Representations of dynamical systems on Banach spaces

Eli Glasner and Michael Megrelishvili

Abstract

We review recent results concerning dynamical systems and their representations on Banach spaces. As the enveloping (or Ellis) semigroup, which is associated to every dynamical system, plays a crucial role in this investigations we also survey the new developments in the theory of these semigroups, complementing the review article [53]. We then discuss some applications of these dynamical results to topological groups and Banach spaces.

Contents

1 Introduction, an overview

Like topological groups, compact dynamical systems, can be represented on Banach spaces. The interrelations between the topological properties of a dynamical system and the geometrical properties of the corresponding Banach space provide new possibilities for the investigation of both source and target. This approach naturally extends some classical research themes and at the same time opens new and sometimes quite unexpected directions. In particular this theory leads to parallel hierarchies in the complexity of both dynamical

systems and Banach spaces. A further indication for the success of this theory is the fact that it provides also a natural environment for the study of representations of topological groups and compact right topological semigroups on Banach spaces.

Throughout our review there are some new results which have not appeared before. Notably we point out the following items: Theorems 5.10, 6.12, 8.8, 8.20, 8.35 and Corollary 9.12.

1.1 Topological prototypes

An important direction in the classical study of (large) compact spaces went via the following general principle: Given a compact space X find a nice class $\mathcal{K}$ of Banach spaces such that there always is an element $V \in \mathcal{K}$ where X can be embedded into V^* equipped with its weak-star topology ?

Eberlein compacta in the sense of Amir and Lindenstrauss [7] are exactly the weakly compact subsets in the class of all (equivalently, reflexive) Banach spaces. If X is a weak* compact subset in the dual V^* of an Asplund space V then, following Namioka [111], X is called a *Radon–Nikodým compactum* (in short: RN). In other words, reflexively representable compact spaces are the Eberlein compacta and Asplund representable compact spaces are the Radon–Nikodým compacta. Hilbert representable compacta are the so-called *uniformly Eberlein* compact spaces. Another interesting class of compact spaces, namely the *weakly Radon–Nikodým* (WRN) compacta, occurs by taking $\mathcal{K}$ to be the class of Rosenthal Banach spaces (i.e. those Banach spaces which do not contain an isomorphic copy of l_1). Comparison of the above mentioned classes of Banach spaces implies the inclusions of the corresponding classes of compact spaces:

$$uEb \subset Eb \subset RN \subset WRN \subset Comp.$$

Note that this classification makes sense only for large compact spaces, where X is not metrizable, in fact, any compact metrizable space is norm embeddedable in a separable Hilbert space.

One of the main directions taken in our survey is the development of a dynamical analog, for compact S-dynamical systems (where S is a semigroup), of the above mentioned classification of large compact spaces (this is made precise in Definition 1.2 and Question 1.3 below). Perhaps the first outstanding feature of this new theory is that, in contrast to the purely topological case (i.e., the case of trivial actions), for dynamical systems, the main interest of the dynamical theory is within the class of metrizable dynamical systems. For example, even for $X := [0, 1]$, the unit interval, the action of the cyclic group $\mathbb{Z}$ on X generated by the map $f(x) = x^2$ is RN and not Eberlein. There exists a compact metric $\mathbb{Z}$-system which is reflexively but not Hilbert representable, i.e., Eberlein but not uniformly Eberlein (section 6.5). There are compact metric $\mathbb{Z}$-systems which are WRN but not RN, etc.

It turns out that the corresponding classes of metric dynamical systems coincide with well known important classes whose study is well motivated by other independent reasons. For example we have, Eberlein = WAP (weakly almost periodic systems), RN = HNS (hereditarily non-sensitive), WRN = tame systems. The investigation of Hilbert representable (i.e., "uniformly Eberlein") systems is closely related to the study of unitary and reflexive representability of groups.

Another remarkable feature of the new theory is the fact that the correspondence goes both ways. Thus, for example, every metric WRN but not RN $\mathbb{Z}$-system leads to an example of a separable Rosenthal Banach space which is not Asplund (see section 1.9).

1.2 The hierarchy of Banach representations

With every Banach space V one may naturally associate several structures which are related to the theories of topological dynamics, topological groups and compact right topological semigroups:

Definition 1.1   *1. Iso (V) is the group of linear onto self-isometries of V. It is a topological (semitopological) group with respect to the strong (respectively, weak) operator topology. It is naturally included in the semigroup $\Theta(V) := \{\sigma \in L(V,V) : ||\sigma|| \leq 1\}$ of non-expanding linear operators. The latter is a topological (semitopological) monoid with respect to the strong (respectively, weak) operator topology. Notation: $\Theta(V)_s$, Iso $(V)_s$ (respectively, $\Theta(V)_w$, Iso $(V)_s$) or simply $\Theta(V)$ and Iso (V), where the topology is understood.*

2. *For every subsemigroup $S \leq \Theta(V)^{op}$ the pair (S, B^*) is a dynamical system, where B^* is the weak star compact unit ball in the dual space V^*, and $\Theta(V)^{op}$ is the opposite semigroup (which can be identified with the adjoint) to $\Theta(V)$. The action is jointly (separately) continuous where S carries the strong (weak) operator topology.*

3. *The enveloping semigroup $E(S, B^*)$ of the system (S, B^*) is a compact right topological semigroup. In particular, $\mathcal{E}(V) := E(\Theta(V)^{op}, B^*)$ will be called the enveloping semigroup of V. Its topological center is just $\Theta(V)_w^{op}$ which is densely embedded into $\mathcal{E}(V)$. (See subsection 4.4 below for the general definition of enveloping semigroups and more about $\mathcal{E}(V)$.)*

Definition 1.2 *[97, 56, 57, 58] A representation of a dynamical system (G, X) on a Banach space V is given by a pair (h, α), where $h : G \to$ Iso (V) is a co-homomorphism of the group G into the group Iso (V) of linear isometries of V, and $\alpha : X \to V^*$ is a weak* continuous bounded G-map with respect to the dual action of $h(G)$ on V^*. For semigroup actions (S, X) we consider the co-homomorphisms $h : S \to \Theta(V)$. If α is a topological embedding then we say that (h, α) is* faithful.

General questions:

Query 1.3 *Let $\mathcal{K}$ be a "nice" class of Banach spaces.*

1. *Which dynamical S-systems X admit a faithful representation on some Banach space $V \in \mathcal{K}$?*

2. *Which topological groups can be embedded into Iso (V) for some $V \in \mathcal{K}$?*

3. *Which compact right topological semigroups (in particular, which enveloping semigroups of dynamical systems) can be embedded into $\mathcal{E}(V)$ for some $V \in \mathcal{K}$?*

An old observation of Teleman [138] (see also the survey of Pestov [117] for a detailed discussion) is that every (Hausdorff) topological group can be embedded into Iso (V) for some Banach space V (namely, one can take $V :=$ RUC(G)). Furthermore, every continuous dynamical system (G, X) has a faithful representation on $V := C(X)$, where one can identify $x \in X$ with the point mass δ_x viewed as an element of $C(X)^*$. This is true also for continuous semigroup actions (Remark 4.4.2).

The geometry of $C(X)$, in general, is bad. For example, a typical disadvantage here is the norm discontinuity of the dual action of G on $C(X)^*$. In contrast, if V is an Asplund Banach space then the dual group action on V^* is norm continuous.

In the following table we encapsulate some features of the trinity: dynamical systems, enveloping semigroups, and Banach representations. Here X is a compact metrizable G-space and $E(X)$ denotes the corresponding enveloping semigroup. The symbol f stands for an arbitrary function in $C(X)$ and $fG = \{f \circ g : g \in G\}$ denotes its orbit. Finally, cls (fG) is the pointwise closure of fG in $\mathbb{R}^X$.

DS	Dynamical characterization	Enveloping semigroup	Banach representation
WAP	cls (fG) is a subset of $C(X)$	Every element is continuous	Reflexive
HNS	cls (fG) is metrizable	$E(X)$ is metrizable	Asplund
Tame	cls (fG) is Fréchet	Every element is Baire 1	Rosenthal

Table 1: The hierarchy of Banach representations

1.3   WAP systems and Reflexive Banach spaces

The theory of weakly almost periodic (WAP) functions on semitopological semigroups was developed by W. F. Eberlein, [32], A. Grothendieck, [68], I. Glicksberg and K. de Leeuw, [30].

By [134, 94] for any compact Hausdorff semitopological semigroup S there exists a reflexive Banach space V such that S is topologically isomorphic to a closed subsemigroup of $\Theta(V)$. For every semitopological semigroup S the algebra WAP(S) of weakly almost periodic functions on S, induces the universal compact semitopological semigroup compactification $S \to S^{WAP}$. Thus, S is embedded into $\Theta(V)_w$ for some reflexive Banach space V iff the WAP functions on S separate points and closed subsets. If $S = G$ is a group then this is equivalent to the embeddability of G into Iso (V). Note that for every reflexive space V the weak and strong operator topologies on Iso (V) are the same (see Corollary 3.13). For the Polish group $G := H_+[0,1]$ of orientation preserving homeomorphisms of the closed interval, every WAP function is constant and every continuous representation on a reflexive space is trivial.

By [97], a compact metrizable dynamical system is Eberlein (= reflexively representable) iff it is WAP in the sense of Ellis-Nerurkar [37]. General WAP dynamical systems are characterized as those systems that have sufficiently many representations on reflexive Banach spaces. This result gives an easy geometric proof of the Lawson-Ellis theorem 5.7, which in turn is a generalization of a classical theorem of Ellis: any compact semitopological group is a topological group. Another generalization of Ellis' theorem is presented in Theorem 9.11, which asserts that any tame compact right topological group is a topological group.

1.4   HNS systems and Asplund Banach spaces

In several recent works new and perhaps unexpected connections between the (lack of) chaotic behavior of a dynamical system and the existence of linear representations of the system on an Asplund Banach space were discovered. The property of *sensitive dependence on initial conditions* appears as a basic constituent in several definitions of "chaos". Hereditary non-sensitive (in short: HNS) dynamical systems, were introduced in [56]. This notion, which was motivated by earlier results of Akin-Auslander-Berg (on almost equicontinuous systems [4]) and Glasner-Weiss [63, 64] (on locally equicontinuous systems), is much more flexible than mere nonsensitivity and is preserved by subsystems as well as factors.

Asplund Banach spaces and the closely related *Radon–Nikodým property* are main themes in Banach space theory. Recall that a Banach space V is called *Asplund* if the dual of every separable linear subspace is separable, iff every bounded subset A of the dual space V^* is (weak*,norm)-*fragmented*, iff V^* has the Radon–Nikodým property. Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund.

As we already mentioned in section 1.1, a dynamical system is said to be a *Radon-Nikodým system* (RN) if it is Asplund representable. Now it turns out that for metrizable compact G-systems the three classes of RN, HNS and HAE (hereditarily almost equicontinuous) dynamical systems coincide. See Theorem 7.3.

1.5   Tame systems and Rosenthal Banach spaces

Tame dynamical systems appeared first in the work of Köhler [83] under the name of *regular* systems. In [56] we formulated a dynamical version of the Bourgain-Fremlin-Talagrand (in short: BFT) dichotomy (Theorem 8.4 below). According to this dichotomy an enveloping semigroup is either *tame*: has cardinality $\leq 2^{\aleph_0}$, or it is topologically wield and contains a copy of $\beta\mathbb{N}$, the Čech-Stone compactificaltion of a discrete countable set. In subsection 8.3 we also consider the so called Todorčević trichotomy and hint at some possible applications to dynamics.

Rosenthal's celebrated dichotomy theorem asserts that every bounded sequence in a Banach space either has a weak Cauchy subsequence or it admits a subsequence equivalent to the unit vector basis of l_1 (an l_1-*sequence*). Thus, a Banach space V does not contain an isomorphic copy of l_1 if and only if every bounded sequence in V has a weak-Cauchy subsequence [127]. We call a Banach space satisfying these equivalent conditions a *Rosenthal space*.

Now it is shown in [58, 60] (see Theorem 8.1 below) that a compact metric S-system X is tame if and only if (S, X) admits a faithful representation on a separable Rosenthal space. Also, the dynamical BFT dichotomy combined with a characterization of Rosenthal Banach spaces, Theorem 9.3, lead to a dichotomy theorem for Banach spaces (Theorem 8.6).

As a general principle one can measure the usefulness of a new mathematical notion by the number of seemingly unrelated ways by which it can be characterized. According to this principle the notion of tameness stands rather high. In addition to those characterizations already mentioned, tameness can also be characterized by the lack of an "independence" property (section 8.6), where combinatorial Ramsey type arguments take a leading role, by the fact that the elements of the enveloping semigroup of a tame system are Baire class 1 maps, and, using results of Talagrand, as systems in which the topological Glivenko-Cantelli property is satisfied (section 8.4). Also, for abelian acting groups there is now a complete structure theorem available for tame minimal systems (section 8.8).

Tame and HNS dynamical systems were further investigated in several recent publications. See for example the papers by Huang [72], Huang-Ye [73] and Kerr-Li [82].

1.6   The crucial role of the Davis-Figiel-Johnson-Pelczyński theorem

In all the main three cases where an equivalence of a dynamical property to the representability on a class of Banach spaces was established (namely the cases of WAP, HNS and Tame dynamical systems) a crucial role in the proof is played by a well known method of construction of Banach spaces known as the Davis-Figiel-Johnson-Pelczyński theorem, [27].

1.7   Representations of groups and functions

Every (Hausdorff) topological group G can be represented as a subgroup of $\mathrm{Iso}\,(V)_s$ and of $\mathrm{Homeo}\,(X)$ for some Banach space V and compact X. (In fact, one may take X to be the weak* compact unit ball $X := B(V^*)$.) Dynamical systems representation theory provides key ideas for constructing distinguishing examples of topological groups. For example, one can build natural examples of Polish topological groups which are Rosenthal representable but not reflexively (or, even, Asplund) representable. Also groups which are reflexively but not Hilbert representable, [96]. Such Polish groups can even be monothetic, [66].

One of the most fruitful approaches to the study of topological groups is via the analysis of certain classes of functions on the group. For example when G is a compact group then, by the classical Peter-Weyl theorem, every $f \in C(G)$ can be uniformly approximated by matrix

coefficients of finite-dimensional unitary representations of G. Recent research shows that many natural classes of functions on groups can be characterized as matrix coefficients of group representations on suitable classes of Banach spaces. For example, WAP, Asplund, and tame functions on G, correspond to matrix coefficients of representations of G on reflexive, Asplund and Rosenthal spaces, respectively. WAP, Asplund, and tame functions on G are functions which come from WAP, HNS (hereditarily non-sensitive), and tame compact G-compactifications of G (i.e., G-ambits), respectively.

Every (Hausdorff) topological group admits a right topological semigroup compactification $\alpha : G \to P$ such that α is an embedding. For example, one may take the greatest ambit, the semigroup G-compactification of G induced by the algebra RUC(G). See for example, [145, 117, 53].

Query 1.4 *Which Polish groups admit faithful right topological semigroup compactifications $\alpha : G \to P$ such that P is: (a) semitopological; (b) metrizable; (c) Fréchet-Urysohn ?*

Here and below *faithful* will mean a topological embedding. Equivalently one can ask when the algebras WAP(G), Asp(G) and Tame(G), respectively, generate the topology of G, see Remark 4.14. For the group $G := H_+[0,1]$ the algebra Asp(G) (hence also WAP(G)) consist only of constant functions ([95, 57, 60]). This implies that *metrizable* semigroup compactifications of G and Asplund (e.g., reflexive) representations are trivial. In contrast, quite unexpectedly, for this group Tame(G) generates the topology of G. This shows that $H_+[0,1]$ is Rosenthal representable and admits a faithful right topological semigroup compactification $\alpha : G \to P$ such that P is Fréchet (see Theorem 10.3). It is still unknown if the latter is true for every Polish group.

