SUFFICIENT CONDITIONS UNDER WHICH A TRANSITIVE SYSTEM IS CHAOTIC

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Abstract. Let (X,T) be a topologically transitive dynamical system. We show that if there is a subsystem (Y,T) of (X,T) such that $(X\times Y, T\times T)$ is transitive, then (X,T) is strongly chaotic in the sense of Li and Yorke. We then show that many of the known sufficient conditions in the literature, as well as a few new results, are corollaries of this statement. In fact the kind of chaotic behavior we deduce in these results is a much stronger variant of Li-Yorke chaos which we call uniform chaos. For minimal systems we show, among other results, that uniform chaos is preserved by extensions and that a minimal system which is not uniformly chaotic is PI.

Contents

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Introduction

The presence or the lack of chaotic behavior is one of the most prominent traits of a dynamical system. However, by now there exists in the literature on dynamical systems a plethora of ways to define chaos. In 1975, Li and Yorke introduced a notion of chaos [LY75], known now as Li-Yorke chaos, for interval maps. With a small modification this notion can be extended to any metric space. Another notion was introduced later by Devaney [D89].

In [GW93] the authors suggested using positive topological entropy as the defining criterion for chaotic behavior. More recently it was shown that both Devaney chaos [HY02], and positive entropy [BGKM02] imply Li-Yorke chaos. We remark that weak mixing as well (or even scattering) implies Li-Yorke chaos. Thus, in a certain sense Li-Yorke chaos is the weakest notion of chaos. We refer the reader to the recent monograph [AAG08] and the review [GY08] on local entropy theory, which include discussions of the above notions.

It is natural to ask which transitive systems are chaotic and this is the main theme of this work. In Section 1 we introduce our terminology and review some basic facts. In Section 2 we first prove, the somewhat surprising fact (Theorem 2.11) that every transitive system is "partially rigid". This is then used in Section 3 to deduce the following criterion. For a transitive topological dynamical system (X, T) if there is a subsystem (Y, T) of (X, T) (i.e. Y is a non-empty closed and T-invariant subset of X) such that $(X \times Y, T \times T)$ is transitive, then (X, T) is strongly Li-Yorke chaotic. As we will see many of the known sufficient conditions in the literature, as well as a few new results, are corollaries of this fact. In fact the kind of chaotic behavior we deduce in these results is a much stronger variant of Li-Yorke chaos which we call uniform chaos. In Section 4 we reexamine these results in view of the Kuratowski-Mycielski theory. In Section 5 we specialize to minimal dynamical systems. After reviewing some structure theory we show, among other results that for minimal systems uniform chaos is preserved by extensions, and that if a minimal system is not uniformly chaotic then it is a PI system. We also show that a minimal *strictly PI* system which is not point distal admits a proximal scrambled Mycielski set. This perhaps suggests that a minimal system which does not contain such a set is actually point distal, but we have to leave that issue as an open problem.

Throughout the paper and mostly in Section 5 we make heavy use of structure theory and of the extension to $\beta\mathbb{N}$ of the $\mathbb{N}$ action given by T to. We refer, for example, to the sources [G76], [V77], [Au88], and [Ak97] for the necessary background.

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1. Preliminary definitions and results

In this section we briefly review some basic definitions and results from topological dynamics. Relevant references are [GM89], [GW93], [AAB96], [Ak97], [AG01], [HY02], [Ak03], [AAG08]. The latter is perhaps a good starting point for a beginner. One can also try to trace the historical development of these notions from that source and the reference list thereof.

1.1. **Transitivity and ideas of chaos.** We write $\mathbb{Z}$ to denote the integers, $\mathbb{N}$ for the natural numbers, i.e the positive integers. Throughout this paper a *topological dynamical system* (TDS for short) is a pair (X, T), where X is a non-vacuous compact metric space with a metric d and T is a continuous surjective map from X to itself. A non-vacuous closed invariant subset $Y \subset X$ defines naturally a *subsystem* (Y, T) of (X, T).

For subsets $A, B \subset X$ we define for a TDS (X, T) the *hitting time set* $N(A, B) := \{n \in \mathbb{N} : A \cap T^{-n}B \neq \emptyset\}$. When $A = \{x\}$ is a singleton we write simply $N(x, B)$ and if moreover B is a neighborhood of x we refer to $N(x, B)$ as the set of *return times* to B.

Recall that (X, T) is called *topologically transitive* (or just *transitive*) if for every pair of nonempty open subsets U and V, the set $N(U, V)$ is non-empty.

Let $\omega(x, T)$ be the set of the limit points of the orbit of x,

$$Orb(x, T) := \{x, T(x), T^2(x), \ldots\}.$$

A point $x \in X$ is called a *transitive point* if $\omega(x, T) = X$. It is easy to see that if (X, T) is transitive then the set of all transitive points, denoted $Trans_T$, is a dense G_δ set of X.

Remark 1.1. Notice that if (X, T) is transitive with $z \in Trans_T$ and X has an isolated point x then $x = T^k(z)$ for arbitrarily large values of k. Hence, x is periodic and is a transitive point. That is, X consists of a single periodic orbit. In particular, since we have defined $N(A, B) \subset \mathbb{N}$ it follows that the compactifications of the translation map $n \mapsto n + 1$ on $\mathbb{Z}$ or $\mathbb{N}$ are <u>not</u> transitive systems.

If $Trans_T = X$ then we say that (X, T) is *minimal*. Equivalently, (X, T) is minimal if and only if it contains no proper subsystems. By the well-known Zorn's Lemma argument any dynamical system (X, T) contains some minimal subsystem, which is called a *minimal set* of X. Each point belonging to some minimal set of X is called a *minimal point*.

A TDS (X,T) is *(topologically) weakly mixing* if the product system$(X\times X, T\times T)$ is transitive. Furstenberg's *2 implies n Theorem*, [F67], says that the product system $(X^n, T\times ...\times T)$ is then transitive for every positive integer n.

A pair $(x,y)\in X\times X$ is said to be *proximal* if $\liminf_{n\to+\infty} d(T^n x, T^n y)=0$ and it is called *asymptotic* when $\lim_{n\to+\infty} d(T^n x, T^n y)=0$. If in addition $x\neq y$, then (x,y) is a *proper* proximal (or asymptotic) pair. The sets of proximal pairs and asymptotic pairs of (X,T) are denoted by $P(X,T)$ and $Asym(X,T)$ respectively. A point $x\in X$ is a *recurrent point* if there are $n_i\nearrow+\infty$ such that $T^{n_i}x\to x$. A pair $(x,y)\in X^2$ which is not proximal is said to be *distal*. A pair is said to be a *Li-Yorke pair* if it is proximal but not asymptotic. A pair $(x,y)\in X^2\setminus\Delta_X$ is said to be a *strong Li-Yorke pair* if it is proximal and is also a recurrent point of X^2. Clearly a strong Li-Yorke pair is a Li-Yorke pair. A system without proper proximal pairs (Li-Yorke pairs, strong Li-Yorke pairs) is called *distal* (*almost distal*, *semi-distal* respectively). It follows directly from the definitions that a distal system is almost distal and an almost distal system is semi-distal.

One can localize the concept of distality. A point x is called a *distal point* if its proximal cell $P[x]=\{x'\in X:(x,x')\in P(X,T)\}=\{x\}$. A theorem of Auslander says that any proximal cell contains some minimal point (see below). Hence, a distal point is necessarily a minimal point. A minimal system (X,T) is called *point distal* if it contains a distal point. A theorem of Ellis [E73] says that in a metric minimal point distal system the set of distal points is dense and G_δ.

A dynamical system (X,T) is *equicontinuous* if for every $\epsilon>0$ there is $\delta>0$ such that $d(x,y)<\delta$ implies $d(T^n x, T^n y)<\epsilon$, for every $n\in\mathbb{N}$. It can be shown, see e.g. [Ak96], that if T is an equicontinuous surjection then it is an isometry of a metric topologically equivalent to d. It clearly follows that an equicontinuous system is distal.

As with distality, the notion of equicontinuity can be localized. A point $x\in X$ is called an *equicontinuity point* if for every $\epsilon>0$ there is $\delta>0$ such that $d(x,y)<\delta$ implies $d(T^n x, T^n y)<\epsilon$ for all $n\in\mathbb{N}$. We denote by Eq_T the set of equicontinuity points of (X,T). Notice that the term equicontinuity is used because it describes exactly the usual analysts' notion of equicontinuity for the set of iterates $\{T^n : n\in\mathbb{N}\}$. Eq_T is always a G_δ set. A TDS (X,T) is called *almost equicontinuous* if Eq_T is dense and so is residual. When $Eq_T=X$ then the usual compactness argument implies that the system is equicontinuous.

A TDS (X,T) is called *sensitive* if there is an $\epsilon>0$ such that whenever U is a nonempty open set there exist $x,y\in U$ such that $d(T^n x, T^n y)>\epsilon$ for some $n\in\mathbb{N}$. While this clearly implies that $Eq_T=\emptyset$, the converse is not true for general systems.

If the system (X,T) is transitive then the *Auslander-Yorke Dichotomy Theorem* says that either the system is sensitive and so $Eq_T=\emptyset$ or else $Eq_T=Trans_T$ and so the system is almost equicontinuous. In particular a minimal system is either equicontinuous or sensitive (see [AY80], [GW93] and [AAB96]). For more recent results on sensitivity, see [HLY, YZ08] and the references therein.

A *homomorphism* $\pi : (X,T) \to (Y,S)$ is a continuous map from X to Y such that $S \circ \pi = \pi \circ T$. It is called a *factor map* or an *extension* when π is onto, in which case we say that (X,T) an *extension* of (Y,S) and that (Y,S) is a *factor* of (X,T). An extension π is determined by the corresponding closed invariant equivalence relation $R_\pi = \{(x_1,x_2) : \pi x_1 = \pi x_2\} = (\pi \times \pi)^{-1}\Delta_Y \subset X \times X$.

An extension $\pi : (X,T) \to (Y,S)$ is called *asymptotic* if $R_\pi \subset Asmp(X,T)$. Similarly we define *proximal, distal* extensions. We define π to be an *equicontinuous* extension if for every $\epsilon > 0$ there is $\delta > 0$ such that $(x,y) \in R_\pi$ and $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \epsilon$, for every $n \in \mathbb{N}$. The extension π is called *almost one-to-one* if the set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is a dense G_δ subset of X.

A subset $A \subset X$ is called *scrambled* (*strongly scrambled*) if every pair of distinct points in A is Li-Yorke (strong Li-Yorke). The system (X,T) is said to be *Li-Yorke chaotic* (*strong Li-Yorke* chaotic) if it contains an uncountable scrambled (strongly scrambled) set.

A TDS (X,T) is said to be *chaotic in the sense of Devaney* (or an infinite *P-system*) if it is transitive and X is infinite with a dense set of periodic points. Such a system is always sensitive (see [BBCDS92] and [GW93]).

1.2. **Ellis semigroups.** An *Ellis semigroup* is a semigroup equipped with a compact Hausdorff topology such that for every $p \in E$ the map $R_p : E \to E$ defined by $R_p(q) = qp$ is continuous. (This is sometimes called a right topological, or a left topological, or a right semi-topological semigroup. Here we try to use a non-ambiguous term which we hope will standardize the terminology.)

A subset I of E is called an *ideal* (= left ideal) when it is nonempty and $q \in E, p \in I$ implies $qp \in I$. For example, if $p \in E$ then Ep is a closed ideal. A subset H of E is called a *co-ideal* when it is nonempty and for $q \in E, p \in H$ we have $qp \in H$ if and only if $q \in H$. So both ideals and co-ideals are nonempty subsemigroups of E. An *Ellis action* is an action of an Ellis semigroup E on a compact Hausdorff space X such that for every $x \in X$ the map $R_x : E \to X$ defined by $R_x(q) = qx$ is continuous. For example, the semigroup multiplication on an Ellis semigroup E is an action of E on itself called the translation action. For an Ellis action of E on X and $x \in X$, the *isotropy set* $H_x := \{p \in E : px = x\}$ is a co-ideal if it is nonempty.

The main example we will use is the Čech-Stone compactification of the positive integers, $\beta\mathbb{N}$. The translation map g on $\mathbb{N}$ defined by $n \mapsto n+1$ extends to a continuous (not surjective) map T_0 on $\beta\mathbb{N}$. If T is any continuous map on a compact Hausdorff space X then for each $x \in X$ the map given by $n \mapsto T^n(x)$ extends to a continuous map from $\beta\mathbb{N}$ to X. Concatenating these maps, we obtain a map $\beta\mathbb{N} \times X \to X$. In particular, from T_0 we get a map $\beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ which is associative and so gives $\beta\mathbb{N}$ the structure of an Ellis semigroup. For a TDS (X,T) the map $\beta\mathbb{N} \times X \to X$ is an Ellis action extending the $\mathbb{N}$ action. We let $\beta^*\mathbb{N}$ denote the closed subset $\beta\mathbb{N} \setminus \mathbb{N}$. A closed subset J of $\beta\mathbb{N}$ is an ideal if and only if $J \subset T_0(J)$. In particular, $\beta^*\mathbb{N}$ is an ideal and the map T_0 restricts to a homeomorphism on $\beta^*\mathbb{N}$. For any $x \in X$ observe that $\beta\mathbb{N}x = R_x(\beta\mathbb{N})$ is the orbit-closure of x in X and $\beta^*\mathbb{N}x = \omega(x,T)$. For the $\beta\mathbb{N}$ action on X associated with a TDS (X,T) and $x \in X$

we use $H_x := \{p \in \beta^*\mathbb{N} : px = x\}$, the isotropy set with respect to the closed ideal $\beta^*\mathbb{N}$. Thus, H_x is nonempty exactly when x is recurrent.

