach algebra) of all functions on X which come from S-compactifications. The algebra $\mathrm{RUC}(X)$ is the set of all functions which come from jointly continuous S-compactifications.

Remark 1.18. Let X be an S-system.
  (1) For every S-invariant normed subspace V of $\mathrm{WRUC}(X)$ we have the regular weakly continuous V-representation (h, α) of (S, X) on V defined by $\alpha(x)(f) = f(x)$, $f \in V$ and the corresponding S-compactification $\alpha : X \to Y := \overline{\alpha(X)}$. The action of S on Y is continuous iff $V \subset \mathrm{RUC}(X)$.
  (2) Let (h, α) be a representation of the S-system X on a Banach space V. The inclusion $\alpha(X) \subset V^*$ induces a *restriction operator*

$$r : V \to C(X), \ r(v)(x) = \langle v, \alpha(x) \rangle.$$

  Then r is a linear S-operator (between right actions) with $||r|| \le 1$. If h is weakly (strongly) continuous then $r(V) \subset \mathrm{WRUC}(X)$ (respectively, $r(V) \subset \mathrm{RUC}(X)$).
  (3) For every topological space X the classical order preserving Gelfand-Kolmogoroff correspondence between compactifications of X and unital subalgebras has a natural S-space generalization. More precisely, if X is an S-space then S-invariant unital Banach subalgebras F of $\mathrm{RUC}(X)$ (resp., $\mathrm{RMC}(X)$) control the S-compactifications $X \to Y$ with (resp., separately) continuous actions $S \times Y \to Y$.

The correspondence described in Remark 1.18.3 for Banach subalgebras F of $\mathrm{RUC}(X)$ is well known for topological group actions, [72]. One can easily extend it to the case of topological

semigroup actions [5, 50]. Compare this also to the description of jointly continuous *affine S-compactifications* (Section 3) in terms of S-invariant closed linear unital subspaces of $\mathrm{RUC}(X)$.

Regarding a description of separately continuous S-compactifications via subalgebras of $\mathrm{RMC}(X)$ and for more details about Remarks 1.17, 1.18 see, for example, [49, 50] and, also, Remark 3.12 below.

A word of caution about our notation of $\mathrm{WRUC}(S), \mathrm{RUC}(S), \mathrm{RMC}(S)$. Note that in [7] the corresponding notation is $\mathrm{WLUC}(S)$, $\mathrm{LUC}(S)$, $\mathrm{LMC}(S)$ (and sometimes $\mathrm{WLC}(S)$, $\mathrm{LC}(S)$, [8]).

Remark 1.19. Let $\mathcal{P}$ be a class of compact separately continuous S-dynamical systems. The subclass of S-systems with continuous actions will be denoted by $\mathcal{P}_c$. Assume that $\mathcal{P}$ is closed under products, closed subsystems and S-isomorphisms. In such cases (following [73, Ch. IV]) we say that $\mathcal{P}$ is *suppable*. Let X be a not necessarily compact S-space and let $\mathcal{P}(X)$ be the collection of functions on X coming from systems having property $\mathcal{P}$. Then, as in the case of jointly continuous actions (see [25, Prop. 2.9]), there exists a universal S-compactification $X \to X^{\mathcal{P}}$ of X such that $(S, X) \in \mathcal{P}$. Moreover, $j(C(X^{\mathcal{P}})) = \mathcal{P}(X)$. In particular, $\mathcal{P}(X)$ is a uniformly closed, S-invariant subalgebra of $C(X)$. Analogously, one defines $\mathcal{P}_c(X)$. Again it is a uniformly closed, S-invariant subalgebra of $C(X)$, which is in fact a subalgebra of $\mathrm{RUC}(X)$. For the corresponding S-compactification $X \to X^{\mathcal{P}_c}$ the action of S on $X^{\mathcal{P}_c}$ is continuous.

In particular, for the left action of S on itself we get the definitions of $\mathcal{P}(S)$ and $\mathcal{P}_c(S)$. As in [25, Prop. 2.9] one may show that $\mathcal{P}(S)$ and $\mathcal{P}_c(S)$ are m-introverted Banach subalgebras of $C(S)$ and they define the $\mathcal{P}$-universal and $\mathcal{P}_c$-universal semigroup compactifications $S \to S^{\mathcal{P}}$ and $S \to S^{\mathcal{P}_c}$.

In the present paper we are especially interested in the following classes of compact S-systems: a) Tame systems (Definition 6.14); b) Hereditarily Non-Sensitive, HNS in short (Definition 6.8); c) Weakly Almost Periodic, WAP in short (Section 6.2). See Sections 6.2, 6.3, 6.4 and also [25, 26, 27].

For the corresponding algebras, defined by Remark 1.19, we use the following notation: $\mathrm{Tame}(X)$, $\mathrm{Asp}(X)$, $\mathrm{WAP}(X)$. Note that the Tame (respectively, HNS, WAP) systems are exactly the compact systems which admit sufficiently many representations on Rosenthal (respectively, Asplund, reflexive) Banach spaces (Section 7).

Lemma 1.20. (1) *For every S-space X we have $\mathcal{P}_c(X) \subset \mathrm{RUC}(X) \subset \mathrm{WRUC}(X) \subset \mathrm{RMC}(X)$ and $\mathcal{P}_c(X) \subset \mathcal{P}(X) \cap \mathrm{RUC}(X)$. If $\mathcal{P}$ is preserved by factors then $\mathcal{P}_c(X) = \mathcal{P}(X) \cap \mathrm{RUC}(X)$.*
  (2) *If X is a compact S-system with continuous action then $\mathcal{P}_c(X) = \mathcal{P}(X)$, $\mathrm{RUC}(X) = \mathrm{WRUC}(X) = \mathrm{RMC}(X) = C(X)$.*
  (3) *If $S = G$ is a Čech-complete semitopological group then for every G-space X we have $\mathcal{P}_c(X) = \mathcal{P}(X)$, $\mathrm{RUC}(X) = \mathrm{WRUC}(X) = \mathrm{RMC}(X)$; in particular, $\mathrm{RUC}(G) = \mathrm{WRUC}(G) = \mathrm{RMC}(G)$.*
  (4) $\mathrm{WAP}_c(G) = \mathrm{WAP}(G)$ *remains true for every semitopological group G.*
  (5) [8, p. 173] *If S is a k-space as a topological space then $\mathrm{WRUC}(X) = \mathrm{RMC}(X)$.*

Proof. (1) is straightforward. In order to check the less obvious part $\mathcal{P}_c(X) \supset \mathcal{P}(X) \cap \mathrm{RUC}(X)$ we use a fundamental property of cyclic compactifications (see Remark 3.12.1).

(2) easily follows from (1). (3) follows from Fact 1.3, and (4) from Fact 6.5. (5) is a generalized version of [8, Theorem 5.6] and easily follows from Grothendieck's Lemma [8, Cor. A6]. □

Definition 1.21. Let X be a compact space with a separately continuous action $\pi : S \times X \to X$. We say that X is WRUC-*compatible* (or that X is WRUC) if $C(X) = \mathrm{WRUC}(X)$. An equivalent condition is that the induced action $\pi_P : S \times P(X) \to P(X)$ be separately continuous (Lemma 3.8).

Remark 1.22. We mention three useful sufficient conditions for being WRUC-compatible (compare [49, Def. 7.6] where this concept appears under the name *w-admissible*): a) the action $S \times X \to X$ is continuous; b) S, as a topological space, is a k-space (e.g., metrizable); c) (S, X) is WAP. Below in Proposition 7.5 we show that $\mathrm{Tame}(X) \subset \mathrm{WRUC}(X)$ for every S-space X. In particular, it follows that every *compact* tame (hence, every WAP) S-system is WRUC-compatible.

1.6. Semigroup compactifications.

Definition 1.23. Let S be a semitopological semigroup.
  (1) [8, p. 105] A *right topological semigroup compactification* of S is a pair (γ, T) such that T is a compact right topological semigroup, and γ is a continuous semigroup homomorphism

from S into T, where $\gamma(S)$ is dense in T and the left translation $\lambda_s : T \to T$, $x \mapsto \gamma(s)x$ is continuous for every $s \in S$, that is, $\gamma(S) \subset \Lambda(T)$.

It follows that the *associated action*

$$\pi_\gamma : S \times T \to T, \quad (s,x) \mapsto \gamma(s)x = \lambda_s(x)$$

is separately continuous.

(2) [65, p. 101] A *dynamical right topological semigroup compactification* of S is a right topological semigroup compactification (γ, T) in the sense of (1) such that, in addition, γ is a jointly continuous S-compactification, i.e., the action $\pi_\gamma : S \times T \to T$ is jointly continuous.

If S is a monoid (as we require in the present paper) with the neutral element e then it is easy to show that necessarily T is a monoid with the neutral element $\gamma(e)$. For a discrete semigroup S, (1) and (2) are equivalent. Directly from Lawson's theorem mentioned above (Fact 1.3) we have:

Fact 1.24. *Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then $\gamma : G \to T$ is a right topological semigroup compactification of G if and only if γ is a dynamical right topological semigroup compactification of G.*

For every semitopological semigroup S there exists a maximal right topological (dynamical) semigroup compactification. The corresponding algebra is $\mathrm{RMC}(S)$ (respectively, $\mathrm{RUC}(S)$). If in the definition of a semigroup compactification (γ, T) we remove the condition $\gamma(S) \subset \Lambda(T)$ then maximal compactifications (in this setting) need not exist (See [7, Example V.1.11] which is due to J. Baker).

Let $\mathcal{A}$ be a closed unital subalgebra of $C(X)$ for some topological space X. We let $\nu_\mathcal{A} : X \to X^\mathcal{A}$ be the associated compactification map (where, as before, $X^\mathcal{A}$ is the maximal ideal space of $\mathcal{A}$). For instance, the *greatest ambit* (see, for example, [71, 73]) of a topological group G is the compact G-space $G^{\mathrm{RUC}} := G^{\mathrm{RUC}(G)}$. It defines the *universal dynamical semigroup compactification* of G. For $\mathcal{A} = \mathrm{WAP}(G)$ we get the *universal semitopological compactification* $G \to G^{WAP}$ of G, which is the universal WAP compactification of G (see [16]). Note that by [52] the projection $q : G^{\mathrm{RUC}} \to G^{WAP}$ is a homeomorphism iff G is precompact.

Remark 1.25. (1) Recall that $\mathrm{RUC}(G)$ generates the topology of G for every topological group G. It follows that the corresponding canonical representation (*Teleman's representation*)

$$(h, \alpha_{\mathrm{RUC}}) : (G, G) \rightrightarrows (\Theta(V)^{op}, B^*)$$

on $V := \mathrm{RUC}(G)$ is faithful and h induces a topological group embedding of G into $\mathrm{Iso}\,(V)$. See [59] for details.

(2) There exists a nontrivial Polish group G whose universal semitopological compactification G^{WAP} is trivial. This is shown in [48] for the Polish group $G := H_+[0,1]$ of orientation preserving homeomorphisms of the unit interval. Equivalently: every (weakly) continuous representation $G \to \mathrm{Iso}\,(V)$ of G on a reflexive Banach space V is trivial.

(3) A stronger result is shown in [26]: every continuous representation $G \to \mathrm{Iso}\,(V)$ of G on an Asplund space V is trivial and every Asplund function on G is constant (note that $Asp_c(G) = \mathrm{Asp}(G)$ for Polish G by Lemma 1.20.3). Every nontrivial right topological semigroup compactification of the Polish topological group $G := H_+[0,1]$ is not metrizable [30]. In contrast we show in Theorem 7.8 that G is Rosenthal representable.

1.7. Enveloping semigroups. Let X be a compact S-system with a separately continuous action. Consider the natural map $j : S \to C(X, X), s \mapsto \lambda_s$. As usual denote by $E(X) = cl_p(j(S)) \subset X^X$ the enveloping (Ellis) semigroup of (S, X). The associated homomorphism $j : S \to E(X)$ is a right topological semigroup compactification (say, *Ellis compactification*) of S, $j(e) = id_X$ and the associated action $\pi_j : S \times E(X) \to E(X)$ is separately continuous. Furthermore, if the S-action on X is continuous then π_j is continuous, i.e., $S \to E(X)$ is a dynamical semigroup compactification.

Lemma 1.26. (1) *Let X be a compact semitopological S-space and L a subset of $C(X)$ such that L separates points of X. Then the Ellis compactification $j : S \to E(X)$ is equivalent to the compactification of S which corresponds to the subalgebra $\mathcal{A}_L := \langle m(L, X) \rangle$, the smallest*

norm closed S-invariant unital subalgebra of $C(S)$ which contains the family

$$\{m(f,x): S \to \mathbb{R}, \ s \mapsto f(sx)\}_{f \in L, \ x \in X}.$$

(2) *Let $q: X_1 \to X_2$ be a continuous onto S-map between compact S-spaces. There exists a (unique) continuous onto semigroup homomorphism $Q: E(X_1) \to E(X_2)$ with $j_{X_1} \circ Q = j_{X_2}$.*

(3) *Let Y be a closed S-subspace of a compact S-system X. The map $r_X: E(X) \to E(Y)$, $p \mapsto p|_Y$ is the unique continuous onto semigroup homomorphism such that $r_X \circ j_X = j_Y$.*

(4) *Let $\alpha: S \to P$ be a right topological compactification of a semigroup S. Then the enveloping semigroup $E(S,P)$ of the semitopological system (S,P) is naturally isomorphic to P.*

(5) *If X is metrizable then $E(X)$ is separable. Moreover, $j(S) \subset E(X)$ is separable.*

Proof. (1) The proof is straightforward using the Stone-Weierstrass theorem.

(2) By Remark 1.16 it suffices to show that the compactification $j_{X_1}: S \to E(X_1)$ dominates the compactification $j_{X_2}: S \to E(X_2)$. Equivalently we have to verify the inclusion of the corresponding algebras. Let $q(x) = y, f_0 \in C(X_2)$ and $f = f_0 \circ q$. Observe that $m(f_0, y) = m(f, x)$ and use (1).

(3) Is similar to (2).

(4) Since $E(S,P) \to P, a \mapsto a(e)$ is a natural homomorphism, $j_P: S \to E(S,P)$ dominates the compactification $S \to P$. So it is enough to show that, conversely, $\alpha: S \to P$ dominates $j_P: S \to E(S,P)$. By (1) the family of functions

$$\{m_{f,x}: S \to \mathbb{R}\}_{f \in C(P), \ x \in P}$$

generates the Ellis compactification $j_P: S \to E(S,P)$. Now observe that each $m_{f,x}: S \to \mathbb{R}$ can be extended naturally to the function $P \to \mathbb{R}$, $p \mapsto f(px)$ which is *continuous*.

(5) Since X is a metrizable compactum, $C(X,X)$ is separable and metrizable in the compact open topology. Then $j(S) \subset C(X,X)$ is separable (and metrizable) in the same topology. Hence, the dense subset $j(S) \subset E(X)$ is separable in the pointwise topology. This implies that $E(X)$ is separable. $\qquad\square$

Remark 1.27. Every enveloping semigroup $E(S,X)$ is an example of a compact right topological admissible semigroup. Conversely, every compact right topological *admissible* semigroup P is an enveloping semigroup (of $(\Lambda(P), P)$, as it follows from Lemma 1.26.4).

2. Operator compactifications

Operator compactifications provide an important tool for constructing and studying semigroup compactifications via representations of semigroups on Banach spaces (or, more generally, on locally convex vector spaces). In classical works by Eberlein, de Leeuw and Glicksberg, it was shown that weakly almost periodic Banach representations of a semigroup S induce semitopological compactifications of S. In general the situation is more complicated and we have to deal with right topological semigroup compactifications of a semigroup S. We refer to the papers of Witz [75] and Junghenn [40]. We note also that in his book [17] R. Ellis builds his entire theory of abstract topological dynamics using the language of operator representations.

First we reproduce the construction of Witz with some minor changes. Let $h: S \to L(V)$ be a weakly continuous, equicontinuous representation (co-homomorphism) of a semitopological semigroup S into the space $L(V)$ of continuous linear operators on a locally convex vector space V. "Equicontinuous" here means that the subset $h(S) \subset L(V)$ is an equicontinuous family of linear operators. Then the weak* operator closure $\overline{h(S)^{op}}$ of the adjoint semigroup $h(S)^{op} \subset L(V)^{op} = adj(L(V))$ in $L(V^*)$ is a right topological semigroup compactification of S. We obtain the compactification

$$S \to P := \overline{h(S)^{op}} \subset L(V^*)$$

which, following Junghenn, we call an *operator compactification* of S (induced by the representation h). The weak* operator topology on $L(V^*)$ is the weakest topology generated by the system

$$\widetilde{m}(v, \psi): L(V^*) \to \mathbb{R}, \ p \mapsto \langle v, p\psi \rangle = \langle vp, \psi \rangle$$

of maps, where $v \in V, \psi \in V^*$, $vp = p^*(v) \in V^{**}$ and $p^*: V^{**} \to V^{**}$ is the adjoint of p.

In fact, the semigroup P can be treated also as the weak* operator closure of $h(S)$ in $L(V, V^{**})$. The latter version is found mainly in [75] and [40].

The *coefficient algebra* A_h (respectively, *coefficient space* M_h) of the representation $h : S \to \Theta(V)$ is the smallest norm closed, unital subalgebra (respect., subspace) of $C(S)$ containing all the matrix coefficients of h

$$m(V, V^*) = \{m(v, \psi) : S \to \mathbb{R}, \quad s \mapsto \langle vh(s), \psi \rangle | \ v \in V, \ \psi \in V^* \}.$$

That is, according to our notation $A_h = \langle m(V, V^*) \rangle$ and $M_h = \overline{sp}^{norm}(m(V, V^*) \cup \{\mathbf{1}\})$.

Lemma 2.1. *Let $S \to P := \overline{h(S)^{op}} \subset L(V^*)$ be the operator compactification induced by a weakly continuous equicontinuous representation $h : S \to L(V)$ on a locally convex space V. The algebra of this compactification is just the coefficient algebra A_h.*

Proof. For every $(v, \psi) \in V \times V^*$ the function $m(v, \psi) : S \to \mathbb{R}$ is a restriction of the continuous map $\widetilde{m}(v, \psi)|_P : P \to \mathbb{R}, \quad p \mapsto \langle v, p\psi \rangle = \psi(vp)$. Such maps separate points of P. Now use the Stone-Weierstrass theorem. $\square$

2.1. **The enveloping semigroup of a Banach space.** Let V be a Banach space and $\Theta(V)$ the semigroup of all non-expanding operators from V to itself. As in Section 1.2 consider the natural left action of $\Theta(V)^{op}$ on the weak* compact unit ball B^*. This action is separately continuous when $\Theta(V)^{op}$ carries the weak operator topology.

Definition 2.2. Given a Banach space V we denote by $\mathcal{E}(V)$ the enveloping semigroup of the dynamical system $(\Theta(V)^{op}, B^*)$. We say that $\mathcal{E}(V)$ is the *enveloping semigroup of V*.

Always, $\mathcal{E}(V)$ is a compact right topological admissible affine semigroup. The corresponding Ellis compactification $j : \Theta(V)^{op} \to \mathcal{E}(V)$ is a topological embedding. Alternatively, $\mathcal{E}(V)$ can be defined as the weak* operator closure of the adjoint monoid $\Theta(V)^{op}$ in $L(V^*)$ (Lemma 2.4.2). So it is the operator compactification of the semigroup $\Theta(V)$.

If V is separable then $\mathcal{E}(V)$ is separable by Lemma 1.26.5 because B^* is metrizable. $\mathcal{E}(V)$ is metrizable iff V is separable Asplund, Theorem 6.11.

Every weakly continuous representation $h : S \to \Theta(V)$ of a semitopological semigroup on a Banach space V (by non-expanding operators) gives rise to a right topological semigroup compactification

$$h : S \to \overline{h(S)^{op}} \subset \mathcal{E}(V)$$

where $\overline{h(S)^{op}}$ is the closure in $\mathcal{E}(V)$. We sometimes call it the *standard operator compactification* of S (generated by the representation h).

Definition 2.3. Let $\alpha : P \to K$ be a continuous (not necessarily onto) homomorphism between compact right topological admissible semigroups. Suppose that S is a dense subsemigroup of $\Lambda(P)$. We say that:
  (1) α is *S-admissible* if $\alpha(S) \subset \Lambda(K)$.
  (2) α is *admissible* if it is S-admissible with respect to some dense subsemigroup $S \subset \Lambda(P)$.
  (3) P is *representable* on a Banach space V if there exists an admissible embedding α of P into $\mathcal{E}(V)$. If V is Rosenthal (Asplund, reflexive) then we say that P is Rosenthal (Asplund, reflexively) representable.

Every standard operator compactification generated by a representation h of S on V induces an admissible embedding of $\overline{h(S)^{op}}$ into $\mathcal{E}(V)$ because, $h(S) \subset \Theta(V)^{op}$ (and $\Theta(V)^{op} = \Lambda(\mathcal{E}(V))$, Lemma 2.6.5). In the next lemma, as before, given a subset $A \subset C(S)$, we let $\langle A \rangle$ denote the closed unital subalgebra of $C(S)$ generated by A.