By a result of Ferri and Galindo [41] the group c_0 is not reflexively representable. This answers some questions from [99]. It is not yet well understood which abelian Polish groups are Asplund representable, [57], or which admit continuous injective reflexive representations [41].

1.8   Sensitivity, fragmentability and fixed point theorems

The link between the various dynamical aspects of almost equicontinuity on the one hand and the geometrical Banach space RN properties on the other hand is the versatile notion of *fragmentability*. It played a central role in the works on RN compacta (see e.g. Namioka [111]). For some applications of fragmentability for topological transformation groups, see [93, 94, 97, 56, 136].

In Section 11 we study the relevance of nonsensitivity and fragmentability to fixed point theory in affine dynamical systems. Our approach involves a topological version of non-sensitivity for families of functions which, in turn, gives another interpretation to the fragmentability concept. For details see [59]. We discuss a fixed point theorem 11.2 which extends Ryll-Nardzewski's theorem and some of its generalizations. The existence of a fixed point in an affine dynamical system can be deduced from the existence of an invariant measure. Using topological sensitivity it is then possible to reduce the problem at hand to the situation where the existence of an invariant measure follows from a well known theorem of Furstenberg [42]: Every distal compact dynamical system admits an invariant probability measure.

1.9   Some connections to Banach space theory

One of the important questions in Banach space theory until the mid 70's was to construct a separable Rosenthal space which is not Asplund. The first counterexamples were constructed independently by James [75] and Lindenstrauss and Stegall [87]. In view of Theorem 8.1 we

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now see that a fruitful way of producing such distinguishing examples comes from dynamical systems. Just consider a compact metric tame G-system which is not HNS and then apply Theorem 8.1. We have several examples of dynamical systems of this type; e.g. the Sturmian cascades (Examples 8.31 (2), (3)) or the actions of $GL_n(\mathbb{R})$ on the sphere or the projective space, Examples 8.31 (6).

One may make this result sharper by using Theorem 10.3 and Remark 10.4.1. There exists a separable Rosenthal space V without the *adjoint continuity property*. Indeed, by Theorem 10.3 the Polish group $G := H_+[0, 1]$, which admits only trivial Asplund (and, hence, reflexive) representations, is however Rosenthal representable.

Finally, let us mention yet another potentially interesting direction of research, which may lead to a new classification of Rosenthal Banach spaces. This arises from the fine structure of Rosenthal compacta — as described for example in Debs' review [28] — applied to the Rosenthal compacta of the form $\mathcal{E}(V)$ (Definition 4.19). See Theorem 4.20 and Section 8.3.

2 Preliminaries

Topological spaces are always assumed to be Hausdorff and completely regular. The closure of a subset $A \subset X$ is denoted by $\overline{A}$ or $\mathrm{cls}\,(A)$. Banach spaces and locally convex vector spaces are over the field $\mathbb{R}$ of real numbers. For a subset A of a Banach space we denote by $sp(A)$ and $\overline{sp}^{norm}(A)$ the linear span and the norm-closed linear span of A respectively. We denote by $co(A)$ and $\overline{co}(A)$ the convex hull and the closed convex hull of a set A, respectively. If $A \subset V^*$ is a subset of the dual space V^* we we usually use the weak* topology on A and $\overline{co}(A)$ or $\overline{co}^{w^*}(A)$ will denote the w^*-closure of $co(A)$ in V^*. For a topological space X we denote by $C(X)$ the Banach algebra of real valued continuous and bounded functions equipped with the supremum norm. For a subset $A \subset C(X)$ we denote by $\langle A \rangle$ the smallest unital (i.e., containing the constants) closed subalgebra of $C(X)$ containing A. In this section we give some background material mainly based on [60, 58].

2.1 Semigroups and actions

Let P be a semigroup which is also a topological space. By $\lambda_a : P \to P, x \mapsto ax$ and $\rho_a : P \to P, x \mapsto xa$ we denote the left and right a-transitions. The subset $\Lambda(P) := \{a \in P : \lambda_a \text{ is continuous}\}$ is called the *topological center* of P.

Definition 2.1 *A semigroup P as above is said to be:*

1. *a* right topological semigroup *if every ρ_a is continuous.*

2. semitopological *if the multiplication $P \times P \to P$ is separately continuous.*

3. *[107]* admissible *if P is right topological and $\Lambda(P)$ is dense in P.*

Let A be a subsemigroup of a right topological semigroup P. If $A \subset \Lambda(P)$ then the closure $\mathrm{cls}\,(A)$ is a right topological semigroup. In general, $\mathrm{cls}\,(A)$ is not necessarily a subsemigroup of P (even if P is compact right topological and A is a left ideal). Also $\Lambda(P)$ may be empty for general compact right topological semigroup P. See [19, p. 29].

Let S be a semitopological semigroup with a neutral element e. Let $\pi : S \times X \to X$ be a *left action* of S on a topological space X. This means that $ex = x$ and $s_1(s_2 x) = (s_1 s_2) x$ for all $s_1, s_2 \in S$ and $x \in X$, where as usual, we write sx instead of $\pi(s, x) = \lambda_s(x) = \rho_x(s)$.

We say that X is a *dynamical S-system* (or an *S-space*) if the action π is separately continuous (that is, if all orbit maps $\rho_x : S \to X$ and all translations $\lambda_s : X \to X$ are continuous). We often write (S, X) to denote an S dynamical system.

Given $x \in X$, its *orbit* is the set $Sx = \{sx : s \in S\}$ and the closure of this set, $\mathrm{cls}\,(Sx)$, is the *orbit closure* of x. A point x with $\mathrm{cls}\,(Sx) = X$ is called a *transitive point*, and the set of transitive points is denoted by X_{tr}. We say that the system is *point transitive* when $X_{tr} \neq \emptyset$. The system is called *minimal* if $X_{tr} = X$.

Let $S \times X \to X$ and $S \times Y \to Y$ be two actions. A map $f : X \to Y$ between S-spaces is an *S-map* if $f(sx) = sf(x)$ for every $(s,x) \in S \times X$.

When we talk about a *continuous S-space* we require that the action π is jointly continuous.

A point transitive dynamical system (S, X) together with a distinguished point $x_0 \in X_{tr}$, so that $X = \mathrm{cls}\,Sx_0$, is sometimes called an *ambit*. Finally when the acting group is the group of integers $\mathbb{Z}$, we sometimes refer to $\mathbb{Z}$-systems as *cascades*.

Theorem 2.2 *[86] Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then every separately continuous action of G on a compact space X is continuous.*

Notation: *All semigroups S are assumed to be monoids, i.e, semigroups with a neutral element which will be denoted by e. Also actions are* monoidal *(meaning $ex = x, \forall x \in X$) and separately continuous. We reserve the symbol G for the case when S is a group.*

3 Fragmentability and representations on Banach spaces

The concept of fragmentability originally comes from Banach space theory and has several applications in Topology, and more recently also in Topological Dynamics.

Definition 3.1 *Let (X, τ) be a topological space and let (Y, μ) a uniform space.*

1. *[76] X is (τ, μ)-fragmented by a (typically, not continuous) function $f : X \to Y$ if for every nonempty subset A of X and every $\varepsilon \in \mu$ there exists an open subset O of X such that $O \cap A$ is nonempty and the set $f(O \cap A)$ is ε-small in Y. We also say in that case that the function f is* fragmented. *Notation: $f \in \mathcal{F}(X, Y)$, whenever the uniformity μ is understood. If $Y = \mathbb{R}$ then we write simply $\mathcal{F}(X)$.*

2. *We say that a family of functions $F = \{f : (X, \tau) \to (Y, \mu)\}$ is fragmented if condition (1) holds simultaneously for all $f \in F$. That is, $f(O \cap A)$ is ε-small for every $f \in F$.*

3. *We say that F is an* eventually fragmented family *if every countable infinite subfamily $C \subset F$ contains an infinite fragmented subfamily $K \subset C$.*

Remark 3.2 *[56, 58]*

1. *It is enough to check the condition of Definition 3.1 for closed subsets $A \subset X$ and for $\epsilon \in \mu$ from a subbase γ of μ (that is, the finite intersections of the elements of γ form a base of the uniform structure μ).*

2. *When X and Y are Polish spaces, $f : X \to Y$ is fragmented iff f is a Baire class 1 function.*

3. *When X is compact and (Y, ρ) metrizable uniform space then $f : X \to Y$ is fragmented iff f has a* point of continuity property *(i.e., for every closed nonempty $A \subset X$ the restriction $f_{|A} : A \to Y$ has a continuity point).*

4. *When Y is compact with its unique compatible uniformity μ then $p : X \to Y$ is fragmented if and only if $f \circ p : X \to \mathbb{R}$ has the point of continuity property for every $f \in C(Y)$.*

The first assertion in the following lemma can be proved using Namioka's joint continuity theorem.

Lemma 3.3 *[62, 58]*

1. *Suppose F is a compact space, X is Čech-complete, Y is a uniform space and we are given a separately continuous map $w : F \times X \to Y$. Then the naturally associated family $\tilde{F} := \{\tilde{f} : X \to Y\}_{f \in F}$ is fragmented, where $\tilde{f}(x) = w(f, x)$.*

2. *Suppose F is a compact and metrizable space, X is hereditarily Baire and M is separable and metrizable. Assume we are given a map $w : F \times X \to M$ such that every $\tilde{x} : F \to M, f \mapsto w(f, x)$ is continuous and $y : X \to M$ is continuous at every $\tilde{y} \in Y$ for some dense subset Y of F. Then the family $\tilde{F}$ is fragmented.*

For some applications of the fragmentability concept for topological transformation groups, see [93, 94, 97, 56, 59, 58, 60].

3.1 Banach space classes defined by fragmentability

We recall the definitions of three important classes of Banach spaces: Asplund, Rosenthal and PCP. Each of them can be characterized in terms of fragmentability.

3.1.1 Asplund Banach spaces

Recall that a Banach space V is an *Asplund space* if the dual of every separable linear subspace is separable.

When V is a Banach space we denote by B, or B_V, the closed unit ball of V. $B^* = B_{V^*}$ and $B^{**} := B_{V^{**}}$ will denote the weak* compact unit balls in the dual V^* and second dual V^{**} of V respectively. In the following result the equivalence of (1), (2) and (3) is well known and (4) is a reformulation of (3) in terms of fragmented families.

Theorem 3.4 *[114, 111] Let V be a Banach space. The following conditions are equivalent:*

1. *V is an Asplund space.*

2. *V^* has the Radon-Nikodým property.*

3. *Every bounded subset A of the dual V^* is (weak*,norm)-fragmented.*

4. *B is a fragmented family of real valued maps on the compactum B^*.*

Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. Namioka's Joint Continuity Theorem implies that every weakly compact set in a Banach space is norm fragmented, [111]. This explains why every reflexive space is Asplund. For more details cf. [111, 23, 40].

3.1.2 Banach spaces not containing l_1

We say that a Banach space V is *Rosenthal* if it does not contain an isomorphic copy of l_1. Clearly, every Asplund space is Rosenthal.

Definition 3.5 *Let X be a topological space. We say that a subset $F \subset C(X)$ is a* Rosenthal *family (for X) if F is norm bounded and the pointwise closure $\mathrm{cls}_{\mathrm{p}}(F)$ of F in $\mathbb{R}^X$ consists of fragmented maps, that is, $\mathrm{cls}_{\mathrm{p}}(F) \subset \mathcal{F}(X)$.*

Theorem 3.6 *[58] Let X be a compact space and $F \subset C(X)$ a bounded subset. The following conditions are equivalent:*

1. *F does not contain a subsequence equivalent to the unit basis of l_1.*

2. *F is a Rosenthal family for X.*

3. *F is an eventually fragmented family.*

We will use some well known characterizations of Rosenthal spaces.

Theorem 3.7 *Let V be a Banach space. The following conditions are equivalent:*

1. *V is a Rosenthal Banach space.*

2. *(E. Saab and P. Saab [133]) Each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B^*. Equivalently, $B^{**} \subset \mathcal{F}(B^*)$.*

3. *(Haydon [70, Theorem 3.3]) For every weak* compact subset $Y \subset V^*$ the weak* and norm closures of the convex hull $co(Y)$ in V^* coincide: $\mathrm{cls}_{w^*}(co(Y)) = \mathrm{cls}_{norm}(co(Y))$.*

4. *B is an eventually fragmented family of maps on B^*.*

Condition (2) is a reformulation (in terms of fragmented maps) of a criterion from [133] which was originally stated in terms of the point of continuity property. The equivalence of (1) and (4) follows from Theorem 3.6.

3.1.3 Banach spaces with PCP

A Banach space V is said to have the *point of continuity property* (PCP for short) if every bounded weakly closed subset $C \subset V$ admits a point of continuity of the identity map $(C, weak) \to (C, norm)$ (see for example Edgar-Wheeler [33] and [77]). Every Banach space with RNP has PCP. In particular, this is true for the duals of Asplund spaces and for reflexive spaces. This concept was studied, among others, by Bourgain and Rosenthal. They show, for instance, that there are separable Banach spaces with PCP which do not satisfy RNP.

Theorem 3.8 *(Jayne and Rogers [77]) Let V be a Banach space. The following conditions are equivalent:*

1. *V has PCP.*

2. *Every bounded subset $A \subset V$ is $(weak, norm)$-fragmented.*

3.2 Representations of dynamical systems on Banach spaces

A *representation* of a semigroup S on a normed space V is a co-homomorphism $h : S \to \Theta(V)$, where $\Theta(V) := \{T \in L(V) : \|T\| \leq 1\}$ and $h(e) = id_V$. Here $L(V)$ is the space of continuous linear operators $V \to V$ and id_V is the identity operator. This is equivalent to the requirement that $h : S \to \Theta(V)^{op}$ be a monoid homomorphism, where $\Theta(V)^{op}$ is the opposite semigroup of $\Theta(V)$. If $S = G$, is a group then $h(G) \subset \mathrm{Iso}\,(V)$, where $\mathrm{Iso}\,(V)$ is the group of all linear isometries from V onto V. The adjoint operator induces an injective co-homomorphism $adj : \Theta(V) \to \Theta(V^*), adj(s) = s^*$. We will identify $adj(\Theta(V)) \subset L(V^*)$ and the opposite semigroup $\Theta(V)^{op}$. Mostly we use the same symbol s instead of s^*. Since $\Theta(V)^{op}$ acts from the right on V and from the left on V^* we sometimes write vs for $h(s)(v)$ and $s\psi$ for $h(s)^*(\psi)$.

A pair of vectors $(v, \psi) \in V \times V^*$ defines a function (called a *matrix coefficient* of h)

$$m_{v,\psi} : S \to \mathbb{R}, \quad s \mapsto \psi(vs) = \langle vs, \psi \rangle = \langle v, s\psi \rangle.$$

The *weak operator topology* on $\Theta(V)$ (similarly, on $\Theta(V)^{op}$) is the weak topology generated by all matrix coefficients. So $h : S \to \Theta(V)^{op}$ is *weakly continuous* iff $m_{v,\psi} \in C(S)$ for every $(v, \psi) \in V \times V^*$. The *strong operator topology* on $\Theta(V)$ (and on $\Theta(V)^{op}$) is the pointwise topology with respect to its left (respectively, right) action on the Banach space V.