At times it is useful that the semigroup act *faithfully* on X, i.e. distinct elements act by different maps on X. We then use the *enveloping semigroup* $E = E(X,T)$ which is the closure in X^X (with its compact, usually non-metrizable, pointwise convergence topology) of the set $\{T^n : n \in \mathbb{N}\}$. With the operation of composition of maps this is an Ellis semigroup and the operation of evaluation is an Ellis action of $E(X,T)$ on X which extends the action of $\mathbb{N}$ via T. The map $n \mapsto T^n$ extends to define a continuous map $\beta\mathbb{N} \to X^X$ whose image is $E(X,T)$.

The elements of $E(X,T)$ may behave very badly as maps of X into itself; usually they are not even Borel measurable. However our main interest in this Ellis action lies in its algebraic structure and its dynamical significance. A key lemma in the study of this algebraic structure is the following:

Lemma 1.2 (Ellis)**.** *If E is an Ellis semigroup, then E contains an* idempotent*; i.e., an element v with $v^2 = v$.*

If $v \in E$ is an idempotent then $v \in H_v$ and so H_v is a co-ideal.

In the next proposition we state some basic properties of Ellis semigroups.

Proposition 1.3. *Let E be an Ellis semigroup.*

(1) *A nonempty closed subsemigroup H of E is a minimal closed co-ideal if and only if $H = H_u$ for every idempotent $u \in H$.*
(2) *A nonempty subset I of E is a minimal ideal of the semigroup E if and only if for every $I = Ep$ for every $p \in I$. In particular a minimal ideal is closed. We will refer to it simply as a* minimal ideal. *Minimal ideals I in E exist and for each such ideal the set of idempotents in I, denoted by $J = J(I)$, is non-empty.*
(3) *Let I be a minimal ideal and J its set of idempotents then:*
   (a) *For $v \in J$ and $p \in I$, $pv = p$.*
   (b) *For each $v \in J$, $vI = \{vp : p \in I\} = \{p \in I : vp = p\}$ is a subgroup of I with identity element v. For every $w \in J$ the map $p \mapsto wp$ is a group isomorphism of vI onto wI.*
   (c) *$\{vI : v \in J\}$ is a partition of I. Thus if $p \in I$ then there exists a unique $v \in J$ such that $p \in vI$.*
(4) *Let $K, L,$ and I be minimal ideals of E. Let v be an idempotent in I, then there exists a unique idempotent v' in L such that $vv' = v'$ and $v'v = v$. (We write $v \sim v'$ and say that v' is* equivalent *to v.) If $v'' \in K$ is equivalent to v', then $v'' \sim v$.*

Let (X,T) be a TDS and let $(p,x) \mapsto px$ be the induced Ellis action of $\beta\mathbb{N}$ on X.

(4) *I is a minimal ideal of the semigroup $\beta\mathbb{N}$ if and only if it is a minimal subsystem of $(\beta^*\mathbb{N}, T_0)$. If I and L are minimal ideals of $\beta\mathbb{N}$ and $v \in J(I)$ then The map $p \mapsto pv$ of L to I is an isomorphism of dynamical systems.*
(5) *A pair $(x,x') \in X \times X$ is proximal if and only if $px = px'$ for some $p \in E$, if and only if there exists a minimal ideal I in E with $px = px'$ for every $p \in I$.*

(6) *If* (X,T) *is minimal, then the proximal cell of* x

$$P[x] = \{x' \in X : (x,x') \in P\} = \{vx : v \in \hat{J}\},$$

where $\hat{J} = \bigcup\{J(I) : I$ *is a minimal left ideal in* $\beta\mathbb{N}\}$ *is the set of minimal idempotents in* $\beta\mathbb{N}$.

Finally, a homomorphism $\pi : (X,T) \to (Y,S)$ is an action map for the $\beta\mathbb{N}$ Ellis actions, i.e. $\pi(px) = p\pi(x)$ for all $(p,x) \in \beta\mathbb{N} \times X$.

We refer to [G76], [Au88], [Ak97] and [G03] for more details.

1.3. **Families and filters.** We say that a collection $\mathcal{F}$ of subsets of $\mathbb{N}$ (or $\mathbb{Z}$) is a *a family* if it is hereditary upward, i.e. $F_1 \subseteq F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is called *proper* if it is neither empty nor the entire power set of $\mathbb{N}$, or, equivalently if $\mathbb{N} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any nonempty collection $\mathcal{A}$ of subsets of $\mathbb{N}$ generates a family $\mathcal{F}(\mathcal{A}) := \{F \subseteq \mathbb{N} : F \supset A$ for some $A \in \mathcal{A}\}$.

If a family $\mathcal{F}$ is closed under finite intersections and is proper, then it is called a *filter*. A collection of nonempty subsets $\mathcal{B}$ is a *filter base* if for every $B_1, B_2 \in \mathcal{B}$ there is $B_3 \in \mathcal{B}$ with $B_3 \subset B_1 \cap B_2$. Clearly, $\mathcal{B}$ is a filter base if and only if the family $\mathcal{F}(\mathcal{B})$ is a filter. A maximal filter is called an *ultrafilter*. By Zorn's lemma every filter is contained in an ultrafilter.

For a family $\mathcal{F}$ its *dual* is the family $\mathcal{F}^* := \{F \subseteq \mathbb{N} : F \cap F' \neq \emptyset \ for\ all\ F' \in \mathcal{F}\}$.

The collection of ultrafilters on $\mathbb{N}$ can be identified with the Čech-Stone compactification $\beta\mathbb{N}$, where to $n \in \mathbb{N}$ corresponds the principle ultrafilter $\{A : n \in A \subset \mathbb{N}\}$. (see Subsection 1.2 above).

Lemma 1.4. *Let* (X,T) *be a transitive TDS. Then the collection of sets*

$$\mathcal{A} = \{N(U,U) : U \text{ is a nonempty open subset of } X\}$$

is a filter base, whence the family $\mathcal{F}(\mathcal{A})$ *is a filter.*

Proof. Let U_1 and U_2 be nonempty open subsets of X. As (X,T) is transitive, there is an $n \in \mathbb{N}$ such that $U_3 = U_1 \cap T^{-n}U_2 \neq \emptyset$. Then

$$\begin{aligned} N(U_3,U_3) &\subseteq N(U_1,U_1) \cap N(T^{-n}U_2, T^{-n}U_2) \\ &= N(U_1,U_1) \cap N(T^nT^{-n}U_2, U_2) \\ &= N(U_1,U_1) \cap N(U_2,U_2), \end{aligned}$$

and our claim follows. □

A subset of F of $\mathbb{N}$ is called *thick* when it contains arbitrarily long runs. That is, for every $n \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $i, i+1, \dots, i+n \in F$. With g the translation map $n \mapsto n+1$ on $\mathbb{N}$, $F \subseteq \mathbb{N}$ is thick exactly when any finite intersection of translates $\{g^{-i}(F) : i = 0, 1, ...\}$ is nonempty. In that case these intersections generate a filter of thick subsets. We will then denote this filter by $\mathcal{F}_F$. It is a theorem of Furstenberg [F67] that a TDS (X,T) is weak mixing if and only if $N(U_1,U_2)$ is thick for every pair of nonempty open $U_1, U_2 \subseteq X$.

For a TDS (X,T) and a point $x \in X$ define

$$\mathcal{I}_x = \{N(x,U) : U \text{ is a neighborhood of } x\}.$$

A point x is *recurrent* for (X,T) if and only if each such return time set $N(x,U)$ is nonempty and so if and only if $\mathcal{I}_x$ is a filter base. For a pair $(x_1,x_2) \in X \times X$ define

$$\mathcal{P}_{(x_1,x_2)} = \{N((x_1,x_2),V) : V \text{ is a neighborhood of the diagonal in } X \times X\}.$$

A pair (x_1,x_2) is proximal if and only if each such $N((x_1,x_2),V)$ is nonempty and so if and only if $\mathcal{P}_{(x_1,x_2)}$ is a filter base.

2. Transitivity, recurrence and proximality

2.1. Recurrent and proximal sets.

Definition 2.1. Let (X,T) be a TDS and $K \subseteq X$.

(1) We say that K is *pointwise recurrent* if every $x \in K$ is a recurrent point.
(2) K is *uniformly recurrent* if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ with $d(T^n x, x) < \epsilon$ for all x in K.
(3) K is *recurrent* if every finite subset of K is uniformly recurrent.

In [GM89], these concepts were introduced for the total space X. When X is uniformly recurrent, the system (X,T) is called *uniformly rigid* and when X is recurrent, the system is called *weakly rigid*. (X,T) is called *rigid* when the identity on X is the pointwise limit of some *sequence* of iterates $\{T^{n_k}\}$ with $n_k \nearrow \infty$.

There are related concepts for proximality.

Definition 2.2. Let (X,T) be a TDS and $K \subset X$.

(1) A subset K of X is called *pairwise proximal* if every pair $(x,x') \in K \times K$ is proximal.
(2) The subset K is called *uniformly proximal* if for every $\epsilon > 0$ there is $n \in \mathbb{N}$ with $\operatorname{diam} T^n K < \epsilon$.
(3) A subset K of X is called *proximal* if every finite subset of K is uniformly proximal.

Remark 2.3. Let $(\Omega = \{1,2,3,4\}^{\mathbb{Z}}, \sigma)$ be the Bernoulli shift on four symbols. Let a_n, b_n, c_n and d_n denote the blocks of n consecutive 1s, 2s, 3s and 4s, respectively. Let $\omega_1 \in \Omega$ be defined on $\mathbb{N}$ as the concatenation of the blocks $a_n b_n c_n d_n$, $n = 1,2,3,\dots$. Then, let $\omega_1(-j) = \omega_1(j)$. Define ω_2 similarly but with blocks $a_n c_n d_n b_n$, and ω_3 using $c_n b_n d_n a_n$. Clearly the set $\{\omega_1,\omega_2,\omega_3\}$ is pairwise proximal but not proximal. It seems that, e.g. by using the method of *concatenation flows* developed in [GM89], it is possible to construct a minimal subsystem of the shift $([0,1]^{\mathbb{Z}}, \sigma)$ which contains such a three point set as required. However, we have not worked out the details of such a construction.

Observe that

$$\text{uniform recurrence} \Rightarrow \text{recurrence} \Rightarrow \text{pointwise recurrence},$$

and

$$\text{uniform proximality} \Rightarrow \text{proximality} \Rightarrow \text{pairwise proximality}.$$

Remark 2.4. Clearly, if K satisfies any of these properties then each subset of K satisfies the corresponding property. If K is either uniformly recurrent or uniformly proximal then its closure $\overline{K}$ satisfies the corresponding property.

Recurrence and proximality are properties of *finite type*, that is, they hold for K if and only if they hold for every finite subset of K. It follows that if $\mathcal{K}$ is a chain of recurrent subsets (or proximal subsets) then $\bigcup \mathcal{K}$ is recurrent (resp. proximal). It then follows from Zorn's Lemma that any recurrent/proximal set is contained in a maximal recurrent/proximal set.

For a TDS (X, T) and $n \geq 1$, we define subsets of X^n

$$Recur_n(X) = \{(x_1, \ldots, x_n) : \forall \epsilon > 0,\ \exists k \in \mathbb{N} \text{ with } d(T^k x_i, x_i) < \epsilon, \forall i\}.$$

$$Prox_n(X) = \{(x_1, x_2, \cdots, x_n) : \forall \epsilon > 0\ \exists k \in \mathbb{N}\text{with } d(T^k x_i, T^k x_j) < \epsilon, \forall i, j\}.$$

Clearly, $Recur_n(X)$ and $Prox_n(X)$ are G_δ subsets of X^n.

For $K \subset X$ we use the Ellis action of $\beta\mathbb{N}$ on X to define subsets of $\beta\mathbb{N}$

$$H_K = \{p \in \beta^*\mathbb{N} : px = x \text{ for all } x \in K\},$$

$$A_K = \{p \in \beta^*\mathbb{N} : px_1 = px_2 \text{ for all } x_1, x_2 \in K\}.$$

If $K = \{x\}$ then the three notions of recurrence agree for K and they hold if and only if x is a recurrent point. Since a point x is a recurrent point if and only if $x \in \omega(x, T) = \beta^*\mathbb{N}x$, it is recurrent exactly when there exists $p \in \beta^*\mathbb{N}$ such that $px = x$ and so when H_K is nonempty. On the other hand, $(x_1, ..., x_n)$ is an element of $Recur_n(X)$ if and only if it is a recurrent point for the n-fold product $(X^n, T \times \cdots \times T)$.

If $K = \{x_1, ..., x_n\}$ then K is proximal if and only if $(x_1, ..., x_n) \in Prox_n(X)$ in which case proximality is automatically uniform. Furthermore, K is proximal if and only if there exists $p \in \beta^*\mathbb{N}$ such that pK is a singleton and so if and only if A_K is nonempty.

Proposition 2.5. *Let (X, T) be a TDS and $K \subset X$.*

(a) *When it is nonempty, the subset H_K is a closed co-ideal in $\beta^*\mathbb{N}$.*
The following are equivalent.
  (1) *K is recurrent.*
  (2) *For every $n \geq 1$ $K^n \subset Recur_n(X)$.*
  (3) *H_K is a nonempty subset of $\beta^*\mathbb{N}$.*
  (4) *There exists an idempotent $u \in \beta^*\mathbb{N}$ such that $ux = x$ for all $x \in K$.*

(b) *When it is nonempty, the subset A_K is a closed ideal in $\beta^*\mathbb{N}$. The following are equivalent.*
  (1) *K is proximal.*
  (2) *For every $n \geq 1$ $K^n \subset Prox_n(X)$.*

(3) *A_K is a nonempty subset of $\beta^*\mathbb{N}$.*
(4) *There exists an idempotent $u \in \beta^*\mathbb{N}$ such that uK is a singleton subset of X.*

Proof. It is clear that H_K is a closed subsemigroup and that A_K is an ideal when it is nonempty (by convention an ideal is required to be nonempty). Furthermore, if $L_1, L_2 \subset X$ then

$$\begin{aligned} H_{L_1 \cup L_2} &= H_{L_1} \cap H_{L_2}, \\ A_{L_1 \cup L_2} &\subset A_{L_1} \cap A_{L_2}. \end{aligned}$$

(a) (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) These conditions are clearly equivalent when K is finite and conditions (1) and (2) hold if and only if they hold for every finite subset of K. Now assume that H_L is nonempty for every finite subset L of K. The above equation implies that the collection $\{H_L : L \subset K \text{ and } L \text{ finite}\}$ is a filterbase and so its intersection H_K is nonempty.

