THE UNIVERSAL MINIMAL SPACE FOR GROUPS OF HOMEOMORPHISMS OF H-HOMOGENEOUS SPACES

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To Anatoliy Stepin with great respect.

Abstract. Let X be a h-homogeneous zero-dimensional compact Hausdorff space, i.e. X is a Stone dual of a homogeneous Boolean algebra. It is shown that the universal minimal space $M(G)$ of the topological group $G = \mathrm{Homeo}(X)$, is the space of maximal chains on X introduced in [Usp00]. If X is metrizable then clearly X is homeomorphic to the Cantor set and the result was already known (see [GW03]). However many new examples arise for non-metrizable spaces. These include, among others, the generalized Cantor sets $X = \{0,1\}^\kappa$ for non-countable cardinals κ, and the *corona* or *remainder* of ω, $X = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers.

1. Introduction

The existence and uniqueness of a universal minimal G dynamical system, corresponding to a topological group G, is due to Ellis (see [Ell69], for a new short proof see [GL11]). He also showed that for a discrete infinite G this space is never metrizable, and the latter statement was generalized to the locally compact non-compact case by Kechris, Pestov and Todorcevic in the appendix to their paper [KPT05]. For Polish groups this is no longer the case and we have such groups with $M(G)$ being trivial (groups with the fixed point property or extremely amenable groups) and groups with metrizable, easy to compute $M(G)$, like $M(G) = S^1$ for the group $G = \mathrm{Homeo}_+(S^1)$

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([Pes98]) and $M(G) = LO(\omega)$, the space of linear orders on a countable set, for $S_\infty(\omega)$ ([GW02]).

Following Pestov's work Uspenskij has shown in [Usp00] that the action of a topological group G on its universal minimal system $M(G)$ (with $\operatorname{card} M(G) \geq 3$) is never 3-transitive so that, e.g., for manifolds X of dimension > 1 as well as for $X = Q$, the Hilbert cube, and $X = K$, the Cantor set, $M(G)$ can not coincide with X. Uspenskij proved his theorem by introducing the space of maximal chains $\Phi(X)$ associated to a compact space X. In [GW03] the authors then showed that for X the Cantor set and $G = \text{Homeo}(X)$, in fact, $M(G) = \Phi$. It turns out that this group G is a closed subgroup of $S_\infty(\omega)$ and in [KPT05] Kechris, Pestov and Todorcevic unified and extended these earlier results and carried out a systematic study of the spaces $M(G)$ for many interesting closed subgroups of S_∞.

In the present work we go back to [GW03] and generalize it in another direction. We consider the class of h-homogeneous spaces X and show that for every space in this class the universal minimal space $M(G)$ of the topological group $G = \text{Homeo}(X)$ is again Uspenskij's space of maximal chains on X. If X is metrizable then clearly X is homeomorphic to the Cantor set and the result of [GW03] is retrieved (although even in this case our proof is new, as we make no use of a fixed point theorem). However, many new examples arise when one considers non-metrizable spaces. These include, among others, the generalized Cantor sets $X = \{0,1\}^\kappa$ for non-countable cardinals κ, and the widely studied **corona** or **remainder of** ω, $X = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers. As in [GW03] the main combinatorial tool we apply is the dual Ramsey theorem.

1.1. **H-homogeneous spaces and homogeneous Boolean algebras.** The following definitions are well known (see e.g. [HNV04] Section H-4):

(1) A zero-dimensional compact Hausdorff topological space X is called **h-homogeneous** if every non-empty clopen subset of X is homeomorphic to the entire space X.
(2) A Boolean algebra B is called **homogeneous** if for any nonzero element a of B the relative algebra $B|a = \{x \in B : x \leq a)$ is isomorphic to B.

Using Stone's Duality Theorem (see [BS81] IV§4) a zero-dimensional compact Hausdorff h-homogeneous space X is the Stone dual of a homogeneous Boolean Algebra, i.e. any such space is realized as the space of ultrafilters B^* over a homogeneous Boolean algebra B equipped with the topology given by the base $N_a = \{U \in B^* : a \in U\}$, $a \in B$. Here are some examples of h-homogeneous spaces (see [ŠR89]):

(1) The countable atomless Boolean algebra is homogeneous. It corresponds by Stone duality to the Cantor space $K = \{0,1\}^{\mathbb{N}}$.

(2) Every infinite free Boolean algebra is homogeneous. These Boolean algebras correspond by Stone duality to the generalized Cantor spaces, $\{0,1\}^{\kappa}$, for infinite cardinals κ

(3) Let $P(\omega)$ be the Boolean algebra of all subsets of ω (the first infinite cardinal) and let $fin \subset P(\omega)$ be the ideal comprising the finite subsets of ω. Define the equivalence relations $A \sim_{fin} B$, $A, B \in P(\omega)$, if and only if $A \triangle B$ is in fin. The quotient Boolean algebra $P(\omega)/fin$ is homogeneous. This Boolean algebra corresponds by Stone duality to the **corona** $\omega^* = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of ω.

(4) A topological space X is called a **Parovičenko space** if:

   (a) X is a zero-dimensional compact space without isolated points and with weight $\mathbf{c}$,

   (b) every two disjoint open F_σ subsets in X have disjoint closures, and

   (c) every non-empty G_δ subset of X has non-empty interior.

   Under CH Parovičenko proved that every Parovičenko space is homeomorphic to ω^* ([Par63]).

   In [DM78] van Douwen and van Mill show that under $\neg$ CH, there are two non-homeomorphic Parovičenko spaces. Their second example of a Parovičenko space is the corona $X = \beta Y \setminus Y$, where Y is the σ-compact space $\omega \times \{0,1\}^{\mathbf{c}}$. It is not hard to see that in Y the clopen sets are of the form $L = \bigcup_{a \in A} \{a\} \times C_a$ for some $A \subset \omega$, where for all $a \in A$, C_a is non-empty and clopen. If $|A| = \infty$ then $L \cong Y$ and if $|A| < \infty$ then $Cl_{\beta Y}(L) \subset Y$. These facts imply in a straightforward manner that X is h-homogeneous. In [DM78] it is pointed out that under MA X is not homeomorphic to ω^*. Thus under $\neg$ CH+MA, this example provides another weight $\mathbf{c}$ h-homogeneous space.

