THE UNIVERSAL MINIMAL SPACE FOR GROUPS OF HOMEOMORPHISMS OF H-HOMOGENEOUS SPACES

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To Anatoliy Stepin with great respect.

Abstract. Let $X$ be a h-homogeneous zero-dimensional compact Hausdorff space, i.e. $X$ is a Stone dual of a homogeneous Boolean algebra. It is shown that the universal minimal space $M(G)$ of the topological group $G = \text{Homeo}(X)$, is the space of maximal chains on $X$ introduced in [Usp00]. If $X$ is metrizable then clearly $X$ is homeomorphic to the Cantor set and the result was already known (see [GW03]). However many new examples arise for non-metrizable spaces. These include, among others, the generalized Cantor sets $X = \{0, 1\}^\kappa$ for non-countable cardinals $\kappa$, and the corona or remainder of $\omega$, $X = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers.

1. Introduction

The existence and uniqueness of a universal minimal $G$ dynamical system, corresponding to a topological group $G$, is due to Ellis (see [Ell69], for a new short proof see [GL11]). He also showed that for a discrete infinite $G$ this space is never metrizable, and the latter statement was generalized to the locally compact non-compact case by Kechris, Pestov and Todorcevic in the appendix to their paper [KPT05]. For Polish groups this is no longer the case and we have such groups with $M(G)$ being trivial (groups with the fixed point property or extremely amenable groups) and groups with metrizable, easy to compute $M(G)$, like $M(G) = S^1$ for the group $G = \text{Homeo}_+(S^1)$.
Following Pestov’s work Uspenskij has shown in [Usp00] that the action of a topological group $G$ on its universal minimal system $M(G)$ (with $\text{card} \ M(G) \geq 3$) is never 3-transitive so that, e.g., for manifolds $X$ of dimension $> 1$ as well as for $X = Q$, the Hilbert cube, and $X = K$, the Cantor set, $M(G)$ can not coincide with $X$. Uspenskij proved his theorem by introducing the space of maximal chains $\Phi(X)$ associated to a compact space $X$. In [GW03] the authors then showed that for $X$ the Cantor set and $G = \text{Homeo}(X)$, in fact, $M(G) = \Phi$. It turns out that this group $G$ is a closed subgroup of $S_\infty(\omega)$ and in [KPT05] Kechris, Pestov and Todorcevic unified and extended these earlier results and carried out a systematic study of the spaces $M(G)$ for many interesting closed subgroups of $S_\infty$.

In the present work we go back to [GW03] and generalize it in another direction. We consider the class of h-homogeneous spaces $X$ and show that for every space in this class the universal minimal space $M(G)$ of the topological group $G = \text{Homeo}(X)$ is again Uspenskij’s space of maximal chains on $X$. If $X$ is metrizable then clearly $X$ is homeomorphic to the Cantor set and the result of [GW03] is retrieved (although even in this case our proof is new, as we make no use of a fixed point theorem). However, many new examples arise when one considers non-metrizable spaces. These include, among others, the generalized Cantor sets $X = \{0,1\}^\kappa$ for non-countable cardinals $\kappa$, and the widely studied corona or remainder of $\omega$, $X = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers. As in [GW03] the main combinatorial tool we apply is the dual Ramsey theorem.

1.1. H-homogeneous spaces and homogeneous Boolean algebras. The following definitions are well known (see e.g. [HNV04] Section H-4):

1. A zero-dimensional compact Hausdorff topological space $X$ is called **h-homogeneous** if every non-empty clopen subset of $X$ is homeomorphic to the entire space $X$.

2. A Boolean algebra $B$ is called **homogeneous** if for any nonzero element $a$ of $B$ the relative algebra $B|a = \{ x \in B : x \leq a \}$ is isomorphic to $B$. 
Using Stone’s Duality Theorem (see [BS81] IV§4) a zero-dimensional compact Hausdorff h-homogeneous space $X$ is the Stone dual of a homogeneous Boolean Algebra, i.e. any such space is realized as the space of ultrafilters $B^*$ over a homogeneous Boolean algebra $B$ equipped with the topology given by the base $N_a = \{U \in B^* : a \in U\}$, $a \in B$. Here are some examples of h-homogeneous spaces (see [ŠR89]):

(1) The countable atomless Boolean algebra is homogeneous. It corresponds by Stone duality to the Cantor space $K = \{0, 1\}^\mathbb{N}$.

(2) Every infinite free Boolean algebra is homogeneous. These Boolean algebras correspond by Stone duality to the generalized Cantor spaces, $\{0, 1\}^\kappa$, for infinite cardinals $\kappa$.

(3) Let $P(\omega)$ be the Boolean algebra of all subsets of $\omega$ (the first infinite cardinal) and let $fin \subset P(\omega)$ be the ideal comprising the finite subsets of $\omega$. Define the equivalence relations $A \sim_{fin} B$, $A, B \in P(\omega)$, if and only if $A \Delta B$ is in $fin$. The quotient Boolean algebra $P(\omega)/fin$ is homogeneous. This Boolean algebra corresponds by Stone duality to the corona $\omega^* = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of $\omega$.

(4) A topological space $X$ is called a Parovičenko space if:

(a) $X$ is a zero-dimensional compact space without isolated points and with weight $c$,

(b) every two disjoint open $F_\sigma$ subsets in $X$ have disjoint closures, and

(c) every non-empty $G_\delta$ subset of $X$ has non-empty interior.

Under CH Parovičenko proved that every Parovičenko space is homeomorphic to $\omega^*$ ([Par63]).

In [DM78] van Douwen and van Mill show that under $\neg$ CH, there are two non-homeomorphic Parovičenko spaces. Their second example of a Parovičenko space is the corona $X = \beta Y \setminus Y$, where $Y$ is the $\sigma$-compact space $\omega \times \{0, 1\}^c$. It is not hard to see that in $Y$ the clopen sets are of the form $L = \bigcup_{a \in A} \{a\} \times C_a$ for some $A \subset \omega$, where for all $a \in A$, $C_a$ is non-empty and clopen. If $|A| = \infty$ then $L \cong Y$ and if $|A| < \infty$ then $Cl_{\beta Y}(L) \subset Y$. These facts imply in a
straightforward manner that $X$ is $h$-homogeneous. In [DM78] it is pointed out that under MA $X$ is not homeomorphic to $\omega^*$. Thus under $\neg \text{CH} + \text{MA}$, this example provides another weight $c$ $h$-homogeneous space.