Definition 3.9 *(See [97, 56]) Let X be a dynamical S-system.*

1. *A* representation *of (S, X) on a normed space V is a pair*

$$(h, \alpha) : S \times X \rightrightarrows \Theta(V) \times V^*$$

where $h : S \to \Theta(V)$ is a co-homomorphism of semigroups and $\alpha : X \to V^*$ is a weak* continuous bounded S-mapping with respect to the dual action

$$S \times V^* \to V^*, \ (s\varphi)(v) := \varphi(h(s)(v)).$$

We say that the representation is weakly (strongly) continuous if h is weakly (strongly) continuous. A representation (h, α) is said to be faithful *if α is a topological embedding.*

2. *If $S := G$ is a group then a representation of (S, X) on V is a pair (h, α), where α is as above and $h : G \to \mathrm{Iso}\,(V)$ is a group co-homomorphism.*

3. *If $\mathcal{K}$ is a subclass of the class of Banach spaces, we say that a dynamical system (S, X) is weakly (respectively, strongly) $\mathcal{K}$-representable if there exists a weakly (respectively, strongly) continuous faithful representation of (S, X) on a Banach space $V \in \mathcal{K}$.*

4. *A subdirect product, i.e. an S-subspace of a direct product, of weakly (strongly) $\mathcal{K}$-representable S-spaces is said to be weakly (strongly) $\mathcal{K}$-approximable.*

Lemma 3.10 *Let $h : G \to \mathrm{Iso}\,(V)$ be a given weakly continuous co-representation of a topological group G on V. For every $\psi \in V^*$ and $v \in V$ define the operators*

$$L_\psi : V \to C(G), \quad R_v : V^* \to C(G), \quad \text{where} \quad L_\psi(v) = R_v(\psi) = m_{v,\psi}.$$

Then

1. *L_ψ and R_v are linear bounded operators.*

2. *If ψ (respectively: $v \in V$) is norm G-continuous, then $m_{v,\psi}$ is left (respectively: right) uniformly continuous on G.*

3. *(Eberlein) If V is reflexive, then $m_{v,\psi} \in \mathrm{WAP}(G)$.*

Proof. See [97, Fact 3.5] and also Example 5.3. □

Assertion (3) of this lemma comes from Eberlein. The converse is also true: every wap function is a matrix coefficient of some (co-)representation on a reflexive space.

3.3 When does weak imply strong ?

Recall that a Banach space V has the *Kadec property* if the weak and norm topologies coincide on the unit (or some other) sphere of V. Let us say that a subset X of a V is a *Kadec subset* (*light subset* in [94]) if the weak topology coincides with the norm topology. *Light* linear subgroups $G \leq GL(V)$ (with respect to the weak and strong *operator topologies*) can be defined Analogously. Clearly, if G is *orbitwise Kadec* on V that is, all orbits Gv are light in V, then G is necessarily light. The simplest examples are the spheres (orbits of the unitary group $\mathrm{Iso}\,(H)$) in Hilbert spaces H.

In general, $\mathrm{Iso}\,(V)$ need not be light. Indeed, $\mathrm{Iso}\,(C([0,1]^2))$ is not light, [94]. The following results show that linear actions frequently are "orbitwise Kadec" in the presence of fragmentability properties.

A not necessarily compact G-system X is called *quasiminimal* if $\mathrm{int}(\mathrm{cls}\,(Gz)) \neq \emptyset$ for every $z \in X$. 1-orbit systems and compact minimal G-systems are quasiminimal.

Theorem 3.11 *Let $G \leq GL(V)$ be a bounded subgroup, X a bounded, $(weak, norm)$-fragmented G-invariant subset of a Banach space V. Then every, not necessarily closed, quasiminimal G-subspace (e.g., the orbits) Y of X is a Kadec subset.*

This result together with a characterization of PCP (Theorem 3.8) yield:

Theorem 3.12 *[94, 97] Let V be a Banach space with PCP (e.g., reflexive, RNP, or the dual of Asplund). Then for any bounded subgroup G of $GL(V)$ the weak and strong operator topologies coincide. Moreover, the orbit Gv is a light subset in V for every vector $v \in V$. Hence every weakly continuous (co)homomorphism $h : G \to GL(V)$ where V has PCP and $h(G)$ is bounded, is strongly continuous.*

Corollary 3.13 *The weak and the strong operator topologies coincide on $\text{Iso}(V)$ for every Banach space V with PCP (e.g., for reflexive V).*

In contrast, for $V := C[0,1]^2$ the weak and strong topologies on $\text{Iso}(V)$ are different, [94]. Some boundedness condition is necessary even for l_2. In fact, $GL(l_2)$ is not light. The group G may not be replaced in general by semigroups. Indeed, the semigroup $\Theta(l_2)$ is weakly compact but not strongly compact.

Regarding Theorem 3.12 note that, making use the technique from [94], Shtern proves [135, Theorem 1] that for locally uniformly bounded representations of topological groups on Banach spaces with PCP, weak continuity implies strong continuity.

A topological space X is called a *Namioka space* if for every compact space Y and separately continuous map $\gamma : Y \times X \to \mathbb{R}$, there exists a dense subset $P \subset X$ such that γ is jointly continuous at every $(y,p) \in Y \times P$. A topological space is said to be Čech-*complete* if it can be represented as a G_δ-subset of a compact space. Every Čech-complete (e.g., locally compact or Polish) space is a Namioka space.

Proposition 3.14 *Let G be a semitopological group, X a semitopological G-space and $f \in C(X)$. Suppose that $(\text{cls}_w(fG), weak)$ is a Namioka space. Then fG is light in $C(X)$.*

Proof. If a bounded subset of a Banach space is a Namioka space under the weak topology then it is (weak,norm)-fragmented. So we can complete the proof by applying Theorem 3.11 (for right actions). $\square$

Theorem 3.15 *Let G be a semitopological group. Then for every semitopological G-space X and every $f \in \text{WAP}(X)$ the pointwise and norm topologies coincide on the orbit fG. The pointwise and the norm topologies coincide on fG and Gf for every $f \in \text{WAP}(G)$. In particular, $\text{WAP}(X) \subset \text{RUC}(X)$ and $\text{WAP}(G) \subset \text{RUC}(G)$.*

Proof. Proposition 3.14 guarantees that fG is (weak, norm)-Kadec in $C(X)$. On the other hand, the weak and pointwise topologies coincide on the weak compact set $\text{cls}_w(fG) \subset C(X)$. $\square$

Now we turn to the weak* version of the lightness concept. Let V be a Banach space. Let's say that a subset A of the dual V^* is *weak* light* if the weak* and the norm topologies coincide on A. If G is a subgroup of $GL(V^*)$, then the weak* (resp., norm*) topology on G is the weakest topology which makes all orbit maps $\{\tilde{\psi} : G \to V^* : \psi \in V^*\}$ weak* (resp., norm) continuous.

Theorem 3.16 *[97] Suppose that V is an Asplund space, $G \leq GL(V)$ is a bounded subgroup, and $X \subset V^*$ is a bounded G-invariant subset.*

(i) If (X, w^*) is a quasiminimal (e.g., 1-orbit) G-subset, then X is weak light.*

(ii) The weak and strong* operator topologies coincide on G (e.g., on $\mathrm{Iso}\,(V)$).*

Definition 3.17 *Let $\pi : G \times V \to V$ be a continuous left action of G on V by linear operators. The* adjoint (or, dual) right *action $\pi^* : V^* \times G \to V^*$ is defined by $\psi g(v) := \psi(gv)$. The corresponding* adjoint (dual) left *action is $\pi^* : G \times V^* \to V^*$, where $g\psi(v) := \psi(g^{-1}v)$.*

A natural question here is whether the dual action π^* of G on V^* is jointly continuous with respect to the norm topology on V^*. When this is the case we say that the action π (and, also the corresponding representation $h : G \to \mathrm{Iso}\,(V)$, when π is an action by linear isometries) is *adjoint continuous.*

Theorem 3.16 implies that any strongly continuous group representation $h : G \to \mathrm{Iso}\,(V)$ on an Asplund space V is adjoint continuous. More generally, this is true for any continuous linear topological group action (not necessarily by isometries).

Theorem 3.18 *[97, Corollary 6.9] Let V be an Asplund Banach space and $\pi : G \times V \to V$ a linear jointly continuous action. Then the dual action $\pi^* : G \times V^* \to V^*$ is also jointly continuous.*

The regular representation $\mathbb{T} \to \mathrm{Iso}\,(V)$ of the circle group $G := \mathbb{T}$ on $V := C(\mathbb{T})$ is continuous but not adjoint continuous. Consider the Banach space $V := l_1$ and the topological subgroup $G := S(\mathbb{N})$ ("permutations of coordinates") of $\mathrm{Iso}\,(l_1)$. Then we have a natural continuous representation of the symmetric topological group $S(\mathbb{N})$ on l_1 which is not adjoint continuous.

One more application is a quick proof of Helmer's theorem, [71], $\mathrm{WAP}(G) \subset \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$. In fact, we can show more (see Remark 4.6 for definitions of $\mathrm{Asp}(G)$ and $\mathrm{Asp}_s(G)$).

Theorem 3.19 *[97] $\mathrm{WAP}(G) \subset \mathrm{Asp}_s(G) \subset \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$ For every semitopological group G.*

Proof. For $\mathrm{WAP}(G) = \mathrm{WAP}_s(G) \subset \mathrm{Asp}_s(G)$ see Lemmas 4.7.4 and 9.2. Let $f \in \mathrm{Asp}_s(G)$. By Theorem 9.7.3 the function f coincides with a matrix coefficient $m_{v,\psi}$ for a suitable strongly continuous antihomomorphism $h : G \to \mathrm{Iso}\,(V)_s$, where V is Asplund. In particular, v is a norm continuous vector. By Theorem 3.16 (or, 3.18) the orbit $G\psi$ is light. Hence, ψ is a norm continuous vector. By Lemma 3.10.2, $f = m_{v,\psi}$ is both left and right uniformly continuous. $\square$

Remark 3.20 *Theorem 3.16 can be extended to a locally convex version of Asplund Banach spaces. Following [97] we say that an l.c.s. V is a Namioka-Phelps space ($V \in NP$) if every equicontinuous subset $X \subset V^*$ is (w^*, μ^*)-fragmented. The class NP is closed under subspaces, products and l.c. sums, and includes: Asplund Banach spaces, semireflexive l.c.s. and Nuclear l.c.s.*

For more results about actions on Banach and locally convex spaces we refer to [71, 86, 97, 136].

4 Compactifications and functions

A *compactification* of a topological space X is a pair (ν, Y) where Y is a compact (Hausdorff) space and $\nu : X \to Y$ a continuous map with a dense range.

The Gelfand-Kolmogoroff theory establishes an order preserving bijective correspondence (up to equivalence of compactifications) between Banach *unital* subalgebras $\mathcal{A} \subset C(X)$ and compactifications $\nu : X \to Y$ of X. The Banach unital S-subalgebra $\mathcal{A}$ induces the *canonical $\mathcal{A}$-compactification* $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$, where $X^{\mathcal{A}}$ is the spectrum (or the Gelfand space — the collection of continuous multiplicative functionals on $\mathcal{A}$). The map $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}} \subset \mathcal{A}^*$ is defined by the *Gelfand transform*, the evaluation at x functional, $\alpha_{\mathcal{A}}(x)(f) := f(x)$. Conversely, every compactification $\nu : X \to Y$ is equivalent to the *canonical $\mathcal{A}_\nu$-compactification* $\alpha_{\mathcal{A}_\nu} : X \to X^{\mathcal{A}_\nu}$, where the algebra $\mathcal{A}_\nu$ is defined as the image $j_\nu(C(Y))$ of the embedding $j_\nu : C(Y) \hookrightarrow C(X), \ \phi \mapsto \phi \circ \nu$.

Definition 4.1 *Let X be an S-system. An S-compactification of X is a continuous S-map $\alpha : X \to Y$, with a dense range, into a compact S-system Y. An S-compactification is said to be* jointly continuous *(respectively,* separately continuous*) if the action $S \times Y \to Y$ is jointly continuous (respectively, separately continuous).*

An S-system X with jointly continuous action is said to be S-Tykhonov (or, compactifiable) if it admits a proper S-compactification with jointly continuous action.

By S_d we denote a discrete copy of S.

Remark 4.2 *If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two compactifications, then ν_2 dominates ν_1, that is, $\nu_1 = q \circ \nu_2$ for some (uniquely defined) continuous map $q : Y_2 \to Y_1$ iff $\mathcal{A}_{\nu_1} \subset \mathcal{A}_{\nu_2}$. If in addition, X, Y_1 and Y_2 are S_d-systems (i.e., all the s-translations on X, Y_1 and Y_2 are continuous) and if ν_1 and ν_2 are S-maps, then q is also an S-map. Furthermore, if the action on Y_1 is (separately) continuous then the action on Y_2 is (respectively, separately) continuous. If ν_1 and ν_2 are homomorphisms of semigroups then q is also a homomorphism. See [152, App. D].*

4.1 From representations to compactifications

Representations of dynamical systems (S, X) lead to S-compactifications of X. Let V be a normed space and let

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, V^*)$$

be a representation of (S, X), where α is a weak* continuous map. Consider the induced compactification $\alpha : X \to Y := \overline{\alpha(X)}$, the weak* closure of $\alpha(X)$. Clearly, the induced natural action $S \times Y \to Y$ is well defined and every left translation is continuous. So, Y is an S_d-system.

Remark 4.3 *1. The induced action $S \times Y \to Y$ is separately continuous iff the matrix coefficient $m_{v,y} : S \to \mathbb{R}$ is continuous $\forall \ v \in V, \ y \in Y$.*

  2. If h is strongly (weakly) continuous then the induced dual action of S on the weak compact unit ball B^* and on Y is jointly (respectively, separately) continuous.*

To every S-space X we associate the *regular representation* on the Banach space $V := C(X)$ defined by the pair (h, α) where $h : S \to \Theta(V), s \mapsto L_s$ (with $L_s f(x) = f(sx)$) is the natural co-homomorphism and $\alpha : X \to V^*, x \mapsto \delta_x$ is the evaluation map $\delta_x(f) = f(x)$. Denote by (WRUC(X)) RUC(X) the set of all (weakly) right uniformly continuous functions. That is functions $f \in C(X)$ such that the orbit map $\tilde{f} : S \to C(X), s \mapsto fs = L_s(f)$

is (weakly) norm continuous. Then RUC(X) and WRUC(X) are norm closed S-invariant unital linear subspaces of $C(X)$ and the restriction of the regular representation is continuous on RUC(X) and weakly continuous on WRUC(X). Furthermore, RUC(X) is a Banach subalgebra of $C(X)$. If $S \times X \to X$ is continuous and X is compact then $C(X) =$ RUC(X). In particular, for the left action of S on itself $X := S$ we write simply RUC(S) and WRUC(S). If $X := G$ is a topological group with the left action on itself then RUC(G) is the usual algebra of right uniformly continuous functions on G. Note that WRUC(S) plays a major role in the theory of semigroups, being the largest left introverted linear subspace of $C(S)$ (Rao's theorem; see for example, [18]).

Remark 4.4 *Let X be an S-system.*

1. *For every S-invariant normed subspace V of* WRUC(X) *we have the regular weakly continuous V-representation, (h, α) of (S, X) on V defined by $\alpha(x)(f) = f(x)$, $f \in V$, and the corresponding S-compactification $\alpha : X \to Y := \overline{\alpha(X)}$. The action of S on Y is continuous iff $V \subset$ RUC(X).*

2. *For every continuous dynamical system (S, X) we have $C(X) =$ RUC(X). So there exists a faithful representation of (S, X) on $C(X)$.*

In this way one gets a description of jointly continuous S-compactifications of an S-space X in terms of subalgebras of RUC(X) in the spirit of the Gelfand-Kolmogoroff theorem. This theory is well known for topological group actions, [150]. One can easily extend it to the case of topological semigroup actions (Ball and Hagler [14] and [98]).

We say that a function $f \in C(X)$ on an S-space X *comes from* an S-compactification $\nu : X \to Y$ (recall that we require only that the actions on X, Y are separately continuous) if there exists $\tilde{f} \in C(Y)$ such that $f = \tilde{f} \circ \nu$. Denote by RMC(X) the set (in fact a unital Banach algebra) of all (right multiplicatively continuous) functions on X which come from S-compactifications. The algebra RUC(X) is the set of all functions which come from jointly continuous S-compactifications. Regarding a description of separately continuous S-compactifications via subalgebras of RMC(X) and for more details about Remarks 4.3, 4.4 see, for example, [97, 98].