(5) Let κ be a cardinal. By a well-known theorem of Kripke ([Kri67]) there is a homogeneous countably generated complete Boolean algebra, the so called **collapsing algebra** $C(\kappa)$ such that if A is a Boolean algebra with a dense subset of power at most κ, then there is a complete embedding of A in $C(\kappa)$.

(6) It is not hard to check that the product of any number of h-homogeneous spaces is again h-homogeneous.

1.2. **The universal minimal space.** A compact Hausdorff G-space X is said to be **minimal** if X and $\emptyset$ are the only G-invariant closed subsets of X. By Zorn's lemma each G-space contains a minimal G-subspace. These minimal objects are in some sense the most basic ones in the category of G-spaces. For various topological groups G they have been the object of intensive study. Given a topological group G one is naturally interested in describing all of them up to isomorphism. Such a description is given (albeit in a very weak sense) by the following construction: as was mentioned in the introduction one can show there exists a minimal G-space $M(G)$ unique up to isomorphism such that if X is a minimal G-space then X is a factor of $M(G)$, i.e., there is a continuous G-equivariant mapping from $M(G)$ onto X. $M(G)$ is called the **universal minimal G-space**. Usually this minimal universal space is huge and an explicit description of it is hard to come by.

1.3. **The space of maximal chains.** Let K be a compact Hausdorff space. We denote by $\mathrm{Exp}(K)$ the space of closed subsets of K equipped with the Vietoris topology. A subset $C \subset \mathrm{Exp}(K)$ is a **chain** in $\mathrm{Exp}(K)$ if for any $E, F \in C$ either $E \subset F$ or $F \subset E$. A chain is **maximal** if it is maximal with respect to the inclusion relation. One verifies easily that a maximal chain in $\mathrm{Exp}(K)$ is a closed subset of $\mathrm{Exp}(K)$, and that $\Phi(K)$, the space of all maximal chains in $\mathrm{Exp}(K)$, is a closed subset of $\mathrm{Exp}(\mathrm{Exp}(K))$, i.e. $\Phi(K) \subset \mathrm{Exp}(\mathrm{Exp}(K))$ is a compact space. Note that a G-action on K naturally induces a G-action on $\mathrm{Exp}(K)$ and $\Phi(K)$. This is true in particular for

$K = M(G)$. As the G-space $\Phi(M(G))$ contains a minimal subsystem it follows that there exists an injective continuous G-equivariant mapping $f : M(G) \to \Phi(M(G))$. By investigating this mapping Uspenskij in [Usp00] showed that for every topological group G, the action of G on the universal minimal space $M(G)$ is not 3-transitive. As a direct consequence of this theorem only rarely the natural action of the group $G = \mathrm{Homeo}(K)$ on the compact space K coincides with the universal minimal G-action (as is the case for $X = S^1$). In [Gut08] it was shown that for $G = \mathrm{Homeo}(X)$, where X belongs to a large family of spaces that contains in particular the Hilbert cube, the action of G on the universal minimal space $M(G)$ is not 1-transitive.

It is easy to see that every $c \in \Phi(K)$ has a first element F which is necessarily of the form $F = \{x\}$. Moreover, calling $x \triangleq r(c)$ the **root** of the chain c, it is clear that the map $\pi : \Phi(K) \to K$, sending a chain to its root, is a homomorphism of dynamical systems.

1.4. **The main result.** In [GW03] it was shown that the universal minimal space of the group of homeomorphisms of the Cantor set, equipped with the compact-open topology, is the space of maximal chains over the Cantor set. Our goal is to prove the following generalization:

Theorem. *Let X be a h-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \mathrm{Homeo}(X)$ equipped with the compact-open topology, then $M(G) = \Phi(X)$, the space of maximal chains on X.*

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2. Preliminaries

2.1. **Clopen covers.** Let X be a zero-dimensional compact Hausdorff space. Denote by $\mathcal{D}$ ($\tilde{\mathcal{D}}$) the directed set (semilattice) consisting of all finite ordered (unordered) clopen partitions of X which are necessarily of the form $\alpha = (A_1, A_2 \ldots, A_m)$ ($\tilde{\alpha} = \{A_1, A_2 \ldots, A_m\}$), where $\cup_{i=1}^{n} A_i = X$ (disjoint union). The relation is given by refinement: $\alpha \preceq \beta$ ($\tilde{\alpha} \preceq \tilde{\beta}$) iff for any $B \in \beta$ ($B \in \tilde{\beta}$), there is $A \in \alpha$ ($A \in \tilde{\alpha}$) so that $B \subset A$. The join (least upper bound) of α and β, $\alpha \vee \beta = \{A \cap B : \ A \in \alpha, \ B \in \beta\}$, where the ordering of indices is given by the lexicographical order on the indices of α and β ($\tilde{\alpha} \vee \tilde{\beta} = \{A \cap B : \ A \in \tilde{\alpha}, \ B \in \tilde{\beta}\}$). It is convenient to introduce the notations $\mathcal{D}_k = \{\alpha \in \mathcal{D} : \ |\alpha| = k\}$ and $\tilde{\mathcal{D}}_k = \{\alpha \in \mathcal{D} : \ |\tilde{\alpha}| = k\}$. We denote the natural map $(A_1, A_2 \ldots, A_m) \mapsto \{A_1, A_2 \ldots, A_m\}$ by $\tilde{t} : \mathcal{D} \to \tilde{\mathcal{D}}$. There is a natural G-action on $\mathcal{D}$ ($\tilde{\mathcal{D}}$) given by $g(A_1, A_2 \ldots, A_m) = (g(A_1), g(A_2) \ldots, g(A_m))$ ($g\{A_1, A_2 \ldots, A_m\} = \{g(A_1), g(A_2) \ldots, g(A_m)\}$). Let S_k denote the group of permutations of $\{1, \ldots, k\}$. S_k acts naturally on $\mathcal{D}_k$ by $\sigma(B_1, B_2 \ldots, B_k) = (B_{\sigma(1)}, B_{\sigma(2)} \ldots, B_{\sigma(k)})$ for any $\beta = (B_1, B_2 \ldots, B_k) \in \mathcal{D}_k$ and $\sigma \in S_k$. This action commutes with the action of G, i.e. $\sigma g \beta = g \sigma \beta$ for any $\sigma \in S_k$ and $g \in G$. Notice that one can identify $\tilde{\mathcal{D}}_k = \mathcal{D}_k / S_k$.