(5) Let $\kappa$ be a cardinal. By a well-known theorem of Kripke ([Kri67]) there is a homogeneous countably generated complete Boolean algebra, the so called collapsing algebra $C(\kappa)$ such that if $A$ is a Boolean algebra with a dense subset of power at most $\kappa$, then there is a complete embedding of $A$ in $C(\kappa)$.

(6) It is not hard to check that the product of any number of $h$-homogeneous spaces is again $h$-homogeneous.

1.2. The universal minimal space. A compact Hausdorff $G$-space $X$ is said to be minimal if $X$ and $\emptyset$ are the only $G$-invariant closed subsets of $X$. By Zorn’s lemma each $G$-space contains a minimal $G$-subspace. These minimal objects are in some sense the most basic ones in the category of $G$-spaces. For various topological groups $G$ they have been the object of intensive study. Given a topological group $G$ one is naturally interested in describing all of them up to isomorphism. Such a description is given (albeit in a very weak sense) by the following construction: as was mentioned in the introduction one can show there exists a minimal $G$-space $M(G)$ unique up to isomorphism such that if $X$ is a minimal $G$-space then $X$ is a factor of $M(G)$, i.e., there is a continuous $G$-equivariant mapping from $M(G)$ onto $X$. $M(G)$ is called the universal minimal $G$-space. Usually this minimal universal space is huge and an explicit description of it is hard to come by.

1.3. The space of maximal chains. Let $K$ be a compact Hausdorff space. We denote by $\text{Exp}(K)$ the space of closed subsets of $K$ equipped with the Vietoris topology. A subset $C \subset \text{Exp}(K)$ is a chain in $\text{Exp}(K)$ if for any $E, F \in C$ either $E \subset F$ or $F \subset E$. A chain is maximal if it is maximal with respect to the inclusion relation. One verifies easily that a maximal chain in $\text{Exp}(K)$ is a closed subset of $\text{Exp}(K)$, and that $\Phi(K)$, the space of all maximal chains in $\text{Exp}(K)$, is a closed subset of $\text{Exp}(\text{Exp}(K))$, i.e. $\Phi(K) \subset \text{Exp}(\text{Exp}(K))$ is a compact space. Note that a $G$-action on $K$ naturally induces a $G$-action on $\text{Exp}(K)$ and $\Phi(K)$. This is true in particular for
$K = M(G)$. As the $G$-space $\Phi(M(G))$ contains a minimal subsystem it follows that there exists an injective continuous $G$-equivariant mapping $f : M(G) \to \Phi(M(G))$. By investigating this mapping Uspenskij in [Usp00] showed that for every topological group $G$, the action of $G$ on the universal minimal space $M(G)$ is not 3-transitive. As a direct consequence of this theorem only rarely the natural action of the group $G = \text{Homeo}(K)$ on the compact space $K$ coincides with the universal minimal $G$-action (as is the case for $X = S^1$). In [Gut08] it was shown that for $G = \text{Homeo}(X)$, where $X$ belongs to a large family of spaces that contains in particular the Hilbert cube, the action of $G$ on the universal minimal space $M(G)$ is not 1-transitive.

It is easy to see that every $c \in \Phi(K)$ has a first element $F$ which is necessarily of the form $F = \{x\}$. Moreover, calling $x \triangleq r(c)$ the root of the chain $c$, it is clear that the map $\pi : \Phi(K) \to K$, sending a chain to its root, is a homomorphism of dynamical systems.

1.4. **The main result.** In [GW03] it was shown that the universal minimal space of the group of homeomorphisms of the Cantor set, equipped with the compact-open topology, is the space of maximal chains over the Cantor set. Our goal is to prove the following generalization:

**Theorem.** Let $X$ be a $h$-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \text{Homeo}(X)$ equipped with the compact-open topology, then $M(G) = \Phi(X)$, the space of maximal chains on $X$.

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2. Preliminaries

2.1. Clopen covers. Let \( X \) be a zero-dimensional compact Hausdorff space. Denote by \( \mathcal{D} \) (\( \tilde{\mathcal{D}} \)) the directed set (semilattice) consisting of all finite ordered (unordered) clopen partitions of \( X \) which are necessarily of the form \( \alpha = (A_1, A_2, \ldots, A_m) \) (\( \tilde{\alpha} = \{A_1, A_2, \ldots, A_m\} \)), where \( \cup_{i=1}^{m} A_i = X \) (disjoint union). The relation is given by refinement: \( \alpha \leq \beta \) (\( \tilde{\alpha} \leq \tilde{\beta} \)) iff for any \( B \in \beta \) (\( B \in \tilde{\beta} \)), there is \( A \in \alpha \) (\( A \in \tilde{\alpha} \)) so that \( B \subset A \). The join (least upper bound) of \( \alpha \) and \( \beta \), \( \alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\} \), where the ordering of indices is given by the lexicographical order on the indices of \( \alpha \) and \( \beta \) (\( \tilde{\alpha} \vee \tilde{\beta} = \{A \cap B : A \in \tilde{\alpha}, B \in \tilde{\beta}\} \)). It is convenient to introduce the notations \( \mathcal{D}_k = \{\alpha \in \mathcal{D} : |\alpha| = k\} \) and \( \tilde{\mathcal{D}}_k = \{\alpha \in \tilde{\mathcal{D}} : |\tilde{\alpha}| = k\} \). We denote the natural map \( (A_1, A_2, \ldots, A_m) \mapsto \{A_1, A_2, \ldots, A_m\} \) by \( \tilde{i} : \mathcal{D} \to \tilde{\mathcal{D}} \). There is a natural \( G \)-action on \( \mathcal{D} \) (\( \tilde{\mathcal{D}} \)) given by \( g(A_1, A_2, \ldots, A_m) = (g(A_1), g(A_2), \ldots, g(A_m)) \) \( (g(A_1), g(A_2), \ldots, g(A_m)) = \{g(A_1), g(A_2), \ldots, g(A_m)\} \). Let \( S_k \) denote the group of permutations of \( \{1, \ldots, k\} \). \( S_k \) acts naturally on \( \mathcal{D}_k \) by \( \sigma(B_1, B_2, \ldots, B_k) = (B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(k)}) \) for any \( \beta = (B_1, B_2, \ldots, B_k) \in \mathcal{D}_k \) and \( \sigma \in S_k \). This action commutes with the action of \( G \), i.e. \( \sigma g \beta = g \sigma \beta \) for any \( \sigma \in S_k \) and \( g \in G \). Notice that one can identify \( \tilde{\mathcal{D}}_k = \mathcal{D}_k/S_k \).