Lemma 2.4. (1) *Every standard operator compactification $h : S \to \overline{h(S)^{op}} \subset \mathcal{E}(V)$ is equivalent to the Ellis compactification $j : S \to E = E(S, B^*)$. The algebra of these compactifications is the coefficient algebra $A_h = \langle m(V, V^*) \rangle$.*
  (2) *$\overline{\Theta(V)^{op}}$ is isomorphic to $\mathcal{E}(V)$ and the algebra of the compactification $j : \Theta(V)^{op} \hookrightarrow \mathcal{E}(V)$ is the coefficient algebra A_h for $h : \Theta(V) \hookrightarrow \overline{\Theta(V)^{op}} = \mathcal{E}(V)$.*
  (3) *The natural inclusion $\alpha : E(S, B^*) \hookrightarrow \mathcal{E}(V)$ is $j(S)$-admissible.*

Proof. (1) Both of these compactifications have the same algebra A_h. Indeed Lemma 2.1 implies this for the compactification $h : S \to \overline{h(S)^{op}} \subset \mathcal{E}(V)$. For $j : S \to E(S, B^*)$ use Lemma 1.26.1.

Note that (2) is a particular case of (1) for $S = \Theta(V)$.

(3) is trivial because $j(S)$ is dense in $\Lambda(E(S, B^*))$ and $\alpha(j(S)) = h(S) \subset \Theta(V)^{op}$. $\qquad\square$

Proposition 2.5. *Every semigroup compactification is a factor of an operator semigroup compactification.*

Proof. Let (γ, P) be a semigroup compactification of S. Take a faithful Banach representation of the S-flow P on V. For example, one can take the regular representation of P on $V := C(P)$. Now the enveloping semigroup $E(S, P)$ is a factor of $E(S, B^*)$ which is an operator semigroup compactification of S (Lemma 2.4) and $E(S, P)$ is naturally isomorphic to P (Lemma 1.26.4). $\qquad\square$

In Example 5.6 we show that there exists a right topological semigroup compactifications of the group $\mathbb{Z}$, which is not an operator compactifications. It follows that compact right topological *operator* semigroups are not closed under factors. Indeed the compactification $\mathbb{Z} \to \beta\mathbb{Z} = \mathbb{Z}^{\mathrm{RUC}}$ is an operator compactification (by Remarks 4.16.1 below) and it is the universal $\mathbb{Z}$-ambit.

Not every admissible compact right topological (semi)group admits a representation on a Banach space (see Theorem 5.6). On the other hand we will later investigate the question when a "good" semigroup compactification can be realized as a standard operator compactification on "good" Banach spaces (see Section 8).

In the sequel whenever V is understood we use the following simple notations $\mathcal{E} := \mathcal{E}(V)$, $\Theta := \Theta(V)$, $\Theta^{op} := \Theta(V)^{op}$. By S_V we denote the unit sphere of V.

Lemma 2.6. *For every Banach space V, every $v \in S_V$ and $\psi \in S_{V^*}$ we have*

(1) $\Theta v = B$.
(2) $v\mathcal{E} = B^{**}$.
(3) $cl_{w^*}(\Theta^{op}\psi) = B^*$.
(4) $\mathcal{E}\psi = B^*$.
(5) $\Lambda(\mathcal{E}) = \Theta^{op}$.

Proof. (1) Take $f \in S_{V^*}$ such that $f(v) = 1$. For every $z \in B$ define the rank 1 operator

$$A(f, z) : V \to V, \quad x \mapsto f(x)z.$$

Then $A(f, z)(v) = z$ and $A(f, z) \in \Theta$ since $||A(f, z)|| = ||f|| \cdot ||z|| = ||z|| \leq 1$.

(2) By (1), $v\Theta^{op} = \Theta v = B$ which is pointwise dense in B^{**} by Goldstine theorem. So, $v\mathcal{E} = B^{**}$ because $\mathcal{E} \to (V^{**}, w^*), p \mapsto vp$ is continuous and $\mathcal{E} = \overline{\Theta^{op}}$.

(3) We can suppose that V is infinite-dimensional (use (1) for the finite-dimensional case). Then the unit sphere S_{V^*} is norm (hence, weak*) dense in B^*. So it is enough to prove that the weak* closure of $\Theta^{op}\psi$ contains S_{V^*}. Let $\phi \in S_{V^*}$. We have to show that for every $\varepsilon > 0$ and $v_1, v_2, \cdots, v_n \in V$ there exists $s \in \Theta$ such that $||s^*\psi(v_i) - \phi(v_i)|| < \varepsilon$ for every $i = 1, 2, \cdots, n$, where $s^* \in \Theta^{op}$ is the adjoint of the operator s. Since $\psi \in V^*$ and $||\psi|| = 1$ one may choose $z \in B_V$ such that

$$|\phi(v_i)(\psi(z) - 1)| < \varepsilon$$

for every $i = 1, 2, \cdots, n$. Define $s := A(\phi, z)$. Then

$$|(s^*\psi)(v_i) - \phi(v_i)| = |\psi(sv_i) - \phi(v_i)| = |\psi(\phi(v_i)z) - \phi(v_i)| = |\phi(v_i)(\psi(z) - 1)| < \varepsilon$$

for every i.

(4) Follows from (3) because $\mathcal{E}$ is the weak* operator closure of Θ^{op}.

(5) Trivially, $\Lambda(\mathcal{E}) \supseteq \Theta^{op}$. Conversely, let $\sigma \in \Lambda(\mathcal{E})$. Then $\sigma \in L(V^*)$ with $||\sigma|| \leq 1$. Consider the adjoint operator $\sigma^* : V^{**} \to V^{**}$. We have to show that $\sigma^*(v) \in V \subset V^{**}$, for every $v \in \Lambda$, where we treat V as a Banach subspace of V^{**}. By Fact 1.13 it is enough to show that $\sigma^*(v)|_{B^*} : B^* \to \mathbb{R}$ is w^*-continuous. By our assumption, $\sigma \in \Lambda(\mathcal{E})$. That is, the left translation $l_\sigma : \mathcal{E} \to \mathcal{E}$ is continuous. Choose a point $z \in S_{V^*}$ and consider the orbit map $\tilde{z} : \mathcal{E} \to B^*, p \mapsto pz$. Then, $\tilde{z} \circ l_\sigma = \sigma|_{B^*} \circ \tilde{z}$. By (4) we have $\mathcal{E}z = B^*$, hence, $\tilde{z} : \mathcal{E} \to B^*$ is onto. Since $\mathcal{E}$ is compact, it follows that the map $\sigma|_{B^*} : B^* \to B^*$ is continuous. This implies that $\sigma^*(v)|_{B^*} : B^* \to \mathbb{R}$ is w^*-continuous (for any $v \in V$), as desired. $\qquad\square$

3. Affine compactifications of dynamical systems and introversion

3.1. **Affine compactifications in terms of state spaces.** Let S be a semitopological semigroup. An S-system Q is an *affine S-system* if Q is a convex subset of a locally convex vector space and each $\lambda_s : Q \to Q$ is affine. If in addition $S = Q$ acts on itself by left translations and if right translations are also affine maps then S is said to be an *affine semigroup*. For every compact affine S-system Q each element of its enveloping semigroup is a (not necessarily, continuous) affine self-map of Q.

Definition 3.1. (1) [8, p. 123] An *affine semigroup compactification* of a semitopological semigroup S is a pair (ψ, Q), where Q is a compact right topological affine semigroup and $\psi : S \to Q$ is a continuous homomorphism such that $co(\psi(S))$ is dense in Q and $\psi(S) \subset \Lambda(Q)$.

(2) By an *affine S-compactification* of an S-space X we mean a pair (α, Q), where $\alpha : X \to Q$ is a continuous S-map and Q is a convex compact affine S-flow such that $\alpha(X)$ *affinely generates* Q, that is $\overline{co}(\alpha(X)) = Q$ (see [20]).

(3) In particular, for a trivial action (or for the trivial semigroup S) we retrieve in (2) the notion of an *affine compactification* of a topological space X.

An affine S-compactification $\alpha : X \to Q$ induces the S-compactification $\alpha : X \to Y := \overline{\alpha(X)} \subset Q$ of X. Of course we have $Y = \overline{\alpha(X)} = \alpha(X)$ when X is compact. Definition 3.1.2 is a natural extension of Definition 3.1.1.

Remarks 3.2. (1) For any Banach space V, Θ is an affine semitopological semigroup, (Θ^{op}, B^*) is an affine system and the inclusion $\Theta^{op} \hookrightarrow \mathcal{E}$ is an affine semigroup compactification.

(2) Not every semigroup compactification (in contrast to affine semigroup compactifications) comes as an operator compactification. See Theorem 5.6 (and Proposition 4.8) below.

(3) For every continuous compact S-system X, the weak* compact unit ball $B^* \subset C(X)^*$ and its closed subset $P(X)$ of all probability measures, are affine S-systems (Proposition 3.9.2).

(4) Every Banach representation (h, α) of an S-flow X naturally induces an S-affine compactification $X \to Q := \overline{co} \, \alpha(X)$ (Section 4). Conversely, every affine compactification of an S-space X comes from a Banach representation of the S-space X on the Banach space $V \subset C(S)$ which is just the *affine compactification space* (see Lemmas 3.6 and 3.8).

As in the case of compactifications of flows one defines notions of preorder, factors and isomorphisms of affine compactifications. More explicitly, we say that for two affine compactifications, $\alpha_1 : X \to Q_1$ *dominates* $\alpha_2 : X \to Q_2$ if there exists a continuous affine map (a *morphism*) $q : Q_1 \to Q_2$ such that $q \circ \alpha_1 = \alpha_2$. Notation: $\alpha_1 \succeq \alpha_2$. If one may choose q to be a homeomorphism then we say that q is an *isomorphism* of affine compactifications. Notation: $\alpha_2 \cong \alpha_1$. It is easy to see that $\alpha_2 \cong \alpha_1$ iff $\alpha_2 \succeq \alpha_1$ and $\alpha_1 \succeq \alpha_2$.

Lemma 3.3. *If $q : Q_1 \to Q_2$ is a morphism between two S_d-affine compactifications $\alpha_1 : X \to Q_1$ and $\alpha_2 : X \to Q_1$ then q is an S-map.*

Proof. Since the s-translations in Q_1 and Q_2 are affine it easily follows that the inclusion maps $co(\alpha_1(X)) \hookrightarrow Q_1$, $co(\alpha_2(X)) \hookrightarrow Q_2$ are S_d-compactifications and also the restriction map $q : co(\alpha_1(X)) \to co(\alpha_2(X))$ is an onto S-map. The induced map $co(\alpha_1(X)) \to Q_2$ defines an S_d-compactification. Now Remark 1.16 yields that $q : Q_1 \to Q_2$ is an S-map. $\square$

Recall that for a normed unital subspace F of $C(X)$ the *state space* of F is the w^*-compact subset

$$M(F) := \{\mu \in F^* : \ \|\mu\| = \mu(\mathbf{1}) = 1\}$$

of all *means* on F. If in addition $F \subset C(X)$ is a subalgebra, we denote by $MM(F)$ the compact set of all multiplicative means on F. For a compact space X and for $F = C(X)$ the state space $M(C(X))$ is the space of all probability measures on X which we denote as usual by $P(X)$.

Lemma 3.4. ([60, 7, 8] *For every topological space X we have:*

(1) *State space $M(F)$ is convex and weak* compact in the dual F^* of F.*

(2) *The map $\delta : X \to M(F)$, $\delta(x)(f) = f(x)$, is affine and weak*-continuous, and its image $\delta(X)$ affinely generates $M(F)$ (i.e. $\overline{co}^{w^*}(\delta(X)) = M(F)$).*

(3) *Every $\mu \in F^*$ is a finite linear combination of members of $M(F)$.*

(4) *If $F \subset C(X)$ is a subalgebra then $\delta(X)$ is dense in $MM(F)$.*

Thus $\delta : X \to M(F)$ is an affine compactification of X. We call it the *canonical F-affine compactification* of X. The induced compactification $\delta : X \to \overline{\delta(X)} = Y$ is said to be the *canonical F-compactification* of X. By Stone-Weierstrass theorem it follows that $C(Y)$ is naturally isometrically isomorphic to $\mathcal{A}_\delta := \langle F \rangle$, the closed unital subalgebra of $C(X)$ generated by F.

For every compact convex subset Q of a locally convex vector space V we denote by $\mathrm{A}(Q)$ the Banach unital subspace of $C(Q)$ consisting of the affine continuous functions on Q. Of course $f|_Q : Q \to \mathbb{R}$ is affine and continuous for every $f \in V^*$. So by the Hahn-Banach theorem $\mathrm{A}(Q)$ always separates points of Q. It is well known (see [4, Cor. 4.8]) that the subspace

$$\mathrm{A}_0(Q) := \{f|_Q + c : \quad f \in V^*, \ c \in \mathbb{R}\}$$

is uniformly dense in $\mathrm{A}(Q)$. If V is a Banach space then by Fact 1.13 every w^*-continuous functional on V^* is the evaluation at some point $v \in V$. This implies the following useful lemma.

Lemma 3.5. *For every Banach space V and a weak* compact convex set $Q \subset V^*$ the subspace*

$$\mathrm{A}_0(Q) := \{\tilde{v}|_Q + c : \quad v \in V, \ c \in \mathbb{R}\} = r_Q(V) + \mathbb{R} \cdot \mathbf{1}$$

is uniformly dense in $\mathrm{A}(Q)$, where $\tilde{v}(\varphi) = \langle v, \varphi \rangle$ and $r_Q : V \to C(Q)$ is the restriction operator.

Next we classify the affine compactifications of a topological space X in terms of unital closed subspaces of $C(X)$, in the spirit of the Gelfand-Kolmogoroff theorem (compare Remark 1.18.3). At least for compact spaces X and point-separating subspaces $F \subset C(X)$ versions of Lemma 3.6 below can be found in several classical sources. See for example [60, Ch. 6], [13, Ch. 6, §29], [3, Theorem II.2.1], [67, Ch. 6, §23], [4, Ch. 1, §4]. For affine bi-compactifications of transformation semigroups it remains true in a suitable setting, [37, Remark 3.2].

Lemma 3.6. *Let X be a topological space. The assignment $\Upsilon : F \mapsto \delta_F$, where $\delta_F : X \to M(F))$ is the canonical F-affine compactification, defines an order preserving bijective correspondence between the collection of unital Banach subspaces F of $C(X)$ and the collection of affine compactifications of X (up to equivalence). In the converse direction, to every affine compactification $\alpha : X \to Q$ corresponds the unital Banach subspace $F := A(Q)|_X \subset C(X)$ (called the* affine compactification space*). Then the canonical affine compactification $\delta_F : X \to M(F)$ is affinely equivalent to $\alpha : X \to Q$.*

Proof. For a Banach unital subspace F of $C(X)$ define $\Upsilon(F) = (\delta_F, M(F))$ as the canonical F-affine compactification $\delta_F : X \to M(F)$.

Surjectivity of Υ:

Every affine compactification $\alpha : X \to Q$, up to equivalence, is a canonical F-affine compactification. In order to show this consider the set $A(Q)$ of all continuous affine functions on Q, viewed as a (Banach unital) subspace of $C(Q)$. Let $A(Q)|_X$ be the set of all functions on X which are α-extendable to a continuous affine function on Q. Thus $A(Q)|_X := \alpha_\sharp(A(Q)) \subset C(X)$, where $\alpha_\sharp : C(Q) \to C(X), f \mapsto f \circ \alpha$ is the natural linear operator induced by $\alpha : X \to Q$. Every such operator has norm 1. Moreover, since $\alpha(X)$ affinely generates Q and the functions in $A(Q)$ are affine, it follows that $\alpha_\sharp : A(Q) \to C(X)$ is a linear isometric embedding. Denote by F the Banach unital subspace $\alpha_\sharp(A(Q)) = A(Q)|_X$ of $C(X)$. We are going to show that the affine compactifications $\delta_F : X \to M(F)$ and $\alpha : X \to Q$ are isomorphic. Define the evaluation map

$$e : Q \to M(F) \subset F^*, \quad e(q)(f) = \tilde{f}(q),$$
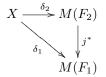
where $\tilde{f} := \alpha_\sharp^{-1}(f) \in A(Q)$ is the uniquely defined extension of $f \in F := A(Q)|_X$. Since $\alpha_\sharp^{-1} : F \to A(Q)$ is a linear isometry we easily obtain that $e(q) \in F^*$. Clearly, $||e(q)|| = e(q)(\mathbf{1}) = 1$ for every $q \in Q$. Hence, indeed $e(q) \in M(F)$ and the map $e : Q \to M(F)$ is well-defined. Since, $\tilde{f} : Q \to \mathbb{R}$ is an affine map for every $f \in F$, it easily follows that $e : Q \to M(F)$ is an affine map. For every $x \in X$ we have $e(\alpha(x))(q) = \tilde{f}(\alpha(x)) = f(x)$. So, $\delta_F = e \circ \alpha$. It is also clear that e is w^*-continuous. Since $\delta_F(X)$ affinely generates $M(F)$ (Lemma 3.4), it follows that $e(Q) = M(F)$. Always, $\mathrm{A}(Q)$ separates points of Q. This implies that $e : Q \to M(F)$ is injective, hence a homeomorphism.

The *Injectivity* and *order-preserving* properties of Υ:

These properties follow from the next claim.

Claim: *If $\alpha_1 : X \to Q_1$ and $\alpha_2 : X \to Q_2$ are two affine compactifications then α_2 dominates α_1 if and only if $F_2 \supset F_1$, where F_1 and F_2 are the corresponding affine compactification spaces.*

Suppose $F_2 \supset F_1$ and let $j : F_1 \hookrightarrow F_2$ be the inclusion map. Then the restricted adjoint map $j^* : M(F_2) \to M(F_1)$ is a weak* continuous affine map and the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\;\delta_2\;} & M(F_2) \\
& {\scriptstyle \delta_1}\searrow & \big\downarrow{\scriptstyle j^*} \\
& & M(F_1)
\end{array}
$$

Moreover j^* is onto (use for example Lemma 3.4.2). The second direction is trivial. $\qquad\square$

Corollary 3.7. *For every compact space X the Banach space $C(X)$ determines the universal (greatest) affine compactification $\alpha_b : X \to P(X) = M(C(X))$. For any other affine compactification $X \to Q$ we have a uniquely determined natural affine continuous onto map, called the* barycenter *map, $b : P(X) \to Q$, such that $\alpha_b = b \circ \delta$.*

Next we deal with affine compactifications of S-systems.

Lemma 3.8. *Assume that X is endowed with a semigroup action $S \times X \to X$ with continuous translations, i.e., X is an S_d-space. Let $\alpha : X \to Q$ be an affine compactification of the S_d-space X. Denote by F the corresponding affine compactification space.*

(1) *$S \times Q \to Q$ is naturally topologically S-isomorphic to the action $S \times M(F) \to M(F)$.*
(2) *$S \times Q \to Q$ is separately continuous iff $F \subset \mathrm{WRUC}(X)$.*
(3) *$S \times Q \to Q$ is continuous iff $F \subset \mathrm{RUC}(X)$.*

Proof. (1) All the translations $\lambda_s : X \to X$ are continuous and $F = A(Q)|_X$ is an S-invariant subset of $C(X)$. So it is clear that the natural dual action $S \times F^* \to F^*$ is well defined and that every translation $\lambda_s : F^* \to F^*$ is weak* continuous. Now observe that $S \times M(F) \to M(F)$ is a restriction of the action $S \times F^* \to F^*$. Since $X \to Q$ and $X \to M(F)$ are S_d-affine compactifications and the evaluation map $e : Q \to M(F)$ from the proof of Lemma 3.6 is an isomorphism of affine compactifications, we obtain by Lemma 3.3 that $e : Q \to M(F)$ is an S-map.

(2) Use Lemma 3.4.3 and the restriction operator $r : F \to C(X)$ (Remark 1.18).
(3) Use Remark 1.18.1. $\qquad\square$

Proposition 3.9. (1) *If X is an S-space then the same assignment Υ, as in Lemma 3.6, establishes an order preserving bijection between the collection of S-invariant unital Banach subspaces F of $WRUC(X)$ and (equivalence classes of) S-affine compactifications of the S-system X. Furthermore, the subspaces of $RUC(X)$ correspond exactly to the S-affine compactifications $X \to Q$ with continuous actions $S \times Q \to Q$.*

(2) *Let X be a compact S-system and $\alpha_b : X \to P(X) = M(C(X))$ be the universal affine compactification of the space X.*

   (a) *For every S_d-affine compactification $\alpha : X \to Q$ the barycenter map $b : P(X) \to Q$ is an S-map.*
   (b) *$S \times P(X) \to P(X)$ is separately continuous iff $C(X) = \mathrm{WRUC}(X)$ (iff X is WRUC).*
   (c) *$S \times X \to X$ is continuous iff $S \times P(X) \to P(X)$ is continuous iff $C(X) = \mathrm{RUC}(X)$.*

Proof. (1) Use Lemma 3.8.
(2) (a) Apply Lemmas 3.3 and Corollary 3.7. For (b) and (c) use Lemma 3.8. $\qquad\square$

Remark 3.10. (1) As we already mentioned every S-affine compactification $\alpha : X \to Q$ induces the S-compactification $\alpha_0 : X \to Y := \overline{\alpha(X)}$ of X. Always the affine compactification space $F := A(Q)|_X \subset C(X)$ generates the induced compactification algebra $\mathcal{A}$ of α_0. That is, $\langle F \rangle = \mathcal{A}$. Indeed, affine continuous functions on Q separate the points. Hence, by Stone-Weierstrass theorem $A(Q)|_Y$ generates $C(Y)$. It follows that $\alpha_0^*(A(Q)|_Y) = F$ generates $\alpha_0^*(C(Y)) = \mathcal{A}$, where $\alpha_0^* : C(Y) \to C(X)$ is the induced map.