2.2. **Partition Homogeneity.** Let us introduce a new definition:

Definition 2.1. A zero-dimensional compact Hausdorff space X is called **partition-homogeneous** if for every two finite ordered clopen partitions of the same cardinality, $\alpha, \beta \in \mathcal{D}_m$, $\alpha = (A_1, A_2 \ldots, A_m)$, $\beta = (B_1, B_2 \ldots, B_m)$ there is $h \in \mathrm{Homeo}(X)$ such that $hA_i = B_i$, $i = 1, \ldots, k$.

Proposition 2.2. *Let X be an infinite zero-dimensional compact Hausdorff space. X is h-homogeneous iff X is partition-homogeneous.*

Proof. Assume X is h-homogeneous. Let $\alpha, \beta \in \mathcal{D}_m$, $\alpha = (A_1, A_2 \ldots, A_m)$, $\beta = (B_1, B_2 \ldots, B_m)$. Select homeomorphisms $h_{A,i}, h_{B,i}$, $i = 1, \ldots, m$ with $h_{A,i} : A_i \to X$, $h_{B,i} : B_i \to X$. Define $g \in \mathrm{Homeo}(X)$ by $g(x) = h_{B,i}^{-1} \circ h_{A,i}(x)$ for $x \in A_i$. Trivially $gA_i = B_i$. Assume now X is partition-homogeneous. Let $A \neq X$ be a clopen set in X. We distinguish between two cases:

(1) A is a singleton. As X is partition-homogeneous there exists $h \in \mathrm{Aut}(X)$ with $hA = A^c$ and $hA^c = A$. We conclude X is a two point space contradicting the assumption that X is infinite.
(2) A is not a singleton. Because X is a compact Hausdorff zero-dimensional space we can find disjoint clopen sets A_1, A_2 such that $A = A_1 \cup A_2$. Let $h_1 \in G$ so that $h_1 A_1 = A_1 \cup A^c$ and $h_1 A_1^c = A_2$. Define the homeomorphism $h : A \to X$.

$$h(x) = \begin{cases} h_1(x) & x \in A_1 \\ x & x \in A_2 \end{cases}$$

□

3. Basic properties of h-homogeneous spaces

3.1. **Induced orders.** Let X be a compact Hausdorff zero-dimensional h-homogeneous space and denote $G = \mathrm{Homeo}(X)$. As X is either trivial or infinite, we will assume from now onward, w.l.og. that X is infinite. Let $\upsilon \in \Phi(X)$ and $D \subset X$ a closed set. Define

$$D_\upsilon = \bigcap_{A \in \upsilon : A \cap D \neq \emptyset} A$$

By maximality of υ, one has $D_\upsilon \in \upsilon$. By a standard compactness argument $D_\upsilon \cap D \neq \emptyset$ and trivially it is the minimal element of υ that intersects D. Similarly for $D \subset X$ a closed set with $r(\upsilon) \in D$, define:

$$D^\upsilon = \overline{\bigcup_{A \in \upsilon : A \subset D} A}$$

The maximal element of υ that is contained in D.

Definition 3.1. Let $\upsilon \in \Phi(X)$ and $\tilde{\alpha} = \{A_1, A_2 \ldots, A_m\} \in \tilde{\mathcal{D}}$. Define $<_{\upsilon|\tilde{\alpha}} = <_\upsilon$, the **induced order** on $\tilde{\alpha}$ by υ:

$$A_i <_\upsilon A_j \Leftrightarrow (A_i)_\upsilon \subseteq (A_j)_\upsilon$$

Similarly for $\upsilon \in \Phi(X)$ and $\alpha \in \mathcal{D}$, define the induced order $<_{\upsilon|\alpha} = <_{\upsilon|\tilde{t}(\alpha)}$. Denote by $t_\upsilon^* : \tilde{\mathcal{D}} \to \mathcal{D}$ the map $\{A_1, A_2 \ldots, A_m\} \mapsto (A_1, A_2 \ldots, A_m)$ where $i < j$ if and only if

$A_i <_{\upsilon|\alpha} A_j$. For $\beta \in \mathcal{D}$, define $t^*_\upsilon(\beta) = t^*_\upsilon(\tilde{t}(\beta))$. Notice that for all $\sigma \in S_k$, $\upsilon \in \Phi(X)$ and $\beta \in \mathcal{D}$,

$$t^*_\upsilon(\sigma t^*_\upsilon(\beta)) = t^*_\upsilon(\beta)$$

Lemma 3.2. $gt^*_\upsilon(\tilde{\beta}) = t^*_{g\upsilon}(g\tilde{\beta})$.

Proof. Let $\alpha = (A_1, A_2 \ldots, A_m) = t^*_\upsilon(\tilde{\beta})$. By definition $i < j$ if and only if $(A_i)_\upsilon \subseteq (A_j)_\upsilon$. Notice $(gA_i)_{g\upsilon} = \bigcap_{gA \in g\upsilon : gA \cap gA_i \neq \emptyset} gA = g \bigcap_{A \in \upsilon : A \cap A_i \neq \emptyset} A = g(A_i)_\upsilon$. Therefore $i < j$ if and only if $(gA_i)_{g\upsilon} \subseteq (gA_j)_{g\upsilon}$, and we conclude $g\alpha = t^*_{g\upsilon}(g\tilde{\beta})$. □

Proposition 3.3. *Let $\upsilon \in \Phi(X)$ and $\tilde{\alpha} = \{A_1, A_2 \ldots, A_m\} \in \tilde{\mathcal{D}}$. $<_{\upsilon|\tilde{\alpha}}$ is a linear order on $\tilde{\alpha}$. The ordering $A_{i_1} <_{\upsilon|\tilde{\alpha}} A_{i_2} <_{\upsilon|\tilde{\alpha}} \ldots <_{\upsilon|\tilde{\alpha}} A_{i_m}$ is characterized by $(A_{i_k})_\upsilon \setminus (A_{i_1} \cup \ldots \cup A_{i_{k-1}})^\upsilon = \{x_k\}$ for $k = 1, 2, \ldots m$ and suitable $x_k \in A_k$.*