2.2. Partition Homogeneity. Let us introduce a new definition:

**Definition 2.1.** A zero-dimensional compact Hausdorff space \( X \) is called **partition-homogeneous** if for every two finite ordered clopen partitions of the same cardinality, \( \alpha, \beta \in \mathcal{D}_m \), \( \alpha = (A_1, A_2, \ldots, A_m) \), \( \beta = (B_1, B_2, \ldots, B_m) \) there is \( h \in \text{Homeo}(X) \) such that \( hA_i = B_i \), \( i = 1, \ldots, k \).

**Proposition 2.2.** Let \( X \) be an infinite zero-dimensional compact Hausdorff space. \( X \) is \( h \)-homogeneous iff \( X \) is partition-homogeneous.

**Proof.** Assume \( X \) is \( h \)-homogeneous. Let \( \alpha, \beta \in \mathcal{D}_m \), \( \alpha = (A_1, A_2, \ldots, A_m) \), \( \beta = (B_1, B_2, \ldots, B_m) \). Select homeomorphisms \( h_{A_i}, h_{B_i}, i = 1, \ldots, m \) with \( h_{A_i} : A_i \to X \), \( h_{B_i} : B_i \to X \). Define \( g \in \text{Homeo}(X) \) by \( g(x) = h_{B_i}^{-1} \circ h_{A_i}(x) \) for \( x \in A_i \). Trivially \( gA_i = B_i \). Assume now \( X \) is partition-homogeneous. Let \( A \neq X \) be a clopen set in \( X \). We distinguish between two cases:
(1) $A$ is a singleton. As $X$ is partition-homogeneous there exists $h \in \text{Aut}(X)$ with $hA = A^c$ and $hA^c = A$. We conclude $X$ is a two point space contradicting the assumption that $X$ is infinite.

(2) $A$ is not a singleton. Because $X$ is a compact Hausdorff zero-dimensional space we can find disjoint clopen sets $A_1, A_2$ such that $A = A_1 \cup A_2$. Let $h_1 \in G$ so that $h_1A_1 = A_1 \cup A_2^c$ and $h_1A_2^c = A_2$. Define the homeomorphism $h : A \to X$.

$$h(x) = \begin{cases} h_1(x) & x \in A_1 \\ x & x \in A_2 \end{cases}$$

\[ \square \]

3. Basic properties of $h$-homogeneous spaces

3.1. Induced orders. Let $X$ be a compact Hausdorff zero-dimensional $h$-homogeneous space and denote $G = \text{Homeo}(X)$. As $X$ is either trivial or infinite, we will assume from now onward, w.l.o.g. that $X$ is infinite. Let $v \in \Phi(X)$ and $D \subset X$ a closed set. Define

$$D_v = \bigcap_{A \in v: A \cap D \neq \emptyset} A$$

By maximality of $v$, one has $D_v \in v$. By a standard compactness argument $D_v \cap D \neq \emptyset$ and trivially it is the minimal element of $v$ that intersects $D$. Similarly for $D \subset X$ a closed set with $r(v) \in D$, define:

$$D^v = \bigcup_{A \in v: A \subset D} A$$

The maximal element of $v$ that is contained in $D$.

**Definition 3.1.** Let $v \in \Phi(X)$ and $\alpha = \{A_1, A_2, \ldots, A_m\} \in \mathcal{D}$. Define $<_{v|\alpha} = <_v$, the induced order on $\alpha$ by $v$:

$$A_i <_v A_j \iff (A_i)_v \subseteq (A_j)_v$$

Similarly for $v \in \Phi(X)$ and $\alpha \in \mathcal{D}$, define the induced order $<_{v|\alpha} = <_{v|\alpha}$, the induced order on $\alpha$. Denote by $t^*_v : \mathcal{D} \to \mathcal{D}$ the map $\{A_1, A_2, \ldots, A_m\} \mapsto (A_1, A_2, \ldots, A_m)$ where $i < j$ if and only if
$A_i <_{v|\alpha} A_j$. For $\beta \in D$, define $t^*_v(\beta) = t^*_v(\tilde{\beta}(\beta))$. Notice that for all $\sigma \in S_k$, $v \in \Phi(X)$ and $\beta \in D$,

$$t^*_v(\sigma t^*_v(\beta)) = t^*_v(\beta)$$

**Lemma 3.2.** $gt^*_v(\tilde{\beta}) = t^*_v(g\tilde{\beta})$.

**Proof.** Let $\alpha = (A_1, A_2, \ldots, A_m) = t^*_v(\tilde{\beta})$. By definition $i < j$ if and only if $(A_i)_v \subseteq (A_j)_v$. Notice $(gA_i)_v = \bigcap_{gA \supseteq gA, gA \notin \emptyset} gA = g\bigcap_{A \subseteq v: A \notin \emptyset} A = g(A_i)_v$. Therefore $i < j$ if and only if $(gA_i)_v \subseteq (gA_j)_v$, and we conclude $g\alpha = t^*_v(g\tilde{\beta})$.

**Proposition 3.3.** Let $v \in \Phi(X)$ and $\tilde{\alpha} = \{A_1, A_2, \ldots, A_m\} \in \tilde{D}$. $<_{v|\tilde{\alpha}}$ is a linear order on $\tilde{\alpha}$. The ordering $A_{i_1} <_{v|\tilde{\alpha}} A_{i_2} <_{v|\tilde{\alpha}} \ldots <_{v|\tilde{\alpha}} A_{i_m}$ is characterized by $(A_{i_k})_v \setminus (A_{i_1} \cup \ldots \cup A_{i_{k-1}})_v = \{x_k\}$ for $k = 1, 2, \ldots, m$ and suitable $x_k \in A_k$.