By Lemma 1.2 H_K contains an idempotent when it is nonempty and so (3) $\Rightarrow$ (4). Finally (4) $\Rightarrow$ (1) is clear.

The proof of (b) is completely analogous. □

Proposition 2.6. *Let (X, T) be a topologically transitive TDS. For every $n \geq 1$ $Recur_n(X)$ is dense in X^n.*

Proof. Fix n and define for $\kappa \in (\mathbb{N})^n$, $T^\kappa : X \to X^n$ by $T^\kappa(x) = (T^{\kappa_1}(x), ..., T^{\kappa_n}(x))$. This defines a homomorphism from (X, T) to the n-fold product $(X^n, T \times \cdots \times T)$.

Let $z \in X$ be a transitive point for (X, T). For any $\epsilon > 0$ and any $(x_1, ..., x_n) \in X^n$ there exists κ such that $d(T^{\kappa_i}(z), x_i) < \epsilon$ for all i. Since z is a recurrent point, $T^\kappa(z)$ is a recurrent point ϵ close to $(x_1, ..., x_n)$. Thus, $Recur_n(X)$ is dense in X^n. □

As we will now see, there is no analogue of Furstenberg's $2 \Rightarrow n$ theorem for density of recurrent points.

If (X, T) is a TDS then $A \subseteq X$ is called *wandering* when the sets $\{T^{-k}(A) : k = 0, 1, 2, \dots\}$ are pairwise disjoint. It is clear that a wandering open set contains no recurrent points.

Lemma 2.7. *For any $n \geq 2$ there exists a finite collection of TDS, $\mathcal{W} = \{(X_1, T_1), ..., (X_n, T_n)\}$ such that for each $i = 1, ..., n$ the $n-1$-fold product system (Y_i, S_i) which omits the factor (X_i, T_i) is weak mixing, but the n-fold product $(X_1 \times \cdots \times X_n, T_1 \times \cdots \times T_n)$ contains a nonempty wandering open set U.*

Proof. We uses a construction due to Weiss, see the Appendix in [AG01]. Recall from Section 1.3 that a subset of F of $\mathbb{N}$ is called *thick* when each finite intersection of the translates $\{g^{-i}(F) : i = 0, 1, ...\}$ is nonempty. The intersections generate a filter of thick sets denoted $\mathcal{F}_F$. Furthermore, a TDS (X, T) is weak mixing if and only if $N(U, V)$ is thick for every pair of nonempty open sets $U, V \subseteq X$ by a theorem of Furstenberg.

On the other hand, given a thick $F \subset \mathbb{N}$ the Weiss construction yields a TDS (X, T) and a nonempty open $U \subseteq X$ such that $N(U, U) = F$ and, in addition, $N(V_1, V_2) \in \mathcal{F}_F$ for every pair of nonempty open $V_1, V_2 \subseteq X$. In particular, the system is weak mixing.

Now choose $R_1, ..., R_n$ pairwise disjoint thick subsets of $\mathbb{N}$ with union F and for $k = 1, ..., n$ let $F_k = F \setminus R_k$. Apply the Weiss construction to F_k to define (X_k, T_k) and $U_k \subseteq X_k$. For nonempty open $V_1, V_2 \subseteq X_k$, $N(V_1, V_2) \in \mathcal{F}_{F_k} \subseteq \mathcal{F}_{R_i}$ provided $i \neq k$. Since $\mathcal{F}_{R_i}$ is a filter it easily follows that for nonempty open $W_1, W_2 \subseteq Y_i$ $N(W_1, W_2) \in \mathcal{F}_{R_i}$. Hence, each (Y_i, S_i) is weak mixing.

On the other hand, for $U = U_1 \times \cdots \times U_n \subseteq X_1 \times \cdots \times X_n$, we have $N(U, U) = \emptyset$ which says that U is wandering. □

Remark 2.8. Note that this can not happen when all the systems in $\mathcal{W}$ are minimal, since it was proved in [HY04] that if (X, T) is minimal and weakly mixing then $N(A, B)$ has lower Banach density 1 for each pair of non-empty open subsets of X.

Theorem 2.9. *For any $n \geq 2$ there exists a TDS (X, T) such that $Recur_k(X)$ is dense in X^k for all $k < n$ but $Recur_n(X)$ is not dense in X^n.*

Proof. Let $\mathcal{W}$ be the list of n weak mixing systems given by Lemma 2.7 and let (X, T) be the co-product, i.e. the disjoint union of the (X_i, T_i)'s. For $k < n$ the product X^k is the disjoint union of n^k clopen invariant subsets on each of which the induced subsystem is weak mixing. Hence, $Recur_k(X)$ is dense in X^k. However, X^n contains a clopen invariant subset on which the subsystem is isomorphic to the product $(X_1 \times \cdots \times X_n, T_1 \times \cdots \times T_n)$ which contains a wandering nonempty open set U. U is disjoint from $Recur_n(X)$. □

Remark 2.10. We remark that (1) if (X, T) is transitive, then $Recur_k(X)$ is dense in X^k for all $k \in \mathbb{N}$, see Proposition 2.6, and (2) if the TDS (X_i, T_i) has a dense set of minimal points for each $1 \leq i \leq k$, then the set of minimal points of $(X_1 \times \ldots \times X_k, T_1 \times \ldots \times T_k)$ is also dense, see [AG01].

2.2. **Transitivity implies dense recurrence.** A nonempty subset K of a compact space X is a *Mycielski* set if it is a countable union of Cantor sets. In the following theorem we show that every transitive TDS contains a dense recurrent Mycielski subset. While we will later derive this result, and more, from the Kuratowski-Mycielski Theorem, we include here a direct proof which employs an explicit construction rather than an abstract machinery (see Theorem 4.7 below).

Theorem 2.11. *Let (X, T) be a transitive TDS without isolated points. Then there are Cantor sets $C_1 \subseteq C_2 \subseteq \cdots$ such that $K = \bigcup_{i=1}^{\infty} C_n$ is a dense recurrent subset of X and for each $N \in \mathbb{N}$, C_N is uniformly recurrent.*

If in additional, for each $n \in \mathbb{N}$, $Prox_n(X)$ is dense in X^n, then we can require that for each $N \in \mathbb{N}$, C_N is uniformly proximal, whence K is a proximal set.

Proof. Let $Y = \{y_1, y_2, \dots\}$ be a countable dense subset of X and for each $n \geq 1$ let $Y_n = \{y_1, y_2, \dots, y_n\}$. Let $\mathcal{F}$ be the smallest family containing the collection

$$\{N(U,U) : U \text{ is a nonempty open subset of } X\}.$$

Since (X,T) is transitive, $\mathcal{F}$ is a filter by Lemma 1.4. Let $a_0 = 0$ and $V_{0,1} = X$. We have the following claim.

Claim: For each $S \in \mathcal{F}^*$ there are sequences $\{a_n\} \subseteq \mathbb{N}$, $\{k_n\} \subseteq S$, and sequences $\{U_n\}_{n=1}^{\infty}$ and $\{V_{n,1}, V_{n,2}, \cdots, V_{n,a_n}\}_{n=1}^{\infty}$ of nonempty open subsets of X with the following properties:

(1) $2a_{n-1} \leq a_n \leq 2a_{n-1} + n$.
(2) $diam\, V_{n,i} < \frac{1}{n}$, $i = 1, 2, \dots, a_n$.
(3) The closures $\{\overline{V_{n,i}}\}_{i=1}^{a_n}$ are pairwise disjoint.
(4) $\overline{V_{n,2i-1}} \cup \overline{V_{n,2i}} \subset V_{n-1,i}$, $i = 1, 2, \cdots, a_{n-1}$.
(5) $Y_n \subset B(\bigcup_{i=1}^{a_n} V_{n,i}, \frac{1}{n})$, where $B(A, \epsilon) := \{x \in X : d(x, A) < \epsilon\}$.
(6) $T^{k_n}(V_{n,2i-1} \cup V_{n,2i}) \subseteq V_{n-1,i}$, $i = 1, 2, \cdots, a_{n-1}$.

Proof of Claim: For $j = 1$, take $a_1 = 2$, $k_1 = 1$, and $V_{1,1}, V_{1,2}$ any two nonempty open sets of diameter < 1 with disjoint closures such that $y_1 \in B(V_{1,1} \cup V_{1,2}, 1)$. Suppose now that for $1 \leq j \leq n-1$ we have $\{a_j\}_{j=1}^{n-1}$, $\{k_j\}_{j=1}^{n-1}$ and $\{V_{j,1}, V_{j,2}, \cdots, V_{j,a_j}\}$, satisfying conditions (1)-(6).

Choose $2a_{n-1} \leq a_n \leq 2a_{n-1} + n$ and nonempty open subsets $V_{n,1}^{(0)}, V_{n,2}^{(0)}, \cdots, V_{n,a_n}^{(0)}$ of X such that:

(a) $diam\, V_{n,i}^{(0)} < \frac{1}{2n}$, $i = 1, 2, \cdots, a_n$.
(b) The closures $\{\overline{V_{n,i}^{(0)}}\}_{i=1}^{a_n}$ are pairwise disjoint.
(c) $\overline{V_{n,2i-1}^{(0)}} \cup \overline{V_{n,2i}^{(0)}} \subset V_{n-1,i}$, $i = 1, 2, \cdots, a_{n-1}$.
(d) $Y_n \subset B(\bigcup_{i=1}^{a_n} V_{n,i}^{(0)}, \frac{1}{2n})$.

As $N(V_{n,i}^{(0)}, V_{n,i}^{(0)}) \in \mathcal{F}$ for each $1 \leq i \leq a_n$, $\bigcap_{i=1}^{a_n} N(V_{n,i}^{(0)}, V_{n,i}^{(0)}) \in \mathcal{F}$. Take $k_n \in S \cap \bigcap_{i=1}^{a_n} N(V_{n,i}^{(0)}, V_{n,i}^{(0)})$. Hence there are nonempty open sets $V_{n,i}^{(1)} \subseteq V_{n,i}^{(0)}$, $1 \leq i \leq a_n$, such that

(e) $T^{k_n}(V_{n,2i-1}^{(1)} \cup V_{n,2i}^{(1)}) \subseteq V_{n-1,i}$, $i = 1, 2, \cdots, a_{n-1}$.

Let $V_{n,i} = V_{n,i}^{(1)}$, $1 \leq i \leq a_n$. Then the conditions $(1) - (6)$ hold for n. By induction we have the claim.

Let $C_n = \bigcap_{j=n}^{\infty} \bigcup_{i=1}^{2^{j-n}a_n} \overline{V_{j,i}}$. Then $C_1 \subseteq C_2 \subseteq \cdots$, and by $(1) - (4)$, C_n is a Cantor set. By (2),(4) and (5), $K = \bigcup_{n=1}^{\infty} C_n$ is dense in X. For each $N \in \mathbb{N}$, by (6), C_N is uniformly recurrent.

Finally, if in addition for each $n \in \mathbb{N}$, $Prox_n(X)$ is dense in $X^{(n)}$ then we can in the above construction, when choosing the subsets $V_{n,i}^{(1)} \subseteq V_{n,i}^{(0)}$, $1 \leq i \leq a_n$, add the following condition to the claim above:

$$(7) \text{ for each } n \in \mathbb{N} \text{ there is } t_n \in \mathbb{N} \text{ such that } \operatorname{diam} T^{t_n}(\bigcup_{i=1}^{a_n} \overline{V_{n,i}}) < \frac{1}{n}.$$

By the requirement (7) we obtain that for each $N \in \mathbb{N}$, C_N is uniformly proximal. □

Remark 2.12. Using the fact that the set $Trans_T$ of transitive points is dense in X we can in the above construction, when choosing the subsets $V_{n,i}^{(1)} \subseteq V_{n,i}^{(0)}$, $1 \le i \le a_n$, add the following condition to the claim in the proof:

$$(8)\ Y_n \subseteq B(Orb(x,T), \frac{1}{n}) \ \text{ for each } x \in \bigcup_{i=1}^{a_n} \overline{V_{n,i}}.$$

It then follows that every point in $\bigcup\limits_{i=1}^{\infty} C_n$ is a transitive point.

Motivated by Theorem 2.11 we define uniformly chaotic set as follows:

Definition 2.13. Let (X,T) be a TDS. A subset $K \subseteq X$ is called a *uniformly chaotic set* if there are Cantor sets $C_1 \subseteq C_2 \subseteq \cdots$ such that

1. $K = \bigcup\limits_{i=1}^{\infty} C_n$ is a recurrent subset of X and also a proximal subset of X;
2. for each $N \in \mathbb{N}$, C_N is uniformly recurrent; and
3. for each $N \in \mathbb{N}$, C_N is uniformly proximal.

(X,T) is called *(densely) uniformly chaotic*, if (X,T) has a (dense) uniformly chaotic subset.

Remark 2.14. In fact condition (1) actually follows from (2) and (3), see Remark 2.4.

Actually, it follows that the inclusion of K is a pointwise limit of a sequence of restrictions $\{T^{n_k} \restriction K\}$. Let $J_N = \{n : d(T^n x, x) < 1/N \text{ for all } x \in C_N\}$. Each J_N is nonempty by assumption and $J_{N+1} \subset J_N$. Choose $n_k \in J_i$. As $k \to \infty$, $T^{n_k}x \to x$ for all $x \in \bigcup_{i=1}^{\infty} C_i$.

Obviously, a uniformly chaotic set is an uncountable strongly scrambled set, hence every uniformly chaotic system is strongly Li-Yorke chaotic. Restating Theorem 2.11 we have:

Theorem 2.15. *Let (X,T) be a nontrivial transitive TDS. If for each $n \in \mathbb{N}$, $Prox_n(X)$ is dense in X^n, then (X,T) is densely uniformly chaotic. In particular every such system is strongly Li-Yorke chaotic.*

Notice that if X has an isolated point then it is a single periodic orbit and so the system is distal. So $Prox_n(X)$ is not dense except when X is a singleton.