Proof. Let $D \subset X$ be clopen so that $r(\upsilon) \in D$, then it is easy to see that $\upsilon_{|D^c} \triangleq \{A \setminus D |\, D^\upsilon \subsetneq A \in \upsilon\}$ is a maximal chain in D^c and in particular has a root $r(\upsilon_{|D^c}) = x_0 \in D^c$. Let i_1 be such that $r(\upsilon) \in A_{i_1}$. Inductively let i_{k+1} be such that $r(\upsilon_{|(A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k})^c}) \in A_{i_{k+1}}$. It is easy to see $A_{i_1} <_\upsilon A_{i_2} <_\upsilon \ldots <_\upsilon A_{i_m}$. This implies both that $<_\upsilon$ is a linear order and $(A_{i_k})_\upsilon \setminus (A_{i_1} \cup \ldots \cup A_{i_{k-1}})^\upsilon = \{x_k\}$ for some $x_k \in A_k$, $k = 1, 2, \ldots m$. □

3.2. **Minimality and proximality of natural actions.** The basis for the Vietoris topology for the compact Hausdorff space $\mathrm{Exp}(X)$ is given by open sets of the form:

$$\mathcal{U} = \langle A_1, \ldots, A_k \rangle = \{F \in \mathrm{Exp}(X) : \ \forall i\, F \cap A_i \neq \emptyset \text{ and } F \subset \bigcup A_i\}$$

where $A_i \subset X$ is clopen. It is easy to see that a basis of clopen neighborhood of a maximal chain $\upsilon \in \Phi(X)$ is given by

$$\mathfrak{U}_\alpha = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$$

where $\alpha = (A_1, A_2 \ldots, A_n) \in \mathcal{D}$ and

$$\mathcal{U}_j = \langle A_1, \ldots, A_j \rangle, \qquad j = 1, 2, \ldots, n,$$

The following lemma is straightforward:

Lemma 3.4. *Let $\alpha = (A_1, A_2 \dots, A_n) \in \mathcal{D}$ and $\upsilon \in \Phi(X)$. Let $<_{\upsilon|\alpha}$ be the induced order of υ on α, then $\upsilon \in \mathfrak{U}_\alpha$ if and only if $<_\upsilon = <$, where $<$ is the usual order on $\{1, 2 \dots, n\}$. In particular $\upsilon \in \mathfrak{U}_{t^*_\upsilon(\alpha)}$.*

Theorem 3.5.
(1) *The system (X, G) is minimal.*
(2) *The system (X, G) is extremely proximal; i.e. for every closed set $\emptyset \neq F \subsetneqq X$ there exists a net $\{g_i\}_{i \in I}$ in G such that we have $\lim_{i \in I} g_i F = \{x_0\}$ for some point $x_0 \in X$ (see* [Gla74]*).*
(3) *The minimal system (X, G) is not isomorphic to the universal minimal system $(M(G), G)$.*
(4) *$(\Phi(X), G)$ is minimal.*
(5) *$(\Phi(X), G)$ is proximal.*

Proof.

(1) Since X is h-homogeneous, then by Proposition 2.2, G acts transitively on non-trivial (i.e. not $\emptyset, X$) clopen sets. Since G acts transitively on the above mentioned basis, it follows that for every $U \in \mathcal{U}$ we have $\cup\{\alpha(U) : \alpha \in G\} = X$. This property is equivalent to the minimality of the system (X, G).
(2) Fix some x_0 in X such that $x_0 \notin F$. For an arbitrary basic clopen neighborhood $U = A$ of x_0 which is disjoint from F choose $\alpha_U \in G$ such that $\alpha_U(A^c) = A$. Then α satisfies $\alpha_U(F) \subset U$. Clearly now $\{\alpha_U : U \text{ a neighborhood of } x_0\}$ is the required net.
(3) As the system (X, G) is certainly 3-transitive this claim follows from Uspenskij's theorem [Usp00]. For completeness we provide a direct proof. Suppose (X, G) is isomorphic to the universal minimal G system. Let $Y \subset \Phi$ be a minimal subset of Φ. Then, by the coalescence of the universal minimal system (every G-endomorphism $\phi : (M(G), G) \to (M(G), G)$ (which is necessarily onto) is an isomorphism, see [GL11] and [Usp00]), the restriction $\pi : Y \to X$, sending a chain to its root, is an isomorphism. Fix $c_0 \in Y$ and let $p_0 \in X$ be its root; i.e. $\pi(c_0) = p_0$. Let $H = \{\alpha \in G : \alpha p_0 = p_0\}$, the stability group of p_0. Since π is an isomorphism we also have $H = \{\alpha \in G : \alpha c_0 = c_0\}$. Choose

$F \in c_0$ such that $\{p_0\} \subsetneqq F \subsetneqq X$ and let $p_0 \neq a \in F$ (recall X is infinite). Choose a clopen partition of (P, A, B) of X with $B \cap F = \emptyset$, $P \cap F \neq \emptyset$ and $A \cap F \neq \emptyset$. Using the fact that X is partition homogeneous, one can find $g \in G$ so that $gP = P$, $gA = B$ and $gB = A$. One redefines g so that $g_{|P} = Id$. As $g(A \cup P) \cap A = \emptyset$, we have $F \setminus gF \neq \emptyset$. As $gA = B$ we have $gF \setminus F \neq \emptyset$. Conclude that F and gF are not comparable. On the other hand $g(p_0) = p_0$ means $g \in H$ whence also $gc_0 = c_0$. In particular $gF \in c_0$ and as c_0 is a chain one of the inclusions $F \subset gF$ or $gF \subset F$ must hold. This contradiction shows that (X, G) cannot be the universal minimal G-system.

(4) Let $\upsilon', \upsilon \in \Phi(X)$ and $\upsilon' \in \mathfrak{U}_\alpha$ for some $\alpha = (A_1, A_2 \ldots, A_n) \in \mathcal{D}$. Let $<_\upsilon$ be the induced order of υ on α. Let $\sigma \in S_n$ be such that for any $i < j$, $A_{\sigma(i)} <_\upsilon A_{\sigma(j)}$. As X is partition homogeneous we can choose $g \in G$ so that $gA_{\sigma(i)} = A_i$. Clearly $g\upsilon \in \mathfrak{U}_\alpha$.