**Proof.** Let $D \subset X$ be clopen so that $r(v) \in D$, then it is easy to see that $v_{|D^c} \triangleq \{A \setminus D \setminus D^c \subseteq A \in v\}$ is a maximal chain in $D^c$ and in particular has a root $r(v_{|D^c}) = x_0 \in D^c$. Let $i_1$ be such that $r(v) \in A_{i_1}$. Inductively let $i_{k+1}$ be such that $r(v_{|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}}) \in A_{i_{k+1}}$. It is easy to see $A_{i_1} <_v A_{i_2} <_v \ldots <_v A_{i_m}$. This implies both that $<_v$ is a linear order and $(A_{i_k})_v \setminus (A_{i_1} \cup \ldots \cup A_{i_{k-1}})_v = \{x_k\}$ for some $x_k \in A_k$, $k = 1, 2, \ldots, m$.

3.2. **Minimality and proximality of natural actions.** The basis for the Vietoris topology for the compact Hausdorff space $\text{Exp}(X)$ is given by open sets of the form:

$$U = \langle A_1, \ldots, A_k \rangle = \{F \in \text{Exp}(X) : \forall i F \cap A_i \neq \emptyset \text{ and } F \subseteq \bigcup A_i\}$$

where $A_i \subset X$ is clopen. It is easy to see that a basis of clopen neighborhood of a maximal chain $v \in \Phi(X)$ is given by

$$U_\alpha = \langle U_1, \ldots, U_n \rangle$$

where $\alpha = (A_1, A_2, \ldots, A_n) \in D$ and

$$U_j = \langle A_1, \ldots, A_j \rangle, \quad j = 1, 2, \ldots, n,$$

The following lemma is straightforward:
Lemma 3.4. Let $\alpha = (A_1, A_2, \ldots, A_n) \in \mathcal{D}$ and $\nu \in \Phi(X)$. Let $<_{\nu|\alpha}$ be the induced order of $\nu$ on $\alpha$, then $\nu \in U_\alpha$ if and only if $<_{\nu} = <$, where $<$ is the usual order on $\{1, 2, \ldots, n\}$. In particular $\nu \in U_{\nu|\alpha}(\alpha)$.

Theorem 3.5. (1) The system $(X,G)$ is minimal.

(2) The system $(X,G)$ is extremely proximal; i.e. for every closed set $\emptyset \neq F \subsetneq X$ there exists a net $\{g_i\}_{i \in I}$ in $G$ such that we have $\lim_{i \in I} g_i F = \{x_0\}$ for some point $x_0 \in X$ (see [Gla74]).

(3) The minimal system $(X,G)$ is not isomorphic to the universal minimal system $(M(G),G)$.

(4) $(\Phi(X),G)$ is minimal.

(5) $(\Phi(X),G)$ is proximal.

Proof.

(1) Since $X$ is h-homogeneous, then by Proposition 2.2, $G$ acts transitively on non-trivial (i.e. not $\emptyset, X$) clopen sets. Since $G$ acts transitively on the above mentioned basis, it follows that for every $U \in \mathcal{U}$ we have $\cup \{\alpha(U) : \alpha \in G\} = X$. This property is equivalent to the minimality of the system $(X,G)$.

(2) Fix some $x_0$ in $X$ such that $x_0 \notin F$. For an arbitrary basic clopen neighborhood $U = A$ of $x_0$ which is disjoint from $F$ choose $\alpha_U \in G$ such that $\alpha_U(A^c) = A$. Then $\alpha$ satisfies $\alpha_U(F) \subset U$. Clearly now $\{\alpha_U : U$ a neighborhood of $x_0\}$ is the required net.

(3) As the system $(X,G)$ is certainly 3-transitive this claim follows from Uspenskij’s theorem [Usp00]. For completeness we provide a direct proof. Suppose $(X,G)$ is isomorphic to the universal minimal $G$ system. Let $Y \subset \Phi$ be a minimal subset of $\Phi$. Then, by the coalescence of the universal minimal system (every $G$-endomorphism $\phi : (M(G),G) \to (M(G),G)$ (which is necessarily onto) is an isomorphism, see [GL11] and [Usp00]), the restriction $\pi : Y \to X$, sending a chain to its root, is an isomorphism. Fix $c_0 \in Y$ and let $p_0 \in X$ be its root; i.e. $\pi(c_0) = p_0$. Let $H = \{\alpha \in G : \alpha p_0 = p_0\}$, the stability group of $p_0$. Since $\pi$ is an isomorphism we also have $H = \{\alpha \in G : \alpha c_0 = c_0\}$. Choose
Choose a clopen partition of \((P,A,B)\) of \(X\) with \(B \cap F = \emptyset\), \(P \cap F \neq \emptyset\) and \(A \cap F \neq \emptyset\). Using the fact that \(X\) is partition homogeneous, one can find \(g \in G\) so that \(gP = P\), \(gA = B\) and \(gB = A\). One redefines \(g\) so that \(g_P = Id\). As \(g(A \cup P) \cap A = \emptyset\), we have \(F \setminus gF \neq \emptyset\). As \(gA = B\) we have \(gF \setminus F \neq \emptyset\). Conclude that \(F\) and \(gF\) are not comparable. On the other hand \(g(p_0) = p_0\) means \(g \in H\) whence also \(gc_0 = c_0\). In particular \(gF \in c_0\) and as \(c_0\) is a chain one of the inclusions \(F \subset gF\) or \(gF \subset F\) must hold. This contradiction shows that \((X,G)\) cannot be the universal minimal \(G\)-system.

(4) Let \(\nu', \nu \in \Phi(X)\) and \(\nu' \in \mathcal{U}_\alpha\) for some \(\alpha = (A_1, A_2, \ldots, A_n) \in \mathcal{D}\). Let \(<_\nu\) be the induced order of \(\nu\) on \(\alpha\). Let \(\sigma \in S_n\) be such that for any \(i < j\), \(A_{\sigma(i)} <_\nu A_{\sigma(j)}\). As \(X\) is partition homogeneous we can choose \(g \in G\) so that \(gA_{\sigma(i)} = A_i\). Clearly \(gv \in \mathcal{U}_\alpha\).