3. A criterion for chaos and applications

3.1. A criterion for chaos.

Theorem 3.1 (A criterion for chaos)**.** *Let (X,T) be a nontrivial transitive TDS. If there is some subsystem (Y,T) of (X,T) such that $(X\times Y,T)$ is transitive, then (X,T) is densely uniformly chaotic.*

Proof. By Theorem 2.15, it suffices to show that for each $n\in\mathbb{N}$, $Prox_n(X)$ is dense in $X^{(n)}$. For a fixed $n\in\mathbb{N}$ and any $\epsilon>0$ let

$$P_n(\epsilon)=\{(x_1,x_2,\dots,x_n): \exists m\in\mathbb{N} \text{ such that } \operatorname{diam}(\{T^m x_1,\dots,T^m x_n\})<\epsilon\}.$$

Thus $Prox_n(X)=\bigcap_{m=1}^{\infty}P_n(\frac{1}{m})$ and by Baire's category theorem it is enough to show that for every $\epsilon>0$, $P_n(\epsilon)$ is a dense open subset of X^n.

Fix $\epsilon>0$, let $U_1,U_2,\cdots,U_n$ be a sequence of nonempty open subsets of X, and let W be a nonempty open subset of Y with $\operatorname{diam}(W)<\epsilon$. By assumption $(X\times Y,T)$ is transitive, whence

$$N(U_1\times W,U_2\times W)=N(U_1,U_2)\cap N(W\cap Y,W\cap Y)\neq\emptyset.$$

Let m_2 be a member of this intersection. Then

$$U_1\cap T^{-m_2}U_2\neq\emptyset \quad\text{and}\quad W\cap T^{-n_2}W\cap Y\neq\emptyset.$$

By induction, we choose natural numbers $m_3,m_4,\cdots,m_n$ such that

$$U_1\cap\bigcap_{i=2}^{n}T^{-m_i}U_i\neq\emptyset \quad\text{and}\quad W\cap\bigcap_{i=2}^{n}T^{-m_i}W\cap Y\neq\emptyset.$$

Since (X,T) is transitive, there is a transitive point $x\in U_1\cap\bigcap_{i=2}^{n}T^{-m_i}U_i$ and let $y\in W\cap\bigcap_{i=2}^{n}T^{-m_i}W$. Since x is a transitive point, there exists a sequence l_k such that $\lim_{k\to\infty}T^{l_k}x=y$. Thus, $\lim_{k\to\infty}T^{l_k}(T^{m_i}x)=T^{m_i}y$ for each $2\le i\le n$. Since $\{y,T^{m_2}y,\dots,T^{m_3}y\}\subset W$ and $\operatorname{diam}(W)<\epsilon$, for large enough l_k, we have

$$\operatorname{diam}(\{T^{l_k}x,T^{l_k}(T^{m_2}x),\dots,T^{l_k}(T^{m_n}x)\})<\epsilon.$$

That is, $(x,T^{m_2}x,\dots,T^{m_n}x)\in P_n(\epsilon)$. Noting that $(x,T^{m_2}x,\dots,T^{m_n}x)\in U_1\times U_2\times\cdots\times U_n$, we have shown that

$$P_n(\epsilon)\cap U_1\times U_2\times\cdots\times U_n\neq\emptyset.$$

As $U_1,U_2,\cdots,U_n$ are arbitrary, $P_n(\epsilon)$ is indeed dense in $X^{(n)}$. □

3.2. Some applications.

In the rest of this section we will obtain some applications of the above criterion. First, we need to recall some definitions (see [BHM02, HY02]).

Two topological dynamical systems are said to be *weakly disjoint* if their product is transitive. Call a TDS (X,T):

- *scattering* if it is weakly disjoint from every minimal system;
- *weakly scattering* if it is weakly disjoint from every minimal equicontinuous system;

- *totally transitive* if it is weakly disjoint from every periodic system. (Note that this is equivalent to the usual definition which requires that (X, T^n) be transitive for all $n \geq 1$.)

Using this terminology and applying Theorem 2.15 we easily obtain the following:

Corollary 3.2. *If (X, T) is a TDS without isolated points and one of the following properties, then it is densely uniformly chaotic:*

(1) *(X, T) is transitive and has a fixed point;*
(2) *(X, T) is totally transitive with a periodic point;*
(3) *(X, T) is scattering;*
(4) *(X, T) is weakly scattering with an equicontinuous minimal subset;*
(5) *(X, T) is weakly mixing.*

Finally

(6) *If (X, T) is transitive and has a periodic point of order d, then there is a closed T^d-invariant subset $X_0 \subset X$, such that (X_0, T^d) is densely uniformly chaotic and $X = \bigcup_{j=0}^{d-1} T^j X_0$. In particular (X, T) is uniformly chaotic.*

Proof. The only claim that needs a proof is (6). Suppose $y_0 \in X$ is a periodic point of period d and let x_0 be a transitive point; so that $\overline{Orb_T(x_0)} = X$. Set $\overline{Orb_{T^d}(x_0)} = X_0$ (this may or may not be all of X). In any case the dynamical system (X_0, T^d) is transitive, and has a fixed point. Thus, by case (1), it is densely uniformly chaotic for T^d. Both uniform proximality and uniform recurrence of subsets go over to (X, T), hence (X, T) is uniformly chaotic. Clearly $X = \bigcup_{j=0}^{d-1} T^j X_0$. □

Part (6) provides a new proof of a result of J-H. Mai [Mai04], and as in Mai's paper we have the following corollary.

Theorem 3.3. *Devaney chaos implies uniform chaos.*

Remark 3.4. If (X, T) is minimal and $Prox(X)$ is dense then (X, T) is weak mixing (as obviously it admits no nontrivial equicontinuous factors). Thus in the minimal case, dense uniform chaos is equivalent to weak mixing.

One can strengthen Corollary 3.2 (2) in the following way.

Definition 3.5. Let (X, T) be a TDS. A point $x \in X$ is *regularly almost periodic* if for each neighborhood U of x there is some $k \in \mathbb{N}$ such that $k\mathbb{N} \subseteq N(x, U)$. Note that such a point is in particular a minimal point (i.e. its orbit closure is minimal).

Remark 3.6. Let (X, T) be a minimal system. Then (X, T) contains a regularly almost periodic point if and only if it is an almost one-to-one extension of an adding machine. If in addition (X, T) is a subshift then it is isomorphic to a Toeplitz system (see e.g. [MP80]).

Next we recall the following definition from [AG01].

Definition 3.7. A property of topological dynamical systems is said to be *residual* if it is non-vacuous and is inherited by factors, almost one-to-one lifts, and inverse limits.

It is not hard to check that being weakly disjoint from a fixed TDS (X, T) is a residual property (see [AG01]). One can also show that the smallest class of TDS which contains the periodic orbits and is closed under inverse limits and almost one-to-one extensions is exactly the class of almost one-to-one extensions of adding machines. It now follows that a TDS is totally transitive if and only if it is weakly disjoint from every almost one-to-one extension of an adding machine.

Corollary 3.8. *If (X, T) is totally transitive with a regularly almost periodic point, then it is densely uniformly chaotic.*

In a similar way we see that a TDS is weakly scattering if and only if it is weakly disjoint from every system which is an almost one-to-one extension of a minimal equicontinuous system (these systems are also called *almost automorphic*). Thus we also have a stronger version of Corollary 3.2 (4)

Corollary 3.9. *If (X, T) is weakly scattering and has an almost automorphic subsystem then it is densely uniformly chaotic.*

The following example shows that we can not weaken the condition "total transitivity" to "transitivity".

Example. Let (X, T) be a Toeplitz system and let $\pi : X \to Z$ be the corresponding almost one-to-one factor map from X onto its maximal adding machine factor. Clearly then every proximal set in X is contained in a fiber $\pi^{-1}(z)$ for some $z \in Z$. Suppose now that $|\pi^{-1}(z)| < \infty$ for every $z \in Z$, and that for some $z \in Z$ there are points $x, y \in \pi^{-1}(z)$ such that (x, y) is a recurrent pair and therefore a strong Li-Yorke pair (one can easily construct such systems, see e.g. [W84]). Let $Y = \overline{Orb((x, y), T)} \subseteq X \times X$. By assumption the point (x, y) is recurrent in $X \times X$ and forms a proximal pair. Thus the system (Y, T) is transitive and, as one can easily check, has Δ_X as its unique minimal subset. Since (X, T) is an almost one to one extension of an adding machine the diagonal $\Delta_X \subset Y$ contains regularly almost periodic points. However (Y, T) can not be Li-Yorke chaotic because our assumption implies that every proximal set in Y is finite.

This example also shows the existence of a non-minimal transitive system which is not Li-Yorke chaotic.

4. The Kuratowski-Mycielski Theory

Let X be a compact metric space. We recall that a subset $A \subset X$ is called a *Mycielski set* if it is a union of countably many Cantor sets. (This definition was introduced in [BGKM02]. Note that in [Ak03] a Mycielski set is required to be dense.) The notion of independent sets and the corresponding topological machinery were introduced by Marczewski [Mar61], and Mycielski [M64]. This theory was further developed by Kuratowski in [K73]. The first application to dynamics is due to Iwanik [I89]. Consequently it was used as a main tool in [BGKM02], where among other results the authors showed that positive entropy implies Li-Yorke chaos. See [Ak03] for a comprehensive treatment of this topic.

In this section we first review the Kuratowski-Mycielski theory, mainly as developed in [Ak03], and then consider the results of Sections 2 and 3 in view of this theory.

4.1. **The Kuratowski-Mycielski Theorem.** We begin by citing two classical results.

Theorem 4.1 (Ulam)**.** *Let $\phi : X \to Y$ be a continuous open surjective map with X and Y metric compact spaces. If R is a dense G_δ subset of X, then*

$$Y_0 = \{y \in Y : \phi^{-1}(y) \cap R \text{ is dense in } \phi^{-1}(y)\}$$

is a dense G_δ subset of Y.

Theorem 4.2 (Mycielski)**.** *Let X be a complete metric space with no isolated points. Let $r_n \nearrow \infty$ be a sequence of positive integers and for every n let R_n be a meager subset of X^{r_n}. Let $\{O_i\}_{i=1}^{\infty}$ be a sequence of nonempty open subsets of X. Then there exists a sequence of Cantor sets $C_i \subset O_i$ such that the corresponding Mycielski set $K = \bigcup_{i=1}^{\infty} C_i$ has the property that for every n and every $x_1, x_2, \dots, x_{r_n}$, distinct elements of K, $(x_1, x_2, \dots, x_{r_n}) \notin R_n$.*

An especially useful instance of Mycielski's theorem is obtained as follows (see [Ak03, Theorem 5.10], and [AAG08, Theorem 6.32]). Let W be a symmetric dense G_δ subset of $X \times X$ containing the diagonal Δ_X, and let $R = X \times X \setminus W$. Let $r_n = n$ and set

$$R_n = \{(x_1, \dots, x_n) : (x_i, x_j) \notin W, \ \forall\, i \neq j\}.$$

Theorem 4.3. *Let X be a perfect compact metric space and W a symmetric dense G_δ subset of $X \times X$ containing the diagonal Δ_X. There exists a dense Mycielski subset $K \subset X$ such that $K \times K \subset W$.*

We collect some notation and results from Akin [Ak03].

For X a compact metric space we denote by $C(X)$ the compact space of closed subsets of X equipped with the Hausdorff metric. Since $\emptyset$ is an isolated point, $C'(X) = C(X) \setminus \{\emptyset\}$ is compact as well.

We call a collection of sets $Q \subset C'(X)$ *hereditary* if it is hereditary downwards, that is, $A \in Q$ implies $C'(A) \subset Q$ and, in particular, every finite subset of A is in Q. For a hereditary subset Q we define $R_n(Q) = \{(x_1, \dots, x_n) \in X^n : \{x_1, \dots, x_n\} \in Q\} = i_n^{-1}(Q)$ where $i_n : X^n \to C'(X)$ is the continuous map defined by $i_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$. In particular, if Q is a G_δ subset of $C'(X)$ then $R_n(Q)$ is a G_δ subset of X^n for all n. Call A a $\{R_n(Q)\}$ set if $A^n \subset R_n(Q)$ for all $n = 1, 2, \dots$ or, equivalently, if every finite subset of A lies in Q. Clearly, the union of any chain of $\{R_n(Q)\}$ sets is an $\{R_n(Q)\}$ set and so every $\{R_n(Q)\}$ set is contained in a maximal $\{R_n(Q)\}$ set.

If $D \subset X$ we define $Q(D) = \{A \in C'(X) : A \subset D\}$, for which $R_n = D^n$. If $B \subset X \times X$ is a subset which satisfies

$$(x, y) \in B \qquad \Longrightarrow \qquad (y, x), (x, x) \in B$$

then we define $Q(B) = \{A \in C'(X) : A \times A \subset B\}$, for which $R_2 = B$ and $(x_1, \ldots, x_n) \in R_n$ if and only if $(x_i, x_j) \in B$ for all $i, j = 1, \ldots, n$.

If D (or B) is a G_δ then so is $Q(D)$ (resp. $Q(B)$). Because the finite sets are dense in $C'(X)$ it follows that if D is dense in X (or B is G_δ and dense in $X \times X$) then $Q(D)$ (resp. $Q(B)$) is dense in $C'(X)$.

Examples: (1) Let $Q(Recur) = \{A \in C'(X) : A$ is uniformly recurrent$\}$. The set $Recur_n(X)$ defined in Section 2 is exactly $R_n(Q(Recur)) = \{(x_1, \ldots, x_n) :$ recurrent in $X^n\}$. The $\{Recur_n\}$ subsets are the recurrent subsets. For fixed n and ϵ the condition $d(T^n x, x) < \epsilon$ for all $x \in A$ is an open condition on $A \in C'(X)$. Hence, $Q(Recur)$ and $Recur_n$ are G_δ sets.