(2) For every weakly continuous representation $h : S \to \Theta(V)$ on a Banach space V we have the associated affine semigroup compactification $h : S \to Q := \overline{co}(h(S)^{op}) \subset \mathcal{E}$. Observe that since $co(h(S)^{op}) \subset \Theta^{op} = \Lambda(\mathcal{E})$, the closure $Q := \overline{co}(h(S)^{op})$ in $\mathcal{E}$ is a semigroup. In this case we say that $S \to Q$ is a *standard affine semigroup compactification*. Every affine semigroup compactification can be obtained in this way (see Proposition 3.18.2).

(3) More generally, every operator compactification (Section 2) $h : S \to P := \overline{h(S)^{op}} \subset L(V^*)$ of S on a locally convex vector space V induces an affine semigroup compactification $\alpha_h : S \to Q := \overline{co}(h(S)^{op} \subset L(V^*)$. The affine compactification space $\mathcal{A}(Q)|_S$ coincides with the coefficient space M_h. Indeed, $M_h \subset \mathcal{A}(Q)|_S$ because every matrix coefficient $m(v, \psi) : S \to \mathbb{R}$ is a restriction of the map $\widetilde{m}(v, \psi) : Q \to \mathbb{R}$ which is continuous and affine. On the other hand the collection $\{\widetilde{m}(v, \psi)\}_{v \in V, \psi \in V^*}$ separates the points of Q. It follows that the Banach unital subspaces M_h and $\mathcal{A}(Q)|_S$ of $C(S)$ induce the isomorphic affine compactifications of S. By Lemma 3.6 we obtain that $M_h = \mathcal{A}(Q)|_S$.

3.2. Cyclic affine S-compactifications.

Let X be a (not necessarily compact) S-system. For every $f \in \mathrm{WRUC}(X)$ denote by $\mathcal{A}_f := \langle fS \cup \{\mathbf{1}\} \rangle \subset C(X)$ the smallest S-invariant unital Banach subalgebra which contains f. The corresponding Gelfand compactification is an S-compactification $\alpha_f : X \to |\mathcal{A}_f| \subset \mathcal{A}_f^*$. We call it the *cyclic compactification* of X (induced by f). Now consider $V_f := \overline{sp}^{norm}(fS \cup \{\mathbf{1}\})$ — the smallest closed linear unital S-subspace of $\mathrm{WRUC}(X)$ generated by f. By Proposition 3.9.1 we have the affine S-compactification $\delta_f : X \to M(V_f) \subset V_f^*$ (where, $\delta_f(x)(\varphi) = \varphi(x)$ for every $\varphi \in V_f$) which we call the *cyclic affine S-compactification* of X. A natural idea is to reconstruct α_f from δ_f restricting the codomain of δ_f. In the following technical lemma we also give a useful realization (up to isomorphisms) of these two compactifications in $C(S)$ with the pointwise convergence topology. Note that we have also a left action of S on $C(S)$ defined by $S \times C(S) \to C(S), (sf)(t) = (ts) = R_s f$.

Lemma 3.11. (1) *The following map*
$$T_f : V_f^* \to C(S), \quad T_f(\psi) = m(f, \psi).$$
is a well defined linear bounded weak*-pointwise continuous S-map between left S-actions.

(2) *The restriction $T_f : M(V_f) \to Q_f$ is an isomorphism of the affine S-compactifications $\delta_f : X \to M(V_f)$ and $\pi_f : X \to Q_f$, where $\pi_f := T_f \circ \delta_f$ and $Q_f := T_f(M(V_f)) \subset C(S)$.*

(3) *Consider*
$$X_f := \overline{\pi_f(X)}^p = cl_p(\{m(f, \delta_f(x))\}_{x \in X}) \subset Q_f.$$
The restriction of the codomain leads to the S-compactification $\pi_f : X \to X_f$ which is isomorphic to the cyclic compactification $\alpha_f : X \to |\mathcal{A}_f|$.

Proof. (1) $m(f, \psi) \in C(S) \; \forall \psi \in V_f^*$ because $f \in \mathrm{WRUC}(X)$. Other conditions are also easy.

(2) $T_f : M(V_f) \to Q_f$ is a morphism of the affine S-compactifications $\delta_f : X \to M(V_f)$ and $\pi_f : X \to Q_f$, where $\pi_f := T_f \circ \delta_f$. So, $\delta_f \succeq \pi_f$. In order to establish that $\pi_f \succeq \delta_f$ it is enough to show that our original function $f : X \to \mathbb{R}$ belongs to the affine compactification space of π_f. This follows from the observation that the evaluation at e functional, $\widehat{e} : C(S) \to \mathbb{R}$, restricted to $\overline{co}(X_f) \subset C(S)$, is an affine function such that $f = \widehat{e} \circ \pi_f$.

(3) By Remark 3.10.1 the algebra of the cyclic compactification $\pi_f : X \to X_f$ is just $\langle V_f \rangle$, but this is exactly $\mathcal{A}_f$, the algebra of the compactification $\alpha_f : X \to |\mathcal{A}_f|$. $\square$

Remark 3.12. (1) Note that $f = F_e \circ \pi_f$, where $F_e := \widehat{e}|_{X_f}$. So f comes from the S-system X_f. Moreover, if f comes from an S-system Y and an S-compactification $\nu : X \to Y$, then there exists a continuous onto S-map $\alpha : Y \to X_f$ such that $\pi_f = \alpha \circ \nu$. The action of S on X_f is continuous iff $f \in \mathrm{RUC}(X)$ (see Remark 1.18.3).

(2) Moreover, $S\widehat{e} = \{\widehat{s}\}_{s \in S}$ separates points of Q_f (where $\widehat{s}$ is the evaluation at s functional) and $(X_f)_{F_e} = X_f$.

(3) The action of S on X_f is separately continuous iff $f \in \mathrm{RMC}(X)$ (use again Remark 1.18.3). By definition, $Q_f = \overline{co}^p(X_f)$ in $C(S)$. At the same time the extended action of S on Q_f need not be separately continuous for $f \in \mathrm{RMC}(X)$. This is a reflection of the fact that in general $\mathrm{RMC}(X)$ is strictly larger than $\mathrm{WRUC}(X)$ (see [8, p. 219]). However by Lemma 1.20 we

know that $\mathrm{WRUC}(X) = \mathrm{RMC}(X)$ in many natural cases. Also, $\mathrm{Tame}(X) \subset \mathrm{WRUC}(X)$ for every S-system X by Proposition 7.5.

Lemma 3.13. *Let V be a Banach S-invariant unital subspace of $\mathrm{WRUC}(X)$ and $f \in V$. Then*

(1) $X_f := \overline{\pi_f(X)}^p = cl_p(m(f, \delta_f(X)))$.
(2) $Q_f = m(f, M(V_f)) = m(f, M(V)) = \overline{co}^p(X_f)$.
(3) *In the particular case of $X := S$, with the left action of S on itself, we have $X_f = \overline{Sf}^p$.*

Proof. Straightforward, using Lemma 3.11. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Some other useful properties of cyclic compactifications can be found in [8, 25, 26].

3.3. **Introversion and semigroups of means.** In this section we assume that F is a normed unital subspace of $C(S)$, where S, as before, is a semitopological monoid. Suppose also that F is left translation invariant, that is, the function $(L_s f)(x) = f(sx)$ belongs to F for every $(f, s) \in F \times S$. Then the dual action $S \times M(F) \to M(F)$ is well defined and each s-translation, being the restriction of the adjoint operator L_s^*, is continuous on $M(F)$.

We recall the fundamental definition of introverted subspaces which was introduced by M.M. Day. We follow [7] and [8].

Definition 3.14. (M.M. Day (1957))

(1) F is *left introverted* if $m(F, F^*) \subset F$ (equivalently (Lemma 3.4.3), $m(F, M(F)) \subset F$).
(2) When F is an algebra then F is said to be *left m-introverted* if $m(F, MM(F)) \subset F$.
    Causion: *We usually say simply that F is introverted (rather than left introverted).*

Fact 3.15. *(Evolution product (in the sense of Pym (1964)) [7], [8, Ch.2.2].*

(1) *If F is an introverted closed subspace of $C(S)$ then F^* is a Banach algebra under the dual space norm and multiplication $(\mu, \varphi) \mapsto \mu \odot \varphi$, where*

$$(\mu \odot \varphi)(f) := \mu(m(f, \varphi)) \quad (f \in F).$$

Furthermore, with respect to the weak* topology F^* is a right topological affine semigroup, $(M(F), \odot)$ is a compact right topological affine subsemigroup, $co(\delta(S)) \subset \Lambda(M(F))$ and $\delta : S \to M(F)$ is an affine semigroup compactification.

(2) *If F is an m-introverted closed subalgebra of $C(S)$ then $(MM(F), \odot)$ is a compact right topological subsemigroup of $(M(F), \odot)$. Furthermore, $\delta : S \to MM(F)$ is a right topological semigroup compactification. See also subsection 3.4 below for another view of m-introverted algebras as the algebras corresponding to enveloping semigroups.*

The following result shows that the m-introverted algebras and introverted subspaces of $C(S)$ correspond to the semigroup compactifications, and affine semigroup compactifications of S respectively (compare Proposition 3.9).

Fact 3.16. *Let S be a semitopological semigroup.*

(1) *[8, p. 108] If (ψ, K) is a semigroup compactification of S then $\psi^* C(K)$ is an m-introverted closed subalgebra of $C(S)$. Conversely, if F is an m-introverted closed subalgebra of $C(S)$ then there exists a unique semigroup compactification (ψ, K) of S such that $\psi^* C(K) = F$. Namely the canonical semigroup compactification $\delta : S \to MM(F)$.*

(2) *[8, p. 123] If (ψ, K) is an affine semigroup compactification of S then $\psi^* A(K)$ is an introverted closed subspace of $\mathrm{WRUC}(S)$. Conversely, if F is an introverted closed subspace of $\mathrm{WRUC}(S)$ then there exists a unique affine semigroup compactification (ψ, K) of S such that $\psi^* A(K) = F$. Namely the canonical affine semigroup compactification $\delta : S \to M(F)$.*

Propositions 3.18.1 and 4.8.1 cover Fact 3.16.2.

Remark 3.17. [8, p. 123 and p. 172] $\mathrm{WRUC}(S)$ is the largest introverted subspace of $C(S)$.

The next proposition demonstrates the universality of the standard operator affine semigroup compactifications.

Proposition 3.18.   (1) *For every introverted closed unital subspace F of $C(S)$ there exists a natural $co(\delta(S))$-admissible affine embedding $M(F) \hookrightarrow \mathcal{E}(F)$ of right topological compact affine semigroups.*

(2) *Every affine semigroup compactification $\alpha : S \to Q$ is equivalent to a standard operator affine semigroup compactification $\alpha' : S \to Q' \subset \mathcal{E}(V)$ for some Banach space V.*

Proof. (1) The following natural map

$$i : M(F) \hookrightarrow \mathcal{E}(F), \quad i(m)(\varphi) = m \odot \varphi \;\; \forall \varphi \in F^*$$

is the desired embedding, where $m \odot \varphi$ is the evolution product (Fact 3.15.1). The continuity is easy to verify, and the injectivity follows from the fact that if e is the neutral element of S then $\delta(e)$ is the neutral element of $M(F)$.

(2) is a conclusion from (1) and Fact 3.16.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.4. **Point-universality of systems.** For a point-transitive compact separately continuous S-system (X, x_0) consider the natural S-compactification map $j_{x_0} : S \to X, s \mapsto sx_0$ and the corresponding Banach algebra embedding $j_{x_0}^* : C(X) \hookrightarrow C(S)$. Denote $\mathcal{A}(X, x_0) = j_{x_0}^*(C(X))$. The enveloping semigroup $(E(X), e)$ is always a point-transitive (separately continuous) S-space. Hence, $\mathcal{A}(E(X), e)$ is well defined and $E(X) \to X, p \mapsto px_0$ is the natural surjective S-map. Clearly $\mathcal{A}(X, x_0) \subset \mathcal{A}(E(X), e)$.

Recall that a point-transitive S-flow (X, x_0) is said to be *point-universal* [25, 26] if it has the property that for every $x \in X$ there is a homomorphism $\pi_x : (X, x_0) \to (\mathrm{cl}\,(Gx), x)$. This is the case iff $S \to X, \; g \mapsto gx_0$ is a right topological semigroup compactification of S; iff (X, x_0) is isomorphic to its own enveloping semigroup, $(X, x_0) \cong (E(X), e)$; iff the algebra $\mathcal{A}(X, x_0)$ is m-introverted.

4. Operator enveloping semigroups

4.1. **The notion of E-compatibility.** In a review article [61, p. 212] J. Pym asks the general question: "how affine flows might be obtained?" and then singles out the canonical construction where, with a given compact S-flow X one associates the induced affine flow on $P(X)$, the compact convex space of probability measures on X, and where X is naturally embedded into $P(X)$ by identifying the points of X with the corresponding dirac measures. Then $P(X)$ is at least S_d-space with respect to the induced affine action $S \times P(X) \to P(X)$. Recall that by Proposition 3.9.2, $P(X)$ is an S-space (i.e., the action is separately continuous) iff X is WRUC.

In turn, $P(X)$ can be viewed (via Riesz' representation theorem) as a part of the weak* compact unit ball B^* in the dual space $C(X)^*$. So we have the embeddings of S_d-systems

$$B^* \supset P(X) \supset X.$$

These embeddings induce the continuous onto homomorphisms of the enveloping semigroups

$$E(B^*) \to E(P(X)) \to E(X).$$

The first homomorphism $E(B^*) \to E(P(X))$ is always an isomorphism (Lemma 3.4.3). Pym asks when the second homomorphism $\phi : E(P(X)) \to E(X)$ is an isomorphism. The first systematic study of this question is to be found in a paper of Köhler [42]. Since ϕ is an isomorphism iff it is injective, following [22], we say that an S-system X (with continuous action) is *injective* when ϕ is an isomorphism. See Definition 4.11 for a more general version.

For cascades (dynamical $\mathbb{Z}$-systems) the first non-injective example was constructed by Glasner [22], answering a question of Köhler [42]. Earlier Immervoll [36] gave an example of a non-injective system (S, X) where S is a some special semigroup S.

Now we turn to a more general question. In the construction above instead of the Banach space $C(X)$ and the natural embeddings $X \subset P(X) \subset B^*$ one may consider representations on general Banach spaces V.

Question 4.1. *When is the enveloping semigroup of an affine compactification of a compact system X, arising from a representation on a Banach space V, isomorphic to the enveloping semigroup of the system itself ?*

More precisely, let
$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*)$$
be a weakly continuous representation of a (not necessarily compact) S-system X on a Banach space V. It induces an S-compactification $X \to Y$, where $Y := \overline{\alpha(X)}$ and an S-affine compactification $X \to Q$, where $Q := \overline{co} \, \alpha(X)$. Since h is weakly continuous it follows that the action of S on the weak* compact unit ball B^* (hence also on Y and Q) is separately continuous.

By our definitions, $\alpha(X)$ and hence, Y and Q are norm bounded. So for some $r > 0$ we have the embeddings of S-systems:
$$rB^* \supset Q \supset Y = \overline{\alpha(X)}.$$
By Lemma 1.26, we get the induced continuous surjective homomorphisms of the enveloping semi-groups (of course, the S-spaces B^* and rB^* are isomorphic):
$$\psi : E(B^*) \to E(Q), \quad \Phi : E(Q) \to E(Y).$$

Lemma 4.2. (1) $m(V, \overline{sp}^{norm}(A)) \subset \overline{sp}^{norm}(m(V, A))$ *for every subset* $A \subset V^*$.
  (2) *The algebra of the compactification* $S \to E(Y)$ *is* $\mathcal{A}(E(Y), e) = \langle m(V, Y) \rangle$.
  (3) *The algebra of the compactification* $S \to E(Q)$ *is* $\mathcal{A}(E(Q), e) = \langle m(V, Q) \rangle$.
  (4) *The algebra of the compactification* $S \to E(B^*)$ *is* $\mathcal{A}(E(B^*), e) = \langle m(V, V^*) \rangle = A_h$, *where* A_h *is the coefficient algebra.*

Proof. (1) is straightforward. For other assertions use (1) and Lemma 1.26.1 taking into account the definitions of Section 3.4. $\qquad\square$

Definition 4.3. Let X be an S-flow.
  (1) We say that an S-affine compactification (Definition 3.1.2) $\alpha : X \to Q$ is *E-compatible* if the map $\Phi : E(Q) \to E(Y)$ is an isomorphism (equivalently, is injective), where $Y := \overline{\alpha(X)}$. By Lemma 4.2 it is equivalent to saying that if $\mathcal{A}(E(Q), e) \subset \mathcal{A}(E(Y), e)$ or if $m(V, Q) \subset \langle m(V, Y) \rangle$.
  (2) We say that a weakly continuous Banach representation $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*)$ of an S-flow X on a Banach space V is:
   (a) *E-compatible* if the map $\Phi : E(Q) \to E(Y)$ is an isomorphism where $Q := \overline{co} \, \alpha(X)$. That is, if the induced affine compactification of the representation (h, α) is E-compatible.
   (b) *strongly E-compatible* if the map $\Phi \circ \Psi : E(B^*) \to E(Y)$ is an isomorphism. It is equivalent to saying that $m(V, V^*) \subset \langle m(V, Y) \rangle$ (equivalently, $m(V, V^*) \subset \mathcal{A}(E(Y), e)$).

We say that $K \subset V^*$ is a *w*-generating* subset of V^* if $sp(\overline{co}^{w^*}(K))$ is norm dense in V^*. A representation (h, α) of a system (S, X) on V is *w*-generating* (or simply *generating*) if $\alpha(X)$ is a w^*-generating subset of V^*. Later on (in the proof of Theorem 7.1) we will have the occasion to use the versatile construction of Davis-Figiel-Johnson-Pelczyński [15]. The second item of the next lemma refers to this construction.

Lemma 4.4. (1) *For every space X and a closed unital linear subspace $V \subset C(X)$ the regular V-representation $\alpha : X \to V^*$ is generating.*
  (2) *Let E be a Banach space and let $\| \, \|_n, n \in \mathbb{N}$, be a sequence of norms on E where each of the norms is equivalent to the given norm of E. For $v \in E$, let*
  $$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad and \quad V := \{v \in E \mid N(v) < \infty\}.$$
  Denote by $j : V \hookrightarrow E$ the inclusion map. Then (V, N) is a Banach space, $j : V \to E$ is a continuous linear injection such that $j^* : E^* \to V^*$ is norm dense. If $E = C(X)$ and $\alpha = j^* \circ \delta : X \to V^*$ is the induced map then $\alpha(X)$ is a w^*-generating subset of V^*.

Proof. (1) Indeed by Lemma 3.4 every $\mu \in V^*$ is a finite linear combination of members of $M(V) = \overline{co}^{w^*}(\alpha(X))$. Hence, $sp(\overline{co}^{w^*}(\alpha(X))) = V^*$.

(2) For the proof that $j^* : E^* \to V^*$ is norm-dense see Fabian [18, Lemma 1.2.2]. Since $\delta(X)$ affinely generates $M(C(X)^*)$ and $M(C(X)^*)$ linearly spans $C(X)^*$ (Lemma 3.4) it follows that $sp(\overline{co}^{w^*}(\alpha(X)))$ is norm dense in V^*, where $\alpha(X) = j^*(\delta(X))$. So, $\alpha(X)$ is w^*-generating in V^*. $\quad\square$

Lemma 4.5. (1) *Suppose $\overline{co}^{w^*}(Y) = \overline{co}^{norm}(Y)$ holds for a weakly continuous representation (h, α) on a Banach space V, where $Y := \overline{\alpha(X)}^{w^*}$. Then the representation is E-compatible.*

(2) *For w^*-generating representations, E-compatibility implies strong E-compatibility.*

(3) *For every regular representation of an S-space X on a closed unital S-invariant linear subspace $V \subset \mathrm{WRUC}(X)$, E-compatibility implies strong E-compatibility.*

(4) *(Monotonicity) Let α_1 and α_2 be two <u>faithful</u> S-affine compactifications of a compact S-space X such that $\alpha_1 \succeq \alpha_2$, then E-compatibility of α_1 implies E-compatibility of α_2.*

Proof. (1) By Lemma 4.2.2 we have $\overline{sp}^{norm}(m(V, Y)) \subset \mathcal{A}(E(Y), e)$. By our assumption on $Q := \overline{co}^{w^*}(Y) = \overline{co}^{norm}(Y)$ and using Lemma 4.2.1 we get

$$m(V, Q) = m(V, \overline{co}^{w^*}(Y)) = m(V, \overline{co}^{norm}(Y)) \subset \overline{sp}^{norm}(m(V, Y)) \subset \mathcal{A}(E(Y), e).$$

Since $\langle m(V, Q) \rangle = \mathcal{A}(E(Q), e)$ (Lemma 4.2.2), we obtain $\mathcal{A}(E(Q), e) \subset \mathcal{A}(E(Y), e)$.

(2) Y is a w^*-generating subset in V^*. Therefore by assertions (1) and (3) of Lemma 4.2 we get that $\psi : E(B^*) \to E(Q)$ is an isomorphism. So $\Phi \circ \Psi : E(B^*) \to E(Y)$ is an isomorphism iff $\Phi : E(Q) \to E(Y)$ is an isomorphism.