(5) Let \(\nu_1, \nu_2 \in \Phi(X)\). Fix some \(\nu' \in \mathcal{U}_\alpha\) for some \(\alpha = (A_1, A_2, \ldots, A_n) \in \mathcal{D}\). Let \(<\) be the usual order on \(\{1, 2, \ldots, n\}\). Inductively we will construct \(g \in G\) so that \(<_{gv_1}\alpha = <_{gv_2}\alpha = <\). Using Lemma 3.4, this implies \(gv_1 \in \mathcal{U}_\alpha\) and \(gv_2 \in \mathcal{U}_\alpha\). As \(\mathcal{U}_\alpha\) is arbitrary, this establishes proximality. Indeed let \(g_1 \in G\) so that \(g_1(r(\nu_1)), g_1(r(\nu_2)) \in A_1\). Assume we have constructed \(g_k \in G\). Define \(g_{k+1} \in G\) so that \(g_{k+1}|_{A_1 \cup A_2 \cup \ldots \cup A_k} = g_k|_{A_1 \cup A_2 \cup \ldots \cup A_k}\) and \(g_{k+1}(r((g_k \nu_1)|(A_1 \cup A_2 \cup \ldots \cup A_k)^c)), g_{k+1}(r((g_k \nu_2)|(A_1 \cup A_2 \cup \ldots \cup A_k)^c)) \in A_{k+1}\). It is easy to see that \(g = g_n\) has the desired properties.

4. Calculation of the Universal Minimal Space

4.1. Overview. The goal of this section is to generalize the main theorem of [GW03]: the universal minimal space of the group of homeomorphisms of the Cantor set, equipped with the compact-open topology, is the space of maximal chains over the Cantor set. We prove the following theorem:
Theorem 4.1. Let $X$ be a $h$-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \text{Homeo}(X)$ equipped with the compact-open topology, then $M(G) = \Phi(X)$.

The proof borrows heavily from the proof in [GW03]. The new ideas (that build on ideas in [GW03]) are presented in subsections 4.2, 4.3, 4.5.

4.2. Order topology. Recall that given a set $Y$ and a linear order $<$ on $Y$ there is a topology generated by the basis of open intervals $(a,b) = \{y \in Y : a < y < b\}$ where $a,b \in Y$ and equality is allowed on the left (right) if $a$ ($b$) is the smallest (biggest) element of $Y$. This topology is called the order topology on $(Y, <)$. For more details see [Mun75] Section 2.3. One of the most important ingredients in the proof in [GW03] is the fact that the topology on the cantor set $K$ is the order topology associated with the natural order $<$ on $K \subset [0,1])$. A natural approach to generalizations of the result in the case of $X = \omega^*$ the corona, is to look for an order that will generate the topology on the corona. However, as the following proposition shows this is impossible.

Proposition 4.2. The topology on $\omega^*$ is not an order topology.

Proof. Assume for a contradiction that the topology on $\omega^*$ is an order topology associated with a linear order $<$. As $\omega^*$ has no isolated points we can find (with no loss of generality) an increasing bounded sequence of points $p_1 < p_2 < p_3 < \cdots < b$. By compactness this sequence admits a least upper bound $p = \lim_{k \to \infty} p_k$, so that the set $\{p_k : k = 1, 2, \ldots\} \cup \{p\}$ is a closed subset of $\omega^*$. However, it is well known that the remainder $\omega^*$ has no nontrivial converging sequences; e.g. one can use the fact that the closure of the set $\{p_k : k = 1, 2, \ldots\}$, like the closure of any infinite discrete countable set in $\beta\omega$, is homeomorphic to $\beta\omega$ (see e.g. [Eng78, Theorem 3.6.14]).

4.3. The spaces $\Omega_k$ and $\tilde{\Omega}_k$ and a cocycle equation. The following subsection is a generalization of Section 3 of [GW03]. Fix $\alpha = (A_1, A_2, \ldots, A_k) \in D_k$ and define the
clopen subgroup \( H_\alpha = \{ g \in G : gA_i = A_i, i = 1, \ldots, k \} \subset G \). Consider the discrete homogeneous space of right cosets \( H_\alpha \backslash G = \{ H_\alpha g : g \in G \} \). There is a natural bijection \( \phi : H_\alpha \backslash G \to D_k \) given by \( \phi(H_\alpha g) = g^{-1}\alpha \). Let \( \Omega_k = \{ 1, -1 \}^{D_k} \) equipped with the product topology. This is a \( G \)-space under the action \( g\omega(\beta) = \omega(g^{-1}\beta) \) for any \( \omega \in \Omega_k, \beta \in D_k \) and \( g \in G \).

Set \( T^k = \{ 1, -1 \}^{S_k} \). We refer to the elements of \( T^k \) as tables. Denote \( \tilde{\Omega}_k = (T^k)^{\hat{D}_k} \) equipped with the product topology. This is a \( G \)-space under the action \( \cdot : G \times \tilde{\Omega}_k \to \tilde{\Omega}_k \) given by \( g : \tilde{\omega}(\beta)(\sigma) = \tilde{\omega}(g^{-1}\beta)(\sigma) \) for any \( \omega \in \Omega_k, \tilde{\beta} \in \tilde{\Omega}_k \) and \( g \in G \).

There is a natural family of homeomorphisms \( \pi_c : \Omega_k \to \tilde{\Omega}_k, c \in \Phi(X) \) given by \( \omega \mapsto \tilde{\omega}^c \), (also denoted \( \tilde{\omega} \) when no confusion arises) where for \( \tilde{\beta} = \{ B_1, B_2, \ldots, B_k \} \in \tilde{D}_k \) and \( \sigma \in S_k \), \( \tilde{\omega}(\beta)(\sigma) = \omega(\sigma^{-1}t^*_c(\tilde{\beta})) \) (\( t^*_c(\cdot) \) is defined after Definition 3.1). In order for \( \pi_c \) to be a \( G \)-homeomorphism we need to equip \( \tilde{\Omega}_k \) with a different \( G \)-action than the natural \( G \)-action mentioned above. Namely \( \bullet_c : G \times \tilde{\Omega}_k \to \tilde{\Omega}_k \), is defined by

\[
g \bullet_c \tilde{\omega}(\beta)(\sigma) = \tilde{\omega}(g^{-1}\beta)(\rho_c(g, \beta)\sigma) = \omega(\sigma^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\beta))
\]

where \( \rho_c : G \times \tilde{\Omega}_k \to S_k \) is defined uniquely by the equation:

\[
\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\beta) = g^{-1}t^*_c(\tilde{\beta})
\]

As \( g \bullet_c \tilde{\omega}(\beta)(\sigma) = \omega(\sigma^{-1}g^{-1}t^*_c(\tilde{\beta})) \), we have the equality \( g \bullet_c \tilde{\omega}(\beta)(\sigma) = \tilde{g}\omega(\tilde{\beta})(\sigma) \) which makes \( \pi_c : (G, \Omega_k) \to (G, \tilde{\Omega}_k) \) a \( G \)-homeomorphism (and formally proves \( g \bullet_c \) is indeed a \( G \)-action).

