

THE GROUP $\text{Aut}(\mu)$ IS ROELCKE PRECOMPACT

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ABSTRACT. Following a similar result of Uspenskij on the unitary group of a separable Hilbert space we show that with respect to the lower (or Roelcke) uniform structure the Polish group $G = \text{Aut}(\mu)$, of automorphisms of an atomless standard Borel probability space (X, μ) , is precompact. We identify the corresponding compactification as the space of Markov operators on $L_2(\mu)$ and deduce that the algebra of right and left uniformly continuous functions, the algebra of weakly almost periodic functions, and the algebra of Hilbert functions on G (i.e. functions on G arising from unitary representations) all coincide. Again following Uspenskij we also conclude that G is totally minimal.

Let (X, μ) be an atomless standard Borel probability space.¹ We denote by $\text{Aut}(\mu)$ the Polish group of measure preserving automorphisms of (X, μ) equipped with the weak topology. If for $T \in G$ we let $U_T : L_2(\mu) \rightarrow L_2(\mu)$ be the corresponding unitary operator (defined by $U_T f(x) = f(T^{-1}x)$), then the map $T \mapsto U_T$ (the *Koopman map*) is a topological isomorphic embedding of the topological group G into the Polish topological group $\mathcal{U}(H)$ of unitary operators on the Hilbert space $H = L_2(\mu)$ equipped with the strong operator topology. The image of G in $\mathcal{U}(H)$ under the Koopman map is characterized as the collection of unitary operators $U \in \mathcal{U}(H)$ for which $U(\mathbf{1}) = \mathbf{1}$ and $Uf \geq 0$ whenever $f \geq 0$; see e.g. [5, Theorem A.11].

It is well known (and not hard to see) that the strong and weak operator topologies coincide on $\mathcal{U}(H)$ and that with respect to the weak operator topology, the group $\mathcal{U}(H)$ is dense in the unit ball Θ of the space $\mathcal{B}(H)$ of bounded linear operators on H . Now Θ is a compact space and as such it admits a unique uniform structure. The trace of the latter on $\mathcal{U}(H)$ defines a uniform structure on $\mathcal{U}(H)$. We denote by \mathcal{J} the collection of Markov operators in Θ , where $K \in \Theta$ is *Markov* if $K(\mathbf{1}) = K^*(\mathbf{1}) = \mathbf{1}$ and $Kf \geq 0$ whenever $f \geq 0$. It is easy to see that \mathcal{J} is a closed subset of Θ . Clearly the image of G in $\mathcal{U}(H)$ is contained in \mathcal{J} and it is well known that this image is actually dense in \mathcal{J} (see e.g. [6] or [7]). Thus, via the embedding of G into \mathcal{J} we obtain also a uniform structure on G . We will denote this uniform space by (G, \mathcal{J}) .

On every topological group G there are two naturally defined uniform structures $\mathcal{L}(G)$ and $\mathcal{R}(G)$. The *lower* or the *Roelcke* uniform structure on G is defined as $\mathcal{U} = \mathcal{L} \wedge \mathcal{R}$, the greatest lower bound of the left and right uniform structures on G . If \mathcal{N} is a base for the topology of G at the neutral element e , then with

$$U_L = \{(x, y) : x^{-1}y \in U\} \quad U_R = \{(x, y) : xy^{-1} \in U\},$$

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¹This assumption may not be an essential one. It seems that the proof of Theorem 1.1 below may hold for a general probability space, but I have not checked the details.

the collections $\{U_L : U \in \mathcal{N}\}$ and $\{U_R : U \in \mathcal{N}\}$ constitute bases for $\mathcal{L}(G)$ and $\mathcal{R}(G)$ respectively. A base for the Roelcke uniform structure is obtained by the collection $\{V_U : U \in \mathcal{N}\}$ where

$$V_U = \{(x, y) : \exists z, w \in U, y = zxw\}.$$

The reader should be warned that, whereas the left and right uniform structures on a subgroup are induced by the corresponding uniform structures of an ambient group, the Roelcke uniform structure does not have a similar property. We refer to the monograph by Roelcke and Dierolf [10] for information about uniform structures on topological groups.

In [12] Uspenskij shows that the uniform structure induced from Θ on $\mathcal{U}(H)$ coincides with the Roelcke structure of this group. In this note we show that the same is true for $G = \text{Aut}(\mu)$ and then, as in [12], deduce that G is totally minimal (see the definition below).