A word of caution about our notation WRUC(S), RUC(S), RMC(S). Note that in [18] the corresponding notation is WLUC(S), LUC(S), LMC(S) (and sometimes WLC(S), LC(S), [19]).

Theorem 4.5 *An S-space X with jointly continuous action is strongly representable on some Banach space V if and only if X is S-Tykhonov.*

Proof. Consider the natural representation of (S, V) on the Banach space $V :=$ RUC(X). $\square$

Remark 4.6 *Let $\mathcal{P}$ be a class of compact separately continuous S-dynamical systems. The subclass of S-systems with continuous actions will be denoted by $\mathcal{P}_c$. Assume that $\mathcal{P}$ is closed under products, closed subsystems and S-isomorphisms. In such cases (following [152, Ch. IV]) we say that $\mathcal{P}$ is* suppable. *Let X be a not necessarily compact S-space and let $\mathcal{P}(X)$ be the collection of functions on X coming from systems having property $\mathcal{P}$. Then, as in the case of jointly continuous actions (see [56, Prop. 2.9]), there exists a universal S-compactification $X \to X^{\mathcal{P}}$ of X such that $(S, X) \in \mathcal{P}$. Moreover, $j(C(X^{\mathcal{P}})) = \mathcal{P}(X)$. In particular, $\mathcal{P}(X)$ is a uniformly closed, S-invariant subalgebra of $C(X)$. Analogously, one defines $\mathcal{P}_c(X)$. Again it is a uniformly closed, S-invariant subalgebra of $C(X)$, which is in fact a subalgebra of* RUC(X). *For the corresponding S-compactification $X \to X^{\mathcal{P}_c}$ the action of S on $X^{\mathcal{P}_c}$ is continuous.*

In particular, for the left action of S on itself we get the definitions of $\mathcal{P}(S)$ and $\mathcal{P}_c(S)$. As in [56, Prop. 2.9] one can show that $\mathcal{P}(S)$ and $\mathcal{P}_c(S)$ are m-introverted Banach subalgebras of $C(S)$ (this property is equivalent to the property of being isomorphic to an enveloping semigroup, see subsection 4.4 below) and that they define the $\mathcal{P}$-universal and $\mathcal{P}_c$-universal semigroup compactifications $S \to S^{\mathcal{P}}$ and $S \to S^{\mathcal{P}_c}$.

In this review we are especially interested in the following classes of compact S-systems: a) Weakly Almost Periodic, WAP in short. b) Hereditarily Non-Sensitive, HNS in short (Definition 7.1); a) Tame systems (Definition 9.1); See sections 5.1, 7, 8 and also [56, 57, 58].

For the corresponding algebras of functions, defined by Remark 4.6, we use the following notation: $\text{WAP}(X)$, $\text{Asp}(X)$ and $\text{Tame}(X)$. Note that the WAP (respectively, HNS, tame) systems are exactly the compact systems which admit sufficiently many representations on reflexive (respectively, Asplund, Rosenthal) Banach spaces (section 9).

Lemma 4.7   *1. For every S-space X we have $\mathcal{P}_c(X) \subset \text{RUC}(X) \subset \text{WRUC}(X) \subset \text{RMC}(X)$ and $\mathcal{P}_c(X) \subset \mathcal{P}(X) \cap \text{RUC}(X)$. If $\mathcal{P}$ is preserved by factors then $\mathcal{P}_c(X) = \mathcal{P}(X) \cap \text{RUC}(X)$.*

   2. If X is a compact S-system with continuous action then $\mathcal{P}_c(X) = \mathcal{P}(X)$, $\text{RUC}(X) = \text{WRUC}(X) = \text{RMC}(X) = C(X)$.

   3. If $S = G$ is a Čech-complete semitopological group then for every G-space X we have $\mathcal{P}_c(X) = \mathcal{P}(X)$, $\text{RUC}(X) = \text{WRUC}(X) = \text{RMC}(X)$; in particular, $\text{RUC}(G) = \text{WRUC}(G) = \text{RMC}(G)$.

   4. $\text{WAP}_c(G) = \text{WAP}(G)$ remains true for every semitopological group G.

Let X be a compact space with a separately continuous action $\pi : S \times X \to X$. Then $C(X) = \text{WRUC}(X)$ iff the induced action $\pi_P : S \times P(X) \to P(X)$ is separately continuous. We say that X is WRUC-*compatible*. We mention three useful sufficient conditions where X is WRUC-*compatible*: a) the action $S \times X \to X$ is continuous; b) S, as a topological space, is a k-space (e.g., metrizable); c) (S, X) is WAP. By Corollary 9.8, $\text{Tame}(X) \subset \text{WRUC}(X)$ for every S-space X. In particular, it follows that every tame (hence, every WAP) *compact S-system is WRUC-compatible*.

4.2   Right topological semigroup compactifications

Another important hierarchy for topological (semi)groups comes from right topological semigroup compactifications.

Definition 4.8 *Let S be a semitopological semigroup.*

   1. [19, p. 105] A right topological semigroup compactification of S is a pair (γ, T) such that T is a compact right topological semigroup and γ is a continuous semigroup homomorphism from S into T, where $\gamma(S)$ is dense in T and the left translation $\lambda_s : T \to T$, $x \mapsto \gamma(s)x$ is continuous for every $s \in S$, that is, $\gamma(S) \subset \Lambda(T)$.
   It follows that the associated action

$$\pi_\gamma : S \times T \to T, \quad (s, x) \mapsto \gamma(s)x = \lambda_s(x)$$

   is separately continuous.

   2. [131, p. 101] A dynamical right topological semigroup compactification of S is a right topological semigroup compactification (γ, T) in the sense of (1) such that, in addition, γ is a jointly continuous S-compactification, i.e., the action $\pi_\gamma : S \times T \to T$ is jointly continuous.

If S is a monoid (as we require in the present paper) with the neutral element e then it is easy to show that necessarily T is a monoid with the neutral element $\gamma(e)$. Directly from Lawson's theorem mentioned above (Theorem 2.2) we have:

Theorem 4.9 *Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then $\gamma : G \to T$ is a right topological semigroup compactification of G if and only if γ is a dynamical right topological semigroup compactification of G.*

A typical example is the enveloping semigroup $E(X)$ of a compact dynamical S-system X together with the natural map $j : S \to E(X)$, section 4.4.

For every semitopological semigroup S there exists a maximal right topological (dynamical) semigroup compactification. The corresponding algebra is (respectively, RUC(S)) RMC(S). If in the definition of a semigroup compactification (γ, T) we remove the condition $\gamma(S) \subset \Lambda(T)$ then maximal compactifications need not exist (See [18, Example V.1.11] which is due to J. Baker).

For more information on right topological semigroups (Ellis semigroups) and their intimate connection to dynamical systems we refer to the monograph [5].

Let $\mathcal{A}$ be a closed unital subalgebra of $C(X)$ for some topological space X. We let $\nu_{\mathcal{A}} : X \to X^{\mathcal{A}}$ be the associated compactification map (where, as before, $X^{\mathcal{A}}$ is the maximal ideal space of $\mathcal{A}$). For instance, the *greatest ambit* (see, for example, [145, 152]) of a topological group G is the compact G-space $G^{\mathrm{RUC}} := G^{\mathrm{RUC}(G)}$. It defines the *universal dynamical semigroup compactification* of G.

For $\mathcal{A} = \mathrm{WAP}(G)$ we get the *universal semitopological compactification* $G \to G^{WAP}$ of G, which is the universal WAP compactification of G (see [30]). Note that by [103] the projection $q : G^{\mathrm{RUC}} \to G^{WAP}$ is a homeomorphism iff G is precompact.

Remark 4.10 *1. Recall that $\mathrm{RUC}(G)$ generates the topology of G for every topological group G. It follows that the corresponding canonical representation (Teleman's representation)*
$$(h, \alpha_{RUC}) : (G, G) \rightrightarrows (\Theta(V)^{op}, B^*)$$
on $V := RUC(G)$ is faithful and h induces a topological group embedding of G into $\mathrm{Iso}\,(V)$. See [117] for details.

2. There exists a nontrivial Polish group G whose universal semitopological compactification G^{WAP} is trivial. This is shown in [95] for the Polish group $G := H_+[0, 1]$. Equivalently: every (weakly) continuous representation $G \to \mathrm{Iso}\,(V)$ of G on a reflexive Banach space V is trivial.

3. A stronger result is shown in [57]: every continuous representation $G \to \mathrm{Iso}\,(V)$ of G on an Asplund space V is trivial and every Asplund function on G is constant (note that $Asp_c(G) = \mathrm{Asp}(G)$ for Polish G by Lemma 4.7.3). Every nontrivial right topological semigroup compactification of the Polish topological group $G := H_+[0, 1]$ is not metrizable [62]. In contrast, Theorem 10.3 shows that G is Rosenthal representable.

4.3   HNS and tame right topological semigroups

It is natural to introduce and study some intermediate subclasses between semitopological compact semigroup and general right topological compact semigroups. The new concepts of HNS and tame semigroups are such examples. Their origin pertains to dynamical systems theory; they have naturally arisen in the study of tame and HNS dynamical systems (sections 5.1, 7, 8).

Definition 4.11 *[56, 58] A compact admissible right topological semigroup P is said to be:*

1. *[58]* tame *if the left translation* $\lambda_a : P \to P$ *is a fragmented map for every* $a \in P$.

2. HNS-semigroup *($\mathcal{F}$-semigroup in [56]) if* $\{\lambda_a : P \to P\}_{a \in P}$ *is a fragmented family of maps.*

These classes are closed under factors. We have the inclusions:

{compact semitopological semigroups} $\subset$ {HNS-semigroups} $\subset$ {Tame semigroups}

Lemma 4.12 *1. Every compact semitopological semigroup P is a HNS-semigroup.*

2. *Every HNS-semigroup is tame.*

3. *If P is a* metrizable *compact right topological admissible semigroup then P is a HNS-semigroup.*

Proof. (2) is trivial. For (1) and (3) apply Lemma 3.3 to $P \times P \to P$. $\square$

Query 4.13 *Which topological groups admit proper right topological semigroup compactifications $\alpha : G \to P$ such that P is: (a) semitopological; (b) tame; (c) HNS; (d) metrizable; (e) Fréchet ?*

Remark 4.14 *For (a), (b) and (c) equivalently, one can ask when the algebras* WAP(G), Tame(G) *and* Asp(G) *respectively, generate the topology of G (recall that always* WAP$(G) =$ WAP$_s(G)$ *and for Polish groups* Tame$_s(G) =$ Tame(G) *and* Asp$_s(G) =$ Asp(G), *Lemma 4.7).*

Always, (e) implies (b) and (d) implies (c). For Polish groups, (c) is equivalent to (d), (b) is equivalent to (e), and (a) is equivalent to the conjunction (a') semitopological & metrizable.

4.4 Enveloping semigroups of dynamical systems

Let X be a compact S-system with a separately continuous action. Consider the natural map $j : S \to C(X, X), s \mapsto \lambda_s$. As usual denote by $E(X) = \mathrm{cls}_p(j(S)) \subset X^X$ the *enveloping semigroup* of (S, X). The associated homomorphism $j : S \to E(X)$ is a right topological semigroup compactification (the *Ellis compactification*) of S, $j(e) = id_X$ and the associated action $\pi_j : S \times E(X) \to E(X)$ is separately continuous. Furthermore, if the S-action on X is continuous then π_j is continuous, i.e., $S \to E(X)$ is a dynamical semigroup compactification. If X is metrizable then $E(X)$ is separable.

Lemma 4.15 *Let $\alpha : S \to P$ be a right topological compactification of a semigroup S. Then the enveloping semigroup $E(S, P)$ of the semitopological system (S, P) is naturally isomorphic to P.*

Remark 4.16 *Every enveloping semigroup $E(S, X)$ is an example of a compact right topological admissible semigroup. Conversely, every compact right topological* admissible *semigroup P is an enveloping semigroup (of $(\Lambda(P), P)$; this follows from Lemma 4.15).*

The following very general questions motivate a substantial part of our survey.

Query 4.17 *Let $E(S, X)$ be the enveloping semigroup of a compact S-system X.*

1. *When does the compact space $E(S, X)$ have nice topological properties ?*

2. *When do the individual maps $p : X \to X$, for $p \in E$, have nice properties ?*

We list here a few results, some of which will be revisited below.

Theorem 4.18 *Let X be a compact S-system and E is its enveloping semigroup.*

1. *(G, X) is equicontinuous (AP) iff E is a group of continuous maps.*

2. *(Ellis [34]) E is a group iff X is a distal system.*

3. *([37, 79]) X is a WAP system iff every $p \in E$ is a continuous map $X \to X$.*

4. *Let (S, X) be point transitive. Then E is a semitopological (topological) semigroup iff X is WAP (resp., AP).*

5. *(Akin-Auslander-Berg [4]) For X metrizable, (G, X) is almost equicontinuous iff there exists a dense G_δ subset $X_0 \subset X$ such that every $p \in E$ is continuous on X_0.*

6. *[56] X is HNS iff E is a fragmented family of maps $X \to X$, iff $j(S)$ is a fragmented family.*

   [62] If X is metrizable then X is HNS iff E is metrizable.

7. *[58] X is tame iff $p : X \to X$, is a fragmented map for every $p \in E$.*

   [62] If X is metrizable then X is tame iff $p : X \to X$, $p \in E$, is of Baire 1 class function.

8. *[60] If E is Fréchet-Urysohn then (S, X) is tame.*

   [56] If X is metrizable then X is tame iff E is Fréchet-Urysohn.

4.5 The enveloping semigroup of a Banach space

Let V be a Banach space and $\Theta(V)$ the semigroup of all non-expanding operators from V to itself. As in section 3.2 consider the natural left action of $\Theta(V)^{op}$ on the weak* compact unit ball B^*. This action is separately continuous when $\Theta(V)^{op}$ carries the weak operator topology.

Definition 4.19   *1. Given a Banach space V we denote by $\mathcal{E}(V)$ the enveloping semigroup of the dynamical system $(\Theta(V)^{op}, B^*)$. We say that $\mathcal{E}(V)$ is the enveloping semigroup of V.*

2. *For every weakly continuous representation $h : S \to \Theta(V)$ of a semigroup on a Banach space V. The closure P_h of S^{op} in $\mathcal{E}(V)$ is a compact right topological semigroup and we get a semigroup compactification $S \to P_h$. We call it an operator compactification of S. Roughly speaking, it is the enveloping semigroup of the representation h.*

Always, $\mathcal{E}(V)$ is a compact right topological admissible affine semigroup. The corresponding Ellis compactification $j : \Theta(V)^{op} \to \mathcal{E}(V)$ is a topological embedding. Alternatively, $\mathcal{E}(V)$ can be defined as the weak* operator closure of the adjoint monoid $\Theta(V)^{op}$ in $L(V^*)$. Note that $\Theta(V)_w^{op}$ is the topological center of $\mathcal{E}$.

If V is separable then $\mathcal{E}(V)$ is separable because B^* is metrizable.

For some Banach spaces V, the topological properties of the right topological compact semigroup $\mathcal{E}(V)$ determine the class $\mathcal{K}$ to which V belongs.