**Lemma 4.3.** \( \rho_c : G \times \tilde{\Omega}_k \to S_k \) obeys the following **cocyly** equation:

\[
\rho_c(gh, \tilde{\beta}) = \rho_c(g, \tilde{\beta})\rho_c(h, g^{-1}\tilde{\beta})
\]

**Proof.** By definition we have \( gh \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \tilde{g}\tilde{h}\omega(\tilde{\beta})(\sigma) = g \bullet_c \tilde{h}\omega(\tilde{\beta})(\sigma) \). Notice

\[
gh \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}\rho_c(gh, \tilde{\beta})^{-1}t^*_c(h^{-1}g^{-1}\tilde{\beta}))
\]

whereas

\[
g \bullet_c \tilde{h}\omega(\tilde{\beta})(\sigma) = h\omega(\sigma^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta})) = \omega(\sigma^{-1}h^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta})).
\]
This implies
\[ \rho_c(gh, \tilde{\beta})^{-1} t_c^*(h^{-1} g^{-1} \tilde{\beta}) = h^{-1} \rho_c(g, \tilde{\beta})^{-1} t_c^*(g^{-1} \tilde{\beta}). \]

As \( \rho_c(h, g^{-1} \tilde{\beta})^{-1} t_c^*(h^{-1} g^{-1} \tilde{\beta}) = h^{-1} t_c^*(g^{-1} \tilde{\beta}) \), we have
\[ \rho_c(gh, \tilde{\beta})^{-1} = \rho_c(h, g^{-1} \tilde{\beta})^{-1} \rho_c(g, \tilde{\beta})^{-1}. \]

Taking the inverses we get \( \rho_c(gh, \tilde{\beta}) = \rho_c(g, \tilde{\beta}) \rho_c(h, g^{-1} \tilde{\beta}) \) \( \square \).

Note that in the end of Section 3 of [GW03] it was mistakenly claimed that \( g \cdot c_0 \cdot \bar{\omega}(\tilde{\beta})(\sigma) = g \cdot \bar{\omega}(\tilde{\beta})(\sigma), \) for \( c_0 = \{[0, t] \cap K \}_{t \in [0, 1]} \) where \( K \), the Cantor set, is embedded naturally in \([0, 1]\).

4.4. The Dual Ramsey Theorem. A partition \( \gamma = (C_1, \ldots, C_k) \) of \( \{1, \ldots, s\} \) into \( k \) nonempty sets is naturally ordered if for any \( 1 \leq i < j \leq k \), \( \min(C_i) < \min(C_j) \).

We denote by \( \Pi^*(\cdot) \) the collection of naturally ordered partitions of \( \{1, \ldots, s\} \) into \( k \) nonempty sets.

**Definition 4.4.** Let \( \beta = (B_1, \ldots, B_s) \in \Pi^k \) and \( \gamma = (C_1, \ldots, C_k) \in \Pi^m \) define the amalgamated partition \( \gamma_\beta = (G_1, \ldots, G_s) \in \Pi^m \) by:
\[ G_j = \bigcup_{i \in B_j} C_i \]

Notice \( \gamma_\beta \) is naturally ordered and \( (\mathcal{P}_\gamma)_\beta = \mathcal{P}_{\gamma_\beta} \). Similarly for \( \alpha = (A_1, A_2 \ldots, A_m) \in \mathcal{D} \) define the amalgamated clopen cover \( \alpha_\gamma = (G_1, G_1, \ldots, G_k) \), where \( G_j = \bigcup_{i \in C_j} A_i \). Notice that \( (\alpha_\gamma)_\beta = \alpha_{x\beta} \).

We denote by \( \tilde{\Pi}^*(\cdot) \) the collection of unordered partitions of \( \{1, \ldots, s\} \) into \( k \) nonempty sets. Notice there is a natural bijection \( \tilde{\Pi}^*(\cdot) \leftrightarrow \Pi^*(\cdot) \).

**Theorem 4.5.** [The dual Ramsey Theorem] Given positive integers \( k, m, r \) there exists a positive integer \( N = DR(k, m, r) \) with the following property: for any coloring of \( \tilde{\Pi}^N_k \) by \( r \) colors there exists a partition \( \alpha = \{A_1, A_2, \ldots, A_m\} \in \tilde{\Pi}^N_m \) of \( N \) into \( m \) non-empty sets such that all the partitions of \( N \) into \( k \) non-empty sets (i.e. elements of \( \tilde{\Pi}^N_k \)) whose atoms are measurable with respect to \( \alpha \) (i.e. each equivalence class is a union of elements of \( \alpha \)) have the same color.
4.5. Minimal symbolic systems. In the beginning of Section 4 of [GW03] a family of mappings $\phi_T : (G, \Phi(X)) \rightarrow (G, \Omega_k), T \in T^k$ are introduced. We will introduce a generalized family but using a different description.

**Definition 4.6.** Let $\beta \in D$ and $c \in \Phi(X)$, define the $\beta$-ratio of $c$, to be the unique element $\theta_\beta(c) \in S_k$ so that:

$$\theta_\beta(c) \beta = t^*_c(\beta)$$

**Lemma 4.7.** The following holds:

1. $\theta_\beta(c) = \theta_{g\beta}(gc)$ for $c \in \Phi(X)$, $g \in G$ and $\beta \in D$.
2. $\theta_{\sigma^{-1}T_\beta}(c) = \sigma$ for $\sigma \in S_k$, $\tilde{\beta} \in \tilde{D}$ and $c \in \Phi(X)$.