(3) Use (2) and Lemma 4.4.1.

(4) Since the affine compactifications are faithful and X is compact, we may identify $E(\alpha_1(X))$ and $E(\alpha_2(X))$ with $E(X)$. Since $\alpha_1 \succeq \alpha_2$ we have the induced homomorphism $E(Q_1) \to E(Q_2)$. The injectivity of the homomorphism $E(Q_1) \to E(X)$ implies the injectivity of $E(Q_2) \to E(X)$. $\square$

Note that (4) need not remain true if we drop the faithfulness of the affine compactifications.

Lemma 4.6. *Let X be an S-space and $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*)$ a weakly continuous representation of (S, X) on a Banach space V. Let $j : S \to E(Y)$ be the Ellis compactification for the S-system $Y := \overline{\alpha(X)}$. The following are equivalent:*

(1) *The representation (h, α) is strongly E-compatible.*

(2) *There exists a $j(S)$-admissible embedding $h' : E(Y) \hookrightarrow \mathcal{E}(V)$ such that $h' \circ j = h$.*

(3) *The semigroup compactifications $j : S \to E(Y)$ and $h : S \to \overline{h(S)}$, where $\overline{h(S)}$ is the closure in $\mathcal{E}(V)$, are naturally isomorphic.*

(4) *$m(V, V^*) \subset \langle m(V, Y) \rangle$ (equivalently, $m(V, V^*) \subset \mathcal{A}(E(Y), e)$).*

Proof. By Lemma 2.4, $E(S, B^*)$ is naturally embedded into $\mathcal{E}(V)$ and the semigroup compactifications $S \to E(S, B^*)$ and $S \to \overline{h(S)}$ are isomorphic. So (1), (2) and (3) are equivalent.

(1) $\Leftrightarrow$ (4): Use Lemma 4.2. $\square$

Proposition 4.7. *The following are equivalent:*

(1) *An affine S-compactification $\alpha : X \to Q$ of an S-space X is E-compatible.*

(2) *The induced representation of (S, X) on the Banach space $V := A(Q)|_X \subset \mathrm{WRUC}(X)$, the affine compactification space of α, is E-compatible (equivalently, strongly E-compatible).*

(3) *$m(V, V^*) \subset \langle m(V, Y) \rangle$, (equivalently, $m(V, M(V)) \subset \langle m(V, Y) \rangle$), where $Y := \overline{\alpha(X)}$.*

(4) *For every $f \in V$ we have*
$$\overline{co(X_f)}^p \subset \langle m(V, Y) \rangle.$$

Proof. (1) $\Leftrightarrow$ (2) By Definition 4.3 and the description of affine compactifications (Lemma 3.6) taking into account Lemma 4.5.3.

(2) $\Leftrightarrow$ (3): Use Lemma 4.6 (taking into account that by Lemma 3.4.3 every $\mu \in V^*$ is a finite linear combination of members of $M(V)$).

(3) $\Leftrightarrow$ (4): Clearly, $m(V, M(V)) = \bigcup_{f \in V} m(f, M(V))$. Recall that, by Lemma 3.13, for each $f \in V$ we have $\overline{co(X_f)}^p = m(f, M(V))$. $\square$

4.2. Affine semigroup compactifications. The second assertion of the next result shows that every affine semigroup compactification is E-compatible.

Proposition 4.8. *Let $\nu : S \to Q$ be an affine semigroup compactification and let $P = \overline{\nu(S)}$. Then*

(1) *[8, p. 123] The space $V = \mathrm{A}(Q)|_S$ is introverted.*

(2) *The affine compactification $\nu : S \to Q$ is E-compatible, that is, the restriction map $E(Q) \to E(P) = P$ is an isomorphism.*

Proof. (1) We have to show that $m(V, M(V)) \subset V$. As in the proof of Lemma 3.6 consider the Banach space $V = A(Q)|_S$ of the affine compactification $S \to Q$. For every $f \in V$ and $\mu \in M(V) = Q$ the corresponding matrix coefficient $m(f, \mu)$ is again in V because $m(f, \mu)$ is a restriction to S of the affine continuous map $Q \to \mathbb{R}$, $q \mapsto \tilde{f}(q \odot \mu)$, where $\tilde{f} \in A(Q)$ with $f = \tilde{f}|_S$ and $q \odot \mu$ is the evolution product (see Fact 3.15.1) in the semigroup $M(V) = Q$.

(2) By (1) we have $m(V, V^*) \subset V$. Observe that $V = A(Q)|_S \subset \mathcal{A}(P, e)$. Since $\mathcal{A}(P, e) \subset \mathcal{A}(E(P), e)$, we get $m(V, V^*) \subset \mathcal{A}(E(P), e) = \langle m(V, P) \rangle$ (Lemma 4.2.2). So, V is E-compatible by Proposition 4.7. $\qquad\square$

Definition 4.9. We say that a subalgebra $\mathcal{A} \subset C(S)$ is *intro-generated* if there exists an introverted subspace $V \subset \mathcal{A}$ such that $\langle V \rangle = \mathcal{A}$.

Below, in Theorem 5.6, we show that the m-introverted algebra of all distal functions $D(\mathbb{Z})$ is not intro-generated. In Example 5.12 we present an m-introverted intro-generated subalgebra $\mathcal{A}$ of $l_\infty(\mathbb{Z}^2)$ which is not introverted.

Proposition 4.10. *Let $\nu : S \to P$ be a right topological semigroup compactification and let $\mathcal{A} = \mathcal{A}(P, e)$ be the corresponding m-introverted subalgebra of $C(S)$. The following are equivalent:*

(1) *The compactification $\nu : S \to P$ is equivalent to a standard operator compactification on a Banach space.*
(2) *The compactification $\nu : S \to P$ is equivalent to an operator compactification on a locally convex vector space.*
(3) *There exists an affine (equivalently, standard affine) semigroup compactification $\psi : S \to Q$ such that the compactification $\psi : S \to \overline{\psi(S)}$ is equivalent to $\nu : S \to P$.*
(4) *The algebra $\mathcal{A}$ of the compactification $\nu : S \to P$ is intro-generated.*
(5) *There exists a Banach unital S-subspace $V \subset \mathrm{WRUC}(S)$ such that $\langle V \rangle = \mathcal{A}$ and for every $f \in V$ we have*
$$\overline{co(Sf)}^p \subset \mathcal{A}.$$

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): By our assumption $\nu : S \to P$ is equivalent to an operator compactification. Therefore there exists a weakly continuous equicontinuous representation $h : S \to L(V)$ of a semitopological semigroup S on a locally convex vector space V such that ν can be identified with $h : S \to P$ where P is the weak* operator closure $\overline{h(S)^{op}}$ of the adjoint semigroup $h(S)^{op} \subset L(V)^{op}$ in $L(V^*)$. Consider the compact subsemigroup $Q := \overline{co}\,\nu(S) = \overline{co}(P)$ of $L(V^*)$ (in weak* operator topology). Then the map $\psi : S \to Q$, $\psi(s) = \nu(s)$ is an affine compactification of S. Indeed, $\psi(S) \subset \Lambda(Q)$ because $\nu(S) \subset L(V)^{op}$. Observe that ψ induces ν because $\overline{\psi(S)} = P$ and $\psi(s) = \nu(s)$ for every $s \in S$. (By Proposition 3.18 we can assume that $\psi : S \to Q$ is a standard affine semigroup compactification.)

(3) $\Rightarrow$ (4): By Proposition 4.8.1 the space $V := A(Q)|_S$ of the affine compactification ψ is introverted. Always, the affine compactification subspace V generates the induced compactification algebra $\mathcal{A}$ (Remark 3.10.1). That is, $\langle V \rangle = \mathcal{A}$. Hence $\mathcal{A}$ is intro-generated.

(4) $\Rightarrow$ (5): If V is an introverted subspace of $\mathcal{A}$ then $m(f, M(V)) \subset V$ for every $f \in V$. Also, $V \subset \mathrm{WRUC}(S)$ by Remark 3.17. So Lemma 3.13 implies $m(f, M(V)) = \overline{co(Sf)}^p$. Hence, $\overline{co(Sf)}^p \subset V \subset \langle V \rangle = \mathcal{A}$.

(5) $\Rightarrow$ (1): Consider the regular representation (h, α) of the S-space $X := S$ on the closed unital S-invariant linear subspace $V \subset \mathrm{WRUC}(S)$. By the Stone-Weierstrass theorem it follows that the algebra of the corresponding S-compactification $\alpha : S \to Y := \overline{\alpha(X)}$ is $\langle V \rangle$. By our assumption $\langle V \rangle = \mathcal{A}$. So we obtain that $\alpha : S \to Y$ can be identified with the original S-compactification $\nu : S \to P$. Recall that $\overline{co(X_f)}^p = \overline{co(Sf)}^p$ (Lemma 3.13) and $\mathcal{A}(E(P), e) = \mathcal{A}(P, e) = \mathcal{A}$ (Lemma 1.26.4). Applying the equivalence (4) $\Leftrightarrow$ (1) of Proposition 4.7 we get that the regular V-representation (h, α) of (S, S) is E-compatible. In fact, strongly E-compatible by Lemma 4.5.3. We obtain that $E(S, B^*) \to E(S, P)$ is an isomorphism. Now observe that $E(S, B^*) \subset \mathcal{E}(V)$ (Lemma 2.4.2) and $E(S, P) = P$ (Lemma 1.26.4). $\qquad\square$

4.3. **Injectivity of compact dynamical systems.** Every continuous action is WRUC (Lemma 1.20.2). So the following naturally extends the definition from [22], mentioned in Section 4.1.

Definition 4.11. We say that a compact WRUC S-system X is *injective* if one (hence all) of the following equivalent conditions are satisfied:

   (1) The greatest affine S-compactification $X \to P(X)$ is E-compatible.
   (2) The regular representation of (S, X) on the Banach space $C(X)$ is (strongly) E-compatible.
   (3) $m(C(X), C(X)^*) \subset \langle m(C(X), X) \rangle$ (equivalently, $m(C(X), C(X)^*) \subset \mathcal{A}(E(X), e)$).
   (4) Every faithful affine S-compactification $X \hookrightarrow Q$ is E-compatible.

Proof. Here we prove that these conditions are equivalent. First of all since X is WRUC the action $S \times P(X) \to P(X)$ is separately continuous by Proposition 3.9.2. Regarding (2) note that by Lemma 4.5.3 every regular E-compatible representation is strongly E-compatible.

   $(1) \Leftrightarrow (2)$ and $(4) \Rightarrow (1)$: Are trivial.
   $(2) \Leftrightarrow (3)$: Use Lemma 4.6.
   $(2) \Rightarrow (4)$: Since $X \hookrightarrow P(X)$ is the greatest S-affine compactification of X we can apply the monotonicity of E-compatibility (Lemma 4.5.4). $\qquad \square$

Proposition 4.12. *Let X be an injective S-system. Then the enveloping semigroup compactification $S \to E(X)$ is equivalent to a (standard) operator compactification and hence all of the equivalent conditions of Proposition 4.10 are satisfied.*

Proof. Since X is injective the regular representation of (S, X) on $C(X)$ is weakly continuous and E-compatible. Now apply Lemma 4.6. $\qquad \square$

Theorem 4.13. *Let $V \subset C(S)$ be an m-introverted Banach S-subalgebra and $\alpha : S \to P$ be the corresponding right topological semigroup compactification. Then the S-system P is injective if and only if V is introverted.*

Proof. By Definition 4.11 the S-flow P is injective iff $m(C(P), C(P)) \subset \mathcal{A}(E(S, P), e)$. By Lemma 1.26.4, $E(S, P)$ is naturally isomorphic to the semigroup P. Hence, $\mathcal{A}(E(P), e) = \mathcal{A}(P.e) = j^*(C(P)) = V$. Observe also that the canonical representations of (S, P) on $C(P)$ and on V are naturally isomorphic. In particular, $m(C(P), C(P)^*) = m(V, V^*)$. Summing up we get: the S-flow P is injective iff $m(V, V^*) \subset V$ iff V is introverted. $\qquad \square$

Lemma 4.14. *Let X be a compact point-transitive S-system. If the enveloping semigroup $E(X)$, as an S-flow, is injective then X is also injective.*

Proof. Let $z \in X$ be a transitive point and let $q : E(X) \to X, q(p) = pz$ be the corresponding onto continuous S-map. It induces the surjective homomorphism $q_E : E(B_{C(E(X))^*}) \to E(B_{C(X)^*})$. By the injectivity of $(S, E(X))$ the S-compactifications $S \to E(E(X))$ and $S \to E(B_{C(E(X))^*})$ are equivalent. On the other hand, $S \to E(E(X))$ is equivalent to $S \to E(X)$. It follows that $S \to E(X)$ dominates $S \to E(B_{C(X)^*})$. Conversely, $S \to E(B_{C(X)^*})$ clearly dominates $S \to E(X)$. Therefore these compactifications are equivalent. $\qquad \square$

4.4. **Some examples of injective dynamical systems.** Let G be a topological group. As in Remark 1.19 a property $\mathcal{P}$ of continuous G-dynamical systems is said to be *suppable* if it is preserved by the operations of taking products and subsystems. To every suppable property $\mathcal{P}$ corresponds a universal point-transitive G-system $(G, G^{\mathcal{P}})$ (see e.g. [25, Proposition 2.9]).

Remark 4.15.   (1) Suppose $\mathcal{P}$ is a suppable property of dynamical systems such that whenever (G, X) has $\mathcal{P}$ then so does $(G, P(X))$. It then follows immediately that the corresponding $\mathcal{P}$-universal point-transitive system $Y = G^{\mathcal{P}}$ is injective. Indeed, then the G-systems $P(Y)$ and $E(P(Y))$ have $\mathcal{P}$. By the universality of Y it is easy to see that Y and $E(P(Y))$ are naturally isomorphic. On the other hand, Y and $E(Y)$ are naturally isomorphic (Lemma 1.26.4). Hence, also $E(P(Y))$ and $E(Y)$ are isomorphic. This way we see that, for example, the universal point-transitive (i) equicontinuous, (ii) WAP, (iii) HNS and (iv) tame dynamical systems, are all injective.

(2) Another application of this principle is obtained by regarding the class of $\mathbb{Z}$-flows having zero topological entropy. It is easy to check that this property is suppable and the fact that it is preserved under the functor $X \mapsto P(X)$ follows from a theorem of Glasner and Weiss [31].

(3) Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the $\{0,1\}$-Bernoulli system on $\mathbb{Z}$. It is well known that the enveloping semigroup of $(\mathbb{Z}, \Omega)$ is $\beta\mathbb{Z}$, the Čech-Stone compactification of $\mathbb{Z}$ (see [21, Exercise 1.25]). Since $\beta\mathbb{Z}$ is the universal enveloping semigroup, it follows that $E(X) = E(\Omega) = \beta\mathbb{Z}$ for every point-transitive system $(\mathbb{Z}, X)$ which admits $(\mathbb{Z}, \Omega)$ as a factor (e.g. every mixing $\mathbb{Z}$-subshift of finite type will have $\beta\mathbb{Z}$ as its enveloping semigroup since it has a Cartesian power which admits Ω as a factor [12]). Moreover we necessarily have in that case that also $E(P(X)) = \beta\mathbb{Z}$. Thus every such X is injective. (See [42] and [9]).

(4) Every tame S-system is injective (Theorem 8.2). This result for metrizable systems was obtained by Köhler [42]. In [22] there is a simple proof which uses the fact that the enveloping semigroup $E(X)$ of a tame metric system X is a Fréchet space.

(5) Every transitive continuous G-system X is a factor of the injective G-system G^{RUC} (see (1)). Thus injectivity is not preserved by factors. The same assertion holds for subsystems (Remark 4.18).

Remark 4.16. (1) Theorem 4.13 shows that injectivity can serve as a key property in providing introverted subalgebras of $C(S)$. In particular, the algebras in Remark 4.15.1 are introverted.

It is well known (see [7, Ch. III, Lemma 8.8]) that every S-invariant unital closed subspace of $\mathrm{WAP}(S)$ is introverted. In particular, $\mathrm{Hilb}(S)$ and $\mathrm{AP}(S)$ are introverted. It is also well known that $\mathrm{RUC}(S)$ is an introverted algebra (see [8, p. 163]). Hence, the corresponding semigroup compactifications, the greatest ambit $S \to S^{\mathrm{RUC}}$, the Bohr compactification $S \to S^{\mathrm{AP}}$ and the universal semitopological semigroup compactification $S \to S^{\mathrm{WAP}}$, respectively, are operator compactifications (Proposition 4.10) and can be extended to affine semigroup compactifications.

(2) In Theorem 8.4, we show that in fact every m-introverted Banach S-subspace of $\mathrm{Tame}(S)$ (hence also of $\mathrm{Asp}(S)$ and $\mathrm{WAP}(S)$) is introverted.

(3) The Roelcke algebra $\mathrm{UC}(G) = \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$ is not even m-introverted in general, [26] for Polish topological groups G.

4.5. **The iteration process.** Starting with an arbitrary topological group G and a compact dynamical system (G, X) (with continuous action) we define inductively a sequence of new systems by iterating the operation of passing to the space of probability measures. Explicitly we let $P^{(1)}(X) = P(X)$ and for $n \geq 1$ we let $P^{(n+1)}(X) = P(P^{(n)}(X))$. Each $P^{(n)}(X)$ is an affine dynamical system and thus the barycenter map $b : P^{(n+1)}(X) \to P^{(n)}(X)$ is a well defined continuous affine homomorphism. Moreover, identifying a measure $\mu \in P^{(n)}(X)$ with the dirac measure $\delta_\mu \in P^{(n+1)}(X)$ we can consider $b : P^{(n+1)}(X) \to P^{(n)}(X)$ as a retract. Next observe that the induced map $b_* : P^{(n+1)}(X) \to P^{(n)}(X)$ coincides with the map $b : P^{(n+1)}(X) \to P^{(n)}(X)$. For convenience we write $Z_n = P^{(n)}(X)$ and we now let $Z = P^{(\infty)}(X) = \varprojlim Z_n$, the inverse limit. We denote by $\pi_n : Z \to Z_n$ the natural projection.

Proposition 4.17. *There is a natural bijection $\alpha : P(Z) \to Z$. In particular (G, Z) is injective.*

Proof. Given $\mu \in P(Z)$ we consider, for each n, its image $z_{n+1} = (\pi_n)_*(\mu) = \mu_n \in P(Z_n) = P^{(n+1)}(X)$. Then,

$$
\begin{aligned}
z_{n+2} = \mu_{n+1} &= (\pi_{n+1})_*(\mu) \\
&= (b \circ \pi_n)_*(\mu) = b_*((\pi_{n+1})_*(\mu)) \\
&= b_*(z_{n+1}) = b(z_{n+1}).
\end{aligned}
$$

Therefore, the sequence (z_n) defines a unique point $z = \alpha(\mu)$ in Z. Clearly $\alpha : P(Z) \to Z$ is a continuous G-map and it is easy to check that it is one-to-one. Finally for $z \in Z$ we have the sequence of measures $z_{n+1} = \pi_{n+1}(z) \in Z_{n+1} = P^{(n+1)}(X) = P(Z_n)$. Because, as maps from $P^{(n+1)}(X)$ to $P^{(n)}(X)$, the maps b and b_* coincide, this choice of measures on the various Z_n is

consistent and defines a measure $\beta(z) \in P(Z)$. One can check now that $\beta \circ \alpha$ is the identity map on $P(Z)$ and our proof is complete. □

Remark 4.18. We note that in the above construction the compact affine space Z is metrizable when X is metrizable. Moreover, via the maps $z \mapsto \delta_z$ we have a natural embedding of X into the space Z. This iterated construction can serve now as a source of examples. Beginning with an arbitrary compact metric system X with a property, say, $\mathcal{R}$ — which is preserved under inverse limits and such that $Y \in \mathcal{R} \Rightarrow P(Y) \in \mathcal{R}$ — we obtain in (Z, T), with $Z = P^{(\infty)}(X)$, a metrizable injective system which contains X as a subsystem and has property $\mathcal{R}$. Some properties $\mathcal{R}$ as above are e.g. "weak-mixing" ([6]), "zero topological entropy" ([31]) and "uniform rigidity" (with respect to a given sequence) ([24]).

5. Examples of non-injective systems

5.1. **Minimal distal non-equicontinuos systems are not injective.** In this section G denotes a semitopological group. Let Q be an affine compact G-system and let $ext\ Q$ denote the set of extreme points of Q. The system Q is said to be *minimally generated* [20] if the G-subsystem $\overline{ext\ Q}$ is minimal. We recall the following theorem.

Fact 5.1. *([20, Theorem 1.1]) Let Q be a minimally generated metric distal affine compact G-system. Then Q is equicontinuous.*

Lemma 5.2. *Let Q be a compact convex affine G-flow and let X be a compact minimal G-subflow. The following are equivalent:*

(1) $X = \overline{ext\ Q}$.
(2) $\overline{co}(X) = Q$.

Proof. (1) $\Rightarrow$ (2): By Krein-Milman theorem (see for example [73, p. 659]), $Q = \overline{co}(ext\ Q)$. Hence, we get $Q = \overline{co}(\overline{ext\ Q}) = \overline{co}X$.