Theorem 4.20 *[60]*

1. *$\mathcal{E}(V)$ is a semitopological semigroup iff V is reflexive.*

2. *$\mathcal{E}(V)$ is a HNS-semigroup iff V is Asplund. $\mathcal{E}(V)$ is metrizable (equivalently, second countable) iff V is separable Asplund.*

3. *$\mathcal{E}(V)$ is a tame semigroup iff V is Rosenthal.*

4. *If the compactum $\mathcal{E}(V)$ is Fréchet then V is a Rosenthal Banach space. For separable V, $\mathcal{E}(V)$ is a Rosenthal compactum iff V is a Rosenthal Banach space.*

5 Reflexive and Hilbert representations

5.1 Compact semitopological semigroups and WAP systems

A function $f \in C(X)$ on an S-space X is called *(weakly) almost periodic* if the orbit $fS := \{fs\}_{s \in S}$ forms a (weakly) precompact subset of $C(X)$. Notation: $f \in \mathrm{AP}(X)$ and $f \in \mathrm{WAP}(X)$ respectively. See for example, [37, 79]. A compact S-space X is said to be *(weakly) almost periodic* if (resp., $C(X) = \mathrm{WAP}(X)$) $C(X) = \mathrm{AP}(X)$. For any S-space X the collections $\mathrm{WAP}(X)$ and $\mathrm{AP}(X)$ are S-invariant subalgebras of $C(X)$ (in fact, of $RMC(X)$). The corresponding Gelfand spaces and compactifications define S-equivariant compactifications $u_{ap} : X \to X^{AP}$ and $u_{wap} : X \to X^{WAP}$. The compactification $S \to S^{WAP}$ (for $X := S$) is the universal semitopological semigroup compactification of S. Furthermore, $\mathrm{WAP}(X)$ and $\mathrm{AP}(X)$ is the set of all functions which come from WAP and AP S-compactifications of X, respectively. When X is compact X^{AP} is the classical *maximal equicontinuous* factor of the system X. A compact S-space X is equicontinuous iff X is almost periodic (AP), that is, iff $C(X) = \mathrm{AP}(X)$.

For every topological group G, treated as a G-space, the corresponding universal AP compactification is the classical *Bohr compactification* $b : G \to bG$, where bG is a compact topological group.

The following characterization of WAP dynamical systems is due to Ellis and Nerurkar [37].

Theorem 5.1 *Let X be a compact S-dynamical system. The following conditions are equivalent.*

1. *(S, X) is WAP.*

2. *The enveloping semigroup $E(S, X)$ consists of continuous maps.*

The proof is based on the following well known lemma.

Theorem 5.2 (Grothendieck's Lemma) *Let X be a compact space. Then a bounded subset A of $C(X)$ is weakly compact iff A is pointwise compact.*

When (S, X) is WAP the enveloping semigroup $E(X)$ is a semitopological semigroup. The converse holds if in addition we assume that (S, X) is point transitive.

Example 5.3 *1. The next example goes back to Eberlein (see also [19, Examples 1.2.f]). If V is reflexive, then every continuous representation (h, α) of a G-system X on V and every pair $(v, \psi) \in V \times V^*$ lead to a weakly almost periodic function $m_{v,\psi}$ on G. This follows easily by the (weak) continuity of the bounded operator $L_\psi : V \to C(G)$, where $L_\psi(v) = m_{v,\psi}$. Indeed if the orbit vG is relatively weakly compact in V (as is the case when V is reflexive), then the same is true for the orbit $L_\psi(vG) = m_{v,\psi}G$ of $m_{v,\psi}$ in $C(G)$. Thus $m_{v,\psi}$ is wap.*

2. Analogously, every $v \in V$ (with reflexive V) defines a wap function $T_v : X \to \mathbb{R}$ on the G-system X which naturally comes from the given dynamical system representation (h, α). Precisely, define

$$T_v : X \to \mathbb{R}, \quad x \mapsto \langle v, \alpha(x) \rangle.$$

Then the set of functions $\{T_v\}_{v \in V}$ is a subset of $\mathrm{WAP}(X)$. If in our example α is an embedding (which implies that X is reflexively representable) then it follows that the collection $\{T_v\}_{v \in V}$ (and hence also $\mathrm{WAP}(X)$) separates the points of X. If, in addition, X is compact it follows that $\mathrm{WAP}(X) = C(X)$ (because $\mathrm{WAP}(X)$ is always a closed subalgebra of $C(X)$). That is, in this case (G, X) is wap in the sense of Ellis and Nerurkar.

The converses of these facts are also true. Every wap function on a G-space X comes from a reflexive representation.

Theorem 5.4 *[97] Let $S \times X \to X$ be a separately continuous action of a semitopological semigroup S on a compact space X. For every $f \in \mathrm{WAP}(X)$ there exist: a reflexive space V, a functional $\phi \in V^*$ and an equivariant pair*

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V), B_V)$$

such that $h : S \to \Theta(V)$ is a weakly continuous homomorphism, $\alpha : X \to B_V$ is a weakly continuous S-map, and $f(x) = \langle \phi, \alpha(x) \rangle = \phi(\alpha(x))$ for every $x \in X$.

If $S = G$ is a semitopological group then one can assume in addition that $h(G) \subset \mathrm{Iso}\,(V)$ and $h : G \to \mathrm{Iso}\,(V)$ is strongly continuous.

Theorem 5.5 *[97, section 4] Let S be a semitopological semigroup.*

1. *A compact (continuous) S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively approximable.*

2. *A compact (continuous) metric S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively representable.*

3. *Every $f \in \mathrm{WAP}(G)$ is a matrix coefficient of a reflexive representation.*

It is important to take into account the following characterization of reflexive spaces.

Lemma 5.6 *Let V be a Banach space. The following conditions are equivalent:*

1. *V is reflexive.*

2. *The compact semigroup $\mathcal{E}$ is semitopological.*

3. *$\mathcal{E} = \Theta^{op}$.*

4. *Θ^{op} is compact with respect to the weak operator topology.*

5. *(Θ^{op}, B^*) is a WAP system.*

We next recall a version of Lawson's theorem [85] and a soft geometrical proof using representations of dynamical systems on reflexive spaces.

Theorem 5.7 (Ellis-Lawson Joint Continuity Theorem) *Let G be a subgroup of a compact semitopological monoid S. Suppose that $S \times X \to X$ is a separately continuous action with compact X. Then the action $G \times X \to X$ is jointly continuous and G is a topological group.*

Proof. A sketch of the proof from [97]: It is easy to see by Grothendieck's Lemma (Theorem 5.2) that $C(X) = \mathrm{WAP}(X)$. Hence (S, X) is a weakly almost periodic system. By Theorem 5.5 the proof can be reduced to the particular case where $(S, X) = (\Theta(V)^{op}, B_{V^*})$ for some reflexive Banach space V with $G := \mathrm{Iso}\,(V)$, where $\Theta(V)^{op}$ is endowed with the weak operator topology. By Corollary 3.13 the weak and strong operator topologies coincide on $\mathrm{Iso}\,(V)$ for reflexive V. In particular, G is a topological group and it acts continuously on B_{V^*}. $\square$

As a corollary one gets the classical result of Ellis.

Theorem 5.8 (Ellis' Theorem) *Every compact semitopological group is a topological group.*

See also [103] and a generalization of Theorem 5.8 in Theorem 9.11.

Another consequence of Theorem 5.5 (taking into account Lemma 5.6) is

Theorem 5.9 ([134] and [94]) *Every compact semitopological semigroup S can be embedded into $\Theta(V) = \mathcal{E}(V^*)$ for some reflexive V.*

Thus, compact semitopological semigroups S can be characterized as closed subsemigroups of $\mathcal{E}(V)$ for reflexive Banach spaces V. By Theorem 9.10 analogous statements (for admissible embeddings) hold for HNS and tame semigroups, where the corresponding classes of Banach spaces are Asplund and Rosenthal spaces respectively.

5.2 Hilbert and Euclidean representations

Theorem 5.10 *Let G be a compact group.*

1. *Every continuous Tykhonov G-space X is Euclidean-approximable.*

2. *Every continuous Tykhonov separable metrizable G-space X is Hilbert representable.*

Proof. (1) It is well known [8] that if G is compact then every Tykhonov G-space X is G-Tykhonov (Definition 4.1). So there exists a proper G-compactification $\nu : X \hookrightarrow Y$. Moreover we can suppose in addition that Y can be G-approximated by metrizable compact G-spaces (see also [102] or [56, Proposition 4.2]). Therefore, in order to prove the claim it suffices to assume that Y itself is a metrizable compact G-system. Now it is enough to show that every metrizable compact G-space X with compact G is Euclidean-approximable. By another well known fact, there exists a unitary linearization of Y (see, for example, [151, Corollary 3.17]). More precisely, there exist: a Hilbert space H, a continuous homomorphism $h : G \to \mathrm{Iso}\,(H)$ and a norm embedding $\alpha : Y \to B_H$ which is equivariant. Applying the Peter-Weyl theorem to the representation h, the Hilbert space H can be represented as a direct sum

$$H = \bigoplus_{j \in J} H_j$$

of irreducible Hilbert G-subspaces H_j where each H_j is finite-dimensional. This implies that the G-system Y (and hence also X) can be approximated by Euclidean G-systems.

(2) Follows from (1) using l_2-sum of representations. $\square$

Theorem 5.10 together with Theorem 4.18.1 show that the equicontinuous (i.e., AP) compact G-systems are exactly the Euclidean-approximable compact G-systems.

Every Euclidean-approximable G-space X is G-Tykhonov and $j(G)$ is a precompact group. The converse is an open question.

Question 5.11 *Let G be a precompact group and X a Tykhonov (say, separable metrizable) G-space. Is it true that (G, X) is Euclidean-approximable, equivalently, is it G-Tykhonov ?*

This question is open even for a cyclic dense subgroup G of the circle group and is closely related to [99, Question 2.3].

Question 5.12 *1. Is there an intrinsic enveloping semigroup characterization of Hilbert representable compact (say, metrizable) G-spaces ?*

2. *Characterize compact semitopological semigroups which can be embedded into the semigroup $\Theta(H)_w$ for some Hilbert space H.*

3. *[99] Is the class of Hilbert representable compact metric G-spaces closed under factors? (See also [66] and Theorem 6.14 below.)*

6 The algebra of Hilbert functions, Eberlein groups and Roelcke precompact groups

6.1 The Fourier-Stietjes algebra

Definition 6.1 *Denote by $B(G)$ the set of matrix coefficients of Hilbert representations for the group G. This is a collection of functions of the form*

$$m_{u,v} : G \to \mathbb{R}, \quad g \mapsto \langle gu, v \rangle$$

where we consider all possible continuous unitary representations $h : G \to U(H)$ *into complex Hilbert spaces H. Then $B(G)$ is a subalgebra of $C(G)$. This algebra is called the* Fourier-Stieltjes algebra *of G (see for example, [39, 91]).*

The algebra $B(G)$ is rarely closed in $C(G)$. More precisely, if G is locally compact then $B(G)$ is closed in $C(G)$ iff G is finite. Clearly, the set $P(G)$ of positive definite functions on G is a subset of $B(G)$ and every $m \in B(G)$ is a linear combination of some elements from $P(G)$. Every positive definite function is wap (see for example [24]). Hence, always, $B(G) \subset \mathrm{WAP}(G)$. The question whether $B(G)$ is dense in $\mathrm{WAP}(G)$ was raised by Eberlein, [129]. This was the motivation for the definition of Eberlein groups, a notion which was originally applied to locally compact groups, see Chou [26] and Mayer [91].

Definition 6.2 *A topological group G is called an* Eberlein group *if the uniform closure* $\mathrm{cls}\,(B(G))$ *is* $\mathrm{WAP}(G)$ *(or, more explicitely, if every wap function on G can be approximated uniformly by Fourier-Stieltjes functions).*

By a result of Rudin [129] the group $\mathbb{Z}$ of all integers and the group $\mathbb{R}$ of all reals are not Eberlein. More generally, Chou [26] proved that every locally compact noncompact nilpotent group is not Eberlein. Some examples of Eberlein groups are $U(H)$, $SL_n(\mathbb{R})$, $H_+[0,1]$, $\mathrm{A}ut\,(\mu)$, $S(\mathbb{N})$. See below Examples 6.9.

Definition 6.3 *Denote by* $\mathrm{H}ilb\,(X)$ *the set of all continuous functions on a G-space X which come from Hilbert representable G-compactifications $\nu : X \to Y$ (see Definition 3.9). In particular, for the canonical left G-space $X := G$ we get the definition of* $\mathrm{H}ilb\,(G)$.

$\mathrm{H}ilb\,(X)$ is a closed G-subalgebra of $C(X)$ and $\mathrm{H}ilb\,(G)$ is a closed G-subalgebra of $\mathrm{RUC}(G)$. If X is compact then it is Hilbert approximable iff $\mathrm{H}ilb\,(X) = C(X)$.

Proposition 6.4 *For every topological group G and a not necessarily compact G-space X we have*
$$\mathrm{AP}(X) \subset \mathrm{H}ilb\,(X) \subset \mathrm{WAP}(X)$$
$$\mathrm{AP}(G) \subset \mathrm{H}ilb\,(G) \subset \mathrm{WAP}(G) \subset \mathrm{UC}(G).$$

The inclusion $\mathrm{AP}(X) \subset \mathrm{H}ilb\,(X)$ can be proved using Theorem 5.10.

Definition 6.5 *1. Let us say that a topological group G is* strongly Eberlein *(SE) if* $\mathrm{H}ilb\,(G) = UC(G)$. [1] *This is equivalent to saying that the positively definite functions on G uniformly approximate every left and right uniformly continuous function $f \in \mathrm{UC}(G)$.*

2. We will say that G is a wap group *when* $\mathrm{WAP}(G) = \mathrm{UC}(G)$.

Thus a topological group is strongly Eberlein iff it is both a wap group and Eberlein.

Theorem 6.6 *[100, Theorem 3.12]* $\mathrm{H}ilb\,(G) = \mathrm{cls}\,(B(G))$ *for every topological group G. So, G is Eberlein iff* $\mathrm{H}ilb\,(G) = \mathrm{WAP}(G)$.

Of course, the Bohr compactification $G \to bG = G^{AP}$ is proper iff G is precompact. G is Hilbert representable iff $B(G)$ (equivalently, the positive definite functions) generates the topology of G. Every locally compact group is Hilbert representable. For example, the regular representation of G on $L_2(G, \lambda)$ (where λ is the Haar measure on G) is an embedding. Moreover, Eymard's technique [39] implies that the semigroup G_∞, the one-point compactification of G, is Hilbert representable and that $C_0(G) \subset \mathrm{H}ilb\,(G)$.

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[1]We warn the reader that in the paper [100] this term was used in a different sense. We adopt here the new terminology which seems to be more appropriate.

6.2 Roelcke precompact groups

On every topological group G there are two naturally defined uniform structures $\mathcal{L}(G)$ and $\mathcal{R}(G)$. The *lower* or the *Roelcke* uniform structure on G is defined as $\mathcal{U} = \mathcal{L} \wedge \mathcal{R}$, the greatest lower bound of the left and right uniform structures on G. We refer to the monograph by Roelcke and Dierolf [125] for information about uniform structures on topological groups.

Definition 6.7 *A topological group G is* Roelcke precompact *(RPC for short) if the lower uniform structure on G is precompact. The corresponding compact completion is the* Roelcke compactification *of G.*

Often the Gelfand space of $\mathcal{U}(G)$ is also called by that name but whereas this Gelfand space exists for every topological group, the property of being Roelcke precompact makes sense only for large groups. For example, a locally compact group is Roelcke precompact iff it is precompact. The same is true for abelian groups. The subject of Roelcke precompact groups was thoroughly studied by Uspenskij [145, 146, 147]. Any topological group can be embedded into Roelcke-precompact group, [147]. Many naturally defined Polish groups are RPC. Among others let us mention, $\mathrm{Iso}\,(\mathbb{U}_1)$, $U(H)$, $\mathrm{Homeo}\,(2^\omega)$, $\mathrm{Homeo}_+[0,1]$, [145, 146] and $S(X)$, $\mathrm{Homeo}\,[0,1]$, [125, p. 169]. See also [57, sections 12 and 13] where a new proof is given for $S(\mathbb{N})$ and $\mathrm{Homeo}\,(2^\omega)$. In [55] Glasner shows that the Polish group $\mathrm{Aut}\,(\mu)$ is RPC (see Example 6.9.3 below).