(5) Let $\upsilon_1, \upsilon_2 \in \Phi(X)$. Fix some $\upsilon' \in \mathfrak{U}_\alpha$ for some $\alpha = (A_1, A_2 \ldots, A_n) \in \mathcal{D}$. Let $<$ be the usual order on $\{1, 2 \ldots, n\}$. Inductively we will construct $g \in G$ so that $<_{g\upsilon_1|\alpha} = <_{g\upsilon_2|\alpha} = <$. Using Lemma 3.4, this implies $g\upsilon_1 \in \mathfrak{U}_\alpha$ and $g\upsilon_2 \in \mathfrak{U}_\alpha$. As $\mathfrak{U}_\alpha$ is arbitrary, this establishes proximality. Indeed let $g_1 \in G$ so that $g_1(r(\upsilon_1)), g_1(r(\upsilon_2)) \in A_1$. Assume we have constructed $g_k \in G$. Define $g_{k+1} \in G$ so that $g_{k+1|A_1 \cup A_2 \cup \ldots \cup A_k} = g_{k|A_1 \cup A_2 \cup \ldots \cup A_k}$ and $g_{k+1}(r((g_k \upsilon_1)_{|(A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k})^c})$, $g_{k+1}(r(g_k \upsilon_2)_{|(A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k})^c}) \in A_{i_{k+1}}$. It is easy to see that $g = g_n$ has the desired properties.

□

4. Calculation of the universal minimal space

4.1. **Overview.** The goal of this section is to generalize the main theorem of [GW03]: the universal minimal space of the group of homeomorphisms of the Cantor set, equipped with the compact-open topology, is the space of maximal chains over the Cantor set. We prove the following theorem:

Theorem 4.1. *Let X be a h-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \mathrm{Homeo}(X)$ equipped with the compact-open topology, then $M(G) = \Phi(X)$.*

The proof borrows heavily from the proof in [GW03]. The new ideas (that build on ideas in [GW03]) are presented in subsections 4.2, 4.3, 4.5

4.2. **Order topology.** Recall that given a set Y and a linear order $<$ on Y there is a topology generated by the basis of open intervals $(a, b) = \{y \in Y : a < y < b\}$ where $a, b \in Y$ and equality is allowed on the left (right) if a (b) is the smallest (biggest) element of Y. This topology is called the **order topology** on $(Y, <)$. For more details see [Mun75] Section 2.3. One of the most important ingredients in the proof in [GW03] is the fact that the topology on the cantor set K is the order topology associated with the natural order $<$ on $K \subset [0, 1]$). A natural approach to generalizations of the result in the case of $X = \omega^*$ the corona, is to look for an order that will generate the topology on the corona. However, as the following proposition shows this is impossible.

Proposition 4.2. *The topology on ω^* is not an order topology.*

Proof. Assume for a contradiction that the topology on ω^* is an order topology associated with a linear order $<$. As ω^* has no isolated points we can find (with no loss of generality) an increasing bounded sequence of points $p_1 < p_2 < p_3 < \cdots < b$. By compactness this sequence admits a least upper bound $p = l.u.b\ \{p_k : k = 1, 2, \dots\}$. It is easy to check that $p = \lim_{k\to\infty} p_k$, so that the set $\{p_k : k = 1, 2, \dots\} \cup \{p\}$ is a closed subset of ω^*. However, it is well known that the remainder ω^* has no nontrivial converging sequences; e.g. one can use the fact that the closure of the set $\{p_k : k = 1, 2, \dots\}$, like the closure of any infinite discrete countable set in $\beta\omega$, is homeomorphic to $\beta\omega$ (see e.g. [Eng78, Theorem 3.6.14]). $\square$

4.3. **The spaces Ω_k and $\tilde{\Omega}_k$ and a cocycle equation.** The following subsection is a generalization of Section 3 of [GW03]. Fix $\alpha = (A_1, A_2 \dots, A_k) \in \mathcal{D}_k$ and define the clopen subgroup $H_\alpha = \{g \in G : gA_i = A_i,\ i = 1, \dots, k\} \subset G$. Consider the discrete homogeneous space of right cosets $H_\alpha \backslash G = \{H_\alpha g : g \in G\}$. There is a natural bijection $\phi : H_\alpha\backslash G \to \mathcal{D}_k$ given by $\phi(H_\alpha g) = g^{-1}\alpha$. Let $\Omega_k = \{1, -1\}^{\mathcal{D}_k}$ equipped with the product topology. This is a G-space under the action $g\omega(\beta) = \omega(g^{-1}\beta)$ for any $\omega \in \Omega_k$, $\beta \in \mathcal{D}_k$ and $g \in G$.

Set $\mathcal{T}^k = \{1, -1\}^{S_k}$. We refer to the elements of $\mathcal{T}^k$ as **tables**. Denote $\tilde{\Omega}_k = (\mathcal{T}^k)^{\tilde{\mathcal{D}}_k}$ equipped with the product topology. This is a G-space under the action $\cdot : G \times \tilde{\Omega}_k \to \tilde{\Omega}_k$ given by $g \cdot \tilde{\omega}(\tilde{\beta})(\sigma) = \tilde{\omega}(g^{-1}\tilde{\beta})(\sigma)$ for any $\omega \in \Omega_k$, $\tilde{\beta} \in \tilde{\Omega}_k$ and $g \in G$.

There is a natural family of homeomorphisms $\pi_c : \Omega_k \to \tilde{\Omega}_k$, $c \in \Phi(X)$ given by $\omega \mapsto \tilde{\omega}^c$, (also denoted $\tilde{\omega}$ when no confusion arises) where for $\tilde{\beta} = \{B_1, B_2 \dots, B_k\} \in \tilde{D}_k$ and $\sigma \in S_k$, $\tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}t^*_c(\tilde{\beta}))$ ($t^*_c(\cdot)$ is defined after Definition 3.1). In order for π_c to be a G-homeomorphism we need to equip $\tilde{\Omega}_k$ with a different G-action than the natural G-action mentioned above. Namely $\bullet_c : G \times \tilde{\Omega}_k \to \tilde{\Omega}_k$, is defined by

$$g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \tilde{\omega}(g^{-1}\tilde{\beta})(\rho_c(g, \tilde{\beta})\sigma) = \omega(\sigma^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta}))$$

where $\rho_c : G \times \tilde{\Omega}_k \to S_k$ is defined uniquely by the equation:

$$\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta}) = g^{-1}t^*_c(\tilde{\beta})$$

As $g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}g^{-1}t^*_c(\tilde{\beta}))$, we have the equality $g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \widetilde{g\omega}(\tilde{\beta})(\sigma)$ which makes $\pi_c : (G, \Omega_k) \to (G, \tilde{\Omega}_k)$ a G-homeomorphism (and formally proves $g\bullet_c$ is indeed a G-action).