If (X, T) is transitive then by Prop. 2.6 $Recur_n(X)$ is dense in X^n. Furthermore, $Trans_T$ is a dense G_δ in X and so $Q(Trans_T) = \{A \in C'(X) : A \subset Trans_T\}$ is a dense G_δ subset of $C'(X)$.

(2) Let $Q(Prox) = \{A \in C'(X) : A$ is uniformly proximal$\}$. The set $Prox_n(X)$ of Section 2 is $R_n(Q(Prox))$. The $\{Prox_n\}$ subsets are the proximal subsets. For fixed n and ϵ the condition $\operatorname{diam} T^n A < \epsilon$ is an open condition on $A \in C'(X)$. Hence, $Q(Prox)$ and $Prox_n$ are G_δ sets.

$Prox_2(X) = P(X, T)$ the set of proximal pairs. The G_δ set $Q(P(X, T))$ is the set of compacta A such that $A \times A \subset P(X, T)$. The $\{R_n(Q(P))\}$ sets are the pairwise proximal sets.

(3) For use below we define for Y a closed subset of X:

$$\begin{aligned} Q(TRANS, Y) = \{A \in C'(X) :\ & \text{for every } \epsilon > 0,\ n \in \mathbb{N}, \text{ pairwise disjoint closed} \\ & A_1, \ldots, A_n \subset A \text{ and } y_1, \ldots, y_n \in Y, \text{ there exists a positive} \\ & \text{integer } k \text{ such that } d(T^k x, y_i) < \epsilon \text{ for all } x \in A_i,\ i = 1, \ldots, n\}. \end{aligned}$$

It is easy to check that $Q(TRANS, Y)$ is a G_δ set, see Akin [Ak03], Lemma 6.6(a). Clearly, $(x_1, \ldots, x_n) \in R_n(Q(TRANS, Y))$ if and only if for every $\epsilon > 0$ and $y_1, \ldots, y_n \in Y$ there exists k such that $d(T^k x_i, y_i) < \epsilon$ for $i = 1, \ldots, n$.

The point of this peculiar condition is given by

Lemma 4.4. *If K is a Cantor set in X, then $K \in Q(TRANS, Y)$ if and only if for every continuous map $h : K \to Y$ and every $\epsilon > 0$ there exists a positive integer k such that $d(T^k x, h(x)) < \epsilon$ for all $x \in K$.*

Proof. Recall that the locally constant functions on K, which are the continuous functions with finite range, form a dense subset of $\mathcal{C}(K, Y)$ the space of continuous functions. It thus suffices to consider such functions h. If $h(K)$ is the set $\{y_1, \ldots, y_n\}$ of n distinct points then $\{A_i = h^{-1}(y_i) : i = 1, \ldots, n\}$ is a clopen partition of K. Hence, $K \in Q$ implies there exists a k such that $T^k \restriction K$ is within ϵ of h.

Conversely, given disjoint closed sets $A_1, \ldots, A_n$ in K and points $y_1, \ldots, y_n \in Y$ there exists a clopen partition $B_1, \ldots, B_n$ of X with $A_i \subset B_i$ for $i = 1, \ldots, n$. The function $h : X \to Y$ with $h(x) = y_i$ for $x \in B_i$ is continuous and approximating it by some $T^k \restriction K$ shows that $K \in Q$. □

For a TDS (X, T) and closed $Y \subset X$, motivated by Lemma 4.4, we will call a Cantor set $K \in Q(TRANS, Y)$ a *Kronecker set for* Y.

Lemma 4.5. *Let (X, T) be a TDS and Y a closed nonempty subset of X. Then any Kronecker set for Y, i.e. a Cantor set $K \in Q(TRANS, Y)$, is uniformly proximal. If, moreover, $K \subset Y$ then K is also uniformly recurrent, hence uniformly chaotic.*

Proof. Apply Lemma 4.4. For the first assertion take $h : K \to Y$ as any constant map $h : K \to Y$, $h(x) = y_0$, $\forall x \in K$. For the second, take $h : K \to Y$ as $h(x) = x$, $\forall x \in K$. □

If X is a perfect, nonempty, compact metric space then $CANTOR(X)$ the set of Cantor sets in X is a dense G_δ subset of $C'(X)$, see e.g. Akin [Ak03, Propsition 4.3(f)].

The importance of all this stems from the *Kuratowski-Mycielski Theorem.* This version comes from Akin [Ak03] Theorem 5.10 and Corollary 5.11.

Theorem 4.6. *For X a perfect, nonempty, compact metric space, let Q be a hereditary G_δ subset of $C'(X)$.*

(a) The following conditions are equivalent

1. *For $n = 1, 2, \dots,$ $R_n(Q)$ is dense in X^n.*
2. *There exists a dense subset A of X which is a $\{R_n(Q)\}$ set, i.e. $A^n \subset R_n(Q)$ for $n = 1, 2, \dots$.*
3. *Q is dense in $C'(X)$.*
4. *$CANTOR(X) \cap Q$ is a dense G_δ subset of $C'(X)$.*
5. *There is a sequence $\{K_i : i = 1, 2, \dots\}$ which is dense in $CANTOR(X)$ such that $\bigcup_{i=1}^n K_i \in Q$ for $n = 1, 2, \dots$.*

(b) The following conditions are equivalent

1. *There is a Cantor set in Q, i.e. $CANTOR(X) \cap Q \neq \emptyset$.*
2. *There is a Cantor set which is an $\{R_n(Q)\}$ set.*
3. *There is an uncountable $\{R_n(Q)\}$ set.*
4. *There is a nonempty $\{R_n(Q)\}$ set with no isolated points.*
5. *There is a nonempty, closed, perfect subset Y of X such that $Y^n \cap R_n(Q)$ is dense in Y^n for $n = 1, 2, \dots$.*

4.2. **Uniform chaos in light of the Kuratowski-Mycielski Theorem.** With this new vocabulary we can reprove Theorem 2.11 by showing that for a transitive system (X, T) the collection $Q(Recur)$, of uniformly recurrent subset, is a dense G_δ subset of $C'(X)$. For the reader's convenience we repeat the statement of the theorem (augmented with a statement about pairwise proximality) and provide a short proof which employs the Kuratowski-Mycielski machinery.

Theorem 4.7. *Let (X, T) be a transitive TDS without isolated points. There are Cantor sets $C_1 \subseteq C_2 \subseteq \cdots$ such that $\bigcup_{i=1}^{\infty} C_n$ is a dense recurrent subset of $Trans_T$ and for each $N \in \mathbb{N}$, C_N is uniformly recurrent.*

- *If in addition, $P(X,T)$ is dense in $X \times X$ then we can require that $\bigcup_{i=1}^{\infty} C_n$ is pairwise proximal.*
- *If in addition, for each $n \in \mathbb{N}$, $Prox_n(X)$ is dense in X^n, then we can require that for each $N \in \mathbb{N}$, C_N is uniformly proximal. Thus under these conditions (X,T) is uniformly chaotic.*

Proof. As described in Example (1) above, $Recur_n(X)$ and $(Trans_T)^n$ are dense G_δ subsets of X^n. Hence, condition (1) of part (a) of the Kuratowski-Mycielski Theorem applies to $Q(Recur) \cap Q(Trans_T)$. The result follows from condition (5) of part (a) with $C_N = \bigcup_{i=1}^N K_i$.

If $P(X,T)$ is dense in X^2 then we can intersect as well with the dense G_δ set $Q(P(X,T))$.

If $Prox_n$ is dense in X^n for every n then $Q(Prox)$ is also a dense G_δ by the Kuratowski-Mycielski Theorem and so we can intersect with it as well. □

Remark 4.8. Notice that in general the collection $Q(Recur)$ of uniformly recurrent subsets of X, is not *finitely determined*; that is, a closed subset $A \subset X$ with $A^n \subset Recur_n$ for every $n \geq 1$ is merely recurrent and need not be uniformly recurrent. Similarly $Q(Prox)$ is not finitely determined and a closed subset $A \subset X$ with $A^n \subset Prox_n$ for every $n \geq 1$ is merely a proximal set and need not be uniformly proximal.

In general, we have

Theorem 4.9. *A TDS (X,T) without isolated points is densely uniformly chaotic if and only if for every $n \geq 1$ the sets $Recur_n(X)$ and $Prox_n(X)$ are both dense in X^n.*

Proof. If $Recur_n(X)$ and $Prox_n(X)$ are both dense in X^n for every n then $Q(Recur) \cap Q(Prox)$ is dense in $C'(X)$. Apply the Kuratowski-Mycielski Theorem as before. □

We do likewise with the criterion for chaos (Theorem 3.1).

Theorem 4.10 (A criterion for chaos)**.** *Let (X,T) be a transitive TDS without isolated points. Assume that (Y,T) is a subsystem of (X,T) such that $(X \times Y, T)$ is transitive. There are Cantor sets $C_1 \subseteq C_2 \subseteq \cdots$ such that*

(1) *$K = \bigcup_{i=1}^{\infty} C_n$ is a dense subset of $Trans_T$ and;*
(2) *for each $N \in \mathbb{N}$, C_N is a Kronecker set for Y and is uniformly recurrent.*

In particular, (X,T) is densely uniformly chaotic.

If (Y,T) is non-trivial, i.e. Y contains at least two points, then (X,T) is sensitive.

Proof. We follow the notation of Examples (1) and (3) above. The work below will be to show that $R_n(Q(TRANS,Y))$ is dense in X^n for $n = 1, 2, \ldots$. We have already seen that $Recur_n(X)$ is dense in X^n. By the Kuratowski-Mycielski Theorem it follows that

$$Q(TRANS, Y) \cap Q(Recur) \cap Q(Trans_T)$$

is dense in $C'(X)$ and that the required sequence of Cantor sets exists.

Fix $\epsilon > 0$ and $y_1, \dots, y_n \in Y$ and choose open subsets $W_1, \dots, W_n$ of diameter less than ϵ with $y_i \in W_i$ for $i = 1, \dots, n$. We will prove that the open set $\bigcup_{k\in\mathbb{N}} T^{-k}W_1 \times \cdots \times T^{-k}W_n$ is dense. Then intersect over positive rational ϵ and $\{y_1, \dots, y_n\}$ chosen from a countable dense subset of Y. The Baire Category Theorem then implies that $R_n(Q(TRANS, Y))$ is a dense G_δ subset of X^n as required.

Let $U_1, \dots, U_n$ be open nonempty subsets of X. Because $X \times Y$ is transitive there exists $r_2 \in N(U_1 \times (W_1 \cap Y), U_2 \times (W_2 \cap Y))$. Let

$$U_{12} \times W_{12} = (U_1 \cap T^{-r_2}U_2) \times (W_1 \cap T^{-r_2}W_2),$$

an open set which meets $X \times Y$. Proceed inductively, finally choosing $r_n \in N((U_{1\dots n-1} \times (W_{1\dots n-1} \cap Y)), U_n \times (W_n \cap Y))$ and let

$$U_{1\dots n} \times W_{1\dots n} = (U_{1\dots n-1} \cap T^{-r_n}U_n) \times (W_{1\dots n-1} \cap T^{-r_n}W_n).$$

Choose $(x, y) \in (U_{1\dots n} \times W_{1\dots n}) \cap (X \times Y)$ with $x \in Trans_T$. Thus, $(x, T^{r_2}x, \dots, T^{r_n}x) \in U_1 \times \cdots \times U_n$ and $(y, T^{r_2}y, \dots, T^{r_n}y) \in W_1 \times \cdots \times W_n$. Since x is a transitive point, we can choose $T^k x$ close enough to y so that $(T^k x, T^{k+r_2}x, \dots, T^{k+r_n}x) \in W_1 \times \cdots \times W_n$. Thus, $(x, T^{r_2}x, \dots, T^{r_n}x) \in (U_1 \times \cdots \times U_n) \cap (T^{-k}W_1 \times \cdots \times T^{-k}W_n)$, as required.

By Lemma 4.5 the system is densely uniformly chaotic.

Now assume that Y has at least two points, so there exists $\epsilon > 0$ such that ϵ is smaller than the diameter of Y. Choose open sets U_1, U_2 and points $y_i \in U_i \cap Y$ ($i = 1, 2$) such $d(z_1, z_2) > \epsilon$ for all $z_i \in U_i$ ($i = 1, 2$). Given V an open set containing x there there exists N such that $C_N \cap V$ is nonempty and so contains distinct points x_1, x_2. Let $h : C_N \to Y$ be a continuous function with $h(x_i) = y_i$ ($i = 1, 2$). Since C_N is a Kronecker set there exists $k \in \mathbb{N}$ such that $z_i = T^k(x_i) \in U_i$ ($i = 1, 2$). Hence, $d(T^k(x_1), T^k(x_2) > \epsilon$. □

Remark 4.11. The condition that Y be nontrivial is necessary for sensitivity. There exist almost equicontinuous transitive systems with fixed points.

From part (b) of Theorem 4.6 we obtain

Theorem 4.12. *Let (X, T) be a TDS.*

(a) *If there exists an uncountable $K \subset X$ which is both proximal and recurrent then there exists a Cantor set $C \subset X$ which is both uniformly proximal and uniformly recurrent, i.e. (X, T) is uniformly chaotic.*
(b) *If there exists an uncountable $K \subset X$ which is strongly scrambled, i.e. (X, T) is strongly Li-Yorke chaotic, then there exists a Cantor set $C \subset X$ which is strongly scrambled.*
(c) *If there exists $\epsilon > 0$ and an uncountable $K \subset X$ which is pairwise proximal and for every $x, y \in K$ $d(T^n x, T^n y) > \epsilon$ for infinitely many $n \in \mathbb{N}$ then there exists a Cantor set $C \subset X$ which satisfies the same property, and so is scrambled.*

Proof. For (a) we apply Theorem 4.6(b) to $Q(Recur) \cap Q(Prox)$. For (b) we apply it to $Q(Recur_2) \cap Q(P(X, T))$.