6.2.1 Roelcke precompact subgroups of $S(\mathbb{N})$

In a recent work [141] Tsankov proves the following result concerning closed subgroups of $S(\mathbb{N})$.

Theorem 6.8 (Tsankov) *For a closed subgroup $G \leq S(\mathbb{N})$, the following conditions are equivalent:*

1. *G is Roelcke precompact;*

2. *For every open subgroup $V \leq G$, the set of double cosets $\{VxV : x \in G\}$ is finite;*

3. *for every continuous action of G on a countable set X with finitely many orbits, the induced action on X^n has finitely many orbits for each $n \in \mathbb{N}$;*

4. *G can be written as an inverse limit of oligomorphic groups.*

Recall that a closed subgroup $G \leq S(\mathbb{N})$ is *oligomorphic* when the condition (3) above is satisfied for the natural action of G on $\mathbb{N}$.

The main results in Tsankov's work though are concerned with the theory of unitary representations. He has given a complete classification of the continuous irreducible unitary representations of the oligomorphic permutation groups (these include, the infinite permutation group $S(\mathbb{N})$, the automorphism group of the countable dense linear order, the homeomorphism group of the Cantor space, the general linear group $GL(\infty, \mathbb{F}_q)$, the group of automorphisms of the random graph, etc.), and more generally of the RPC closed subgroups of $S(\mathbb{N})$. Given such a group $G \leq S(\mathbb{N})$, there are only countably many irreducible representations of G and every unitary representation of G is completely reducible. The irreducible representations have the form $\mathrm{Ind}_{C(V)}^G(\sigma)$, where V is an open subgroup, $C(V)$ its commensurator and σ is an irreducible representation of the finite group $C(V)/V$. He also shows that the Gelfand-Raikov theorem holds for topological subgroups of $S(\mathbb{N})$: for all such groups, continuous irreducible Hilbert representations separate points.

By a result of Pestov [116] the group $L_0([0,1])$ of (classes of) measurable functions from $[0,1]$ to the circle, which has a faithful unitary representation by multiplication on the Hilbert space $L_2([0,1])$, admits no irreducible representations.

6.3 Examples

Next we mention some examples of Eberlein and strong Eberlein groups.

Examples 6.9   *1. The Polish group $G := H_+[0,1]$ is Eberlein but not strong Eberlein. Indeed, we have $WAP(G) = \{constants\}$. Hence also $\mathrm{H}ilb\,(G) = \{constants\}$. Then $\mathrm{H}ilb\,(G) \neq UC(G)$ because $UC(G)$ always generates the topology of G.*

2. The unitary group $\mathrm{Iso}\,(H) = U(H)$ is strongly Eberlein. Uspenskij proves in [144] that the completion of this group with respect to the Roelcke uniformity (= infimum of the left and the right uniformities) is naturally equivalent to the embedding $\mathrm{Iso}\,(H) \to \Theta(H)$ into the compact semitopological semigroup $\Theta(H)$. The action $(\mathrm{Iso}\,(H), \Theta(H))$ is Hilbert approximable, [100]. It follows that a function $f : \mathrm{Iso}\,(H) \to \mathbb{R}$ can be approximated uniformly by matrix coefficients of Hilbert representations if and only if f is left and right uniformly continuous (i.e., $f \in UC(\mathrm{Iso}\,(H))$).

3. Let (X, μ) be an atomless standard Borel probability space. We denote by $G = \mathrm{Aut}\,(\mu)$ the Polish group of measure preserving automorphisms of (X, μ) equipped with the weak topology. If for $T \in G$ we let $U_T : L_2(\mu) \to L_2(\mu)$ be the corresponding unitary operator (defined by $U_T f(x) = f(T^{-1}x)$), then the map $T \mapsto U_T$ (the Koopman map) is a topological isomorphic embedding of the topological group G into the Polish topological group $\mathcal{U}(H)$ of unitary operators on the Hilbert space $H = L_2(\mu)$ equipped with the strong operator topology. The image of G in $\mathcal{U}(H)$ under the Koopman map is characterized as the collection of unitary operators $U \in \mathcal{U}(H)$ for which $U(\mathbf{1}) = \mathbf{1}$ and $Uf \geq 0$ whenever $f \geq 0$; see e.g. [54, Theorem A.11]. In [55] Glasner shows that the Polish group $\mathrm{Aut}\,(\mu)$ is Roelcke precompact and that the corresponding compactification coincides with the collection of Markov operators in $\Theta(H)$, where $K \in \Theta$ is Markov if $K(\mathbf{1}) = K^*(\mathbf{1}) = \mathbf{1}$ and $Kf \geq 0$ whenever $f \geq 0$. It is then shown in [55] that again $\mathrm{H}ilb\,(G) = WAP(G) = UC(G)$; i.e. $\mathrm{Aut}\,(\mu)$ is strongly Eberlein.

4. In [57, sections 12 and 13] we show that the group $G = \mathrm{Homeo}\,(C)$ of self-homeomorphisms of the Cantor set C is Roelcke precompact (actually oligomorphic) with $WAP(G) \subset SUC(G) \subsetneq UC(G)$ (for the definition of $SUC(G)$ see section 7.1 below). Thus, in particular G is not a wap group and a fortiori not strong Eberlein.

5. By a result of Veech [149] every semisimple Lie group G with finite center (e.g., $G := SL_n(\mathbb{R})$) is strongly Eberlein. In fact, $WAP(G) = C_0(G) \oplus \mathbb{R}$ for $G := SL_n(\mathbb{R})$. On the other hand, $C_0(G) \subset \mathrm{H}ilb\,(G)$ for every locally compact group G (as we noticed after Theorem 6.6).

Question 6.10 *Is there a topological group G with $WAP(G) = UC(G)$ but $\mathrm{H}ilb\,(G) \subsetneq WAP(G)$, i.e. G which is a wap group but not Eberlein ?*

6.4 $S(\mathbb{N})$ is strongly Eberlein

In this subsection we will show that $S(\mathbb{N})$ is strongly Eberlein. This generalizes our results in [57, sections 12] where we have shown that it is a wap group.

Lemma 6.11 *Let $G \leq S(\mathbb{N})$ be a closed subgroup of $S(\mathbb{N})$ and $V \leq G$ a clopen subgroup of G, then the indicator function $\mathbf{1}_V$ is positive definite on G.*

Proof. Let $X = G/V$ denote the discrete countable quotient space and $\pi : G \to X$ the corresponding quotient map. Consider $\mathbf{1}_V$ as an element of $\ell_2(X)$ and set $f = \mathbf{1}_V \circ \pi$. Then for every $g \in G$ we have:

$$\langle g\mathbf{1}_V, \mathbf{1}_V \rangle = f(g).$$

□

Theorem 6.12 *The group $G = S(\mathbb{N})$ is strongly Eberlein.*

Proof. Given $f \in UC(G)$ and an $\varepsilon > 0$ there exists a clopen subgroup $V \le G$ such that

$$\sup_{g \in G} \sup_{u,v \in V} |f(ugv) - f(g)| < \varepsilon.$$

Set $\hat{f}(g) = \sup_{u,v \in V} f(ugv)$, then clearly $\hat{f}$, being V-biinvariant, is both right and left uniformly continuous; i.e. $\hat{f} \in UC(G)$, and $\|\hat{f} - f\| \le \varepsilon$ in $UC(G)$. We can assume that $V = H(1, \ldots, k)$ for some $k \in \mathbb{N}$, where $H = H(1, \ldots, k) = \{g \in S(\mathbb{N}) : g(j) = j, \ \forall \ 1 \le j \le k\}$. Let

$$\mathbb{N}_*^k = \{(n_1, n_2, \ldots, n_k) : n_j \in \mathbb{N} \text{ are distinct}\} = \{\text{injections}: \ \{1, 2, \cdots, k\} \to \mathbb{N}\}$$

and let G act on $\mathbb{N}_*^k$ by

$$g(n_1, n_2, \ldots, n_k) = (g^{-1}n_1, g^{-1}n_2, \ldots, g^{-1}n_k).$$

The stability group of the point $(1, \ldots, k) \in \mathbb{N}_*^k$ is just V and we can identify the discrete G-space G/V with $\mathbb{N}_*^k$. Under this identification, to a function $f \in UC(G)$ which is right V-invariant (that is $f(gh) = f(g)$, $\forall g \in G, h \in V$), corresponds a bounded function $\omega_f \in \Omega_k = \mathbb{R}^{\mathbb{N}_*^k}$, namely

$$\omega_f(n_1, n_2, \ldots, n_k) = f(g) \quad \text{iff} \quad g(j) = n_j, \ \forall \ 1 \le j \le k.$$

If we now assume that $f \in UC(G)$ is both right and left V-invariant (so that $f = \hat{f}$) then it is easy to see that f and accordingly its corresponding ω_f, admits only finitely many values. Set $Y_f = Y = \text{cls}\{g\omega_f : g \in G\} \subset \Omega_k = \mathbb{R}^{\mathbb{N}_*^k}$, where the closure is with respect to the pointwise convergence topology. (G, Y_f) is a compact G-system which is isomorphic, via the identification $G/V \cong \mathbb{N}_*^k$, to $X_f \subset \mathbb{R}^G$.

Let now $\mathbb{N}^* = \mathbb{N} \cup \infty$ be the one-point compactification of $\mathbb{N}$. As in [57, section 12] one can check that Y_f is isomorphic to a sub-system of the product system $(G, (\mathbb{N}^*)^k)$ (diagonal action). We observe that (by the Stone Weirstrass theorem) the function $\delta_1 : \mathbb{N}^* \to \mathbb{R}$ (defined by $\delta_1(1) = 1$ and $\delta_1(x) = 0$ otherwise) together with the constant functions generate $C(\mathbb{N}^*)$ (as a closed algebra) and, as $\mathbf{1}_{H(1)} = \delta_1 \circ \pi$ where $\pi : G \to G/H(1) \cong \mathbb{N}$, we conclude, by Lemma 6.11, that the dynamical system $(G, \mathbb{N}^*)$ is Hilbert. Then also $Y_f \subset (\mathbb{N}^*)^k$ is Hilbert and it follows that the function f, which comes from (G, Y_f), is in $\text{H}ilb(G)$. This concludes the proof that $\text{H}ilb(G) = UC(G)$. $\square$

6.5 Systems which are reflexively but not Hilbert representable

For many natural topological groups G (including the discrete group $\mathbb{Z}$ of integers) there exist compact metric G-spaces which are reflexively but not Hilbert representable, [100]. This answers a question of T. Downarowicz. The proof is based on a classical example of W. Rudin: there exists a WAP function on $\mathbb{Z}$ which cannot be uniformly approximated by Fourier-Stieltjes transforms (see also section 6.6 for *recurrent* examples). That is, the fact that $\mathbb{Z}$ is not an Eberlein group. Theorems 5.4 and 5.5 are important steps in the proof. Another distinguishing example from [100] shows that there exists a monothetic compact metrizable semitopological semigroup S which does not admit an embedding into the semitopological compact semigroup $\Theta(H)$ of all contractive linear operators for a Hilbert space H (though, by Theorem 5.9, S admits an embedding into the compact semigroup $\Theta(V)$ for certain reflexive V).

Regarding the question which groups are Reflexively but not Hilbert representable see section 10 below.

6.6 The "Recurrent examples" of Glasner and Weiss

In the paper [66] Glasner and Weiss strengthen an old result of Walter Rudin. They show that there exists a weakly almost periodic function on the group of integers $\mathbb{Z}$ which is not in the norm-closure of the algebra $B(\mathbb{Z})$ of Fourier-Stieltjes transforms of measures on the dual group $\mathbb{T}$ of $\mathbb{Z}$, and which is *recurrent*. Similarly they show the existence of a recurrent function in $H(\mathbb{Z}) \setminus B(\mathbb{Z})$.

They also obtain the following characterization and structure theorem.

Theorem 6.13 (Glasner-Weiss) *A point transitive dynamical system* (X, T) *is* Hilbert representable *iff there exists a* positive definite function $f \in \ell_\infty(\mathbb{Z})$ *such that* $\mathcal{A}_f = \mathcal{A}(X, x_0)$.

Theorem 6.14 (Glasner-Weiss) *Every metrizable recurrent-transitive Hilbert system* (Y, T) *admits a commutative diagram*

$$
\begin{array}{ccc}
(X, U) & & \\
 & \searrow^{\sigma} & \\
\pi \downarrow & & (\tilde{Y}, \tilde{T}) \\
 & \swarrow_{\rho} & \\
(Y, T) & &
\end{array}
$$

with (X, U) *a Hilbert representable dynamical system,* σ *a compact group-extension and* ρ *an almost 1-1 extension.*

Question 6.15 *In Theorem 6.14 we have shown that every metrizable recurrent-transitive Hilbert system admits an almost 1-1 extension which is a group-factor of a Hilbert-representable system. Can one get rid in this structure theorem of either one of these extensions or maybe of both? For example if the answer to Question 5.12.3 is positive then both extensions are redundant.*

7 Asplund spaces and HNS-dynamical systems

The following definition were introduced in [56] for continuous group actions. They easily extend to separately continuous semigroup actions.

Definition 7.1 *We say that a compact S-system X is* hereditarily non-sensitive *(HNS, in short) if any of the following equivalent conditions is satisfied:*

1. *For every closed nonempty subset $A \subset X$ and for every entourage ϵ from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ϵ-small for every $s \in S$.*

2. *The family of translations $\widetilde{S} := \{\tilde{s} : X \to X\}_{s \in S}$ is a fragmented family of maps.*

3. *$E(S, X)$ is a fragmented family of maps from X into itself.*

The equivalence of (1) and (2) is evident from the definitions. Clearly, (3) implies (2). As to the implication (2) $\Rightarrow$ (3), observe that the pointwise closure of a fragmented family is again a fragmented family, [58, Lemma 2.8].

Note that if $S = G$ is a group then in Definition 7.1.1 one may consider only closed subsets A which are G-invariant (see the proof of [56, Lemma 9.4]).

Lemma 7.2 *1. For every S the class of HNS compact S-systems is closed under subsystems, arbitrary products and factors.*

2. *For every HNS compact S-system X the corresponding enveloping semigroup $E(X)$ is HNS both as an S-system and as a semigroup.*

3. *Let P be a HNS-semigroup. Assume that $j : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $j(S) \subset \Lambda(P)$. Then the S-system P is HNS.*

4. *{HNS-semigroups}={enveloping semigroups of HNS systems}.*

A system (G, X) is *equicontinuous at* $x_0 \in X$ if for every $\epsilon > 0$ there exists a neighborhood O of x_0 such that for every $x \in O$ and every $g \in G$ we have $d(gx, gx_0) < \epsilon$. A system is *almost equicontinuous* (AE) if it is equicontinuous at a dense set of points, and *hereditarily almost equicontinuous* (HAE) if every closed subsystem is AE.

Denote by Eq_ϵ the union of all open sets $O \subset X$ such that for every $g \in G$ the set gO has diameter $< \epsilon$. Then Eq_ϵ is open and G-invariant. Let $Eq = \bigcap_{\epsilon>0} Eq_\epsilon$. Note that a system (G, X) is non-sensitive if and only if $Eq_\epsilon \neq \emptyset$ for every $\epsilon > 0$, and (G, X) is equicontinuous at $p \in X$ if and only if $p \in Eq$. Suppose that Eq_ϵ is dense for every $\epsilon > 0$. Then Eq is dense, in virtue of the Baire category theorem. It follows that (G, X) is AE.