Lemma 4.3. $\rho_c : G \times \tilde{\Omega}_k \to S_k$ obeys the following ***cocyle*** equation:

$$\rho_c(gh, \tilde{\beta}) = \rho_c(g, \tilde{\beta})\rho_c(h, g^{-1}\tilde{\beta})$$

Proof. By definition we have $gh \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \widetilde{gh\omega}(\tilde{\beta})(\sigma) = g \bullet_c \widetilde{h\omega}(\tilde{\beta})(\sigma)$. Notice

$$gh \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}\rho_c(gh, \tilde{\beta})^{-1}t^*_c(h^{-1}g^{-1}\tilde{\beta})),$$

whereas

$$g \bullet_c \widetilde{h\omega}(\tilde{\beta})(\sigma) = h\omega(\sigma^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta})) = \omega(\sigma^{-1}h^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta})).$$

This implies

$$\rho_c(gh,\tilde{\beta})^{-1}t_c^*(h^{-1}g^{-1}\tilde{\beta}) = h^{-1}\rho_c(g,\tilde{\beta})^{-1}t_c^*(g^{-1}\tilde{\beta}).$$

As $\rho_c(h,g^{-1}\tilde{\beta})^{-1}t_c^*(h^{-1}g^{-1}\tilde{\beta}) = h^{-1}t_c^*(g^{-1}\tilde{\beta})$, we have

$$\rho_c(gh,\tilde{\beta})^{-1} = \rho_c(h,g^{-1}\tilde{\beta})^{-1}\rho_c(g,\tilde{\beta})^{-1}.$$

Taking the inverses we get $\rho_c(gh,\tilde{\beta}) = \rho_c(g,\tilde{\beta})\rho_c(h,g^{-1}\tilde{\beta})$ $\square$

Note that in the end of Section 3 of [GW03] it was mistakenly claimed that $g \bullet_{c_0} \tilde{\omega}(\tilde{\beta})(\sigma) = g\cdot\tilde{\omega}(\tilde{\beta})(\sigma)$, for $c_0 = \{[0,t]\cap K\}_{t\in[0,1]}$ where K, the Cantor set, is embedded naturally in $[0,1]$.

4.4. **The Dual Ramsey Theorem.** A partition $\gamma = (C_1,\dots,C_k)$ of $\{1,\dots,s\}$ into k nonempty sets is **naturally ordered** if for any $1 \le i < j \le k$, $\min(C_i) < \min(C_j)$. We denote by $\Pi\binom{s}{k}$ the collection of naturally ordered partitions of $\{1,\dots,s\}$ into k nonempty sets.

Definition 4.4. Let $\beta = (B_1,\dots,B_s) \in \Pi\binom{k}{s}$ and $\gamma = (C_1,\dots,C_k) \in \Pi\binom{m}{k}$ define the **amalgamated partition** $\gamma_\beta = (G_1,\dots,G_s) \in \Pi\binom{m}{s}$ by:

$$G_j = \bigcup_{i\in B_j} C_i$$

Notice γ_β is naturally ordered and $(\mathcal{P}_\gamma)_\beta = \mathcal{P}_{\gamma_\beta}$. Similarly for $\alpha = (A_1, A_2\dots,A_m) \in \mathcal{D}$ define the **amalgamated clopen cover** $\alpha_\gamma = (G_1, G_1\dots,G_k)$, where $G_j = \bigcup_{i\in C_j} A_i$. Notice that $(\alpha_\gamma)_\beta = \alpha_{\gamma_\beta}$.

We denote by $\tilde{\Pi}\binom{s}{k}$ the collection of unordered partitions of $\{1,\dots,s\}$ into k nonempty sets. Notice there is a natural bijection $\tilde{\Pi}\binom{s}{k} \leftrightarrow \Pi\binom{s}{k}$.

Theorem 4.5. *[The dual Ramsey Theorem] Given positive integers k,m,r there exists a positive integer $N = DR(k,m,r)$ with the following property: for any coloring of $\tilde{\Pi}\binom{N}{k}$ by r colors there exists a partition $\alpha = \{A_1,A_2,\dots,A_m\} \in \tilde{\Pi}\binom{N}{m}$ of N into m non-empty sets such that all the partitions of N into k non-empty sets (i.e. elements of $\tilde{\Pi}\binom{N}{k}$) whose atoms are measurable with respect to α (i.e. each equivalence class is a union of elements of α) have the same color.*

Proof. This is Corollary 10 of [GR71]. □

4.5. **Minimal symbolic systems.** In the beginning of Section 4 of [GW03] a family of mappings $\phi_T : (G, \Phi(X)) \to (G, \Omega_k), T \in \mathcal{T}^k$ are introduced. We will introduce a generalized family but using a different description.

Definition 4.6. Let $\beta \in \mathcal{D}$ and $c \in \Phi(X)$, define the **β-ratio** of c, to be the unique element $\theta_\beta(c) \in S_k$ so that:

$$\theta_\beta(c)\beta = t_c^*(\beta)$$

Lemma 4.7. *The following holds:*

1. $\theta_\beta(c) = \theta_{g\beta}(gc)$ *for* $c \in \Phi(X)$, $g \in G$ *and* $\beta \in \mathcal{D}$.
2. $\theta_{\sigma^{-1}t_c^*(\tilde{\beta})}(c) = \sigma$ *for* $\sigma \in S_k$, $\tilde{\beta} \in \tilde{\mathcal{D}}$ *and* $c \in \Phi(X)$.

Proof.