**Proof.**

1. By definition $\theta_{g\beta}(gc)g\beta = t^*_c(g\beta)$. By Lemma 3.2, $gt^*_c(\beta) = t^*_c(g\beta)$ and therefore one has $\theta_{g\beta}(gc)g\beta = gt^*_c(\beta)$. As the $G$ and $S_k$ actions commute it implies $\theta_{g\beta}(gc) = t^*_c(\beta)$. By definition $\theta_\beta(c) = t^*_c(\beta)$ and we conclude

$\theta_\beta(c) = \theta_{g\beta}(gc)$.

2. $\theta_{\sigma^{-1}T_\beta}(c)\sigma^{-1}t^*_c(\tilde{\beta}) = t^*_c(\sigma^{-1}t^*_c(\tilde{\beta}))$

Let $T \in T^k$. Define $\phi_T : \Phi(X) \rightarrow \Omega_k$ by

$$\phi_T(c)(\beta) = T(\theta_\beta(c))$$

**Lemma 4.8.** $\phi_T : \Phi(X) \rightarrow \Omega_k$ is continuous and $G$-equivariant.

**Proof.** We start by showing that $\phi_T$ is continuous. Let $n \in \mathbb{N}$, $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{ \pm 1 \}, \beta_1, \beta_2, \ldots, \beta_n \in D_k$. Let $V$ be an open set of $\Omega_k$ so that $V = \{ \omega \in \Omega_k : \omega(\beta_i) = \epsilon_i \}$ and assume $V \neq \emptyset$. Let $c_1 \in \phi_T^{-1}(V)$. Denote $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_{t^*_c(\beta_i)}$. By Lemma 3.4 $c_1 \in \mathcal{U}$ so $\mathcal{U} \neq \emptyset$. We claim $\phi_T(\mathcal{U}) \subset V$. Indeed let $c_2 \in \mathcal{U}$ and fix $i$. By assumption $c_2 \in \mathcal{U}_{t^*_c(\beta_i)}$. By Lemma 3.4 $c_2 \in \mathcal{U}_{t^*_c(\beta_i)}$. Conclude $t^*_c(\beta) = t^*_c(\beta)$, which implies $\theta_{\beta}(c_1) = \theta_{\beta}(c_2)$. This in turn implies $\phi_T(c_1)(\beta_i) = \phi_T(c_2)(\beta_i) = \epsilon_i$. 

---

*Proof.* This is Corollary 10 of [GR71].
To show $G$-equivariance one has to show $g\phi_T(c)(\beta) = \phi_T(c)(g^{-1}\beta) = \phi_T(gc)(\beta)$.

By definition $\phi_T(c)(g^{-1}\beta) = T(\theta_{g^{-1}\beta}(c))$ whereas $\phi_T(gc)(\beta) = T(\theta_{\beta}(gc))$. By Lemma 4.7 $\theta_{g^{-1}\beta}(c) = \theta_g^{-1}(\beta)$. 

Let $c_0 \in \Phi(X)$. We will investigate $\pi_{c_0} \circ \phi_T$. By definition $\tilde{\omega}_c(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}t^*_c(\tilde{\beta}))$ and therefore we have $\tilde{\phi}_T(c)\circ \tilde{\beta}(\sigma) = \tilde{\theta}_\sigma - \tilde{\phi}_T(c_0)\circ \tilde{\beta}(\sigma)$. By Lemma 4.7

$$\tilde{\phi}_T(c_0)\circ \tilde{\beta}(\sigma) = \sigma$$

In particular $\tilde{\phi}_T(c_0)\circ \tilde{\beta}(\sigma)$ does not depend on $\tilde{\beta}$ and we denote it by $\tilde{\omega}_T$.

The following theorem is based on Theorem 4.1 of [GW03]:

**Theorem 4.9.** Every minimal subsystem of $(G, \Omega_k)$ is a factor of $(G, \Phi(X))$.

**Proof.** Fix a minimal subset $\Sigma \subset \Omega_k$. We shall construct a homomorphism $\phi : (G, \Phi(X)) \to (G, \Sigma)$. Moreover it will be shown that $\phi = \phi_T$ for some $T \in T^k$. Fix a point $\omega \in \Sigma$ and $c_0 \in \Phi(X)$. We consider $\tilde{\omega}_c$ as a coloring of elements of $\tilde{\mathcal{D}}_k$ by $r = |T^k|$ where the colors are the tables of $T^k$. For $\tilde{\beta} \in \mathcal{D}_k$, we thus denote by $\tilde{\omega}_c(\tilde{\beta})$ the element in $T^k$. Let $m \in \mathbb{N}$ and fix $\alpha \in \mathcal{D}_m$. Let $\beta \in \mathcal{D}$ such that $\alpha \preceq \beta$, $t^*_c(c_0) = \beta$ and $|\beta| = N = DR(k, m, r)$ as in Theorem 4.5.

We define the coloring map to be $f : \Pi^{(N)}_k \to T^k$ where $\gamma \mapsto \tilde{\omega}_c(\tilde{\iota}(\gamma))$. According to Theorem 4.5 there exists $\eta \in \Pi^{(N)}_m$ and $T_\alpha \in T^k$ so that for any $\tau \in \Pi^{(m)}_k$, $f(\eta_\tau) = T_\alpha$. Let $g_\alpha \in G$ be such that $g_\alpha^{-1}t^*_c(\alpha) = \beta_\eta$. Denote $\tilde{\omega}_g \circ \gamma = g_\alpha \bullet_c \tilde{\omega}_c$. Notice

$$\tilde{\omega}_g(\tilde{\iota}(t^*_c(\alpha)\tau))(\sigma) = \omega(\sigma^{-1}g_\alpha^{-1}(t^*_c(\alpha)\tau))$$

$$= \omega(\sigma^{-1}(g_\alpha^{-1}t^*_c(\alpha)\tau))$$

$$= \omega(\sigma^{-1}(\beta_\eta)\tau) = \omega(\sigma^{-1}\beta_\eta)$$

for any $\tau \in \Pi^{(m)}_k$. We also have

$$T_\alpha = f(\eta_\tau) = \tilde{\omega}_c(\tilde{\iota}(\beta_\eta))(\sigma) = \omega(\sigma^{-1}t^*_c(\tilde{\iota}(\beta_\eta)) = \omega(\sigma^{-1}\beta_\eta)$$