(2) $\Rightarrow$ (1): By Krein-Milman theorem $ext\ Q \neq \emptyset$. Choose $z \in ext\ Q$. The orbit Gz is dense in X by minimality of X. By Milman theorem [73, p. 659], $ext\ Q \subset X$. Since G is a *group* of affine transformations, $gz \in ext\ Q$ for every $g \in G$. So $ext\ Q$ is dense in X. □

Remark 5.3. Theorem 5.1 remains true for non-metrizable Q. Indeed, by [58] the general case can be reduced to the metrizable case using Ellis' construction.

At least for every separable G a more direct argument is as follows. We treat Q as an affine compactification of the minimal system $X := \overline{ext\ Q}$ ((1) $\Rightarrow$ (2) of Lemma 5.2) and observe that one may G-approximate the (faithful) affine compactification $X \hookrightarrow Q$ by metric (not necessarily faithful) affine compactifications $X \to Q_i$. More precisely, let $V \subset C(X)$ be the Banach unital G-invariant space of the affine compactification $X \hookrightarrow Q$. Since G is separable there are sufficiently many *separable* Banach unital G-invariant subspaces V_i to separate points of Q. The corresponding affine distal G-factors $r_i : Q \to Q_i = \overline{co}(X_i)$, $X_i := r_i(X)$ are again minimally generated ((2) $\Rightarrow$ (1) of Lemma 5.2). Since each Q_i is metrizable, it is equicontinuous by Theorem 5.1, and the same is then true for Q.

Proposition 5.4. *Let X be a compact distal minimal G-flow. The following are equivalent:*

(1) *The G-flow X admits an E-compatible faithful affine compactification $\alpha : X \hookrightarrow Q$.*
(2) *The G-flow X admits an E-compatible faithful representation of (S, X) on a Banach space.*
(3) *X is an equicontinuous G-flow.*
(4) *X is an injective G-flow.*

Proof. (1) $\Leftrightarrow$ (2) Follows directly from Proposition 4.7 and Definition 4.3.

(1) $\Rightarrow$ (3) The distality of the G-flow X means, by Ellis result, that the enveloping semigroup $E(X)$ is a group. Since by (1), $E(Q) \to E(X)$ is an isomorphism we obtain that $E(Q)$ is also a group, hence Q is a distal G-flow. By Theorem 5.1, taking into account Lemma 5.2 and Remark 5.3 we conclude that Q, hence also, X are equicontinuous flows.

(3) $\Rightarrow$ (4) This follows, for example, from Theorem 8.2 below.

(4) $\Rightarrow$ (1) Apply Definition 4.11. □

Theorem 5.5. *Let P be a compact right topological <u>group</u> and let $\nu : G \to P$ be a right topological semigroup compactification of a group G. The following are equivalent:*

(1) *The compactification $\nu : G \to P$ is equivalent to an operator compactification.*
(2) *There exists a $\nu(G)$-admissible embedding of P into $\mathcal{E}(V)$ for some Banach space V.*
(3) *P is a topological group.*

Proof. (3) $\Rightarrow$ (1): Every compact topological group is embedded into the unitary group $U(H) \subset \Theta(H)$ for some Hilbert space H and in this case $\Theta(H) = \mathcal{E}(H^*)$.

(1) $\Rightarrow$ (3): By Proposition 4.10, $G \to P$ can be embedded into an affine compactification $G \to Q$, so that $\overline{co}(P) = Q$. The G-flow P is minimal (being distal and point-transitive) and by Lemma 5.2 Q is minimally generated. Now by Fact 5.1 and Remark 5.3 we get that Q is equicontinuous. This implies that (G, P) is equicontinuous and hence $E(G, P) = P$ is a topological group.

(1) $\Leftrightarrow$ (2): Use Proposition 4.10 and Lemma 2.4. $\qquad\square$

Let $\mathrm{D}(\mathbb{Z})$ be the algebra of all distal functions on $\mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z}^{\mathrm{D}(\mathbb{Z})}$ the corresponding semigroup compactification (see [8, p. 178]). The right topological group $\mathbb{Z}^{\mathrm{D}(\mathbb{Z})}$ is not a topological group so by Proposition 5.4 and Theorem 5.5 we get

Theorem 5.6. (1) *The semigroup compactification $\mathbb{Z} \to \mathbb{Z}^{\mathrm{D}(\mathbb{Z})}$ is not an operator compactification.*

(2) *The algebra of all distal functions $\mathrm{D}(\mathbb{Z})$ is not intro-generated.*
(3) *$(\mathbb{Z}, \mathbb{Z}^{\mathrm{D}(\mathbb{Z})})$ is a minimal distal cascade which does not admit faithful E-compatible affine compactifications.*
(4) *The compact right topological group $\mathbb{Z}^{D(\mathbb{Z})}$ does not admit faithful admissible representations on Banach spaces.*

The fact that $\mathrm{D}(\mathbb{Z})$ is not introverted (weaker than Theorem 5.6.2) was shown in [8, p. 179].

5.2. **A Toeplitz non-injective system.** We give here an example of a non-injective metric minimal $\mathbb{Z}$-system which is a transitive almost 1-1 extension of an adding machine. The latter property is actually a characterization of being a *Toepliz dynamical system*. For more details on Toepliz systems we refer to [74].

Theorem 5.7. *There is a Toeplitz non-injective system.*

Proof. Let $y_0 \in \{0, 1\}^{\mathbb{Z}}$ be the "Heaviside" sequence defined by the rule

$$y_0(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0, \end{cases}$$

and let $Y = \bar{\mathcal{O}}_S(y_0)$ be its orbit closure in $\{0, 1\}^{\mathbb{Z}}$ under the shift transformation: $S\omega(n) = \omega(n + 1), \omega \in \{0, 1\}^{\mathbb{Z}}, n \in \mathbb{Z}$. Thus $Y = \{S^n y_0 : n \in \mathbb{Z}\} \cup \{\mathbf{0}, \mathbf{1}\}$ is isomorphic to the 2-point compactification $\mathbb{Z} \cup \{\pm\infty\}$ of the integers.

Let (X, S) be the (necessarily minimal and non-regular) Toeplitz system corresponding to the subshift Y and a suitable sequence of periods $(p_i)_{i \in \mathbb{N}}$ as described by Williams in Section 4 of [74]. Here, as usual, we write (X, S) for the $\mathbb{Z}$-action $(\mathbb{Z}, X)$ generated by S.

By [74, Theorem 4.5] the system (X, S) admits exactly two S-invariant ergodic probability measures (corresponding to the dirac measures $\delta_{\mathbf{0}}$ and $\delta_{\mathbf{1}}$ on Y), which we will denote by μ_0 and μ_1, respectively.

We will show the existence of a probability measure ν on X for which, in the weak* topology on the compact space $P(X)$ of probability measures on X and the action induced by S, we have

(5.1) $$\lim_{n \to -\infty} S^n \nu = \mu_0 \quad \text{and} \quad \lim_{n \to +\infty} S^n \nu = \mu_1.$$

The dynamical system (X, S) has a structure of an almost one-to-one extension of an adding machine, say,

$$\pi : X \to G = \varprojlim \mathbb{Z}/p_i\mathbb{Z}.$$

It follows that the proximal relation $Prox(X)$ on X coincides with the π-relation:

$$R_\pi = \{(x, x') : \pi(x) = \pi(x')\}.$$

In particular this implies that $Prox(X)$ is an equivalence relation. Now this latter condition is equivalent to the fact that $E(X)$, the enveloping semigroup of (X, S) has a unique minimal ideal. Since $E(E(X)) = E(X)$ we conclude that in any dynamical system (X', T) whose enveloping semigroup is isomorphic to $E(X)$ the proximal relation $Prox(X')$ is again an equivalence relation.

However the equations (5.1) clearly show that in the dynamical system $(P(X), S)$ the proximal relation is no longer an equivalence relation. It therefore follows that the natural restriction map $r : E(P(X)) \to E(X)$ is not an isomorphism; i.e. (X, S) is not injective.

It thus remains to construct a measure ν as above. For that purpose let us recall the following objects constructed by Williams. Using the notations of [74] we let $C = \{x \in X : 0 \notin Aper(x)\}$, and let $D \subset X$ be the set of $x \in X$ with $Aper(x)$ a 2-sided infinite sequence. We have $C = \pi^{-1}(\pi(C))$ so that the subset $Aper(x) \subset \mathbb{Z}$ is well defined on G, with $Aper(x) = Aper(\pi(x))$ for every $x \in X$. By definition the fact that X is a non-regular Toeplitz system means that we have $0 < m(\pi(C)) = d < 1$, with m denoting the Haar measure on G. The (Borel) dynamical system $(G \times Y, T)$ is given by the Borel map $T : G \times Y \to G \times Y$ defined as $T(g, y) = (g + \mathbf{1}, S^{\theta(g)}y)$, where $\theta : G \to \{0, 1\}$ is the function $\mathbf{1}_{\pi(C)}$. Williams shows that $\phi(G \times Y) = X$, that $S \circ \phi = \phi \circ T$, and that the restriction of ϕ to the subset $\pi(D) \times Y$ is a Borel isomorphism from $\pi(D) \times Y$ into X. In our case we have

$$\pi(D) \times Y = (\pi(D) \times \{\mathbf{0}\}) \cup \left(\bigcup_{n \in \mathbb{Z}} \pi(D) \times \{S^n y_0\} \right) \cup (\pi(D) \times \{\mathbf{1}\}).$$

We also have $m(\pi(D)) = 1$ and $\phi(m \times \delta_{\mathbf{0}}) = \mu_0$, $\phi(m \times \delta_{\mathbf{1}}) = \mu_1$.

Now let

$$\nu = \phi(m \times \delta_{y_0}).$$

Iterating the map T we see that for $n \in \mathbb{Z}$

$$T^n(g, y) = (g + n\mathbf{1}, S^{\theta_n(g)}y),$$

where for every $g \in G$, $\theta_0(g) = 0$ and

$$\theta_n(g) = \begin{cases} \theta(g) + \theta(g + \mathbf{1}) + \cdots + \theta(g + (n-1)\mathbf{1}) & \text{for } n \geq 1 \\ -\theta(g + n\mathbf{1}) - \theta(g + (n+1))\mathbf{1} - \cdots - \theta(g - \mathbf{1}) & \text{for } n \leq -1. \end{cases}$$

Note that by the ergodic theorem we have

(5.2)
$$\lim_{n \to \pm\infty} \frac{1}{n}\theta_n(g) = m(\pi(C)), \qquad m\text{-a.e.}.$$

Next let us consider the integrals $\int f \, dS^n\nu$ for a fixed continuous real valued function f on X. We have

$$\int_X f(x) \, dS^n\nu(x) = \int_X f(S^n x) \, d\nu(x) = \int_{G \times Y} f \circ S^n \circ \phi(g, y) \, d(m \times \delta_{y_0})$$

$$= \int_{G \times Y} f \circ \phi(T^n(g, y)) \, d(m \times \delta_{y_0})$$

$$= \int_G f \circ \phi((g + n\mathbf{1}, S^{\theta_n(g)}y_0)) \, dm(g)$$

$$= \sum_{j \in \mathbb{Z}} \int_{\{g : \theta_n(g) = j\}} f \circ \phi((g + n\mathbf{1}, S^j y_0)) \, dm(g).$$

If we now further assume that the function f depends only on coordinates i with $|i| \leq N$ for a fixed N then, taking into account the way the map ϕ is defined on $\pi(D) \times Y$ as well as (5.2), we see that indeed

$$\lim_{n \to -\infty} \int f \, dS^n\nu = \int f \, d\mu_0 \quad \text{and} \quad \lim_{n \to +\infty} \int f \, dS^n\nu = \int f \, d\mu_1.$$

Since the collection of functions f depending on finitely many coordinates is uniformly dense in $C(X)$ this proves (5.1) and our proof is complete. $\qquad \square$

In view of the last two sections one would like to know how minimal weakly-mixing systems behave with respect to injectivity.

Problem 5.8. *Construct examples of minimal weakly-mixing $\mathbb{Z}$-flows which are injective (not injective).*

5.3. A non-injective $\mathbb{Z}^2$-dynamical system which admits an E-compatible faithful affine compactification. Let $X = \{0,1\}^{\mathbb{Z}}$ and let σ denote the shift transformation on X. Define two $\mathbb{Z}^2$-actions on X by

$$\Phi_{mn}x = \sigma^m x \quad \text{and} \quad \Psi_{mn}x = \sigma^n x.$$

Since $E(X, \sigma)$ is canonically isomorphic to $\beta\mathbb{Z}$ we clearly have $E(X, \Phi) \cong E(X, \Psi) \cong \beta\mathbb{Z}$. In particular it follows that the two $\mathbb{Z}^2$-systems (X, Φ) and (X, Ψ) are injective. Next consider the product $\mathbb{Z}^2$-system $(X \times X, \Xi)$, where the action is diagonal; i.e.

$$\Xi_{mn}(x, y) = (\Phi_{mn}x, \Psi_{mn}y) = (\sigma^m x, \sigma^n y).$$

The proof of the next claim is straightforward.

Claim 5.9. $E(X \times X, \Xi) \cong E(X, \Phi) \times E(X, \Psi) \cong \beta\mathbb{Z} \times \beta\mathbb{Z}$. Moreover, identifying an element p of $E(X \times X, \Xi)$ with a pair $p = (p_\Phi, p_\Psi)$, we have $p = (p_\Phi, p_\Psi) = (p_\Phi, \mathrm{id}) \circ (\mathrm{id}, p_\Psi)$ and if $\Xi_{m_i n_i} \to p$ then, $(\Phi_{m_i 0}, \mathrm{id}) = (\Phi_{m_i 0}, \Psi_{m_i 0}) = \Xi_{m_i 0} \to (p_\Phi, \mathrm{id})$ and $(\mathrm{id}, \Psi_{0n_i}) = (\Phi_{0n_i}, \Psi_{0n_i}) = \Xi_{0n_i} \to (\mathrm{id}, p_\Psi)$.

Proposition 5.10. *The product dynamical system $(\mathbb{Z}^2, X \times X)$ (of two different $\mathbb{Z}^2$-flows) is not injective.*

Proof. Suppose to the contrary that it is injective, i.e. that we have $E(P(X \times X), \Xi) \cong E(X \times X, \Xi) \cong E(X, \Phi) \times E(X, \Psi)$. Let μ be the Bernoulli measure $(1/2(\delta_0 + \delta_1))^{\mathbb{Z}}$ on X, an element of $P(X)$. We let Δ_μ, an element of $P(X \times X)$, be the corresponding graph measure on $X \times X$ defined as the push-forward of μ via the diagonal map $x \mapsto (x, x)$. Let $A = \{\Xi_{mm} : m \in \mathbb{Z}\}$ and let $p \in \overline{A} \subset E(P(X \times X), \Xi)$ be any element which is not in A so that $p = \lim \Xi_{m_i m_i}$ with $m_i \nearrow \infty$ a net in $\mathbb{Z}$. We let p_Φ denote its projection in $E(P(X), \Phi)$ and p_Ψ its projection in $E(P(X), \Psi)$.

Claim 5.11. (1) $p\Delta_\mu = \Delta_\mu$.
  (2) $(p_\Phi, \mathrm{id})\Delta_\mu = \mu \times \mu$.
  (3) $(\mathrm{id}, p_\Psi)\Delta_\mu = \mu \times \mu$.

Proof. The first equality holds trivially as Δ_μ is A-invariant. The second and third equalities follow from the fact that (X, μ, σ) is mixing as a measure dynamical system. $\qquad\square$

We complete the proof of the proposition by pointing out the following absurd:

$$\Delta_\mu = p\Delta_\mu = (p_\Phi, \mathrm{id}) \circ (\mathrm{id}, p_\Psi)\Delta_\mu = (p_\Phi, \mathrm{id})\mu \times \mu = \mu \times \mu.$$

$\qquad\square$

Example 5.12. (1) The semigroup $\mathbb{Z}^2$-compactification defined naturally by the embedding

$$\nu : \mathbb{Z}^2 \to Y = \beta\mathbb{Z} \times \beta\mathbb{Z}$$

  is not injective but admits an E-compatible faithful affine compactification.
  (2) There exists an intro-generated m-introverted Banach subalgebra of $l_\infty(\mathbb{Z}^2)$ which is not introverted.

Proof. Let V be the Banach subspace of $l_\infty(\mathbb{Z}^2)$ consisting of functions of the form $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ with $f_1, f_2 \in l_\infty(\mathbb{Z})$. It is easy to see that V is an introverted $\mathbb{Z}^2$-subspace of $C(Y)$. Furthermore, by the Stone-Weierstrass theorem, the closed algebra $\langle V \rangle$ generated by V is the algebra $\mathcal{A} \subset l_\infty(\mathbb{Z}^2)$ which corresponds to the compactification $\nu : \mathbb{Z}^2 \to Y = \beta\mathbb{Z} \times \beta\mathbb{Z}$. In particular, by Propositions 4.7 and 4.10, $(\mathbb{Z}^2, Y)$ admits an E-compatible faithful affine compactification. However, the $\mathbb{Z}^2$-flow Y is not injective. Indeed, Y is just the enveloping semigroup $E(\mathbb{Z}^2, X \times X)$ of the $\mathbb{Z}^2$-flow $X \times X$ from Proposition 5.10. Therefore, by Lemma 4.14 the injectivity of Y will imply (observe that Y is transitive) that $X \times X$ is injective contradicting Proposition 5.10. Finally, Theorem 4.13 shows that $\mathcal{A}$ is not introverted. $\qquad\square$

The nonmetrizability of Y in Example 5.12 and of $\mathbb{Z}^{D(\mathbb{Z})}$ in Theorem 5.6 is unavoidable by Theorem 8.3.

Problem 5.13. *Are there examples as above with $\mathbb{Z}$ as the acting group, rather than $\mathbb{Z}^2$?*

6. Tame and HNS systems and related classes of right topological semigroups

6.1. **Some classes of right topological semigroups.** To the basic classes of right topological semigroups listed in 1.1 above, we add the following two which have naturally arisen in the study of tame and HNS dynamical systems.

Definition 6.1. [25, 27] A compact admissible right topological semigroup P is said to be:
   (1) [27] *tame* if the left translation $\lambda_a : P \to P$ is a fragmented map for every $a \in P$.
   (2) *HNS-semigroup* ($\mathcal{F}$-*semigroup* in [25]) if $\{\lambda_a : P \to P\}_{a \in P}$ is a fragmented family of maps.

These classes are closed under factors. We have the inclusions:

$$\{\text{compact semitopological semigroups}\} \subset \{\text{HNS-semigroups}\} \subset \{\text{Tame semigroups}\}$$

Lemma 6.2.   (1) *Every compact semitopological semigroup P is a HNS-semigroup.*
   (2) *Every HNS-semigroup is tame.*
   (3) *If P is a* metrizable *compact right topological admissible semigroup then P is a HNS-semigroup.*

Proof. (1) Apply Lemma 1.8.1 to $P \times P \to P$.
   (2) is trivial.
   (3) Apply Lemma 1.8.2 to $P \times P \to P$. □

If P is Fréchet, as a topological space, then P is a tame semigroup by Corollary 6.20 below.

6.2. **Compact semitopological semigroups and WAP systems.** As usual, a continuous function on a (not necessarily compact) S-space X is said to be *weakly almost periodic* (WAP) if the weakly closure of the orbit fS is weakly compact in $C(X)$. Notation: $f \in \mathrm{WAP}(X)$. It is equivalent that $f = \tilde{f} \circ \alpha$ comes from an S-compactification $\alpha : X \to Y$ such that $\tilde{f} \in \mathrm{WAP}(Y)$. In fact, one may choose the cyclic S-compactification $Y = X_f$. A compact dynamical S-system X is said to be WAP if $C(X) = \mathrm{WAP}(X)$. The latter happens iff every element $p \in E(X)$ is a continuous selfmap of X (Ellis and Nerurkar).

Proposition 6.3. *Let V be a Banach space. The following are equivalent:*
   (1) *V is reflexive.*
   (2) *The compact semigroup $\mathcal{E}$ is semitopological.*
   (3) *$\mathcal{E} = \Theta^{op}$.*
   (4) *Θ is compact with respect to the weak operator topology.*
   (5) *(Θ^{op}, B^*) is a WAP system.*

Proof. Use Lemma 2.6.1 and the standard characterizations of reflexive Banach spaces. □

Fact 6.4. [49, Section 4] *Let S be a semitopological semigroup.*
   (1) *A compact (continuous) S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively approximable.*
   (2) *A compact (continuous) metric S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively representable.*

We next recall a version of Lawson's theorem [43] and its soft geometrical proof using representations of dynamical systems on reflexive spaces.

Fact 6.5. (Ellis-Lawson's Joint Continuity Theorem) *Let G be a subgroup of a compact semitopological monoid S. Suppose that $S \times X \to X$ is a separately continuous action with compact X. Then the action $G \times X \to X$ is jointly continuous and G is a topological group.*

Proof. A sketch of the proof from [49]: It is easy to see by Grothendieck's Lemma that $C(X) = \mathrm{WAP}(X)$. Hence (S, X) is a weakly almost periodic system. By Theorem 6.4 the proof can be reduced to the particular case where $(S, X) = (\Theta(V)^{op}, B_{V^*})$ for some reflexive Banach space V with $G := \mathrm{Iso}\,(V)$, where $\Theta(V)^{op}$ is endowed with the weak operator topology. By [47] the weak and strong operator topologies coincide on $\mathrm{Iso}\,(V)$ for reflexive V. In particular, G is a topological group and it acts continuously on B_{V^*}. □

As a corollary one gets the classical result of Ellis. See also a generalization in Theorem 8.7.