The subject of Roelcke precompact (RPC) groups was thoroughly studied by Uspenskij. In addition to the paper [12] the interested reader can find more information about RPC groups in [13], [14] and [15]. In [13] the author shows that the group $\text{Homeo}(C)$ of self-homeomorphisms of the Cantor set C with the compact-open topology is RPC. See also [8, Sections 12 and 13] where a new proof is given to the fact that the Polish group $S_\infty(\mathbb{N})$ of permutations of the natural numbers is RPC and where an alternative proof for the RPC property of $\text{Homeo}(C)$ is indicated.

1. $\text{Aut}(\mu)$ IS ROELCKE PRECOMPACT

1.1. Theorem. *The uniform structure induced from \mathcal{J} on G coincides with the Roelcke uniform structure $\mathcal{L} \wedge \mathcal{R}$. Thus the Roelcke uniform structure on G is precompact and the natural embedding $G \rightarrow \mathcal{J}$ is a realization of the Roelcke compactification of G .*

Proof. Given $\epsilon > 0$ and a finite measurable partition $\alpha = \{A_1, \dots, A_n\}$ of X we set

$$U_{\alpha, \epsilon} = \{T \in G : \mu(A_i \Delta T^{-1}A_i) < \epsilon, \forall 1 \leq i \leq n\},$$

$$W_{\alpha, \epsilon} = \{(S, T) \in G \times G : |\mu(A_i \cap S^{-1}A_j) - \mu(A_i \cap T^{-1}A_j)| < \epsilon, \forall 1 \leq i, j \leq n\},$$

and

$$\tilde{W}_{\alpha, \epsilon} = \{(S, T) \in G \times G : \exists P, Q \in U_{\alpha, \epsilon}, T = PSQ\}.$$

Note that sets of the form $W_{\alpha, \epsilon}$ constitute a base for the uniform structure on G induced from \mathcal{J} , while the $\tilde{W}_{\alpha, \epsilon}$ form a base for the Roelcke uniform structure on G .

The fact that the identity map $(G, \mathcal{L} \wedge \mathcal{R}) \rightarrow (G, \mathcal{J})$ is uniformly continuous actually follows from Uspenskij's result that the map $(\mathcal{U}(H), \mathcal{L} \wedge \mathcal{R}) \rightarrow (\mathcal{U}(H), \Theta)$ is uniformly continuous. However a direct proof is easy:

If $T = PSQ$ with $P, Q \in U_{\alpha, \epsilon}$ then

$$\begin{aligned} |\mu(A_i \cap T^{-1}A_j) - \mu(A_i \cap S^{-1}A_j)| &= |\mu(A_i \cap Q^{-1}S^{-1}P^{-1}A_j) - \mu(A_i \cap S^{-1}A_j)| \\ &\leq |\mu(A_i \cap Q^{-1}S^{-1}P^{-1}A_j) - \mu(Q^{-1}A_i \cap Q^{-1}S^{-1}P^{-1}A_j)| \\ &\quad + |\mu(A_i \cap S^{-1}P^{-1}A_j) - \mu(A_i \cap S^{-1}A_j)| < 2\epsilon. \end{aligned}$$

Thus $\tilde{W}_{\alpha, \epsilon} \subset W_{\alpha, 2\epsilon}$. This means that the identity map $(G, \mathcal{L} \wedge \mathcal{R}) \rightarrow (G, \mathcal{J})$ is uniformly continuous.

For the other direction we start with a given $\tilde{W}_{\alpha,\epsilon}$. Suppose $(S, T) \in W_{\alpha,\epsilon/n^2}$. Set

$$A_{ij} = A_i \cap T^{-1}A_j, \quad A'_{ij} = A_i \cap S^{-1}A_j.$$

We have $|\mu(A_{ij}) - \mu(A'_{ij})| < \epsilon/n^2$ for every i and j . Define a measure preserving $R \in G$ as follows. For each pair i, j let $B_{ij} \subset A_{ij}$ with $\sum_{i,j} \mu(A_{ij} \setminus B_{ij}) < \epsilon$ and $\mu(B_{ij}) \leq \mu(A'_{ij})$. Next choose a measure preserving isomorphism ϕ_{ij} from B_{ij} onto a subset $B'_{ij} \subset A'_{ij}$ and let $\phi : \cup B_{ij} \rightarrow X$ be the map whose restriction to B_{ij} is ϕ_{ij} . Finally, extend ϕ to an element $R \in G$ by defining R on $X \setminus \cup B_{ij}$ to be any measure preserving isomorphism $X \setminus \cup B_{ij} \rightarrow X \setminus \cup B'_{ij}$.