as $t^*_c(c_0) = \beta$. Conclude $\tilde{\omega}_c(\tilde{\iota}(t^*_c(\alpha)\tau)) = T_\alpha$. Let $\hat{v} \in \Sigma$ be an accumulation point of the net $\{\tilde{\omega}_c\}_{\alpha \in \mathcal{D}}$. Let $\hat{\xi}_1, \hat{\xi}_2 \in \tilde{\mathcal{D}}_k$. Let $\alpha$ be a common ordered refinement. By the
calculations we have just performed for any $\gamma \succeq \alpha, \tilde{\xi}_1 = \tilde{t}(t_0^*(\gamma)_{\tau_1})$ and $\tilde{\xi}_2 = \tilde{t}(t_0^*(\gamma)_{\tau_2})$ for some $\tau_1, \tau_2 \in \Pi(\vert \gamma \vert_k)$, we have $\tilde{\omega}_{c_0}(\tilde{\xi}_1) = \tilde{\omega}_{c_0}(\tilde{\xi}_2)$. This implies there exists $T \in T^k$ such that for any $\tilde{\xi} \in \tilde{D}_k$, $\tilde{\nu}(\tilde{\xi}) = T$, i.e. $\tilde{\nu} = \tilde{\omega}_T$ defined above. We conclude $\Sigma = \phi_T(\Phi(X))$. □

4.6. Calculation of the universal minimal space. We now proceed as in [GW03].

**Lemma 4.10.** If $Y$ is zero-dimensional compact Hausdorff topological space then the topological group $\text{Homeo}(Y)$ equipped with the compact-open topology has a clopen basis at the identity.

**Proof.** See the proof of Lemma 3.2 of [MS01]. The clopen basis is given by $\{H_\alpha\}_{\alpha \in \mathcal{D}}$ where $H_\alpha$ is defined in Subsection 4.3. □

**Theorem 4.11.** Let $H$ be a topological group. If the topology of $H$ admits a basis for neighborhoods at the identity consisting of clopen subgroups, then $M(H)$ is zero dimensional.

**Proof.** This follows from Proposition 3.4 of [Pes98] where it is shown that under the same conditions the greatest ambit of $H$ is zero-dimensional. □

We now give the proof of Theorem 4.1:

**Proof.** The proof is a reproduction of the proof appearing in [GW03] that $M(G) = \Phi(K)$, where $K$ is the Cantor set and $G = \text{Homeo}(K)$ is equipped with the compact-open topology. By Theorem 3.5 $(G, \Phi(X))$ is minimal and therefore there is an epimorphism $\pi : (G, M(G)) \to (G, \Phi(X))$. Fix $c_0 \in \Phi(X)$ and let $m_0 \in M(G)$ so that $\pi(m_0) = c_0$. By Lemma 4.10 and Theorem 4.11 $M(G)$ is zero-dimensional. Let $D \subset M(G)$ be a clopen subset and define the continuous function $F_D = 21_D - 1$, where $1_D$ is the indicator function of $D$. If $H = \{g \in G : gD = D\}$ then $H$ is a clopen subgroup of $G$ and hence it contains $H_\alpha$ for some $\alpha \in \mathcal{D}_k$ for some $k \in \mathbb{N}$ (see proof of Lemma 4.10). It follows that the map $\psi_D(m) = (F_D(gm))_{g \in G}$, $m \in M(G)$ can be defined as a mapping into $\{1, -1\}^{H_\alpha \setminus G} = \Omega_k$ and thus we have
\( \psi_D : (G, M(G)) \to (G, \Omega_k) \), so that if we set \( Y_D = \psi_D(M(G)) \), the system \((Y_D, G)\) is a minimal symbolic subsystem of \( \Omega_k \). Denote \( y_D = \psi_D(m_0) \).

Apply Theorem 4.9 to define a \( G \)-homomorphism \( \phi_D : \Phi \to \Omega_k \), with and \( y_D' = \phi_D(c_0) \). Given a clopen subset \( D \subset M(G) \) consider the following diagram:

\[
\begin{array}{ccc}
(M(G), m_0) & \xrightarrow{\pi} & (\Phi, c_0) \\
\psi_D \downarrow & & \downarrow \phi_D \\
(Y_D, y_D) & \xrightarrow{\phi_D} & (Y_D, y_D')
\end{array}
\]

The image \((\psi_D \times (\phi_D \circ \pi))(M(G), m_0) = (W, (y_D, y_D'))\), with \( W \subset Y_D \times Y_D \), is a minimal subset of the product system \((Y_D \times Y_D, G)\). By Theorem 3.5(5) \((Y_D, G)\) is proximal. Therefore the diagonal \( \Delta = \{(y, y) : y \in Y_D\} \) is the unique minimal subset of the product system and we conclude that \( y_D = y_D' \), so that the above diagram is replaced by

\[
\begin{array}{ccc}
(M(G), m_0) & \xrightarrow{\pi} & (\Phi, c_0) \\
\psi_D \downarrow & & \downarrow \phi_D \\
(Y_D, y_D) & \xrightarrow{\phi_D} & (Y_D, y_D)
\end{array}
\]

Next form the product space

\( \Pi = \prod \{Y_D : D \text{ a clopen subset of } M(G)\} \),

and let \( \psi : M(G) \to \Pi \) be the map whose \( D \)-projection is \( \psi_D \) (i.e. \( (\psi(m))_D = \psi_D(m) \)). We set \( Y = \psi(M(G)) \) and observe that, since clearly the maps \( \psi_D \) separate points on \( M(G) \), the map \( \psi : M(G) \to Y \) is an isomorphism, with \( \psi(m_0) = y_0 \), where \( y_0 \in Y \) is defined by \( (y_0)_D = y_D \). Likewise define \( \phi : \Phi(X) \to Y \) by \( (\phi(m))_D = \phi_D(m) \), so that also \( \phi(c_0) = y_0 \). These equations force the identity \( \psi = \phi \circ \pi \) in the diagram

\[
\begin{array}{ccc}
(M(G), m_0) & \xrightarrow{\pi} & (\Phi, c_0) \\
\psi \downarrow & & \downarrow \phi \\
(Y, y_0) & \xrightarrow{\phi} & (Y, y_0)
\end{array}
\]

Since \( \psi \) is a bijection it follows that so are \( \pi \) and \( \phi \) and the proof is complete. \( \square \)
References


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