1. By definition $\theta_{g\beta}(gc)g\beta = t_{gc}^*(g\beta)$. By Lemma 3.2, $gt_c^*(\beta) = t_{gc}^*(g\beta)$ and therefore one has $\theta_{g\beta}(gc)g\beta = gt_c^*(\beta)$. As the G and S_k actions commute it implies $\theta_{g\beta}(gc)\beta = t_c^*(\beta)$. By definition $\theta_\beta(c)\beta = t_c^*(\beta)$ and we conclude $\theta_\beta(c) = \theta_{g\beta}(gc)$.
2. $\theta_{\sigma^{-1}t_c^*(\tilde{\beta})}(c)\sigma^{-1}t_c^*(\tilde{\beta}) = t_c^*(\sigma^{-1}t_c^*(\tilde{\beta}))$

□

Let $T \in \mathcal{T}^k$. Define $\phi_T : \Phi(X) \to \Omega_k$ by

$$\phi_T(c)(\beta) = T(\theta_\beta(c))$$

Lemma 4.8. $\phi_T : \Phi(X) \to \Omega_k$ *is continuous and* G*-equivariant.*

Proof. We start by showing that ϕ_T is continuous. Let $n \in \mathbb{N}$, $\epsilon_1, \epsilon_2, \dots \epsilon_n \in \{\pm 1\}$, $\beta_1, \beta_2, \dots \beta_n \in \mathcal{D}_k$. Let V be an open set of Ω_k so that $V = \{\omega \in \Omega_k : \omega(\beta_i) = \epsilon_i\}$ and assume $V \neq \emptyset$. Let $c_1 \in \phi_T^{-1}(V)$. Denote $\mathfrak{U} = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathfrak{U}_{t_{c_1}^*(\beta)}$. By Lemma 3.4 $c_1 \in \mathfrak{U}$ so $\mathfrak{U} \neq \emptyset$. We claim $\phi_T(\mathfrak{U}) \subset V$. Indeed let $c_2 \in \mathfrak{U}$ and fix i. By assumption $c_2 \in \mathfrak{U}_{t_{c_1}^*(\beta_i)}$. By Lemma 3.4 $c_2 \in \mathfrak{U}_{t_{c_2}^*(\beta_i)}$. Conclude $t_{c_1}^*(\beta) = t_{c_2}^*(\beta)$, which implies $\theta_{\beta_i}(c_1) = \theta_{\beta_i}(c_2)$. This in turn implies $\phi_T(c_1)(\beta_i) = \phi_T(c_2)(\beta_i) = \epsilon_i$.

To show G-equivariance one has to show $g\phi_T(c)(\beta) = \phi_T(c)(g^{-1}\beta) = \phi_T(gc)(\beta)$. By definition $\phi_T(c)(g^{-1}\beta) = T(\theta_{g^{-1}\beta}(c))$ whereas $\phi_T(gc)(\beta) = T(\theta_\beta(gc))$. By Lemma 4.7 $\theta_\beta(gc) = \theta_{g^{-1}\beta}(c)$. □

Let $c_0 \in \Phi(X)$. We will investigate $\pi_{c_0} \circ \phi_T$. By definition $\tilde{\omega}^{c_0}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}t_c^*(\tilde{\beta}))$ and therefore we have $\widetilde{\phi_T(c)}^{c_0}(\tilde{\beta})(\sigma) = T(\theta_{\sigma^{-1}t^*_{c_0}(\tilde{\beta})}(c))$. By Lemma 4.7

$$\widetilde{\phi_T(c_0)}^{c_0}(\tilde{\beta})(\sigma) = \sigma$$

In particular $\widetilde{\phi_T(c_0)}^{c_0}(\tilde{\beta})(\sigma)$ *does not depend* on $\tilde{\beta}$ and we denote it by $\tilde{\omega}_T$.

The following theorem is based on Theorem 4.1 of [GW03]:

Theorem 4.9. *Every minimal subsystem of* (G, Ω_k) *is a factor of* $(G, \Phi(X))$.

Proof. Fix a minimal subset $\Sigma \subset \Omega_k$. We shall construct a homomorphism $\phi : (G, \Phi(X)) \to (G, \Sigma)$. Moreover it will be shown that $\phi = \phi_T$ for some $T \in \mathcal{T}^k$. Fix a point $\omega \in \Sigma$ and $c_0 \in \Phi(X)$. We consider $\tilde{\omega}^{c_0}$ as a coloring of elements of $\tilde{\mathcal{D}}_k$ by $r = |\mathcal{T}^k|$ where the colors are the tables of $\mathcal{T}^k$. For $\tilde{\beta} \in \mathcal{D}_k$, we thus denote by $\tilde{\omega}^{c_0}(\tilde{\beta})$ the element in $\mathcal{T}^k$. Let $m \in \mathbb{N}$ and fix $\alpha \in \mathcal{D}_m$. Let $\beta \in \mathcal{D}$ such that $\alpha \preceq \beta$, $t^*_{c_0}(\beta) = \beta$ and $|\beta| = N = DR(k, m, r)$ as in Theorem 4.5.

We define the coloring map to be $f : \Pi\binom{N}{k} \to \mathcal{T}^k$ where $\gamma \hookrightarrow \tilde{\omega}^{c_0}(\tilde{t}(\beta_\gamma))$. According to Theorem 4.5 there exists $\eta \in \Pi\binom{N}{m}$ and $T_\alpha \in \mathcal{T}^k$ so that for any $\tau \in \Pi\binom{m}{k}$, $f(\eta_\tau) = T_\alpha$. Let $g_\alpha \in G$ be such that $g_\alpha^{-1}t^*_{c_0}(\alpha) = \beta_\eta$. Denote $\tilde{\omega}^{c_0}_{g_\alpha} = g_\alpha \bullet_{c_0} \tilde{\omega}^{c_0}$. Notice

$$\begin{aligned}
\tilde{\omega}^{c_0}_{g_\alpha}(\tilde{t}(t^*_{c_0}(\alpha)_\tau))(\sigma) &= \omega(\sigma^{-1}g_\alpha^{-1}(t^*_{c_0}(\alpha)_\tau)) \\
&= \omega(\sigma^{-1}(g_\alpha^{-1}t^*_{c_0}(\alpha))_\tau) \\
&= \omega(\sigma^{-1}(\beta_\eta)_\tau) = \omega(\sigma^{-1}\beta_{\eta_\tau})
\end{aligned}$$

for any $\tau \in \Pi\binom{m}{k}$. We also have

$$T_\alpha = f(\eta_\tau) = \tilde{\omega}^{c_0}(\tilde{t}(\beta_{\eta_\tau}))(\sigma) = \omega(\sigma^{-1}t^*_{c_0}(\tilde{t}(\beta_{\eta_\tau}))) = \omega(\sigma^{-1}\beta_{\eta_\tau})$$