While the set of asymptotic pairs is not an F_σ set, $Asymp_\epsilon = \bigcup_{k\in\mathbb{N}}\{(x,y) : d(T^nx,T^ny) \le \epsilon \text{ for all } n \ge k\}$ is. For (c) we apply Theorem 4.6(b) to $Q(Recur_2) \cap Q((X\times X)\setminus Asymp_\epsilon)$. □

Remark 4.13. If K is scrambled then for every $x,y \in K$ there exists $\epsilon > 0$ such that $d(T^nx,T^ny) > \epsilon$ for infinitely many $n \in \mathbb{N}$. However, it is not clear that an ϵ can be chosen to work for all pairs. Thus, part (c) leaves open the question asked in [BHS08] whether (X,T) Li-Yorke chaotic always implies the existence of a scrambled Cantor subset.

5. Chaotic subsets of minimal systems

It is well known that a non-equicontinuous minimal system is sensitive In this section we will have a closer look at chaotic behavior of minimal systems and will examine the relationship between chaos and structure theory.

5.1. **On the structure of minimal systems.** The structure theory of minimal systems originated in Furstenberg's seminal work [F63]. In this subsection we briefly review some of the main results of this theory. It was mainly developed for group actions and accordingly we assume for the rest of the paper that T is a homeomorphism. Much of this work can be done for a general locally compact group actions, but for simplicity we stick to the traditional case of $\mathbb{Z}$-actions. We refer the reader to [G76], [V77], and [Au88] for details.

The notion of relatively incontractible (RIC) extension was introduced in [EGS75]. As the original definition is a bit technical we use here an equivalent one. In an appendix below we remind the reader of the original definition and sketch the proof of the equivalence of the two definitions.

Call an extension $\pi : X \to Y$ of minimal systems *relatively incontractible (RIC) extension* if it is open and for every $n \ge 1$ the minimal points are dense in the relation

$$R^n_\pi = \{(x_1,\dots,x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall\ 1 \le i \le j \le n\}.$$

(See Theorem 7.2 in the appendix below.)

We say that a minimal system (X,T) is a *strictly PI system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota,w_\iota)\}_{\iota\le\eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \le \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that (X,T) is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by almost one-to-one extensions (or by highly proximal extensions in the non-metric case) we get the notion of HPI *systems*. If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have I *systems*. These notions can be easily relativize and we then speak about I, HPI, and PI extensions.

In this terminology Furstenberg's structure theorem for distal systems (Furstenberg [F63]) and the Veech-Ellis structure theorem for point distal systems (Veech [V70], and Ellis [E73]), can be stated as follows:

Theorem 5.1. *A metric minimal system is distal if and only if it is an I-system.*

Theorem 5.2. *A metric minimal dynamical system is point distal if and only if it is an HPI-system.*

Finally we have the structure theorem for minimal systems, which we will state in its relative form (Ellis-Glasner-Shapiro [EGS75], McMahon [Mc76], Veech [V77], and Glasner [G05]).

Theorem 5.3 (Structure theorem for minimal systems)**.** *Given a homomorphism $\pi : X \to Y$ of minimal dynamical system, there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*

$$\begin{array}{ccccccccccccc}
X & \xleftarrow{\theta_0^*} & X_0 & & \xleftarrow{\theta_1^*} & X_1 & \cdots & X_\nu & & \xleftarrow{\theta_{\nu+1}^*} & X_{\nu+1} & \cdots & X_\eta = X_\infty \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_0} & \searrow{\scriptstyle\sigma_1} & & \downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\pi_\nu} & \searrow{\scriptstyle\sigma_{\nu+1}} & & \downarrow{\scriptstyle\pi_{\nu+1}} & & \downarrow{\scriptstyle\pi_\infty} \\
Y & \xleftarrow[\theta_0]{} & Y_0 & \xleftarrow[\rho_1]{} & Z_1 & \xleftarrow[\theta_1]{} Y_1 & \cdots & Y_\nu & \xleftarrow[\rho_{\nu+1}]{} & Z_{\nu+1} & \xleftarrow[\theta_{\nu+1}]{} Y_{\nu+1} & \cdots & Y_\eta = Y_\infty
\end{array}$$

where for each $\nu \leq \eta, \pi_\nu$ is RIC, ρ_ν is isometric, θ_ν, θ_ν^* are proximal and π_∞ is RIC and weakly mixing of all orders. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$. Thus X_∞ is a proximal extension of X and a RIC weakly mixing extension of the strictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if X is a PI-system.

5.2. **Lifting chaotic sets.** Using Ellis semigroup techniques we are able to lift chaotic sets in minimal systems. We begin with a lemma concerning proximal sets in minimal systems.

Let I be an ideal in $\beta^*\mathbb{N}$. For a TDS (X,T) we say that K is an I *pairwise proximal set* if for every pair $x_1, x_2 \in K$ there exists $p \in I$ such that $px_1 = px_2$. Thus, K is pairwise proximal exactly when it is $\beta^*\mathbb{N}$ pairwise proximal.

Recall that K is a proximal set when

$$A_K = \{p \in \beta^*\mathbb{N} : pK \text{ is a singleton}\}$$

is nonempty. In that case A_K is a closed ideal in $\beta^*\mathbb{N}$. See Proposition 2.5 above.

Similarly, if H is a co-ideal in $\beta^*\mathbb{N}$, we say that K is an H *pointwise recurrent set* if for every $x \in K$ there exists $p \in H$ such that $px = x$. Thus, K is pointwise recurrent exactly when it is $\beta^*\mathbb{N}$ pointwise recurrent.

Lemma 5.4. *Let (X,T) be a minimal TDS and $K \subseteq X$.*

(a) *Let $H \subset \beta^*\mathbb{N}$ be a co-ideal minimal in $\beta^*\mathbb{N}$. K is an H pointwise recurrent set if and only if it is a recurrent set and $H \subseteq H_K$.*

(b) *Let $I \subset \beta^*\mathbb{N}$ be a minimal ideal. K is an I pairwise proximal set if and only if it is a proximal set and $I \subseteq A_K$. In that case, there exists for every $x \in K$ an idempotent $v_x \in I$ such that $v_x K = \{x\}$. In particular,*

$$K = \{v_x y : x \in K\} \qquad \text{for any } y \in K,$$

and

$$v_y v_x = v_y \qquad \text{for all } x, y \in K.$$

Proof. (a) Clearly, if H is a co-ideal contained in H_K then K is H pointwise recurrent.

On the other hand, assume H is a minimal co-ideal in $\beta^*\mathbb{N}$ and K is H pointwise recurrent. If $x \in K$ then $\{p \in H : px = x\}$ is nonempty and so is a co-ideal which is contained in H and so equals H by minimality. As $x \in K$ was arbitrary, $H \subseteq H_K$ and so K is recurrent.

(b)If I is any ideal contained in A_K then K is obviously I pairwise proximal.

Now assume that $I \subset \beta^*\mathbb{N}$ is a minimal left ideal, Because (X,T) is minimal, $Ix = X$ for any $x \in X$, and so $I_x = \{p \in I : px = x\}$ is a nonempty closed subsemigroup. By Ellis' Lemma 1.2 there exists an idempotent $v_x \in I_x$. If K is I pairwise proximal and $x, y \in K$ then there exist $p, q \in I$ such that $px = py$ and $q(px) = x$. Hence, $I_{x,y} = \{r \in I : ry = rx = x\}$ contains qp and so is a nonempty closed subsemigroup. Again Lemma 1.2 implies there exists an idempotent $u \in I_{x,y}$. By minimality of I, $I = \beta^*\mathbb{N}u$ and so $su = s$ for all $s \in I$. In particular, $v_x u = v_x$ and so $v_x y = v_x u y = v_x x = x$. As $y \in K$ was arbitrary we have $v_x K = \{x\}$ for any idempotent v_x in I_x. In particular, $v_x \in A_K$ and so K is a proximal set. Since A_K is an ideal and I is minimal, $I = \beta^*\mathbb{N}v_x \subseteq A_K$ and $v_y v_x = v_y$. Clearly, $K = \{v_x y : x \in K\}$ for any $y \in K$. □

Theorem 5.5. *Let $\pi : (X,T) \longrightarrow (Y,S)$ be an extension between minimal systems.*

1. *For any proximal subset K of Y there is a proximal subset K' of X with $\pi(K') = K$.*
2. *If π is a proximal extension and K is a proximal subset of Y, then any set K' of X with $\pi(K') \subseteq K$ is a proximal set.*
3. *For any recurrent subset K of Y, there is a recurrent subset K' of X with $\pi(K') = K$. Moreover if K is both proximal and recurrent then there is a subset K' of X with $\pi(K') = K$ which is both proximal and recurrent. In particular, for any strongly Li-Yorke pair (y, y') in $Y \times Y$ there is a strongly Li-Yorke pair (x, x') in $X \times X$ with $\pi(x) = y, \pi(x') = y'$.*
4. *If π is a distal extension and $K \subset Y$ is a recurrent set of Y, then any set K' of X with $\pi(K') = K$ is a recurrent set.*

In the cases (2) and (3) we have $\pi \restriction K'$ is one-to-one.

Proof. If K is a proximal subset of Y we apply Lemma 5.4 and its proof to define the ideal A_K in $\beta^*\mathbb{N}$, choose a minimal ideal $I \subset A_K$ and idempotents $\{v_y \in I : y \in K\}$ such that $v_y K = \{y\}$ for all $y \in K$.

1. Fix $x_0 \in X$ such that $y_0 = \pi(x_0) \in K$. Assuming that K is a proximal subset we define $j : K \to X$ by $j(y) = v_y x_0$. Observe that $\pi(j(y)) = \pi(v_y x_0) = v_y y_0 = y$. So with $K' = j(K)$ we have $\pi(K') = K$. On the other hand, $z \in K$ implies $v_y v_z = v_y$ and so $v_y j(z) = v_y x_0 = j(y)$ for all $z \in K$. That is, $v_y K'$ is the singleton $\{j(y)\}$ and so K' is proximal.

2. Now assume that π is proximal and $\pi(K') \subseteq K$ with K a proximal subset. Let u be an arbitrary idempotent in I so that uK is a singleton.

For any pair $x_1, x_2 \in K'$ we have

$$\pi(ux_1) = u\pi(x_1) = u\pi(x_2) = \pi(ux_2).$$

As $u(ux_1, ux_2) = (ux_1, ux_2)$ and u is a minimal idempotent, (ux_1, ux_2) is a minimal point. Since π is proximal, we have $ux_1 = ux_2$. Since the pair x_1, x_2 was arbitrary, uK' is a singleton.

3. Assume that K is a recurrent subset. By Proposition 2.5

$$H_K = \{p \in \beta^*\mathbb{N} : py = y \text{ for every } y \in K\}$$

is a nonempty closed subsemigroup. By Lemma 1.2 there is an idempotent $u \in H_K$. Choose for each $y \in K$, $\ell(y) \in \pi^{-1}(y)$. Let

$$K' = \{u\ell(y) : y \in K\}.$$

Since $\pi(u\ell(y)) = u\pi(\ell(y)) = uy = y$ it follows that $\pi(K') = K$. Since u is an idempotent it acts as the identity on K'.

Now assume in addition that K is a proximal subset. $A_K u$ is a closed ideal. Since u acts as the identity on K, it follows that $pu(K)$ is a singleton for every $p \in A_K$, i.e. $A_K u \subset A_K$. If I a minimal ideal in $A_K u$ then $pu = p$ for all $p \in I$. In particular, the idempotents $v_x \in I$ satisfy $v_x u = v_x$ and so $uv_x uv_x = uv_x v_x = uv_x$. That is, uv_x is an idempotent in I. Furthermore, $uv_x(K) = \{ux\} = \{x\}$. Thus, we can replace v_x by uv_x if necessary and so assume that $uv_x = v_x$.

As in (1) define $j(x) = v_x x_0$ to obtain the proximal set $K' = j(K)$. Since $uv_x = v_x$, $uj(x) = j(x)$ and so u acts as the identity on K'. That is, K' is a recurrent set as well.

4. As in part 3. we can pick an idempotent $u \in H_K$. Now for any $x \in X$ with $\pi(x) \in K$ the points x and ux are proximal. But as $\pi(ux) = u\pi(x) = \pi(x)$ and π is a distal extension we conclude that $ux = x$. Thus if $\pi(K') \subseteq K$ then $ux = x$ for every $x \in K'$, whence K' is recurrent. □

Remark 5.6. Notice that the proof of (3) shows that if K is both proximal and recurrent then there exist an idempotent u and a minimal ideal $I \subset \beta^*\mathbb{N}u$ and idempotents $v_x \in J(I)$ for all $x \in K$ such that $uv_x = v_x$ and $v_x(K) = \{x\}$. Conversely, it is clear that the existence of these idempotents implies that K is both proximal and recurrent.

Corollary 5.7. *Let $\pi : (X,T) \to (Y,S)$ be a homomorphism of minimal systems. If Y contains a uniformly chaotic subset then so does X.*

Proof. Theorem 4.12(a) says that a TDS contains a uniformly chaotic set if and only if it contains an uncountable set which is both proximal and recurrent. So the result follows from Theorem 5.5(3). □

Remark 5.8. One would like to prove analogous lifting theorems for Li-Yorke and strong Li-Yorke chaotic sets (i.e. uncountable scrambled and strongly scrambled sets). Unfortunately the collection of closed scrambled sets is not, in general, a G_δ subset of $C'(X)$, and we therefore can not use this kind of argument to show that Li-Yorke chaos lifts under homomorphisms of minimal systems. The problem with lifting closed strongly scrambled sets (which do form a G_δ set) is that we do not know whether an uncountable strongly scrambled set can always be lifted through an extension of minimal systems.

In general, if $\pi : (X,T) \to (Y,S)$ is any factor map and K is a proximal (or recurrent) subset of X then $\pi(K)$ is a proximal (resp. recurrent) subset of Y. In fact,

$$A_K \subseteq A_{\pi(K)} \qquad \text{and} \qquad H_K \subseteq H_{\pi(K)}.$$

Furthermore, if K is uniformly proximal or uniformly recurrent then $\pi(K)$ satisfies the corresponding property. The limitation of these results is that K might lie entirely in a fiber of π in which case $\pi(K)$ is trivial. However we do obtain the following result.