If (G, X) is non-sensitive and $x \in X$ is a transitive point, then for every $\epsilon > 0$ the open invariant set Eq_ϵ meets Gx and hence contains Gx. Thus $x \in Eq$. If, in addition, (G, X) is minimal, then $Eq = X$. Thus minimal non-sensitive systems are equicontinuous, [64, Theorem 1.3], [3], or [56, Corollary 5.15]).

Theorem 7.3 *([56, Theorem 9.14], [62]) For a compact metric G-space X the following conditions are equivalent:*

1. *the dynamical system (G, X) is RN, that is, it admits a proper representation on an Asplund Banach space;*

2. *X is HNS;*

3. *X is HAE;*

4. *every nonempty closed G-subspace Y of X has a point of equicontinuity;*

5. *for any compatible metric d on X the metric $d_G(x, y) := \sup_{g \in G} d(gx, gy)$ defines a separable topology on X;*

6. *the enveloping semigroup $E(X)$ is metrizable.*

One of the equivalences in the following result is a well known characterization of Asplund spaces in terms of fragmentability, Theorem 3.4.

Theorem 7.4 *[60] Let V be a Banach space. The following conditions are equivalent:*

1. *V is an Asplund Banach space.*

2. *(Θ^{op}, B^*) is an HNS system.*

3. *$\mathcal{E}$ is a HNS-semigroup.*

The next theorem is based on ideas from [62].

Theorem 7.5 *Let V be a Banach space. The following conditions are equivalent:*

1. *V is a separable Asplund space.*

2. *$\mathcal{E}$ is metrizable.*

3. *$\mathcal{E}$ is homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$ (for infinite-dimensional V).*

A simplified proof of Theorem 7.3 comes from the following more general result.

Theorem 7.6 *Let X be a compact S-system. Consider the following assertions:*

(a) *$E(X)$ is metrizable.*

(b) *(S, X) is HNS.*

Then:

1. *(a) $\Rightarrow$ (b).*

2. *If X, in addition, is metrizable then (a) $\Leftrightarrow$ (b).*

Proposition 7.7 *[60] Let S be a semitopological semigroup and $\alpha : S \to P$ be a right topological semigroup compactification.*

1. *If P is metrizable then P is a HNS-semigroup and the system (S, P) is HNS.*

2. *Let $V \subset C(S)$ be an m-introverted closed subalgebra of $C(S)$. If V is separable then necessarily $V \subset \mathrm{Asp}(S)$.*

Theorem 7.8 *(Veech, Troallic, Auslander, see [12]) Every WAP compact minimal G-system is equicontinuous.*

For the proof observe that if (G, X) is NS then $Eq(X) \supset Trans(X)$. An alternative proof comes from Theorem 3.16 (a Kadec subsets argument).

Theorem 7.3 leads to the following definitive result which answers negatively Problem 3.3 in [51].

Theorem 7.9 *[62, Theorem 6.2] A metric minimal system (G, X) is equicontinuous if and only if its enveloping semigroup $E(X)$ is metrizable.*

Proof. It is well known that the enveloping semigroup of a metric equicontinuous system is a metrizable compact topological group. Conversely, if $E(X)$ is metrizable then, by Theorem 7.3, (X, G) is HNS, hence also HAE [56], and being also minimal it is equicontinuous. $\square$

By Theorem 9.5 (originally proved in [56]), a metric compact G-system is HNS iff it can be represented on a separable Asplund Banach space V. It follows that the algebra $\mathrm{Asp}(G)$, of functions on a topological group G which come from HNS (jointly continuous) G-systems, coincides with the collection of functions which appear as matrix coefficients of continuous co-representations of G on Asplund Banach spaces. Replacing Asplund by reflexive, we obtain the corresponding characterization (see [97]) of the algebra $\mathrm{WAP}(G)$ of weakly almost periodic functions. Since every reflexive space is Asplund we have $\mathrm{WAP}(G) \subset \mathrm{Asp}(G)$. Refer to [97, 56, 62, 58] and the review article [53] for more details.

From Theorem 7.3 we obtain

Corollary 7.10 *[62] The following three classes of semigroups coincide:*

1. *Metrizable enveloping semigroups of G-systems.*

2. *Enveloping semigroups of HNS (HAE) metrizable G-systems.*

3. *Metrizable right topological semigroup compactifications of G.*

For WAP systems we have an analogous statement:

Corollary 7.11 *[62] The following classes of semigroups coincide:*

1. *Enveloping semigroups of WAP metrizable G-systems.*

2. *Metrizable semitopological semigroup compactifications of G.*

Moreover, when the acting group G is commutative, a point transitive WAP system is isomorphic to its enveloping semigroup, which in this case is a commutative semitopological semigroup. Thus for such G the class of all metric, point transitive, WAP systems coincides with the class of all metrizable, commutative, semitopological semigroup compactifications of G.

7.1 SUC systems

Let G be a topological group. Denote by $\mathcal{L}$ and $\mathcal{R}$ the left and right uniformities on G. We start with a simple observation.

Lemma 7.12 *For every compact G-space X with a continuous action the corresponding orbit maps*

$$\widetilde{x} : (G, \mathcal{R}) \to (X, \mu_X), \quad g \mapsto gx$$

are uniformly continuous for every $x \in X$.

In general, for non-commutative groups, one cannot replace $\mathcal{R}$ by the left uniformity $\mathcal{L}$. We say that (G, X) is SUC (*Strongly Uniformly Continuous*) if every $\widetilde{x}$ ($x \in X$) is also left uniformly continuous (equivalently, uniformly continuous with respect to the Roelcke uniformity). More explicitly we have:

Definition 7.13 *A uniform G-space (X, μ) is strongly uniformly continuous at $x_0 \in X$ (notation: $x_0 \in \mathrm{RUC}_X$) if for every $\varepsilon \in \mu$ there exists a neighborhood U of $e \in G$ such that*

$$(gux_0, gx_0) \in \varepsilon$$

for every $g \in G$ and every $u \in U$. If $\mathrm{RUC}_X = X$ we say that X is strongly uniformly continuous.

A function $f \in C(Y)$ on a G-space Y is *strongly uniformly continuous* (notation: $f \in \mathrm{RUC}(Y)$) if it comes from a SUC compact dynamical system with continuous action. The collection of SUC functions $\mathrm{RUC}(G)$ is then a C^*-subalgebra of $\mathrm{RUC}(G)$.

The algebra $\mathrm{RUC}(G)$ was introduced in [57] where, among others, the following results were obtained.

(i) $\mathrm{RUC}(G)$ is contained in the Roelcke algebra $\mathrm{UC}(G)$. (ii) It contains the algebra $\mathrm{Asp}_c(G)$ and, a fortiori, the algebra $\mathrm{WAP}_c(G) = \mathrm{WAP}(G)$. (iii) For the Polish groups $G = H_+[0,1]$ and $G = \mathbb{U}_1$ (isometries of the Urysohn space of diameter 1), $\mathrm{RUC}(G)$ is trivial. (iv) For the group $G = S(\mathbb{N})$, of permutations of a countable set, one has $\mathrm{WAP}(G) = \mathrm{RUC}(G) = \mathrm{UC}(G)$, and a concrete description of the corresponding metrizable (in fact Cantor) semitopological semigroup compactification is given. (v) For the group $G = H(C)$ of homeomorphisms of the Cantor set, in contrast, $\mathrm{RUC}(G)$ is properly contained in $\mathrm{UC}(G)$. This implies that for the latter group the Gelfand space of the C^*-algebra $\mathrm{UC}(G)$ does not yield a right topological semigroup compactification.

8 Rosenthal Banach spaces and tame systems

8.1 Rosenthal spaces

Rosenthal's celebrated dichotomy theorem asserts that every bounded sequence in a Banach space either has a weak Cauchy subsequence or a subsequence equivalent to the unit vector

basis of l_1 (an l_1-*sequence*). Consequently a Banach space V does not contain an l_1-sequence if and only if every bounded sequence in V has a weak-Cauchy subsequence [127]. In [58] the authors call a Banach space satisfying these equivalent conditions a *Rosenthal space*. There are several other important characterizations of Rosenthal spaces of which we will cite the following two. Rosenthal spaces are exactly those Banach spaces whose dual has the weak Radon-Nikodým property [137, Corollary 7-3-8]. Finally, for a Banach space V with dual V^* and second dual V^{**} one can consider the elements of V^{**} as functions on the weak star compact unit ball $B^* := B_{V^*} \subset V^*$. While the elements of V are clearly continuous on B^* this is not true in general for elements of V^{**}. By a result of Odell and Rosenthal [115], a *separable* Banach space V is Rosenthal iff every element v^{**} from V^{**} is a Baire 1 function on B^*. More generally E. Saab and P. Saab [133] show that V is Rosenthal iff every element of V^{**} has the *point of continuity property* when restricted to B^*; i.e., every restriction of v^{**} to a closed subset of B^* has a point of continuity.

The main result of [58] is that, for metrizable systems, the property of being *tame* is a necessary and sufficient condition for Rosenthal representability. See more general Theorem 9.5.1.

Theorem 8.1 *Let X be a compact metric S-space. The following conditions are equivalent:*

1. (S, X) *is a tame G-system.*

2. (S, X) *is representable on a separable Rosenthal Banach space.*

8.2 A dynamical version of the Bourgain-Fremlin-Talagrand dichotomy

The following theorem of Rosenthal [128], reformulated by Todorčević [139], was the starting point of the work of Bourgain Fremlin and Talagrand [21].

Theorem 8.2 *Let X be a Polish space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on X which is pointwise bounded (i.e., for each $x \in X$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is bounded). Then, either the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a pointwise convergent subsequence, or it contains a subsequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$.*

A sequence $\{(A_{n,0}, A_{n,1})\}_{n\in\mathbb{N}}$ of disjoint pairs of subsets of X is said to be *independent* if for every finite $F \subset I$ and $\sigma : F \to \{0, 1\}$ we have $\bigcap_{n\in F} A_{n,\sigma(n)} \neq \emptyset$. It is said to be *convergent* if for every $x \in X$, either $x \notin A_{n,0}$ for all but finitely many n, or $x \notin A_{n,1}$ for all but finitely many n.

For example, if $\{f_n\}_{n\in\mathbb{N}}$ is a pointwise convergent sequence of continuous functions then for every two real numbers $s < t$ the sequence $\{(f_n^{-1}(-\infty, s], f_n^{-1}[t, \infty))\}_{n\in\mathbb{N}}$ is convergent. On the other hand, for $X = \{0, 1\}^{\mathbb{N}}$ the sequence of pairs $\{(A_{n,0}, A_{n,1})\}_{n\in\mathbb{N}}$, with $A_{n,i} = \{x \in X : x(n) = i\}$, is an independent sequence.

The following claim is the combinatorial essence of Rosenthal's theorem: *A sequence of disjoint pairs $\{(A_{n,0}, A_{n,1})\}_{n\in\mathbb{N}}$ in every set X always contains either a convergent subsequence or an independent subsequence.*

Ideas of independence and ℓ_1 structure were introduced into dynamics by Glasner and Weiss in [65]. First by using the local theory of Banach spaces in proving that if a compact topological $\mathbb{Z}$-system (X, T) has zero topological entropy then so does the induced system $(\mathfrak{M}(X), T_*)$ on the compact space of probability measures on X; and also in providing a characterization of K-systems in terms of *interpolation sets* which are the same as independence

sets in this situation; see section 8.6 below. We refer the reader to [65] and [82] for more information on these notions.

Recall that a topological space K is a *Rosenthal compactum* [67] if it is homeomorphic to a pointwise compact subset of the space $\mathcal{B}_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a (non-hereditarily) separable first countable non-metrizable Rosenthal compactum is the *Helly compact* of all nondecreasing selfmaps of $[0, 1]$ in the pointwise topology. Another Rosenthal compactum is the *two arrows* (or, *split interval*) space of Alexandroff and Urysohn (see Engelking [38]). A topological space K is *Fréchet* (or, *Fréchet-Urysohn*) if for every $A \subset K$ and every $x \in \mathrm{cls}\,(A)$ there exists a sequence of elements of A which converges to x. Every Rosenthal compact space K is Fréchet by a result of Bourgain-Fremlin-Talagrand [21, Theorem 3F], motivated by results of Rosenthal [127] (see also [139]). Clearly, $\beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers $\mathbb{N}$, cannot be embedded into a Fréchet space (in fact, any convergent sequence in $\beta\mathbb{N}$ is eventually constant).

The second assertion (the BFT dichotomy) in the next theorem is presented as it appears in the book of Todorčević [139] (see Proposition 1, section 13).

Theorem 8.3   *1. [21, Theorem 3F] Every Rosenthal compact space K is Fréchet.*

2. *(Bourgain-Fremlin-Talagrand (BFT) dichotomy) Let X be a Polish space and let $\{f_n\}_{n\in\mathbb{N}} \subset C(X)$ be a sequence of real valued functions which is* pointwise bounded. *Let K be the pointwise closure of $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^X$. Then either $K \subset B_1(X)$ (so that K is Rosenthal compact) or K contains a homeomorphic copy of $\beta\mathbb{N}$.*

In general, $E(S, X)$ is not a Fréchet compact space even for metrizable cascades $(\mathbb{Z}, X)$. The following dynamical dichotomy, which is based on the (BFT) dichotomy, was established in [56, 60].

Theorem 8.4 *(A dynamical BFT dichotomy) Let X be a compact metric dynamical S-system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative. Either*

1. *E is a separable Rosenthal compact, hence $\mathrm{card}\,E \leq 2^{\aleph_0}$; or*

2. *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence $\mathrm{card}\,E = 2^{2^{\aleph_0}}$.*

In [51] a dynamical system is called *tame* if the first alternative occurs, i.e. $E(X)$ is Rosenthal compact.

Theorem 8.5 *The following conditions on a metric compact dynamical S-system X are equivalent:*

1. *(S, X) is tame.*

2. *$\mathrm{card}\,(\mathrm{E(X)}) = 2^{\aleph_0}$.*

3. *$E(X)$ is a Fréchet space.*

4. *Every element of $E(X)$ is of Baire class 1.*

5. *Every element of $E(X)$ is Borel measurable.*

6. *Every element of $E(X)$ is universally measurable.*

We can now draw the following corollary (see also Theorem 9.3 below).

Corollary 8.6 (BFT dichotomy for separable Banach spaces) *Let V be a separable Banach space and let $\mathcal{E} = \mathcal{E}(V)$ be its (separable) enveloping semigroup. We have the following alternative. Either*

1. $\mathcal{E}$ is a Rosenthal compactum, hence $\operatorname{card} \mathcal{E} \leq 2^{\aleph_0}$; or

2. the compact space $\mathcal{E}$ contains a homeomorphic copy of $\beta\mathbb{N}$, hence $\operatorname{card} \mathcal{E} = 2^{2^{\aleph_0}}$.

The first possibility holds iff V is a Rosenthal Banach space.

Proof. Recall that $\mathcal{E} = E(\Theta^{op}, B^*)$. By Theorem 9.3, V is Rosenthal iff (Θ^{op}, B^*) is tame. Since V is separable, B^* is metrizable. So we can apply Theorem 8.4. $\square$

8.3 Topological obstructions for being $E(X)$

Which compact spaces K can serve as enveloping semigroups $E(X)$, where (S, X) is a compact metric dynamical system ? Since such $E(X)$ is separable it is necessary to restrict this question to separable compact spaces K. A non-trivial obstruction arises from Theorem 8.4. There are separable compact spaces K which are not Fréchet (and a fortiori not Rosenthal by Theorem 8.3) yet, with cardinality $\operatorname{card} K = 2^{\aleph_0}$, see e.g. [11, §, III, Exercise 134]. Then, by Theorem 8.4, such compact space K cannot be homeomorphic to any $E(X)$ where X is a compact metric dynamical system.