as $t^*_{c_0}(\beta) = \beta$. Conclude $\tilde{\omega}^{c_0}_{g_\alpha}(\tilde{t}(t^*_{c_0}(\alpha)_\tau)) = T_\alpha$. Let $\tilde{v} \in \Sigma$ be an accumulation point of the net $\{\tilde{\omega}^{c_0}_{g_\alpha}\}_{\alpha \in \mathcal{D}}$. Let $\tilde{\xi}_1, \tilde{\xi}_2 \in \tilde{\mathcal{D}}_k$. Let α be a common ordered refinement. By the calculations we have just performed for any $\gamma \succeq \alpha$, $\tilde{\xi}_1 = \tilde{t}(t^*_{c_0}(\gamma)_{\tau_1})$ and $\tilde{\xi}_2 = \tilde{t}(t^*_{c_0}(\gamma)_{\tau_2})$ for some $\tau_1, \tau_2 \in \Pi(\binom{|\gamma|}{k})$), we have $\tilde{\omega}^{c_0}_{g_\gamma}(\tilde{\xi}_1) = \tilde{\omega}^{c_0}_{g_\gamma}(\tilde{\xi}_2)$. This implies there exists $T \in \mathcal{T}^k$ such that for any $\tilde{\xi} \in \tilde{\mathcal{D}}_k$, $\tilde{v}(\tilde{\xi}) = T$, i.e. $\tilde{\nu} = \tilde{\omega}_T$ defined above. We conclude $\Sigma = \phi_T(\Phi(X))$. □

4.6. **Calculation of the universal minimal space.** We now proceed as in [GW03].

Lemma 4.10. *If Y is zero-dimensional compact Hausdorff topological space then the topological group* Homeo(Y) *equipped with the compact-open topology has a clopen basis at the identity.*

Proof. See the proof of Lemma 3.2 of [MS01]. The clopen basis is given by $\{H_\alpha\}_{\alpha\in\mathcal{D}}$ where H_α is defined in Subsection 4.3. □

Theorem 4.11. *Let H be a topological group. If the topology of H admits a basis for neighborhoods at the identity consisting of clopen subgroups, then $M(H)$ is zero dimensional.*

Proof. This follows from Proposition 3.4 of [Pes98] where it is shown that under the same conditions the **greatest ambit** of H is zero-dimensional. □

We now give the proof of Theorem 4.1:

Proof. The proof is a reproduction of the proof appearing in [GW03] that $M(G) = \Phi(K)$, where K is the Cantor set and $G = \text{Homeo}(K)$ is equipped with the compact-open topology. By Theorem 3.5 $(G, \Phi(X))$ is minimal and therefore there is an epimorphism $\pi : (G, M(G)) \to (G, \Phi(X))$. Fix $c_0 \in \Phi(X)$ and let $m_0 \in M(G)$ so that $\pi(m_0) = c_0$. By Lemma 4.10 and Theorem 4.11 $M(G)$ is zero-dimensional. Let $D \subset M(G)$ be a clopen subset and define the continuous function $F_D = 2\mathbf{1}_D - \mathbf{1}$, where $\mathbf{1}_D$ is the indicator function of D. If $H = \{g \in G : gD = D\}$ then H is a clopen subgroup of G and hence it contains H_α for some $\alpha \in \mathcal{D}_k$ for some $k \in \mathbb{N}$ (see proof of Lemma 4.10). It follows that the map $\psi_D(m) = (F_D(gm))_{g\in G}$, $m \in M(G)$ can be defined as a mapping into $\{1, -1\}^{H_\alpha\backslash G} = \Omega_k$ and thus we have

$\psi_D : (G, M(G)) \to (G, \Omega_k)$, so that if we set $Y_D = \psi_D(M(G))$, the system (Y_D, G) is a minimal symbolic subsystem of Ω_k. Denote $y_D = \psi_D(m_0)$.

Apply Theorem 4.9 to define a G-homomorphism $\phi_D : \Phi \to \Omega_k$, with and $y'_D = \phi_D(c_0)$. Given a clopen subset $D \subset M(G)$ consider the following diagram:

$$\begin{tikzcd} (M(G), m_0) \arrow[r, "\pi"] \arrow[d, "\psi_D"'] & (\Phi, c_0) \arrow[d, "\phi_D"] \\ (Y_D, y_D) & (Y_D, y'_D) \end{tikzcd}$$

The image $(\psi_D \times (\phi_D \circ \pi))(M(G), m_0) = (W, (y_D, y'_D))$, with $W \subset Y_D \times Y_D$, is a minimal subset of the product system $(Y_D \times Y_D, G)$. By Theorem 3.5(5) (Y_D, G) is proximal. Therefore the diagonal $\Delta = \{(y, y) : y \in Y_D\}$ is the unique minimal subset of the product system and we conclude that $y_D = y'_D$, so that the above diagram is replaced by

$$\begin{tikzcd} (M(G), m_0) \arrow[rr, "\pi"] \arrow[dr, "\psi_D"'] & & (\Phi, c_0) \arrow[dl, "\phi_D"] \\ & (Y_D, y_D) & \end{tikzcd}$$

Next form the product space

$$\Pi = \prod \{Y_D : D \text{ a clopen subset of } M(G)\},$$

and let $\psi : M(G) \to \Pi$ be the map whose D-projection is ψ_D (i.e. $(\psi(m))_D = \psi_D(m)$). We set $Y = \psi(M(G))$ and observe that, since clearly the maps ψ_D separate points on $M(G)$, the map $\psi : M(G) \to Y$ is an isomorphism, with $\psi(m_0) = y_0$, where $y_0 \in Y$ is defined by $(y_0)_D = y_D$. Likewise define $\phi : \Phi(X) \to Y$ by $(\phi(m))_D = \phi_D(m)$, so that also $\phi(c_0) = y_0$. These equations force the identity $\psi = \phi \circ \pi$ in the diagram

$$\begin{tikzcd} (M(G), m_0) \arrow[rr, "\pi"] \arrow[dr, "\psi"'] & & (\Phi, c_0) \arrow[dl, "\phi"] \\ & (Y, y_0) & \end{tikzcd}$$

Since ψ is a bijection it follows that so are π and ϕ and the proof is complete. $\square$

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