Fact 6.6. (Ellis Theorem) *Every compact semitopological group is a topological group.*

Another consequence of Theorem 6.4 (taking into account Proposition 6.3) is

Fact 6.7. ([68] and [47]) *Every compact semitopological semigroup S is embedded into $\Theta(V) = \mathcal{E}(V^*)$ for some reflexive V.*

Thus, compact semitopological semigroups S can be characterized as closed subsemigroups of $\mathcal{E}(V)$ for reflexive Banach spaces V. We will show below, in Theorem 8.5, that analogous statements (for admissible embeddings) hold for HNS and tame semigroups, where the corresponding classes of Banach spaces are Asplund and Rosenthal spaces respectively.

6.3. **HNS-semigroups and dynamical systems.** The following definition (for continuous group actions) originated in [25] . One may extend it to separately continuous semigroup actions.

Definition 6.8. We say that a compact S-system X is *hereditarily non-sensitive* (HNS, in short) if one of the following equivalent conditions are satisfied:

 (1) For every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε-small for every $s \in S$.
 (2) The family of translations $\widetilde{S} := \{\tilde{s} : X \to X\}_{s \in S}$ is a fragmented family of maps.
 (3) $E(S, X)$ is a fragmented family of maps from X into itself.

The equivalence of (1) and (2) is evident from the definitions. Clearly, (3) implies (2). As to the implication $(2) \Rightarrow (3)$, observe that the pointwise closure of a fragmented family is again a fragmented family, [27, Lemma 2.8].

Note that if $S = G$ is a group then in Definition 6.8.1 one may consider only closed subsets A which are G-invariant (see the proof of [25, Lemma 9.4]).

Lemma 6.9. (1) *For every S the class of HNS compact S-systems is closed under subsystems, arbitrary products and factors.*
 (2) *For every HNS compact S-system X the corresponding enveloping semigroup $E(X)$ is HNS both as an S-system and as a semigroup.*
 (3) *Let P be a HNS-semigroup. Assume that $j : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $j(S) \subset \Lambda(P)$. Then the S-system P is HNS.*
 (4) *$\{$HNS-semigroups$\}=\{$enveloping semigroups of HNS systems$\}$.*

Proof. (1) As in [27] using the stability properties of fragmented families.

(2) (S, E) is a HNS system because HNS is preserved by subdirect products. So, by Definition 6.8, $\{\lambda_a : E \to E\}_{a \in j(S)}$ is a fragmented family of maps. Then its pointwise closure $\{\lambda_a : E \to E\}_{a \in E}$ is also a fragmented family.

(3) Since $j(S) \subset \Lambda(P)$ the closure $\overline{j(S)}$ is a subsemigroup of P. We can assume that $\overline{j(S)} = P$. By Lemma 1.26.4, the enveloping semigroup $E(S, P) \subset P^P$ can be naturally identified with P so that every $a \in E(S, P)$ is identified with the corresponding left translation $\lambda_a : P \to P$. Since P is a HNS-semigroup the set of all left translations $\{\lambda_a : P \to P\}_{a \in E}$ is a fragmented family. Hence, (S, P) is a HNS system (Definition 6.8).

(4) Combine (2) and (3) taking into account Lemma 1.26.4. $\square$

Theorem 6.10. *Let V be a Banach space. The following are equivalent:*

 (1) *V is an Asplund Banach space.*
 (2) *(Θ^{op}, B^*) is a HNS system.*
 (3) *$\mathcal{E}$ is a HNS-semigroup.*

Proof. $(1) \Rightarrow (2)$: Use Definition 6.8.2 and the following well known characterization of Asplund spaces: V is Asplund iff B^* is $(w^*, norm)$-fragmented (Fact 1.9).

$(2) \Rightarrow (1)$ By Fact 1.9 we have to show that B is a fragmented family for B^*. Choose a vector $v \in S_V$. Since Θ^{op} is a fragmented family of self-maps on B^* and as $v : B^* \to \mathbb{R}$ is uniformly

continuous we get that the system $v\Theta^{op} = \Theta v$ of maps from B^* to $\mathbb{R}$ is also fragmented. Now recall that $\Theta v = B$ by Lemma 2.6.1.

(2) $\Rightarrow$ (3): Follows from Lemma 6.9.2 and the fact that $\mathcal{E}$ is the enveloping semigroup $E(\Theta^{op}, B^*)$.

(3) $\Rightarrow$ (2): $\Lambda(\mathcal{E}) = \Theta^{op}$ (Lemma 2.6.5) and $\mathcal{E}$ is a HNS-semigroup. So, $(S, \mathcal{E})$ is HNS by Lemma 6.9.3 with $S = \Theta^{op}$. Take $\psi \in B^*$ with $||\psi|| = 1$. The map $q : \mathcal{E} \to B^*$, $p \mapsto p\psi$ defines a continuous homomorphism of Θ^{op}-systems. By Lemma 2.6.4, we have $\mathcal{E}\psi = B^*$. So q is onto. Now observe that the HNS property is preserved by factors of S-systems (Lemma 6.9.1). $\qquad \square$

Our next theorem is based on ideas from [30].

Theorem 6.11. *Let V be a Banach space. The following are equivalent:*

    (1) *V is a separable Asplund space.*
    (2) *$\mathcal{E}$ is homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$ (for infinite-dimensional V).*
    (3) *$\mathcal{E}$ is metrizable.*

Proof. (1) $\Rightarrow$ (2) Since $\mathcal{E}$ is a compact affine subset in the Fréchet space $\mathbb{R}^{\mathbb{N}}$ we can use Keller's Theorem [11, p. 100].

(2) $\Rightarrow$ (3) Is trivial.

(3) $\Rightarrow$ (1) $\mathcal{E}$ is a HNS-semigroup by Lemma 6.2.3. Now Theorem 6.10 implies that V is Asplund. It is also separable; indeed, by Lemma 2.6.4, B^* is a continuous image of $\mathcal{E}$, so that B^* is also w^*-metrizable, which in turn yields the separability of V. $\qquad \square$

Now in Theorem 6.12.2 we obtain a short proof of one of the main results of [30] (stated there for continuous group actions).

Theorem 6.12. *Let X be a compact S-system. Consider the following assertions:*

    (a) *$E(X)$ is metrizable.*
    (b) *(S, X) is HNS.*

Then:

    (1) *(a) $\Rightarrow$ (b).*
    (2) *If X, in addition, is metrizable then (a) $\Leftrightarrow$ (b).*

Proof. (1) By Definition 6.8 we have to show that $E(X)$ is a fragmented family of maps from X into itself. The unique compatible uniformity on the compactum X is the weakest uniformity on X generated by $C(X)$. Using Remark 1.7.1 one may reduce the proof to the verification of the following claim: $E^f := \{f \circ p : p \in E(X)\}$ is a fragmented family for every $f \in C(X)$. In order to prove this claim apply Lemma 1.8.2 to the induced mapping $E(X) \times X \to \mathbb{R}, (p, x) \mapsto f(px)$ (using our assumption that $E(X)$ is metrizable).

(2) If X is a metrizable HNS S-system then by Theorem 7.1 below, (S, X) is representable on a separable Asplund space V. We can assume that X is S-embedded into B^*. The enveloping semigroup $E(S, B^*)$ is embedded into $\mathcal{E}$ (Lemma 2.4). The latter is metrizable by virtue of Theorem 6.11. Hence $E(S, X)$ is also metrizable, being a continuous image of $E(S, B^*)$. $\qquad \square$

Proposition 6.13. *Let S be a semitopological semigroup and $\alpha : S \to P$ be a right topological semigroup compactification.*

    (1) *If P is metrizable then P is a HNS-semigroup and the system (S, P) is HNS.*
    (2) *Let $V \subset C(S)$ be an m-introverted closed subalgebra of $C(S)$. If V is separable then necessarily $V \subset \mathrm{Asp}(S)$.*

Proof. (1) By Lemma 6.2.3, P is a HNS-semigroup. By Lemma 6.9.3, the system (S, P) is HNS.

(2) By Fact 3.16.1, the algebra V induces a semigroup compactification $S \to P$. Since V is separable, P is metrizable. So by (1), (S, P) is HNS. Therefore, $V \subset \mathrm{Asp}(S)$. $\qquad \square$

6.4. Tame semigroups and tame systems.

Definition 6.14. A compact separately continuous S-system X is said to be *tame* if the translation $\lambda_a : X \to X$, $x \mapsto ax$ is a fragmented map for every element $a \in E(X)$ of the enveloping semigroup.

This definition is formulated in [27] for continuous group actions.

According to Remark 1.19 we define, for every S-space X, the S-subalgebras $\mathrm{Tame}(X)$ and $\mathrm{Tame}_c(X)$ of $C(X)$. Recall that in several natural cases we have $\mathcal{P}_c(X) = \mathcal{P}(X)$ (see Lemma 1.20).

Lemma 6.15. *Every WAP system is HNS and every HNS is tame. Therefore, for every semitopological semigroup S and every S-space X (in particular, for $X := S$) we have*

$$\mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{Tame}(X) \quad \mathrm{WAP}_c(X) \subset \mathrm{Asp}_c(X) \subset \mathrm{Tame}_c(X).$$

Proof. If (S, X) is WAP then $E(X) \times X \to X$ is separately continuous. By Lemma 1.8.1 we obtain that E is a fragmented family of maps from X to X. In particular, its subfamily $\{\tilde{s} : X \to X\}_{s \in S}$ of all translations is fragmented. Hence, (S, X) is HNS.

Directly from the definitions we conclude that every HNS is tame. $\square$

Another proof of Lemma 6.15 comes also from Banach representations theory for dynamical systems because every reflexive space is Asplund and every Asplund is Rosenthal.

By [29], a compact metrizable S-system X is tame iff S is *eventually fragmented* on X, that is, for every infinite (countable) subset $C \subset G$ there exists an infinite subset $K \subset C$ such that K is a fragmented family of maps $X \to X$.

Lemma 6.16.   (1) *For every S the class of tame S-systems is closed under closed subsystems, arbitrary products and factors.*

    (2) *For every tame compact S-system X the corresponding enveloping semigroup $E(X)$ is tame both as an S-system and as a semigroup.*

    (3) *Let P be a tame right topological compact semigroup and let $\nu : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $\nu(S) \subset \Lambda(P)$. Then the S-system P is tame.*

    (4) *{tame semigroups}={enveloping semigroups of tame systems}.*

Proof. (1) As in [27] using the stability properties of fragmented maps.

(2) (S, E) is a tame system because by (1) tameness is preserved by subdirect products. Its enveloping semigroup can be identified with E itself (Lemma 1.26.4), so that $\lambda_p : E \to E$ is fragmented for every $p \in E$.

(3) $\nu(S) \subset \Lambda(P)$, so $\overline{\nu(S)}$ is a semigroup. We can assume that $\overline{\nu(S)} = P$. By Lemma 1.26.4, the enveloping semigroup $E(S, P) \subset P^P$ can be naturally identified with P in such a way that every $a \in E(S, P)$ is identified with the corresponding left translation $\lambda_a : P \to P$ for some $a \in P$. Since P is a tame semigroup every left translation $\lambda_a : P \to P$ is fragmented. Hence, (S, P) is a tame system.

(4) Combine (2) and (3) taking into account Lemma 1.26.4. $\square$

Proposition 6.17. *Let X be a compact S-space and $f \in C(X)$. The following conditions are equivalent:*

    (1) $f \in \mathrm{Tame}(X)$.

    (2) $\mathrm{cl}_p(fS) \subset \mathcal{F}(X)$ *(i.e. the orbit fS is a Rosenthal family for X).*

Proof. See [27, Prop. 5.6]. $\square$

Theorem 6.18. *Let V be a Banach space. The following are equivalent:*

    (1) *V is a Rosenthal Banach space.*

    (2) *(Θ^{op}, B^*) is a tame system.*

    (3) *$p : B^* \to B^*$ is a fragmented map for each $p \in \mathcal{E}$.*

    (4) *$\mathcal{E}$ is a tame semigroup.*

Proof. (2) $\Leftrightarrow$ (3): Follows from the definition of tame flows because $\mathcal{E} = E(\Theta^{op}, B^*)$.

(2) $\Rightarrow$ (4): Since $\mathcal{E} = E(\Theta^{op}, B^*)$, Lemma 6.16.2 applies.

(4) $\Rightarrow$ (2): By our assumption, $\mathcal{E}$ is a tame semigroup. Then by Lemma 6.16.3 the system $(\Theta^{op}, \mathcal{E})$ is tame. Its factor (Lemma 2.6.4) (Θ^{op}, B^*) is tame, too.

(2) $\Rightarrow$ (1): By a characterization of Rosenthal spaces [27, Prop. 4.12] (see also Fact 1.12) it suffices to show that $B^{**} \subset \mathcal{F}(B^*)$. Since (Θ^{op}, B^*) is tame, $p : B^* \to B^*$ is fragmented for every

$p \in E(\Theta^{op}, B^*) = \mathcal{E}$. Pick an arbitrary $v \in B_V$ with $\|v\| = 1$. Then $v\mathcal{E}$ is exactly B^{**} by Lemma 2.6.2. So every $\phi \in B^{**}$ is a composition $v \circ p$, where p is a fragmented map. Since $v : B^* \to \mathbb{R}$ is weak* continuous we conclude that $\phi : B^* \to B^*$ is fragmented.

(1) $\Rightarrow$ (3): We have to show that $\mathcal{E} \subset \mathcal{F}(B^*, B^*)$ for every Rosenthal space V. Let $p \in \mathcal{E}$. Then $p \in \Theta(V^*)$. That is, p is a linear map $p : V^* \to V^*$ with norm ≤ 1. Then, for every vector $f \in V$, the composition $f \circ p : V^* \to \mathbb{R}$ is a linear bounded functional on V^*. That is, $f \circ p \in V^{**}$ belongs to the second dual. Again, by the above mentioned characterization of Rosenthal spaces, the corresponding restriction $f \circ p|_{B^*} : B^* \to \mathbb{R}$ is a fragmented function for every $f \in V$. Since V separates points of B^* we can apply [27, Lemma 2.3.3]. It follows that $p : B^* \to B^*$ is fragmented for every $p \in \mathcal{E}$. $\qquad\square$

6.5. **A dynamical BFT dichotomy.** Recall that a topological space K is a *Rosenthal compactum* [33] if it is homeomorphic to a pointwise compact subset of the space $\mathcal{B}_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compactum is the *Helly compact* of all nondecreasing selfmaps of $[0, 1]$ in the pointwise topology. Recall that a topological space K is *Fréchet* (or, *Fréchet-Urysohn*) if for every $A \subset K$ and every $x \in cl(A)$ there exists a sequence of elements of A which converges to x. Every Rosenthal compact space K is Fréchet by a result of Bourgain-Fremlin-Talagrand [10, Theorem 3F], generalizing a result of Rosenthal.

Theorem 6.19. *If the enveloping semigroup $E(X)$ is a Fréchet (e.g., Rosenthal) space, as a topological space, then (S, X) is a tame system (and $E(X)$ is a tame semigroup).*

Proof. Let $p \in E(X)$. We have to show that $p : X \to X$ is fragmented. By properties of fragmented maps [27, Lemma 2.3.3] it is enough to show that $f \circ p : X \to \mathbb{R}$ is fragmented for every $f \in C(X)$. By the Fréchet property of $E(X)$ we may choose a sequence s_n in S such that the sequence $j(s_n)$ converges to p in E(X). Hence the sequence of continuous functions $f \circ s_n = f \circ j(s_n)$ converges pointwise to $f \circ p$ in $\mathbb{R}^X$. Apply Lemma 1.8.2 to the evaluation map $F \times X \to \mathbb{R}$, where $F := \{f \circ p\} \cup \{f \circ j(s_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^X$ carries the pointwise topology. We conclude that F is a fragmented family. In particular, $f \circ p$ is a fragmented map. ($E(X)$ is a tame semigroup by Lemma 6.16.2.) $\qquad\square$

Corollary 6.20. *Let P be a compact right topological admissible semigroup. If P is Fréchet (e.g., when it is Rosenthal), as a topological space, then P is a tame semigroup.*

Proof. Applying Theorem 6.19 to the system (S, P), with $S := \Lambda(P)$ we obtain that $E(S, P) = P$ is a tame semigroup. $\qquad\square$

The following result was proved in [25, Theorem 3.2] using the Bourgain-Fremlin-Talagrand (BFT) dichotomy in the setting of continuous group actions. The same arguments work also for separately continuous semigroup actions. For the sake of completeness we include a simplified proof.

Fact 6.21 (A dynamical BFT dichotomy)**.** *Let X be a compact metric dynamical S-system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative. Either*

(1) *E is a separable Rosenthal compact, hence $\operatorname{card} E \leq 2^{\aleph_0}$; or*

(2) *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence $\operatorname{card} E = 2^{2^{\aleph_0}}$.*

The first possibility holds iff X is a tame S-system.

Proof. For every $f \in C(X)$ define $E^f := \{f \circ p : p \in E\}$. Then E^f is a pointwise compact subset of $\mathbb{R}^X$, being a continuous image of E under the map $q_f : E \to E^f$, $p \mapsto f \circ p$. Since X is metrizable by Lemma 1.26.5 there exists a sequence $\{s_m\}_{m=1}^\infty$ in S such that $\{j(s_m)\}_{m=1}^\infty$ is dense in $E(X)$. In particular, the sequence of real valued functions $\{f \circ s_m\}_{m=1}^\infty$ is pointwise dense in E^f.

Choose a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C(X)$ which separates the points of X. For every pair s, t of distinct elements of E there exist a point $x_0 \in X$ and a function f_{n_0} such that $f_{n_0}(sx_0) \neq f_{n_0}(tx_0)$. It follows that the continuous diagonal map

$$\Phi : E \to \prod_{n \in \mathbb{N}} E^{f_n}, \qquad p \mapsto (f_1 \circ p, f_2 \circ p, \dots)$$

separates the points of E and hence is a topological embedding. Now if for each n the space E^{f_n} is a Rosenthal compactum then so is $E \cong \Phi(E) \subset \prod_{n=1}^\infty E^{f_n}$, because the class of Rosenthal

compacta is closed under countable products and closed subspaces. On the other hand if at least one $E^{f_n} = cl_p(\{f_n \circ s_m\}_{m=1}^\infty)$ is not Rosenthal then, by a version of the BFT-dichotomy (Todorčević [69, Section 13]) it contains a homeomorphic copy of $\beta\mathbb{N}$ and it is easy to see that so does its preimage E. In fact if $\beta\mathbb{N} \cong Z \subset E^{f_n}$ then any closed subset Y of E which projects onto Z and is minimal with respect to these properties is also homeomorphic to $\beta\mathbb{N}$.

Now we show the last assertion. If X is tame then every $p \in E(X)$ is a fragmented self-map of X. Hence every $f \circ p \in E^f$ is fragmented. By Remark 1.7.2 this is equivalent to saying that every $f \circ p$ is Baire 1. So $E^f \subset \mathcal{B}_1(X)$ is a Rosenthal compactum. Therefore, $E \cong \Phi(E) \subset \prod_{n \in \mathbb{N}} E^{f_n}$ is also Rosenthal. Conversely, if E is a Rosenthal compactum then (S, X) is tame by Theorem 6.19. $\qquad\square$

Theorem 6.22 (BFT dichotomy for Banach spaces)**.** *Let V be a separable Banach space and let $\mathcal{E} = \mathcal{E}(V)$ be its (separable) enveloping semigroup. We have the following alternative. Either*

(1) *$\mathcal{E}$ is a Rosenthal compactum, hence card $\mathcal{E} \le 2^{\aleph_0}$; or*

(2) *the compact space $\mathcal{E}$ contains a homeomorphic copy of $\beta\mathbb{N}$, hence card $\mathcal{E} = 2^{2^{\aleph_0}}$.*

The first possibility holds iff V is a Rosenthal Banach space.

Proof. Recall that $\mathcal{E} = E(\Theta^{op}, B^*)$. By Theorem 6.18, V is Rosenthal iff (Θ^{op}, B^*) is tame. Since V is separable, B^* is metrizable. So we can apply Fact 6.21. $\qquad\square$

6.6. **Amenable affine compactifications.** Let G be a topological group and X a G-space. Let us say that an affine S-compactification $\alpha : X \to Y$ is *amenable* if Y has a G-fixed point. We say that a closed unital linear subspace $\mathcal{A} \subset \mathrm{WRUC}(X)$ is (left) *amenable* if the corresponding affine G-compactification is amenable. By Ryll-Nardzewski's classical theorem $\mathrm{WAP}(G)$ is amenable. Let $f \in \mathrm{RUC}(G)$ and let $\pi_f : X \to Q_f$ be the corresponding cyclic affine G-compactification (Section 3.2). In our recent work [28] we show that $\mathrm{Asp_c}(G)$ is amenable and that for every $f \in \mathrm{Asp_c}(G)$ there exists a G-fixed point (a *G-average* of f) in Q_f. The first result together with Proposition 6.13 yield the following:

Corollary 6.23. *Let G be a topological group and $\mathcal{A}$ a (left) m-introverted closed subalgebra of $\mathrm{RUC}(G)$. If $\mathcal{A}$ is separable then $\mathcal{A}$ is amenable.*

A topological group G is said to be amenable if $\mathrm{RUC}(G)$ is amenable. By a classical result of von Neumann, the free discrete group $\mathbb{F}_2$ on two symbols is not amenable. So, $\mathrm{RUC}(\mathbb{F}_2) = l_\infty(\mathbb{F}_2)$ is not amenable. By [28], $\mathrm{Tame}(\mathbb{F}_2)$ is not amenable. It would be interesting to study for which non-amenable groups G the algebra $\mathrm{Tame_c}(G)$ is amenable and for which $f \in \mathrm{Tame_c}(G)$ there exists a G-fixed point of Q_f.