Theorem 5.9. *Let $\pi : (X,T) \to (Y,S)$ be a distal extension of minimal systems. Y contains a uniformly chaotic subset if and only if X does.*

Proof. If K is a pairwise proximal subset of X then the restriction $\pi \restriction K$ is injective when π is distal. So if K is a union of uniformly recurrent and uniformly proximal Cantor sets then $\pi(K)$ is. The converse is a special case of Corollary 5.7. □

5.3. **Weakly mixing extensions.**

Theorem 5.10. *Let (X,T) be a TDS and $\pi : (X,T) \to (Y,S)$ an open nontrivial weakly mixing extension. Then there is a residual subset $Y_0 \subseteq Y$ such that for every point $y \in Y_0$ the set $\pi^{-1}(y)$ contains a dense strongly scrambled Mycielski subset K such that $K \times K \setminus \Delta_X \subseteq Trans(R_\pi)$. In particular (X,T) is strongly Li-Yorke chaotic.*

If moreover π is weakly mixing and RIC, then there is a residual subset $Y_0 \subseteq Y$ such that for every point $y \in Y_0$ a dense Mycielski set $K \subset \pi^{-1}(y)$, $y \in Y_0$ as above can be found which is uniformly chaotic, whence X is uniformly chaotic.

Proof. Since π is open, it follows that $\pi \times \pi : R_\pi \to Y, (x_1, x_2) \mapsto \pi(x_1)$ is open as well. Since R_π is transitive, the set of transitive points $Trans(R_\pi)$ is a dense G_δ subset of R_π. By Ulam's Theorem there is a residual subset $Y_0 \subseteq Y$ such that for every point $y \in Y_0$,

$$Trans(R_\pi) \cap Recur_2 \cap (\pi_\infty^{-1}(y) \times \pi_\infty^{-1}(y))$$

is dense G_δ in $\pi^{-1}(y) \times \pi^{-1}(y)$.

Now for each $y \in Y_0$, we claim that $\pi^{-1}(y)$ has no isolated points. In fact if this is not true, then there exists $x \in \pi^{-1}(y)$ such that $\{x\}$ is an open subset of $\pi^{-1}(y)$. Moreover, $\{(x,x)\}$ is an open subset of $\pi^{-1}(y) \times \pi^{-1}(y)$. Since $Trans(R_\pi) \cap (\pi^{-1}(y) \times \pi^{-1}(y))$ is dense G_δ in $\pi^{-1}(y) \times \pi^{-1}(y)$, one has $(x,x) \in Trans(R_\pi)$. This shows that $R_\pi = \Delta_X$ which contradicts the fact that π is a non-trivial extension. Finally, by Theorem 4.3 there is a dense s-chaotic subset $K \subseteq \pi^{-1}(y)$ such that

$$K \times K \setminus \Delta_X \subseteq Trans(R_\pi) \cap (\pi^{-1}(y) \times \pi^{-1}(y)) \setminus \Delta_X \subseteq Trans(R_\pi).$$

We now further assume that π is a RIC extension. Then by [G05], Theorem 2.7, π is weakly mixing of all orders (i.e. R_π^n is transitive for all $n \geq 2$) and in particular for every $n \geq 2$,

$$Prox_n \cap Recur_n \cap \pi^{-1}(y)^n$$

is a dense in $\pi^{-1}(y)^n$, for every $y \in Y_0$. Applying Theorem 4.6 we obtain our claim. □

5.4. **The non PI case.** The following theorems of Bronstein [Bro79] and van der Woude [Wo85] give intrinsic characterizations of PI-extensions and HPI-extensions respectively. Recall that a map $\pi : X \to Y$ between compact spaces is called *semi-open* if $\operatorname{int} \pi(U) \neq \emptyset$ for every nonempty open subset $U \subset X$. It was observed by J. Auslander and N. Markley that a homomorphism $\pi : X \to Y$ between minimal systems is always semi-open (see e.g. [G05, Lemma 5.3]).

Theorem 5.11. *Let $\pi : (X,T) \longrightarrow (Y,T)$ be a homomorphism of compact metric minimal systems. Then*

(1) *The extension π is PI if and only if it satisfies the following property: whenever W is a closed invariant subset of R_π which is transitive and has a dense subset of minimal points, then W is minimal.*
(2) *The extension π is HPI if and only if it satisfies the following property: whenever W is a closed invariant subset of R_π which is transitive and the restriction of the projection maps to W are semi-open, then W is minimal.*

Next we show that a minimal system which is a non-PI extension has an s-chaotic subset.

Theorem 5.12. *Let $\pi : (X,T) \longrightarrow (Y,T)$ be a homomorphism of metric minimal systems. If π is a non-PI extension, then there is a dense subset $Y_0 \subset Y$ such that for each $y_0 \in Y_0$, there is a uniformly chaotic subset of $\pi^{-1}(y_0)$. In particular (X,T) is strongly Li-Yorke chaotic.*

Proof. Assume that $\pi : (X,T) \longrightarrow (Y,T)$ is a non-PI extension. Then by Theorem 5.3 there exist $\phi : (X_\infty, T) \to (X,T)$, $\pi_\infty : (X_\infty,T) \to (Y_\infty,T)$ and $\eta : Y_\infty \longrightarrow Y$ such that ϕ is a proximal extension, π_∞ is weakly mixing RIC extension, and η is a PI-extension. As π is non-PI, π_∞ is non-trivial.

Now consider the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\ \phi\ } & X_\infty \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_\infty} \\ Y & \xleftarrow[\ \eta\ ]{} & Y_\infty \end{array}$$

By Theorem 5.10 there is a dense G_δ subset $Y^0_\infty \subset Y_\infty$ such that, for every $y \in Y^0_\infty$, there is a dense uniformly chaotic subset K_y of $\pi^{-1}_\infty(y)$.

Since π is not PI, π_∞ is not proximal. Thus, there is a distal point $(x_1, x_2) \in R_{\pi_\infty} \setminus \Delta_{X_\infty}$. This implies that $\phi(x_1) \neq \phi(x_2)$ as ϕ is a proximal extension. For any $k_1, k_2 \in K = K_y$ with $k_1 \neq k_2$, one has $(k_1, k_2) \in Trans(R_{\pi_\infty})$. As $\phi(x_1) \neq \phi(x_2)$, $(x_1, x_2) \in R_{\pi_\infty}$ and $(k_1, k_2) \in Trans(R_{\pi_\infty})$, one has $\phi(k_1) \neq \phi(k_2)$. That is, $\phi : K \to \phi(K)$ is a bijection. Therefore, as is easy to check, $\phi(K)$ is a uniformly chaotic subset of X. Moreover $\phi(K) \subset \pi^{-1}(\eta(y))$. Finally we let $Y_0 = \eta(Y^0_\infty)$; clearly a dense subset of Y. □

5.5. **The proximal but not almost one-to-one case.** Every extension of minimal systems can be lifted to an open extension by almost one-to-one modifications. To be precise, for every extension $\pi : X \to Y$ of minimal systems there exists a canonically defined commutative diagram of extensions (called the *shadow diagram*)

$$\begin{array}{ccc} X & \xleftarrow{\ \sigma\ } & X^* \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi*} \\ Y & \xleftarrow[\ \tau\ ]{} & Y^* \end{array}$$

with the following properties:

(a) σ and τ are almost one-to-one;
(b) π^* is an open extension;
(c) Y^* is the unique minimal set in the closure of $\{\pi^{-1}(y) : y \in Y\}$ in $C(X)$ and τ is the extension of the uniformly continuous map $\pi^{-1}(y) \mapsto y$.
(d) X^* is the unique minimal set in $R_{\pi\tau} = \{(x, y) \in X \times Y^* : \pi(x) = \tau(y)\}$ and σ and π^* are the restrictions to X^* of the projections of $X \times Y^*$ onto X and Y^* respectively.

We refer to [V70], [G76] and [V77] for the details of this construction.

In [G76] it was shown that a metric minimal system (X, T) with the property that $Prox_n(X)$ is dense in X^n for every $n \geq 2$ is weakly mixing. This was extended by van der Woude [Wo82] as follows (see also [G05]).

Theorem 5.13. *Let $\pi : X \to Y$ be a factor map of the metric minimal system (X, T). Suppose that π is open and that for every $n \geq 2$, $Prox_n(X) \cap R_\pi$ is dense in R_π. Then π is a weakly mixing extension. In particular a nontrivial open proximal extension is a weakly mixing extension.*

Lemma 5.14. *Let $\pi : X \to Y$ be a continuous surjective map between compact metric spaces which is almost one-to-one. If $A \subset X$ is a dense G_δ subset, then $\pi(A)$ contains a dense G_δ subset of Y.*

Proof. Let $A_0 = \{x \in X : \pi^{-1}\pi(x) = \{x\}\}$ and $B_0 = \{y \in Y : \text{Card } \pi^{-1}(y) = 1\}$. Then A_0 (resp. B_0) is a dense G_δ subset of X (resp. Y). Now $A \cap A_0$ is a dense G_δ subset of X, hence a dense G_δ of A_0. As the set of continuity points of $\pi^{-1} : Y \longrightarrow C(X)$ contains B_0, $\pi : A_0 \to B_0$ is a homeomorphism, so $\pi(A \cap A_0)$ is a dense G_δ subset of B_0. Therefore, there exist open subsets U_n of Y such that $\bigcap_{n=1}^{\infty} U_n \cap B_0 = \pi(A \cap A_0)$. This shows that $\pi(A \cap A_0)$ is also a dense G_δ subset of Y. □

Recall that a subset K of X is a proximal set if each finite tuple from K is uniformly proximal (see Definition 2.2). The proof of the following lemma is straightforward.

Lemma 5.15. *Let $\pi : X \longrightarrow Y$ be a proximal extension between minimal systems. Then for each $y \in Y$, $\pi^{-1}(y)$ is a proximal set.*

In the sequel it will be convenient to have the following:

Definition 5.16. Let (X, T) be a TDS.

1. A scrambled Mycielski subset $K \subset X$ will be called a *chaotic subset* of X.
2. A strongly scrambled Mycielski subset will be called an *s-chaotic subset* of X.

We can now prove the following result (see also [AAG08, Theorem 6.33]).

Theorem 5.17. *Let $\pi : X \to Y$ be a proximal but not almost one-to-one extension between minimal systems. Then there is a residual subset $Y_0 \subset Y$ such that for each $y \in Y_0$, $\pi^{-1}(y)$ contains a proximal s-chaotic set K.*

Proof. In the shadow diagram for π, the map π^* is open and proximal. Since π is not almost one-to-one π^* is not trivial. Thus, by Theorem 5.13, π^* is a nontrivial open weakly mixing extension. Hence by Theorem 5.10 there is a residual subset $Y_0^* \subset Y^*$ such that for each $y^* \in Y_0^*$, $\pi^{*-1}(y^*)$ contains an s-chaotic set K^* and $K^* \times K^* \setminus \Delta_{X^*} \subseteq Trans(R_{\pi^*})$. Moreover, π^* being proximal, we have for every $n \geq 2$, $\pi^{*-1}(y^*)^n \subset Prox_n$ and therefore we can require that K^* be proximal as well. Since in the shadow diagram σ and π^* are the restrictions to X^* of the projections of $X \times Y^*$ onto X and Y^* respectively, $\sigma(K^*)$ is an s-chaotic set, as $\pi\sigma(K^*) = \tau\pi^*(K^*) = \{\tau(y^*)\}$, $\sigma(K^*) \subset \pi^{-1}(\tau(y^*))$. Finally, set $Y_0 = \tau(Y_0^*)$. Since τ is almost one-to-one, Y_0 is a residual subset of Y (Lemma 5.14) . □

The following result was first proved in [AAG08].

Corollary 5.18. *Assume that $\pi : X \to Y$ is extension between minimal systems such that the only points $(x_1, x_2) \in Recur_2(X)$ with $\pi(x_1) = \pi(x_2)$ satisfy $x_1 = x_2$, e.g. an asymptotic extension between minimal systems. Then π is almost one-to-one.*

Proof. This follows immediately from Theorem 5.17. Note that π is proximal. If it is not almost one-to-one, then by Theorem 5.17, there are $x_1 \neq x_2, \{x_1, x_2\} \subset K \subset \pi^{-1}(y)$ for some $y \in Y$. □

Remark 5.19. In [GW79] the authors construct an example of a minimal system (X, T) which admits a factor map $\pi : X \to Y$ such that (i) the factor Y is equicontinuous, (ii) the map π is a nontrivial open proximal extension. Now such an X is clearly strictly PI but not HPI. However, according to Theorem 5.13 the extension π is a weakly mixing extension and it follows from Theorem 5.10 that for some $y_0 \in Y$ the fiber $\pi^{-1}(y_0)$ contains a dense proximal s-chaotic subset. Moreover, as suggested to us by Hanfeng Li, using the (Baire category type) method of construction in [GW79], one can enforce on the extension $\pi : X \to Y$ the following additional property: (iii) there is a point $y_0 \in Y$ such that $\pi^{-1}(y_0)$ is a uniformly recurrent subset of X. Of course $\pi^{-1}(y_0)$ is also a proximal set and therefore uniformly chaotic. Thus X is an example of a minimal PI system which is uniformly and a fortiori strongly Li-Yorke chaotic.

Remark 5.20. By [BGKM02], positive topological entropy implies the existence of an s-chaotic subset. In fact in [KL07, Theorem 3.18] Kerr and Li actually prove a much stronger result from which it follows that positive topological entropy implies uniform chaos. Since there are HPI systems with positive entropy (e.g. many Toeplitz systems [W84]) we conclude that there are HPI systems which are uniformly (hence strongly Li-Yorke) chaotic.