It is easy to check that R is in $U_{\alpha,\epsilon}$, and for $TR^{-1}S^{-1}$ we have, up to sets of small measure,

$$\begin{aligned} TR^{-1}S^{-1}(A_i) &= TR^{-1}S^{-1}\left(\bigcup_{j=1}^n A_i \cap SA_j\right) \\ &= TR^{-1}\left(\bigcup_{j=1}^n S^{-1}A_i \cap A_j\right) \\ &= T\left(\bigcup_{j=1}^n T^{-1}A_i \cap A_j\right) = A_i. \end{aligned}$$

Thus also $TR^{-1}S^{-1}$ is in $U_{\alpha,\epsilon}$, whence the equation $T = (TR^{-1}S^{-1})SR$ implies $(S, T) \in \tilde{W}_{\alpha,\epsilon}$. We have shown that $W_{\alpha,\epsilon/n^2} \subset \tilde{W}_{\alpha,\epsilon}$ and it follows that the identity map $(G, \mathcal{J}) \rightarrow (G, \mathcal{L} \wedge \mathcal{R})$ is also uniformly continuous. \square

1.2. Corollary. *The Roelcke and the WAP compactifications of G coincide. Moreover, every bounded right and left uniformly continuous function — and hence also every WAP function — on G can be uniformly approximated by linear combinations of positive definite functions. Or, in other words, every right and left uniformly continuous function arises from a Hilbert representation.*

Proof. Let $\mathcal{R}o(G)$, denote the algebra of bounded right and left uniformly continuous complex-valued functions on G . We write $WAP(G)$ for the algebra of weakly-almost-periodic complex-valued functions on G and finally we let $\mathcal{H}(G)$ be the algebra of Hilbert complex-valued functions on G (i.e. the uniform closure of the algebra of all linear combinations of positive definite functions; the latter is also called the *Fourier-Stieltjes algebra*). We then have

$$\mathcal{R}o(G) \supseteq WAP(G) \supseteq \mathcal{H}(G).$$

By Theorem 1.1 these three algebras coincide for the topological group $G = \text{Aut}(\mu)$. In fact, the functions of the form $F_f(T) = \langle Tf, f \rangle$ with $f \in L_2(\mu)$ and $T \in \mathcal{J}$, when restricted to G , are clearly positive definite. Since these functions generate the algebra $C(\mathcal{J})$, which by Theorem 1.1 is canonically isomorphic to $\mathcal{R}o(G)$, this shows that indeed $\mathcal{R}o(G) = \mathcal{H}(G)$. \square

1.3. Remark. By [12] the same is true for the group $\mathcal{U}(H)$. For more details see [9]. In the literature a topological group G for which

$$WAP(G) = \mathcal{H}(G)$$

is called *Eberlein*. Thus both $\mathcal{U}(H)$ and $\text{Aut}(\mu)$ are Eberlein groups and moreover for these groups

$$\mathcal{R}o(G) = WAP(G) = \mathcal{H}(G).$$

This fact for $\mathcal{U}(H)$ was first shown in [9].

2. $\text{Aut}(\mu)$ IS TOTALLY MINIMAL

A topological group is called *minimal* if it does not admit a strictly coarser Hausdorff group topology. It is *totally minimal* if all its Hausdorff quotient groups are minimal. Stoyanov proved that the unitary group $\mathcal{U}(H)$ is totally minimal [11], [1, Theorem 7.6.18], and Uspenskij provides in [12] an alternative proof based on his identification of Θ as the Roelcke compactification of this group. Using Theorem 1.1 we have the following.

2.1. Theorem. *The topological group $G = \text{Aut}(\mu)$ is totally minimal.*

For completeness we provide a proof of this theorem. It follows Uspenskij's proof with some simplifications. We will use though the next theorem of Uspenskij [12, Theorem 3.2].

2.2. Theorem. *Let S be a compact Hausdorff semitopological semigroup which satisfies the following assumption:*

For every pair of idempotents $p, q \in S$ the conditions $pq = p$ and $qp = p$ are equivalent. (We write $p \leq q$ when $p, q \in S$ satisfy these conditions.)