Example 6.24. (1) Results of [29] show that $\varphi_D(n) = sgn \cos(2\pi n\alpha)$ is a tame function on $\mathbb{Z}$ which is not Asplund.

(2) As a simple illustration of Proposition 6.13 note that the two-point semigroup compactifications of $\mathbb{Z}$ and $\mathbb{R}$ are obviously metrizable. So the characteristic function $\xi_\mathbb{N} : \mathbb{Z} \to \mathbb{R}$ and $arctg : \mathbb{R} \to \mathbb{R}$ are both Asplund. Grothendieck's double limit criterion shows that these functions are not WAP.

7. Representations of semigroup actions on Banach spaces

As was shown in several of our earlier works some properties of dynamical systems are clearly reflected in analogous properties of their enveloping semigroups on the one hand, and in their representations on Banach spaces on the other. Our results from [25, 30, 27] are formulated for group actions. However the main results in these papers remain true for semigroup actions.

For continuous group actions the results (1), (2) of the following theorem were proved respectively in [27] and [25] (compare also with Theorem 6.4). We will show next how the proofs of (1), (2) can be modified to suit the more general case of semigroup actions, obtaining, in fact, also some new results.

Theorem 7.1. *Let S be a semitopological semigroup and X a compact S-system with a separately continuous action.*

(1) (S, X) is a tame (continuous) system if and only if (S, X) is weakly (respectively, strongly) Rosenthal-approximable.

(2) (S, X) is a HNS (continuous) system if and only if (S, X) is weakly (respectively, strongly) Asplund-approximable.

If X is metrizable then in (1) and (2) "approximable" can be replaced by "representable". Moreover, the corresponding Banach space can be assumed to be separable.

Proof. The proof for continuous actions is the same as in [27]. So below we show only how the proof can be adopted for separately continuous actions and weakly continuous representations.

The "only if" part: For (1) use the fact that (Θ^{op}, B^*) is a tame system (Theorem 6.18) for every Rosenthal V and for (2), the fact that (Θ^{op}, B^*) is HNS (Theorem 6.10) for Asplund V.

The "if" part: (1) For every $f \in C(X) = \mathrm{Tame}(X)$ the orbit fS is a Rosenthal family for X (Proposition 6.17). Applying Theorem 7.2 below we conclude that every $f \in C(X) = \mathrm{Tame}(X)$ on a compact G-space X comes from a Rosenthal representation. Since continuous functions separate points of X, this implies that Rosenthal representations of (S, X) separate points of X. So, for (1) it is enough to prove the following result.

Theorem 7.2. *Let X be a compact S-space and let $F \subset C(X)$ be a Rosenthal family for X such that F is S-invariant; that is, $fS \subset F \ \forall f \in F$. Then there exist: a Rosenthal Banach space V, an injective mapping $\nu : F \to B_V$ and a representation*

$$h : S \to \Theta(V), \quad \alpha : X \to V^*$$

of (S, X) on V such that h is weakly continuous, α is a weak* continuous map and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F \ \forall x \in X.$$

Thus the following diagram commutes

(7.1)

$$
\begin{array}{ccc}
F \times X & \longrightarrow & \mathbb{R} \\
\nu \downarrow \quad \alpha \downarrow & & \downarrow id_{\mathbb{R}} \\
V \times V^* & \longrightarrow & \mathbb{R}
\end{array}
$$

If X is metrizable then in addition we can suppose that V is separable.

If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous.

Proof. **Step 1:** The construction of V.

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$, B will denote its unit ball, and B^* will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$. Let W be the symmetrized convex hull of F; that is, $W := \mathrm{co}\,(F \cup -F)$. Consider the sequence of sets

(7.2)
$$M_n := 2^n W + 2^{-n} B.$$

Then W is convex and symmetric. We apply the construction of Davis-Figiel-Johnson-Pelczyński [15] as follows. Let $\| \ \|_n$ be the Minkowski functional of the set M_n, that is,

$$\|v\|_n = \inf\,\{\lambda > 0 \mid v \in \lambda M_n\}.$$

Then $\| \ \|_n$ is a norm on $\mathcal{A}$ equivalent to the given norm of $\mathcal{A}$. For $v \in \mathcal{A}$, set

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and let} \quad V := \{v \in \mathcal{A} \mid N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map. Then (V, N) is a Banach space, $j : V \to \mathcal{A}$ is a continuous linear injection and

(7.3)
$$W \subset j(B_V) = B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B)$$

Step 2: The construction of the representation (h, α) of (S, X) on V.

The given action $S \times X \to X$ induces a natural linear norm preserving continuous right action $C(X) \times S \to C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the construction that W and B are S-invariant subsets in $\mathcal{A}$. This implies that V is an S-invariant subset of $\mathcal{A}$ and the restricted natural linear action $V \times S \to V$, $(v, g) \mapsto vg$ satisfies $N(vs) \leq N(v)$. Therefore, the co-homomorphism $h : S \to \Theta(V)$, $h(s)(v) := vs$ is well defined.

Let $j^* : \mathcal{A}^* \to V^*$ be the adjoint map of $j : V \to \mathcal{A}$. Define $\alpha : X \to V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(x)$. Then (h, α) is a representation of (S, X) on the Banach space V.

By the construction $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

(7.4) $$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F \ \forall x \in X.$$

Step 3: Weak continuity of $h : S \to \Theta(V)$.

By our construction $j^* : C(X)^* \to V^*$, being the adjoint of the bounded linear operator $j : V \to C(X)$, is a norm and weak* continuous linear operator. By Lemma 4.4.2 we obtain that $j^*(C(X)^*)$ is norm dense in V^*. Since V is Rosenthal, Haydon's theorem (Fact 1.12.4) gives $Q := cl_{w^*}(co(Y)) = cl_{norm}(co(Y))$, where $Y := j^*(X)$. Now observe that $j^*(P(X)) = Q$. Since $S \times X \to X$ is separately continuous, every orbit map $\tilde{x} : S \to X$ is continuous, and each orbit map $\widetilde{j^*(x)} : S \to j^*(X)$ is weak* continuous. Then also $\widetilde{j^*(z)} : S \to V^*$ is weak* continuous for each $z \in cl_{norm}(co(j^*(X))) = Q$. Since $sp(Q)$ is norm dense in V^* (and $\|h(s)\| \leq 1$ for each $s \in S$) it easily follows that $\widetilde{j^*(z)} : S \to V^*$ is weak* continuous for every $z \in V^*$. This is equivalent to the weak continuity of h.

If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous. Indeed, by the definition of the norm N, we can show that the action of S on V is norm continuous (use the fact that, for each $n \in \mathbb{N}$, the norm $\|\cdot\|_n$ on $\mathcal{A}$ is equivalent to the given norm on $\mathcal{A}$).

Step 4: V is a Rosenthal space.

By results of [27, Section 4], W is a Rosenthal family for B^* (and X). Furthermore, a deeper analysis shows (we refer to [27, Theorem 6.3] for details) that B_V is a Rosenthal family for B_{V^*}. Thus V is Rosenthal by Fact 1.12.

If the compact space X is metrizable then $C(X)$ is separable and it is also easy to see that (V, N) is separable. $\qquad\square$

This proves Theorem 7.2 and hence also Theorem 7.1.1.

(2) For the "Asplund case" one can modify the proof of (1). The main idea is that the corresponding results of [49, Section 7] and [25, Section 9] can be adopted here, thus obtaining a modification of Theorem 7.2 which replaces a Rosenthal space by an Asplund space, and a "Rosenthal family F" for X by an "Asplund set". The latter means that for every countable subset $A \subset F$ the pseudometric ρ_A on X defined by

$$\rho_A(x, y) := \sup_{f \in A} |f(x) - f(y)|, \ x, y \in X$$

is separable. By [18, Lemma 1.5.3] this is equivalent to saying that $(C(X)^*, \rho_A)$ is separable. Now $co(F \cup -F)$ is an Asplund set for B^* by [18, Lemma 1.4.3]. The rest is similar to the proof of [49, Theorem 7.7]. Checking the weak continuity of h one can apply a similar idea (using again Haydon's theorem as in (1)).

Finally note that if X is metrizable then in (1) and (2) "approximable" can be replaced by "representable" using an l_2-sum of a sequence of separable Banach spaces (see Lemma 1.14.3). $\quad\square$

Remark 7.3. The fundamental DFJP-factorization construction from [15] has an "isometric modification". According to [45] one may assume in Theorem 7.2 that the bounded operator $j : V \to \mathcal{A}$ has the property $\|j\| \leq 1$. More precisely, we can replace in the Equation 7.2 the sequence of sets $M_n := 2^n W + 2^{-n} B$ by $K_n := a^{\frac{n}{2}} W + a^{-\frac{n}{2}} B$, where $2 < a < 3$ is the unique solution of the equation

$\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1$. For details see [45]. Taking into account this modification (which is completely compatible with our S-space setting) for a set $F \subset C(X)$ with $\sup\{|f(x)| : x \in X, f \in F\} \leq 1$ we can assume that $\nu(F) \subset B$ and $\alpha(X) \subset B^*$. Hence the following sharper diagram commutes

(7.5)
$$\begin{array}{ccc} F \times X & \longrightarrow & [-1,1] \\ \nu \downarrow \quad \downarrow \alpha & & \downarrow id \\ B \times B^* & \longrightarrow & [-1,1] \end{array}$$

Note also that this modified version from [45] of the DFJP-construction repairs in particular the proof of [49, Theorem 4.5]. The latter was first corrected in the arxiv version of [49, Theorem 4.5] using, however, diagrams like 7.1, where $\nu(F)$ and $\alpha(X)$ are bounded.

Theorem 7.4. (1) *Let X be a compact S-space. The following conditions are equivalent:*
   (a) *$f \in \text{Tame}(S)$ (respectively, $f \in \text{Tame}_c(S)$).*
   (b) *There exist: a weakly (respectively, strongly) continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v \in V$ such that $f(x) = \langle v, \alpha(x) \rangle \quad \forall\, x \in X$.*
   (2) *Let S be a semitopological semigroup and $f \in C(S)$. The following conditions are equivalent:*
   (a) *$f \in \text{Tame}(S)$ (respectively, $f \in \text{Tame}_c(S)$).*
   (b) *f is a matrix coefficient of a weakly (respectively, strongly) continuous co-representation of S on a Rosenthal space. That is, there exist: a Rosenthal space V, a weakly (respectively, strongly) continuous co-homomorphism $h : S \to \Theta(V)$, and vectors $v \in V$ and $\psi \in V^*$ such that $f(s) = \psi(vs)$ for every $s \in S$.*
   (3) *Similar (to (1) and (2)) results are valid for*
   (a) *Asplund functions and Asplund Banach spaces;*
   (b) *WAP functions and reflexive Banach spaces.*

Proof. (1) (b) $\Rightarrow$ (a): $(\Theta(V)^{op}, B^*)$ is a tame system for every Rosenthal space V by Theorem 6.18. The action is separately (jointly) continuous for the weak (respectively, strong) operator topology on $\Theta(V)^{op}$.

(a) $\Rightarrow$ (b): Let $f \in \text{Tame}(X)$. This means by Proposition 6.17 that the orbit fS is a Rosenthal family for X. Now we can apply Theorem 7.2 to the family $F := fS$ (getting X_f as $\alpha(X)$).

(2) (a) $\Rightarrow$ (b): $f \in \text{Tame}(S)$ (respectively, $f \in \text{Tame}_c(S)$) means that there exist: a tame S-compactification $\gamma : S \to X$ of the S-space S such that $S \times X \to X$ is separately continuous (respectively, jointly continuous) and a continuous function $f_0 : X \to \mathbb{R}$ such that $f = f_0 \circ \nu$. Apply Theorem 7.2 to f_0 getting the desired V and vectors $v := \nu(f)$ and $\psi := \alpha(\gamma(e))$. Now

$$f(s) = \langle v, \alpha(\gamma(s)) \rangle = m(v, \psi)(s) \quad \forall\, s \in S.$$

(b) $\Rightarrow$ (a): Since $h : S \to \Theta(V)$ is weakly (strongly) continuous the natural action of S on the compact space $X := cl_{w^*}(S\psi)$ is separately (respectively, jointly) continuous. Apply Theorem 6.18 to establish that (S, X) is tame. Finally observe that $f(s) = \langle v, s\psi \rangle$ comes from the S-compactification $S \to X, s \mapsto s\psi$.

(3) (a) is similar to (1) using the Asplund version of Theorem 7.2. For (b) note that the case of $f \in \text{WAP}(S)$ was proved in [49, Theorem 5.1]. The case of $f \in WAP_c(S)$ is similar using [49, Theorem 4.6]. $\square$

If in Theorem 7.4, $S := G$ is a semitopological group then for any monoid co-homomorphism $h : G \to \Theta(V)$ we have $h(G) \subset \text{Iso}(V)$. Recall also that $\text{WAP}(G) = \text{WAP}_c(G)$ (Lemma 1.20.4).

Proposition 7.5. *Let $S \times X \to X$ be a separately continuous action. Then:*
   (1) *$\text{Tame}(X) \subset \text{WRUC}(X)$. In particular, $\text{Tame}(S) \subset \text{WRUC}(S)$.*
   (2) *If X is a compact tame (e.g., HNS or WAP) system then (S, X) is WRUC.*

Proof. (1) Let $f \in \text{Tame}(X)$. Then there exist: a compact tame S-system Y, an S-compactification $\nu : X \to Y$ and $\tilde{f} \in C(Y)$ such that $f = \tilde{f} \circ \nu$. By Theorem 7.2, $\tilde{f}$ comes from a weakly continuous representation (h, α) of (S, Y) on a Rosenthal space V. That is, $\tilde{f}(y) = \langle \nu(\tilde{f}), \alpha(y) \rangle \,\forall\, y \in Y$.

Consider the restriction operator (Remark 1.18.2), $r : V \to C(X)$, $r(v)(x) = \langle v, \alpha(x) \rangle$. Then for the vector $r(\nu(\tilde{f})) = f$ the orbit map $S \to C(X), s \mapsto fs$ is weakly continuous.

(2) Since X is tame we have $\mathrm{Tame}(X) = C(X)$. On the other hand, by (1) we have $\mathrm{Tame}(X) \subset \mathrm{WRUC}(X) \subset C(X)$. Hence, $\mathrm{WRUC}(X) = C(X)$. □

Remark 7.6. Proposition 7.5 allows us to strengthen some results of [49]. Namely, in 7.7, 7.11 and 7.12 of [49] one may drop the assumption of WRUC-compatibility of (S, X). Theorem 7.4 unifies and strengthens some earlier results from [49, 27].

7.1. **Representations of topological groups.**

Theorem 7.7. *Let G be a topological group such that $Tame_c(G)$ (respectively, $Asp_c(G)$, $\mathrm{WAP}(G)$) separates points and closed subsets. Then there exists a Rosenthal (respectively, Asplund, reflexive) Banach space V and a topological group embedding $h : G \hookrightarrow \mathrm{Iso}(V)$ with respect to the strong topology.*

Proof. We consider only the case of $\mathrm{Tame}(G)$. Other cases are similar. The case of $\mathrm{WAP}(G)$ is known [49, 51].

For every topological group G the involution $inv : g \mapsto g^{-1}$ defines a topological isomorphism between G and its opposite group G^{op}. So it is equivalent to show that there exists a topological group embedding $h : G \to \mathrm{Iso}(V)^{op}$. Let $\{f_i\}_{i \in I}$ be a collection of tame functions which come from jointly continuous tame G-compactifications of G and separates points and closed subsets. By Theorem 7.4.2 for every $i \in I$ there exist: a Rosenthal space V_i, a strongly continuous co-homomorphism $h_i : G \to \mathrm{Iso}(V_i)$, and vectors $v_i \in V_i$ and $\psi_i \in V_i^*$ such that $f_i(g) = \psi_i(v_i g)$ for every $g \in G$. Consider the l_2-type sum $V := (\Sigma_{i \in I} V_i)_{l_2}$ which is Rosenthal by virtue of Lemma 1.14.2. We have the natural homomorphism $h : G \to \mathrm{Iso}(V)^{op}$ defined by $h(v) = (h_i(v_i))_{i \in I}$ for every $v = (v_i)_{i \in I} \in V$. It is easy to show that h is a strongly continuous homomorphism. Since $\{f_i\}_{i \in I}$ separates points and closed subsets, the family of matrix coefficients $\{m(v_i, \psi_i)\}_{i \in I}$ generates the topology of G. It follows that $h : G \to \mathrm{Iso}(V)^{op}$ is a topological embedding. □

Recall (see Remark 1.25) that for the group $G := H_+[0, 1]$ every Asplund (hence also every WAP) function is constant and every continuous representation $G \to \mathrm{Iso}(V)$ on an Asplund (hence also reflexive) space V must be trivial. In contrast one may show that G is Rosenthal representable

Theorem 7.8. *The group $G := H_+[0, 1]$ is Rosenthal representable.*

Proof. Consider the natural action of G on the closed interval $X := [0, 1]$ and the corresponding enveloping semigroup $E = E(G, X)$. Every element of G is a (strictly) increasing self-homeomorphism of $[0, 1]$. Hence every element $p \in E$ is a nondecreasing function. It follows that E is naturally homeomorphic to a subspace of the Helly compact space (of all nondecreasing selfmaps of $[0, 1]$ in the pointwise topology). Hence E is a Rosenthal compactum. So by the dynamical BFT dichotomy, Fact 6.21, the G-system X is tame. By Theorem 7.1 we have a faithful representation (h, α) of (G, X) on a separable Rosenthal space V. Therefore we obtain a G-embedding $\alpha : X \hookrightarrow (V^*, w^*)$. Then the strongly continuous homomorphism $h : G \to \mathrm{Iso}(V)^{op}$ is injective. Since $h(G) \times \alpha(X) \to \alpha(X)$ is continuous (and we may identify X with $\alpha(X)$) it follows, by the minimality properties of the compact open topology, that h is an embedding. Thus $h \circ inv : G \to \mathrm{Iso}(V)$ is the required topological group embedding. □

Remark 7.9.   (1) Recall that by [46] continuous group representations on Asplund spaces have the *adjoint continuity property*. In contrast this is not true for Rosenthal spaces. Indeed, assuming the contrary we would have, from Theorem 7.8, that the dual action of the group $H_+[0, 1]$ on V^* is continuous, but this is impossible by the following fact [26, Theorem 10.3] (proved also by Uspenskij (private communication)): every adjoint continuous (co)representation of $H_+[0, 1]$ on a Banach space is trivial.
(2) There exists a semigroup compactification $\nu : G = H_+[0, 1] \to P$ into a tame semigroup P such that ν is an embedding. Indeed, the associated enveloping semigroup compactification $j : G \to E$ of the tame system $(G, [0, 1])$ is tame. Observe that j is a topological embedding because the compact open topology on $j(G) \subset \mathrm{Homeo}([0, 1])$ coincides with the pointwise topology.

Question 7.10. *Is it true that every Polish topological group G is Rosenthal representable ? Equivalently, is this true for the universal Polish groups $G = \mathrm{Homeo}\,([0,1]^{\mathbb{N}})$ or $G = Iso(\mathbb{U})$ (the isometry group of the Urysohn space $\mathbb{U}$) ? By Theorem 7.7 a strongly related question is the question whether the algebra $\mathrm{Tame}(G)$ separates points and closed subsets.*

8. Banach representations of right topological semigroups and affine systems

8.1. **Tame representations.**

Theorem 8.1. (1) *Every weakly continuous representation (h, α) of an S-space X on a Rosenthal Banach space is E-compatible.*

(2) *If the representation in (1) is w^*-generating then the representation is strongly E-compatible.*

Proof. (1) Applying Haydon's theorem (Fact 1.12.4) we get by Lemma 4.5.1 that the representation (h, α) is E-compatible.

(2) Use (1) taking into account Lemma 4.5.2. $\qquad\square$

Theorem 8.2. ([42] and [22] for metrizable systems) *Every tame compact S-space X is injective. Hence, every affine S-compactification of a tame system is E-compatible.*

Proof. In view of Definition 4.11 we have to show that $m(f, \phi) \in \mathcal{A}(E(X), e)$ for every $f \in C(X), \phi \in C(X)^*$. By Theorem 7.2, f comes from a Rosenthal representation. There exist: a weakly continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v_0 \in V$ such that

$$f(x) = \langle v_0, \alpha(x) \rangle \quad \forall\, x \in X.$$

Consider the restriction linear S-operator (Remark 1.18.2)

$$r : V \to C(X), \ r(v)(x) = \langle v, \alpha(x) \rangle.$$

Let $r^* : C(X)^* \to V^*$ be the adjoint operator. Since $m(f, \phi) = m(r(v_0), \phi)) = m(v_0, r^*(\phi))$, it is enough to show that $m(v_0, r^*(\phi)) \in \mathcal{A}(E(X), e)$.