5.6. **The PI, non-HPI case.** We have shown (Subsection 5.4) that for a non-PI extension there is a uniformly chaotic set. The natural question now is whether there is a chaotic (s-chaotic, uniformly chaotic) set for a non-HPI extension? (Recall that for a metric X the notions 'HPI extension' and 'point distal extension' coincide.) At present we are unable to answer this question fully. However we will show that the answer is affirmative for a sub-class of non-HPI extensions:

Proposition 5.21. *Let $\pi : X \longrightarrow Y$ be a strictly PI extension but non-HPI extension between minimal systems. Then there is a dense set Y_0 of Y such that for each $y \in Y_0$, $\pi^{-1}(y)$ contains a proximal chaotic set.*

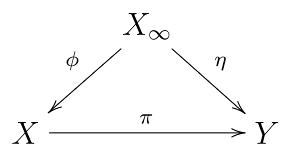
Proof. Since by assumption π is non-HPI, in its strictly PI-tower at least one of the proximal extensions in the canonical PI-tower is not an almost one-to-one extension. Let us denote this segment of the tower by

$$X \xrightarrow{\pi_1} Z_1 \xrightarrow{\pi_2} Z_2 \xrightarrow{\pi_3} Y,$$

with $\pi_1 \circ \pi_2 \circ \pi_3 = \pi$, and where π_1 and π_3 are strictly PI extensions and π_2 is a proximal but not an almost one-to-one extension.

By Theorem 5.17 there exists a dense set $Z_0 \subset Z_2$ such that for each $z \in Z_2$, $\pi_2^{-1}(z)$ contains a proximal s-chaotic set $K \subset Z_1$. By Theorem 5.5 (3), there is a proximal subset $K' \subset X$ with $\pi_1 \circ \pi_1(K') = K$ and as K is s-scarambled, K' is at least scrambled. Now the proposition follows by setting $Y_0 = \pi_3(Z_0)$. □

Now assume that $\pi : X \longrightarrow Y$ is PI and not HPI. This means that in the canonical PI-tower there are maps $\phi : X_\infty \longrightarrow X$ which is proximal and $\eta : X_\infty \longrightarrow Y$ which is strictly PI:



Lemma 5.22. *The extension η is not a strictly HPI extension.*

Proof. Assume η is a strictly HPI-extension. Then by Theorem 5.11.(2) there are no non-minimal transitive subsystem W of R_η such that the coordinate projection $W \to Y$ is semi-open. Now it is easy to see that there is no non-minimal transitive subsystem W' of R_ϕ such that the coordinate projection $W' \to X$ is semi-open. For if W' is a non-minimal transitive subsystem of R_ϕ such that the coordinate projection $W' \to X$ is semi-open, then W' is also a subsystem of R_η. But the composition of two semi-open maps is also semi-open and $\pi : X \to Y$ is semi-open, hence the coordinate projection $W' \to Y$ is semi-open, a contradiction.

Hence using Theorem 5.11.(2) again, this shows that ϕ is an HPI-extension. However ϕ is also proximal, so we conclude that ϕ is almost one-to-one. This shows that π is an HPI-extension contradicting our assumption. □

Thus combining this lemma with Proposition 5.21 we know that X_∞ contains a proximal chaotic subset K. However, we do not know whether its image $\phi(K) \subset X$ is also such a set.

We conclude with the following open problem.

Problem 5.23. *A strictly PI system which is not HPI contains a proximal chaotic subset (Proposition 5.21), is this true also for a PI non-HPI system?*

6. Table

In order to ease the reading of this work we summarize in the table below some of the interrelations between the various kinds of chaos discussed in the paper. In each case the label refers to the existence of a large chaotic set. We write 'ch.' for chaos and 's' for strong. The labels ch. and s-ch. refer to the existence of Mycielski scrambled set and Mycielski strongly scrambled set respectively. Proximal means to say that the chaotic set in question is a proximal set. For most of these arrows it is easy to produce examples which show they can not be reversed. However for some, especially at the bottom of the table, this is no longer so clear and we have not checked what is the exact situation in that respect.

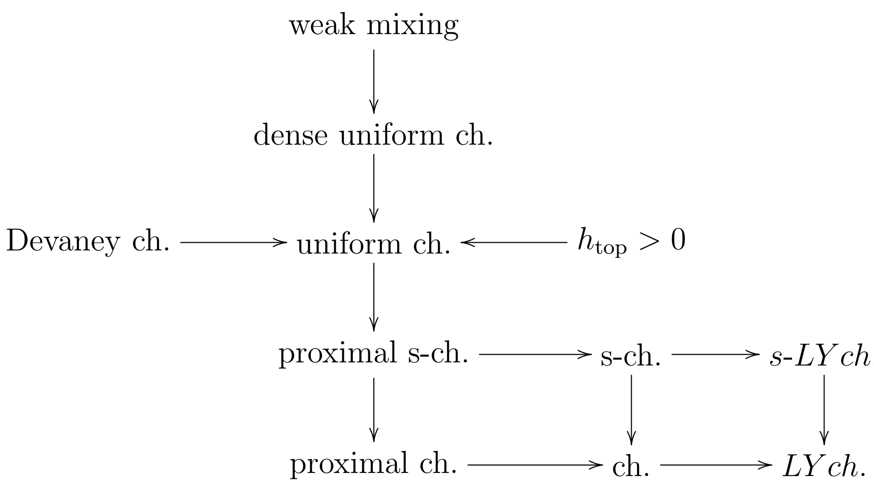


Table 1. Types of chaotic behavior

7. Appendix

7.1. **A characterization of RIC extensions.** Following usual notation we write $\beta\mathbb{Z}$ for the Čech-Stone compactification of the integers, and we fix a minimal left ideal $M \subset \beta\mathbb{Z}$ and an idempotent $u = u^2 \in J(M)$, where $J(M)$ is the nonempty set of idempotents in M. Then the subset $G_u = uM$ is a maximal subgroup of the semigroup M which decomposes as a disjoint union $M = \bigcup\{G_v = vG : v \in J(M)\}$. The group G can be identified with the group of automorphisms of the dynamical system $(M, \mathbb{Z})$ (see e.g [G76] or [Au88]). We also recall that the semigroup $\beta\mathbb{Z}$ is a universal enveloping semigroup and thus "acts" on every compact $\mathbb{Z}$ dynamical system. In particular, when (X, T) is a dynamical system the homeomorphism T defines in a natural way a homeomorphism on $C(X)$, the compact space of closed subsets of X. Now for $p \in \beta\mathbb{Z}$ the "action" of p on the point $A \in C(X)$ is well defined. In order to avoid confusion here we denote the resulting element of $C(X)$ by $p \circ A$ and refer to this action as the *circle operation.* A more concrete definition of this set is

$$p \circ A = \limsup T^{n_i} A,$$

where, denoting by S the generator of $\mathbb{Z}$, S^{n_i} is any net in $\beta\mathbb{Z}$ which converges to p. Thus we always have $pA = \{px : x \in A\} \subset p \circ A$, but usually the inclusion is proper, as often pA is not even a closed subset of X. A *quasifactor* of a system (X, T) is a closed invariant set $\mathcal{M} \subset C(X)$ such that $\bigcup\{A : A \in \mathcal{M}\} = X$.

For an extension $\pi : X \to Y$ of minimal dynamical systems fix $x_0 \in X$ such that $x_0 = ux_0$ and let $y_0 = \pi(x_0)$, and let F denote the subgroup of $G_u = uM$ which fixes y_0. That is, $F = \{\alpha \in G_u : \alpha y_0 = y_0\}$.

We define the minimal quasifactor

$$\mathcal{M} = \{p \circ Fx_0 : p \in M\} \subset C(X).$$

Clearly, $y = py_0$ implies $p \circ Fx_0 \subset \pi^{-1}(y)$ for any $p \in \beta\mathbb{Z}$.

Lemma 7.1. *If $p \in M, v \in J(M)$ with $vp = p$ then $v\pi^{-1}(py_0) = pFx_0$.*

Proof. $pFx_0 \subset p \circ Fx_0 \subset \pi^{-1}(y)$ with $y = py_0$. Since $vp = p$ and $vv = v$, $pFx_0 \subset \{x \in \pi^{-1}(y) : vx = x\} = v\pi^{-1}(y)$.

On the other hand, suppose $\pi(x) = y$ and $vx = x$. There exists $q \in M$ such that $qp = u$ and by replacing q with uq we can assume $uq = q$. Then $qy = qpy_0 = uy_0 = y_0$. Since q, up and u lie in the group G_u with identity u, $u = qp = qup$ implies that q and up are inverses in the group and so $upq = u$. Hence, $pq = vpq = vupq = vu = v$.

Now let $x_1 = qx$. Since $uq = q, ux_1 = x_1$. Also, $\pi(x_1) = \pi(qx) = qy = y_0$. Let $\alpha \in M$ such that $\alpha x_0 = x_1$. By replacing α by $u\alpha$ we can assume $\alpha \in G_u$. Finally, $p\alpha x_0 = px_1 = pqx = vx = x$. That is, $x \in pFx_0$.

□

In [EGS75] the extension π is called *relatively incontractible (RIC)* if for every $p \in M$ we have $p \circ Fx_0 = \pi^{-1}(py_0)$. We show that this definition is equivalent to the one given in Section 5.1. Let us note that in the proof we do not assume the metrizability of the minimal systems and that the same proof works for general (non-commutative) discrete group actions.

Theorem 7.2. *An extension $\pi : X \to Y$ between minimal systems is RIC if and only if π is open and for every $n \geq 1$ the minimal points are dense in the relation*

$$R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall\ 1 \leq i \leq j \leq n\}.$$

Proof. Define the map $\pi^* : Y \to C(X)$ by $\pi^*(y) = \pi^{-1}(y)$, regarded as a point of $C(X)$. π is open if and only if π^* is continuous and so is a homomorphism from Y to $C(X)$. In that case $\pi^{-1}(py) = p \circ \pi^{-1}(y)$ for any $y \in Y$ and $p \in \beta\mathbb{Z}$ and the image $\pi^*(Y) = \{\pi^{-1}(y) : y \in Y\}$ is a minimal quasifactor of X.

On the other hand, it is easy to see that π is RIC exactly when $\pi^*(Y) = \mathcal{M} = \{p \circ Fx_0 : p \in M\}$.

Claim: For an extension $\pi : X \to Y$ between minimal systems, $\pi^*(Y) \subset \mathcal{M}$ if and only if for every $n \geq 1$ the minimal points are dense in the relation $R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall\ 1 \leq i \leq j \leq n\}$.

Assuming the Claim, we complete the proof of the theorem.

If π is RIC then $\pi^*(Y)$ is closed since it equals $\mathcal{M}$. If $\{y_i\}$ converges to y in Y then any limit point of $\{\pi^*(y_i)\}$ is contained in $\pi^{-1}(y)$. Since $\pi^*(Y)$ is closed this limit point must equal $\pi^{-1}(y)$. Hence, π^* is continuous and π is open. The density of minimal points then follows from the Claim.

On the other hand, if π is open then $\mathcal{M}$ and $\pi^*(Y)$ are both minimal quasifactors. By the Claim the density of minimal points implies that they meet and so are equal. So an open extension is RIC when it satisfies the minimal points condition.

Proof of the Claim. Every n-tuple $(x_1, \ldots, x_n)$ with $x_i \in Fx_0,\ i = 1, \ldots, n$ is a minimal point of R_π^n because $x \in Fx_0$ implies $ux = x$. For any t in the acting group $\mathbb{Z}, utx = tux = tx$ (and if the acting group is nonabelian then $u'tx = tux = tx$ using the conjugate idempotent $u' = tut^{-1}$). Hence, if for every $y \in Y$ there exists $p \in \beta\mathbb{Z}$ such that $\pi^{-1}(y) = p \circ Fx_0$ then the density of minimal points in R_π^n follows from the definition of the circle operation.

Conversely, suppose the minimal points are dense in each R^n_π. We show that $\pi^{-1}(y)$ is in $\mathcal{M}$ for an arbitrary $y \in Y$.

(If X is not metrizable let d be any continuous pseudo-metric on X.) Let $(x_1, \dots, x_n)$ be an n-tuple of elements of $\pi^{-1}(y)$ which is ϵ-dense in $\pi^{-1}(y)$ (with respect to d). By compactness, we can choose U a neighborhood of y such that $\pi^{-1}(U)$ is contained in the ϵ neighborhood $B^d_\epsilon(\pi^{-1}(y))$.

By the density of minimal points there is a point $y' \in U$ and an n-tuple $(x'_1, \dots, x'_n)$ of points of $\pi^{-1}(y')$, such that (i) $d(x_i, x'_i) < \epsilon$ for every i, and (ii) $(x'_1, \dots, x'_n)$ is a minimal point of R^n_π.

There is a minimal idempotent $v \in J(M)$ such that $vx'_i = x'_i$ for every i and it follows that $\{x'_1, \dots, x'_n\} \subset v\pi^{-1}(y')$. Note that we must have $vy' = y'$ and there is therefore some $p \in M$ such that $y' = py_0$ and $p = vp$.

By Lemma 7.1

$$\{x'_1, \dots, x'_n\} \subset v\pi^{-1}(y') = pFx_0 \subset p \circ Fx_0.$$

Thus, every point of $\pi^{-1}(y)$ is within 2ϵ of a point in $p \circ Fx_0$ (with respect to d). On the other hand, $p \circ Fx_0 \subset \pi^{-1}(y') \subset \pi^{-1}(U)$ implies that every point of $p \circ Fx_0$ is within ϵ of a point in $\pi^{-1}(y)$. Hence, the d Hausdorff distance between $\pi^{-1}(y)$ and $p \circ Fx_0 \in \mathcal{M}$ is at most 2ϵ. As ϵ is arbitrary (as is d in the nonmetric case) and $\mathcal{M}$ is closed in $C(X)$, it follows that $\pi^{-1}(y) \in \mathcal{M}$. □

Remark 7.3. We see from the proof that if π is open then it is RIC if and only if $\mathcal{M}$ and $\pi^*(Y)$ intersect. In particular, π is RIC if and only if it is open and $u \circ Fx_0 = \pi^{-1}(y_0)$.

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