Then every nonempty closed subsemigroup K of S contains a least idempotent; i.e. an idempotent p such that $p \leq q$ for every idempotent q in K .

It is not hard to check that Θ (and therefore also \mathcal{J}) satisfies the assumption of this theorem.

A proof of Theorem 2.1. Let τ denote the topology of a Hausdorff topological group G and suppose that τ' is a coarser Hausdorff group topology. Then, the identity map $(G, \tau) \rightarrow (G, \tau')$ is continuous and $\tau = \tau'$ iff this map is open. A moment's reflection now shows that in order to prove that G is totally minimal it suffices to check that every surjective homomorphism of Hausdorff topological groups $f : G \rightarrow G'$ is an open map.

So let $f : G \rightarrow G'$ be such a homomorphism and observe that then G' is Roelcke precompact and satisfies $\mathcal{R}o(G') = WAP(G')$ as well. We denote by \mathcal{J}' the corresponding (Roelcke and WAP) compactification of G' and observe that the dynamical systems (\mathcal{J}, G) and (\mathcal{J}', G') are their own enveloping semigroups (see e.g. [5]). Now, (\mathcal{J}, G) and (\mathcal{J}', G') being WAP systems, the latter are compact semitopological semigroups. Moreover, the map $f : G \rightarrow G'$ naturally extends to a continuous homomorphism $F : \mathcal{J} \rightarrow \mathcal{J}'$. (This fact frees us from the need to use Proposition 2.1 from [12].)

Let $K = F^{-1}(e')$, where e' is the neutral element of G' . Clearly then K is a closed subsemigroup of \mathcal{J} . Moreover, we have $gK = Kg = F^{-1}(g')$ whenever $g' = f(g) \in G'$. In fact, clearly $gK \subset F^{-1}(g')$ and if $F(q) = g'$ for some $q \in \mathcal{J}$ then $F(g^{-1}q) = f(g^{-1})F(q) = g'^{-1}g' = e'$, hence $g^{-1}q \in K$ and $q \in gK$. Thus $gK = F^{-1}(g')$ and symmetrically also $Kg = F^{-1}(g')$.

Next observe that $gKg^{-1} = K$ for every $g \in G$. Thus if p is the least idempotent in K , provided by Theorem 2.2, then $gpg^{-1} = p$ for all $g \in G$ and we conclude (an

easy exercise) that either $p = I$, the identity element of G , or p is the projection on the space of constant functions (i.e. the operator of integration on $L_2(X, \mu)$). In the second case we have $e' = F(p) = F(gp) = f(g)F(p) = f(g)e' = f(g)$ for every $g \in G$ and we conclude that $G' = \{e'\}$.

Suppose then that $p = I$. In that case we have $qK \subset K$ for every $q \in K$ and qK being a closed subsemigroup, we conclude that $I \in qK$. Similarly we get $I \in Kq$. Whence q is an invertible element of \mathcal{J} , i.e. an element of G . Thus when $p = I$ we have $K \subset G$.

Let now g be an arbitrary element of G and let $g' = f(g) \in G'$. Suppose $G' \ni f(g_i) = g'_i \rightarrow g'$ is a convergent sequence in G' . With no loss in generality we assume that g_i converges to an element $q \in \mathcal{J}$ and it then follows that $F(q) = g'$. As $F^{-1}(g') = gK$ we conclude that $q \in gK \subset G$. Now $f(g_i q^{-1}g) = f(g_i) = g'_i$ and $g_i q^{-1}g \rightarrow g$. This shows that f is an open map and the proof is complete. \square

2.3. Remark. The group $G = \text{Aut}(\mu)$ is in fact algebraically simple (Fathi [2]). Thus minimality of G implies total minimality. Note that with only slight changes the same proof applies to $\mathcal{U}(H)$ and thus we have here a simplified version of Uspenskij's proof.

2.4. Remark. It is perhaps worthwhile to mention here two other outstanding properties of the Polish group $G = \text{Aut}(\mu)$. The first, due to Giordano and Pestov [3] or [4], is that this group has the *fixed point on compacta property*, i.e. whenever G acts on a compact space it admits a fixed point (this property is also called *extreme amenability*). The second is the fact that the natural unitary representation of G on $L_2^0(X, \mu)$ (the subspace of functions with zero mean) is irreducible (see [5, Theorem 5.14]).

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