Analyzing the proof of Theorem 7.2 we may assume in addition, in view of Lemma 4.4.2, that the representation (h, α) is generating. By Theorem 8.1 the representation (h, α) is strongly E-compatible. So by Lemma 4.6 we have $m(v_0, r^*(\phi)) \in \mathcal{A}(E(\alpha(X)), e)$. Since $\alpha : X \to \alpha(X)$ is a surjective S-map we have the natural surjective homomorphism $E(X) \to E(\alpha(X))$. Hence, $\mathcal{A}(E(\alpha(X)), e) \subset \mathcal{A}(E(X), e)$. Thus, $m(v_0, r^*(\phi)) \in \mathcal{A}(E(X), e)$, as required. $\qquad\square$

Theorem 8.3. *Let $\nu : S \to P$ be a right topological semigroup compactification. If P is a tame semigroup (e.g., HNS-semigroup, semitopological, or metrizable) then the S-system P is injective and the algebra of the compactification ν is introverted (in particular, ν is an operator compactification).*

Proof. By Lemma 6.16.3, P is a tame S-system. Theorem 8.2 guarantees that it is injective. Hence, by Theorem 4.13 the algebra of the compactification ν is introverted (Proposition 4.10 implies that ν is an operator compactification). $\qquad\square$

Theorem 8.4. *Let $V \subset C(S)$ be an m-introverted Banach subalgebra. If $V \subset \mathrm{Tame}(S)$ (e.g., if V is separable) then V is introverted. In particular, $\mathrm{Tame}(S)$, $\mathrm{Asp}(S)$, $\mathrm{WAP}(S)$ are introverted (and the same is true for $\mathrm{Tame}_c(S)$, $\mathrm{Asp}_c(S)$, $\mathrm{WAP}_c(S)$).*

Proof. Consider the corresponding semigroup compactification $\nu : S \to P$. Since $V \subset \mathrm{Tame}(S)$ the system (S, P) is tame. Then its enveloping semigroup $E(S, P)$ is a tame semigroup (Lemma 6.16.2) and $E(S, P)$ can be naturally identified with P (Lemma 1.26.4). Now combine Theorems 8.3 and 4.13. By Remark 1.19 the subalgebras above are m-introverted (if V is separable then $V \subset \mathrm{Asp}(S)$ by Proposition 6.13, hence, $V \subset \mathrm{Tame}(S)$). $\qquad\square$

8.2. **Banach representation of enveloping semigroups.** By Theorem 6.18 the semigroup $\mathcal{E}(V)$ is tame for every Rosenthal space V. We now show that, in the converse direction, every tame (respectively, HNS) semigroup P, or equivalently, every enveloping semigroup of a tame (respectively, HNS) system, admits a faithful representation on a Rosenthal (respectively, Asplund) Banach space V. Fact 6.7 (for semitopological semigroups and reflexive spaces) is a particular case of the following result.

Theorem 8.5. (Enveloping semigroup representation theorem)
(1) *Let P be a tame semigroup. Then there exist a Rosenthal Banach space V and a $\Lambda(P)$-admissible embedding of P into $\mathcal{E}(V)$.*
(2) *If P is a HNS-semigroup then there is a $\Lambda(P)$-admissible embedding of P into $\mathcal{E}(V)$ where V is an Asplund Banach space.*
(3) *If P is a semitopological semigroup then there is an embedding of P into $\Theta(V) = \mathcal{E}(V^*)$ where V is a reflexive Banach space.*

Proof. Let $S = \Lambda(P)$ be the topological center of P. Since P is admissible, S is a dense submonoid of P. Denote by $j : S \to P$ the corresponding inclusion. Now P, as an S-system, is tame (Lemma 6.16.3). By Theorem 7.1 there exists a family of flow representations $\{(h_i, \alpha_i)\}_{i \in I}$

$$h_i : S \to \Theta(V_i)^{op}, \quad \alpha_i : P \to B_{V_i^*}$$

of (S, P) on Rosenthal Banach spaces V_i, where each h_i is a weakly continuous homomorphism and $\{\alpha_i\}_{i \in I}$ separates points of P. As in the proof of Theorem 8.2 we may assume (by Lemma 4.4.2) that these representations are generating. Then, by Theorem 8.1, they are strongly E-compatible.

Consider the l_2-type sum $V := (\Sigma_{i \in I} V_i)_{l_2}$. Then we have the natural l_2-sum of representations $h : S \to \Theta(V)^{op}$ defined by $h(v) = (h_i(v_i))_{i \in I}$ for every $v = (v_i)_{i \in I} \in V$. Since $V^* = (\Sigma_{i \in I} V_i)^*_{l_2} = (\Sigma_{i \in I} V_i^*)_{l_2}$ (Lemma 1.14.1) and each h_i is weakly continuous it is easy to show that h is a weakly continuous homomorphism. We have the corresponding standard operator compactification $j_K : S \to K = \overline{h(S)} \subset \mathcal{E}(V)$. Since $h(S) \subset \Theta^{op}(V) = \Lambda(\mathcal{E}(V))$, the embedding $K \subset \mathcal{E}(V)$ is S-admissible (Definition 2.3). By Lemma 1.14.2 we know that V is Rosenthal. So in order to complete the proof for "Rosenthal case" (other cases are similar) we have to check the following claim.

Claim 8.6. The semigroup compactifications $j : S \to P$ and $j_K : S \to K$ are equivalent.

Proof of the claim: Let $\mathcal{A}_j$ and $\mathcal{A}_K$ be the corresponding subalgebras of $C(S)$. We will show that each of them equals to

$$\mathcal{A} := \langle \bigcup_{i \in I} m(V_i, V_i^*) \rangle.$$

Each $Y_i := \alpha_i(P)$ is an S-factor of P. Consider its enveloping semigroup $E(S, Y_i)$ and the compactification $j_i : S \to E(S, Y_i)$. Since the family of S-maps $\{\alpha_i : P \to Y_i\}_{i \in I}$ separates points of P the induced system of homomorphisms $r_{\alpha_i} : E(S, P) \to E(S, Y_i)$ separates points of $P = E(S, P)$. So, $\langle \cup_{i \in I} \mathcal{A}(E(Y_i), e) \rangle = \mathcal{A}_j$. The representations (h_i, α_i) are strongly E-compatible. By Lemma 4.6 we get $m(V_i, V_i^*) \subset \mathcal{A}(E(Y_i), e)$. By Lemma 4.2, $\mathcal{A}(E(Y_i), e) = \langle m(V_i, Y_i) \rangle$. So, $\langle m(V_i, V_i^*) \rangle = \mathcal{A}(E(Y_i), e) \ \forall i \in I$. This implies that $\langle \cup_{i \in I} m(V_i, V_i^*) \rangle = \langle \cup_{i \in I} \mathcal{A}(E(Y_i), e) \rangle$. Therefore, $\mathcal{A} = \mathcal{A}_j$.

Now we show that $\mathcal{A}_K = \mathcal{A}$. First observe that the set $L := \cup_{i \in I} V_i$ separates points of $V^* = (\Sigma_{i \in I} V_i^*)_{l_2}$ (and hence of B_{V^*}). By Lemma 2.4 .1 the standard operator compactification $j_K : S \to K$ is equivalent to the Ellis compactification $S \to E = E(S, B_{V^*})$. Apply Lemma 1.26.1 to the S-system $X = B_{V^*}$ and L. Then $\mathcal{A}_K = \langle m(L, B_{V^*}) \rangle = \langle m(L, V^*) \rangle$. For every $v \in V_i \subset L$, $\phi \in V^*, s \in S$ we have $\phi(h(s)(v)) = \phi_i(h_i(s)(v))$. So, $m(v, \phi) = m(v, \phi_i)$. Therefore, $m(\cup_{i \in I} V_i, V^*) = \cup_{i \in I} m(V_i, V^*) = \cup_{i \in I} m(V_i, V_i^*)$. It follows that $\mathcal{A}_K = \langle m(L, V^*) \rangle = \langle \cup_{i \in I} m(V_i, V_i^*) \rangle = \mathcal{A}$, as desired. So the claim is proved.

If P is a HNS-semigroup (or a semitopological semigroup) then one may modify our proof accordingly to ensure that V is an Asplund (or a reflexive) Banach space using Theorem 7.1.2 (respectively, 6.4) and Lemma 1.14.2. $\square$

Theorem 8.7. (A generalized Ellis' theorem) *Every tame compact right topological <u>group</u> P is a topological group.*

Proof. By Theorem 8.5 there exists a $\Lambda(P)$-admissible embedding of P into $\mathcal{E}(V)$ for some Rosenthal Banach space V. Since P is a group it is easy to see that its topological center $G := \Lambda(P)$ is a subgroup of P. Now apply Theorem 5.5 to the compactification $\nu : G \hookrightarrow P$ (defined by the natural inclusion) and conclude that P is a topological group. $\qquad\square$

Since every compact semitopological semigroup is tame, Ellis' classical theorem (Fact 6.6) now follows as a special case of Theorem 8.7. (Note that we are not using Ellis' theorem as an intermediate step in the proof of Theorem 8.7.)

Combining Corollary 6.20 and Theorem 8.7 we also have:

Corollary 8.8. *Let P be a compact admissible right topological group. Assume that P, as a topological space, is Fréchet. Then P is a topological group.*

In particular this holds in each of the following cases:
   (1) *(Moors & Namioka [53])* P is first countable.
   (2) *(Namioka [54], Ruppert [64])* P is metrizable.

Corollary 8.9. (Glasner [22] for metrizable X) *A distal minimal (not necessarily, metric) compact G-system is tame if and only if it is equicontinuous.*

Proof. We give the proof for the (non-trivial) "only if" part. When X is distal, E is a group by a well known theorem of Ellis. Also $E := E(X)$ is a tame semigroup by Lemma 6.16.2. By Theorem 8.7 we get that E is a topological group. Finally, X is equicontinuous because X can be identified with the compact coset E-space E/H, where $H = St(x_0)$ is the stabilizer of some point $x_0 \in X$. $\quad\square$

Corollary 8.10. $\mathrm{D}(G) \cap \mathrm{Tame}(G) = \mathrm{AP}(G)$ *for every topological group G.*

Proof. Let $f \in \mathrm{D}(G) \cap \mathrm{Tame}(G)$. Then the cyclic G-space X_f has the following properties: a) distal, b) minimal, c) tame. Indeed, for every distal function on a topological group the cyclic system (G, X_f) is minimal (see [8, p.196]). Now Corollary 8.9 concludes that X_f is equicontinuous. Hence, $f \in AP(G)$. This proves $\mathrm{D}(G) \cap \mathrm{Tame}(G) \subset \mathrm{AP}(G)$. The reverse inclusion is trivial. $\qquad\square$

Remark 8.11. (Non-tame functions)
   (1) Corollary 8.10 implies that $(\mathrm{D}(G) \setminus \mathrm{AP}(G)) \subset (\mathrm{RUC}(G) \setminus \mathrm{Tame}(G))$. Hence any distal function on G which not almost periodic is not tame. As a concrete example for $G = \mathbb{Z}$, take $f(n) = \cos(2\pi n^2 \alpha)$ with α any irrational real number.
   (2) Any function $f \in l_\infty(\mathbb{Z})$ such that the system X_f either has positive entropy, or is minimal and weakly mixing, is non-tame.

8.3. **Haydon's functions.** Recall (Section 3.2) that for every $f \in \mathrm{WRUC}(X)$ on an S-system X we have the cyclic affine S-compactification $\pi_f : X \to Q_f$, where Q_f is the pointwise closure of $co(X_f)$ in $C(S)$ and $X_f := cl_p(\{m(f, \delta_f(x)\}_{x \in X})$ is the cyclic S-system generated by f.

Definition 8.12. We say that $f \in \mathrm{WRUC}(X)$ has the *Haydon's Property* (or is a *Haydon function*) if the pointwise and norm closures of $co(X_f)$ in $C(S)$ (equivalently, in $l_\infty(S)$) coincide. That is, if

$$\overline{co}^{norm}(X_f) = \overline{co}^p(X_f).$$

Proposition 8.13. *Every tame function $f \in \mathrm{Tame}(X)$ has Haydon's property.*

Proof. By Theorem 7.4 there exist a weakly continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v \in V$ such that $f(x) = \langle v, \alpha(x) \rangle \quad \forall\, x \in X$.

Consider the linear bounded S-operator (between left S-actions)

$$T : V^* \to C(S), \ \mu \mapsto m(v, \mu).$$

By Lemma 3.13, $X_f := cl_p(m(f, \delta_f(X)))$. By the choice of $v \in V$ we have $m(f, \delta_f(X)) = m(v, \alpha(X))$. So, $T(\alpha(X)) = \delta_f(X)$. Then $T(Y) = X_f$, where $Y := \overline{\alpha(X)}^{w^*}$. Since T is weak*-pointwise continuous, the compactness argument imply that $T(\overline{co}^{w^*}(Y)) = \overline{co}^p(X_f)$. By Haydon's theorem (Fact 1.12.4), we have $\overline{co}^{norm}(Y) = \overline{co}^{w^*}(Y)$. By the linearity and norm continuity of T we get $T(\overline{co}^{norm}(Y)) \subset \overline{co}^{norm}(T(Y))$. Clearly, $\overline{co}^{norm}(T(Y)) \subset \overline{co}^p(T(Y))$. Summing up (and taking into account that $T(Y) = X_f$) we obtain $\overline{co}^{norm}(X_f) = \overline{co}^p(X_f)$. $\qquad\square$

Example 8.14. Let $\omega \in \Omega = \{0,1\}^{\mathbb{Z}}$ be a transitive point under the shift $\sigma : \Omega \to \Omega$. We consider ω as an element of $l_\infty(\mathbb{Z})$. Then by assumption the cyclic flow $X_\omega = \Omega$, and it can be easily checked that $\overline{co}^{norm}(X_\omega) = \overline{co}^p(X_\omega)$. Thus ω is a Haydon function which is clearly not tame. Thus the converse of Proposition 8.13 is not true. However we do have the following proposition.

Proposition 8.15. *For a Haydon function $f : X \to \mathbb{R}$, the cyclic affine compactification*

$$\alpha : X \to Q_f = \overline{co}^{norm}(X_f) = \overline{co}^p(X_f)$$

is E-compatible.

Proof. By Lemma 3.11, Q_f is a subset of $C(S)$. Therefore, the evaluation map $w : S \times Q_f \to \mathbb{R}$, where $w(s, \phi) = \phi(s)$, is separately continuous. Since $f \in \mathrm{WRUC}(X)$, $\alpha : X \to Q_f$ is an affine S-compactification. In particular, the action $S \times Q_f \to Q_f$ is separately continuous. So, the function

$$m_w(t, y) : S \to \mathbb{R}, \ s \mapsto \tilde{t}(sy) = y(ts)$$

is continuous for every $y \in Q_f$ and $t \in S$. Clearly, S separates points of Q_f. By Lemma 1.26.1, $\langle m_w(S, Q_f) \rangle$ and $\langle m_w(S, X_f) \rangle$ are the algebras of the Ellis compactifications $j_Q : S \to E(Q_f)$ and $j_{X_f} : S \to E(X_f)$, respectively. Since all s-translations on Q_f are affine maps we have $m_w(t, \sum_{i=1}^n c_i q_i) = \sum_{i=1}^n c_i m_w(t, q_i)$ for every $\sum_{i=1}^n c_i = 1$, $c_i > 0$. Also, $|m_w(t, y)(s) - m_w(t, y_0)(s)| \leq ||y - y_0||_\infty$. Since $Q_f = \overline{co}^{norm}(X_f)$, it follows that

$$m_w(S, Q_f) \subset \overline{sp}^{norm}(m_w(S, X_f)) \subset \langle m_w(S, X_f) \rangle.$$

Hence, the Ellis compactifications j_{Q_f} and j_{X_f} are equivalent. $\qquad\qquad\square$

Example 8.16. The distal function $f(n) = \cos(2\pi n^2 \alpha)$ in $l_\infty(\mathbb{Z})$ is not a Haydon function. This follows from Proposition 8.15 and Proposition 5.4.

8.4. **Banach representations of affine S-systems.** As we have already mentioned in Remark 3.2.4, all the affine S-compactifications $\alpha : X \to Q$ of X come, up to equivalence, from representations of dynamical S-systems X on *Banach spaces*. In particular, it follows that Q is *affinely S-isomorphic* to an affine S-subsystem of the weak*-compact unit ball B^* of V^* for some Banach space V. This suggests the following question.

Question 8.17. *Which metric affine S-compactifications $X \to Q$ can be obtained via representations of (S, X) on good Banach spaces V, (say, Rosenthal, Asplund or reflexive) where Q is a w^*-star compact affine S-subset of V^* (as in Section 4.1).*

First note that there is no obstruction in the purely topological case (i.e. for trivial actions). Indeed, by Keller's theorem [11, p. 98] any metric compact convex affine set Q in a locally convex linear space is affinely homeomorphic to a compact convex subset K in the Hilbert space l_2.

Theorem 8.18. (A representation theorem for S-affine compactifications) *Let X be a tame (HNS, WAP) compact metric S-system. Then every S-affine compactification $\gamma : X \to Q$ comes from a weakly continuous representation of (S, X) on a separable Rosenthal (respectively: Asplund, reflexive) Banach space V, where $Q \subset V^*$ is a weak* compact affine subset. If $S \times X \to X$ is continuous we can assume that h is strongly continuous. If $S = G$ is a group then $h(G) \subset \mathrm{Iso}\,(V)^{op} \subset \Theta(V)^{op}$.*

Proof. Let (γ, Q) be an S-affine compactification of a tame system X. As usual let $\mathrm{A}(Q)|_X \subset C(X)$ be the corresponding affine compactification space. $\mathrm{A}(Q)|_X$ is a closed linear unital subspace of $C(X)$. Moreover, it is separable because X is compact metrizable. Choose a countable subset $\{f_n\}_{n \in \mathbb{N}} \subset \mathrm{A}(Q)|_X$ such that $||f_n|| \leq \frac{1}{2^n}$ and $sp(\{f_n\}_{n \in \mathbb{N}})$ is norm dense in $\mathrm{A}(Q)|_X$. We can suppose that $f_1 = \mathbf{1}$.

Since (S, X) is tame, every $f_n \in \mathrm{Tame}(X) = C(X)$. So $f_n S$ is a Rosenthal family for X (Proposition 6.17) for any $n \in \mathbb{N}$. Hence, $f_n S$ is an eventually fragmented family of maps $X \to \mathbb{R}$ by Fact 1.11. Then $F := \cup_{n \in \mathbb{N}}(f_n S)$ is again an eventually fragmented family, as can be shown by diagonal arguments, and the condition $||f_n|| \leq \frac{1}{2^n}$. Hence, F is a Rosenthal family for X by Fact 1.11.

Since F is also S-invariant we can apply Theorem 7.2. We obtain: a Rosenthal space V, an injective continuous operator $j : V \to C(X)$ and a weakly continuous representation (h, α) of (S, X) on the Rosenthal Banach space V.

As we have noticed in the proof of Theorem 7.2, one of the properties of this construction is that $F \subset V$. Hence, $sp(F) \subset V$. Consider the associated S-affine compactification $\gamma_0 : X \to Q_0 \subset V^*$. Here $Q_0 = \overline{co}^{w^*}(\alpha(X))$. We claim that (γ_0, Q_0) is equivalent to (γ, Q). It suffices to show that $A(Q_0)|_X = A(Q)|_X$.

Consider the restriction operators:

$$r_X : V \to A(Q)|_X \subset C(X), \quad r_X(v)(x) := \langle v, \alpha(x)\rangle.$$

$$r_{Q_0} : V \to A(Q_0) \subset C(Q_0), \quad r_Q(v)(y) := \langle v, y\rangle.$$

$$r_0 : C(Q_0) \to C(X), \quad r_0(v)(x) := \langle v, \alpha(x)\rangle = \langle v, \gamma_0(x)\rangle.$$

By the choice of F, clearly, $r_X(sp(F))$ and hence also $r_X(V)$ are norm dense in the Banach space $A(Q)|_X$. Now it suffices to show that $r_X(V)$ is dense also in $A(Q_0)|_X$. First, by Lemma 3.5, $r_{Q_0}(V) + \mathbb{R} \cdot \mathbf{1}$ is dense in $A(Q_0)$. Since $\mathbf{1} = r_{Q_0}(f_1) \in r_{Q_0}(V_0)$ and $r_{Q_0}(V)$ is a linear subspace we conclude that $r_{Q_0}(V) + \mathbb{R} \cdot \mathbf{1} = r_{Q_0}(V)$. Therefore, $r_{Q_0}(V)$ is norm dense in the Banach space $A(Q_0)$. Then $r_0(r_{Q_0}(V))$ is dense in $r_0(A(Q_0)) = A(Q_0)|_X$. Finally, it is easy to check that $r_0(r_{Q_0}(V)) = r_X(V)$. So we can conclude that indeed $r_X(V)$ is dense also in $A(Q_0)|_X$, as desired. This proves the Tame case.

For the Asplund (respectively, reflexive) case we use the corresponding version of Theorem 7.2 as explained in the proof of Theorem 7.1.2 (respectively, [49, Theorem 4.5]). $\square$

Theorem 8.18 can be extended to general (not necessarily metrizable) S-systems X under the assumption that the space $A(Q)|_X$ of the affine compactification $\gamma : X \to Q$ is S-separable. The latter condition means that there exists a countable subset $C \subset A(Q)|_X$ such that $sp(CS)$ is dense in $A(Q)|_X$. In this general case the corresponding Rosenthal space V is not necessarily separable.

Since the space V_f of any cyclic affine S-compactification $\pi_f : S \to Q_f$ is always S-separable we conclude that π_f can be affinely S-represented on a Rosenthal space for every $f \in \text{Tame}(S)$.

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