On the tame side of Theorem 8.4 we cite Todorčević's trichotomy theorem, augmented by the work of Argyros-Dodos-Kanellopoulos [9], as presented in [28, Theorem 10.1]. (See also the discussion of Todorčević's trichotomy and its consequences in [56] and [46].)

Theorem 8.7 *[140], [9] Let K be a separable Rosenthal compactum; then K satisfies exactly one of the following alternatives:*

(i) K is not first countable (and then it contains a copy of $A(2^\omega)$, the Alexandrov compact-ification, of a discrete space of size continuum).

(ii) K is first countable but non-hereditarily separable (and then K contains a copy of $D(2^\omega)$, the duplicate of the Cantor set).

(iii) K is hereditarily separable and non-metrizable (and then K contains a copy of the split interval).

As it turns out however, this trichotomy does not yield any further obstruction on the topological nature of enveloping semigroups in tame systems.

Theorem 8.8 *There are examples of tame metric dynamical systems of all three types. That is, for each of the cases in Theorem 8.7 there are tame metric dynamical systems (G, X) such that $E(X)$ is a separable Rosenthal compactum of type (i), (ii) and (iii), respectively.*

Proof. See Example 8.31.2 for (iii). Example 8.31.6 below provides actions of the remaining types (i) and (ii). Ellis' example of the projective space $\mathbb{P}^{n-1}$ with the action of $G := GL(n, \mathbb{R})$ is of type (i) because $E(\mathbb{P}^{n-1})$ is Fréchet but not first countable, hence of type (i). Finally Akin's example of the sphere $\mathbb{S}^{n-1}$ with the same acting group $G := GL(n, \mathbb{R})$ is of type (ii). In fact, $E(\mathbb{S}^{n-1})$ is first countable but can not be hereditarily separable. To see this observe that if it were hereditarily separable then so would be its factor $E(\mathbb{P}^{n-1})$, but according to the trichotomy Theorem 8.7, this, in turn, would imply that the separable Rosenthal compactum $E(\mathbb{P}^{n-1})$ is first countable, a contradiction. $\square$

Remark 8.9 *Theorems 8.7 and 8.8 suggest an analogous hierarchy of tame dynamical systems, and as a by-product also of separable Rosenthal Banach spaces, by analyzing the topological properties of $\mathcal{E}(V)$ the enveloping semigroups of a Banach space V.*

Answering a question of Talagrand [137, Problem 14-2-41], R. Pol [123] gave an example of a separable compact Rosenthal space K which cannot be embedded in $\mathcal{B}_1(X)$ for any compact metrizable X. In [58] we say that a compact space K is *strongly Rosenthal* if it is homeomorphic to a subspace of $\mathcal{B}_1(X)$ for a compact metrizable X; and that it is *admissible* if there exists a metrizable compact space X and a bounded subset $Z \subset C(X)$ with $K \subset \mathrm{cls}\,_p(Z)$, such that the pointwise closure $\mathrm{cls}\,_p(Z)$ of Z in $\mathbb{R}^X$ consists of Baire 1 functions. Clearly every admissible compactum is strongly Rosenthal.

Theorem 8.10 *[58] Let X be a compact metrizable S-system. Then (S, X) is tame iff the compactum $K := E(X)$ is Rosenthal iff $E(X)$ is admissible.*

Thus, Pol's separable compactum mentioned above cannot be of the form $E(X)$. We do not know if every separable strongly Rosenthal space is admissible. If the answer to this question is in the negative, then this will yield another topological obstruction on being an enveloping semigroup.

Finally, as a consequence of the representation theorem 9.5.1 below we obtain the following result: *A compact space K is an admissible Rosenthal compactum iff it is homeomorphic to a weak* closed bounded subset in the second dual of a separable Rosenthal Banach space V.*

Essentially the same result (using different terminology and setting) was obtained earlier by Marciszewski (see [89, Section 6.2] and [90, Theorem 8.2]).

8.4 Some theorems of Talagrand and applications to $\mathbb{Z}$-systems

The following claim is an exercise in outer measures:

Theorem 8.11 *Let X be a compact metric space and μ a probability measure on X. A function $g : X \to \mathbb{R}$ is non-measurable iff there exist a Borel set $A \subset X$ and $\alpha < \beta$ with*

$$\mu^*(\{g < \alpha\} \cap A) = \mu^*(\{g > \beta\} \cap A) = \mu(A) > 0.$$

Let $\mathcal{F}$ be a pointwise bounded family of functions in $\mathbb{R}^X$. The next theorem is a direct consequence of the above claim and the definition of the pointwise convergence topology (which we denote by τ_p).

Theorem 8.12 *If there is a function $g \in \mathrm{cls}\,_{\tau_p}\mathcal{F}$ which is non-measurable then there are $\alpha < \beta$ and a Borel set $A \subset X$ such that for every k, l:*

$$0 < \mu(A)^{k+l} = (\mu^{k+l})^* \left(\bigcup_{f \in \mathcal{F}} (\{f < \alpha\} \cap A)^k \times (\{f > \beta\} \cap A)^l \right)$$

In fact, by Fubini and the above theorem

$$(\mu^{k+l})^*(\{g < \alpha\} \cap A)^k \times (\{g > \beta\} \cap A)^l = \mu(A)^{k+l}$$

and if $(s_1, \ldots, s_k, t_1, \ldots, t_l)$ is an arbitrary point in

$$(\{g < \alpha\} \cap A)^k \times (\{g > \beta\} \cap A)^l$$

then there is an $f \in \mathcal{F}$ with
$$f(s_i) < \alpha \text{ and } f(t_j) > \beta$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, l$.

Definition 8.13   *1. We say that a Borel set $A \subset X$ is μ-critical for $\mathcal{F}$ if it satisfies the above condition.*

    2. We say that a pointwise bounded family $\mathcal{F} \subset \mathbb{R}^X$ is μ-stable if it does not admit a μ-critical set.

Let $(X, \mathcal{B}, \mu)$ be a standard probability space. On the space $M_\mu(X) \subset \mathbb{R}^X$ of μ-measurable functions we let τ_p denote the pointwise convergence topology and τ_μ the topology of convergence in measure.

Theorem 8.14 (Talagrand) *Let $\mathcal{F} \subset \mathbb{R}^X$ be a μ-stable family then:*

    1. Every member of $\mathcal{F}$ is μ-measurable.

    2. The τ_p-compact set $\mathrm{cls}_{\tau_p} \mathcal{F}$ is also μ-stable.

    3. The map $\mathrm{id} : (\mathcal{F}, \tau_p) \to (\mathcal{F}, \tau_\mu)$ is continuous. In particular $(\mathcal{F}, \tau_\mu)$ is totally bounded.

Definition 8.15   *1. We say that a closed set $A \subset X$ is topologically critical for $\mathcal{F}$ if there are $\alpha < \beta$ such that for every k, l the set*

$$\bigcup_{f \in \mathcal{F}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}$$

    is dense in A^{k+l}.

    2. $\mathcal{F}$ is topologically stable if it does not admit a topologically critical set.

Theorem 8.16 (Talagrand) *A uniformly bounded family $\mathcal{F} \subset C(X)$ is topologically stable iff it is μ-stable for every Borel probability measure on X.*

Definition 8.17 *Let (X, T) be a metric dynamical system and $\mu \in M_T(X)$.*

    1. We say that the associated system (X, μ, T) is μ-stable if for every $f \in C(X)$ the family $\mathcal{F} = \{f \circ T^n : n \in \mathbb{Z}\}$ is μ-stable.

    2. (X, T) is topologically stable if for every $f \in C(X)$ the family $\mathcal{F} = \{f \circ T^n : n \in \mathbb{Z}\}$ is topologically stable.

Definition 8.18 *Let X be a compact metric space and μ a probability measure on X. A uniformly bounded family $\mathcal{F} \subset C(X)$ is*

    1. μ-Glivenko-Cantelli if λ-a.e.

$$\lim_{n \to \infty} \sup_{f \in \mathcal{F}} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(\omega(i)) - \int f \, d\mu \right| = 0.$$

    Here $\omega = (\omega(0), \omega(1), \omega(2), \dots)$ is an i.i.d. process with common distribution μ; i.e. a point in $X^{\mathbb{N}}$ distributed according to the product measure $\lambda = \mu^{\mathbb{N}}$.

    2. topological Glivenko-Cantelli if it is μ-Glivenko-Cantelli for every Borel probability measure on X.

Theorem 8.19 *Suppose (X, μ, T) is μ-stable. Then*

    1. (X, μ, T) is measure theoretically a Kronecker system.

    2. For every $f \in C(X)$ the family $\mathcal{F} = \{f \circ T^n : n \in \mathbb{Z}\}$ is μ-Glivenko-Cantelli.

We can now add two more characterizations to the list in Theorem 8.5:

Theorem 8.20 *The following conditions on a metric dynamical system (X, T) are equivalent:*

    1. (X, T) is tame.

    2. (X, T) is topologically stable.

    3. For every $f \in C(X)$ the family $\mathcal{F} = \{f \circ T^n : n \in \mathbb{Z}\}$ is topologically Glivenko-Cantelli.

8.5 WRN dynamical systems

The "real" definition of Banach spaces with the Radon–Nikodým property is as follows:

Definition 8.21 *A Banach space V has the* Radon–Nikodým property (RNP) *if for every bounded (in total variation) vector valued measure $\nu : \mathcal{B}[0,1] \to V$ which is absolutely continuous with respect to Lebesgue measure μ on $[0,1]$, there exists a* Bochner integrable RN-derivative; *i.e. a function $f \in L_1(\mu, V)$ such that for every measurable A: $\nu(A) = (B)\int_A f\, d\mu$, hence for every $\phi \in V^*$.*

$$\langle \nu(A), \phi \rangle = \int_A \langle f, \phi \rangle \, d\mu \tag{8.1}$$

Definition 8.22 *A Banach space V has the* weak RN property (WRNP) *if the condition in the above definition holds for the Pettis integral: $\nu(A) = (P)\int_A f\, d\mu$. (The latter is defined by the formula*

$$\left\langle (P)\int_A f\, d\mu, \phi \right\rangle = \int_A \langle f, \phi \rangle\, d\mu, \tag{8.2}$$

for every $\phi \in V^*$.)

We then have the following statements:

Theorem 8.23   *1. For a separable Banach space $(P)\int_A f\, d\mu = (B)\int_A f\, d\mu$.*

    2. A Banach space V has the Radon–Nikodým property iff it is of the form V^*, where V is an Asplund space.

    3. A Banach space V does not contain an isomorphic copy of $\ell_1(\mathbb{N})$ (i.e. it is Rosenthal) iff its dual V^* has the WRN property.

If we now say, in analogy with Theorem 8.1, that a dynamical system (S, X) is WRN when it has a faithful representation on a Banach space whose dual has the WRN property (i.e. on a Rosenthal Banach space), then we can view this property as yet another characterization of tameness: *a metric dynamical system is tame iff it is WRN*. See Theorem 8.1.

8.6 Independence characterizations

In this subsection our dynamical systems are compact and metrizable cascades.

Huang [72] and Kerr & Li [82], following the works of Rosenthal [127] and Glasner-Weiss [65], base their works on the notion of independence

Definition 8.24 *(Independence characterizations) For a pair (A_0, A_1) of subsets of X, a subset $J \subset \mathbb{Z}_+$ is an* independence set *if for every nonempty finite subset $I \subset J$ and $s \in \{0,1\}^I$, we have $\bigcap_{i \in I} T^{-i} A_{s(i)} \neq \emptyset$. We call a pair $(x_0, x_1) \in X^k$*

    1. an IE-tuple *if for every product neighborhood $U_0 \times U_1$ of (x_0, x_1) the pair (U_0, U_1) has an independence set of positive density.*

    2. an IT-tuple *if it has an infinite independence set.*

    3. an IN-tuple *if it has arbitrarily long finite independence sets.*

Following preliminary results by Glasner-Weiss and Huang-Ye, Kerr and Li completed the picture with the following succinct characterizations:

Theorem 8.25 *(Kerr-Li [82])*

1. *A system (X, T) has zero topological entropy if and only if it has no (non-diagonal) IE-pairs.*

2. *(X, T) is tame if and only if it has no IT-pairs.*

3. *A system (X, T) is null if and only if it has no IN-pairs. (A system is null if it has zero sequence topological entropy with respect to every subsequence $n_i \nearrow \infty$.)*

8.7 Injective dynamical systems and affine compactifications

An S-system Q is an *affine S-system* if Q is a convex subset of a locally convex vector space and each $\lambda_s : Q \to Q$ is affine. By an *affine S-compactification* of an S-space X we mean a pair (α, Q), where $\alpha : X \to Q$ is a continuous S-map and Q is a convex compact affine S-flow such that $\alpha(X)$ *affinely generates* Q, that is $\overline{co}(\alpha(X)) = Q$. We say that an *$S$-affine compactification* $\alpha : X \to Q$ is *E-compatible* if the restriction map $\Phi : E(Q) \to E(Y)$ is an isomorphism (equivalently, is injective), where $Y := \overline{\alpha(X)}$.

This concept is a refinement of the notion of "injectivity". The latter was introduced by Köhler [83] and examined systematically in [51, 52]. A compact dynamical G-system X is called *injective* if the canonical (restriction) homomorphism $r : E(P(X)) \to E(X)$ — where $E(X)$ denotes the enveloping semigroup of the system (G, X) and $P(X)$ is the compact space of probability measures on X — is an injection, hence an isomorphism. The refinement we investigate in the recent work [60] is the following one. Instead of considering just the space $P(X)$ we consider any faithful affine compactification $(G, X) \hookrightarrow (G, Q)$ and we say that this embedding is *E-compatible* if the homomorphism $r : E(Q) \to E(X)$ is injective (hence an isomorphism).

Obviously any compact system (G, X) whose enveloping semigroup $E(G, X)$ coincides with the universal G-ambit is necessarily injective. Thus, for example, for the group of integers the full Bernoulli shift as well as any mixing subshift of finite type are injective, see Examples 12.1, 12.2 below. In [83] there are several other cases where systems are shown to be injective and the author raises the question whether this is always the case. As she points out this question was posed earlier by J. S. Pym (see [124]).

The following theorem was first proved by Köhler [83] for metric systems. For a different proof see [51, Theorem 1.5] and also [54, Lemma 1.49]. The proof of the general case, [60] is based on Rosenthal-approximability of tame systems and functions, Theorem 9.7, and on Haydon's characterization of Rosenthal Banach spaces, Theorem 3.7.3.

Theorem 8.26 *([83] and [51, 54]) Every tame compact S-space X is injective.*

A fortiori, every affine S-compactification of a tame system is E-compatible. Distal affine dynamical systems have quite rigid properties. It was shown in [49] that a minimally generated metric distal affine G-flow is equicontinuous. Using a version of this result one may show that for a minimal distal dynamical system E-compatibility in any faithful affine compactification implies equicontinuity. Thus such embedding is never E-compatible when the system is distal but not equicontinuous. This result is one of the crucial steps towards the generalized Ellis theorem 9.11.

The second part of the next theorem answers the question of J. S. Pym and A. Köhler posed for cascades (see also S. Immervoll [74] for a special acting semigroup).

Theorem 8.27 *(Glasner [51] for metrizable systems) A compact minimal distal (not necessarily, metric) dynamical system is*

1. *tame if and only if it is equicontinuous;*

2. *injective if and only if it is equicontinuous.*

In particular, in this way we obtain [60] a concrete example of a semigroup compactification which is not an *operator compactification* (Definition 4.19). More precisely, for the algebra $D(\mathbb{Z})$ of all distal functions on $\mathbb{Z}$ the corresponding semigroup compactification $\alpha : \mathbb{Z} \to \mathbb{Z}^{D(\mathbb{Z})}$ is not an operator compactification. That is, there is no representation $h : \mathbb{Z} \to \mathrm{Iso}\,(V)$ on a Banach space V such that α is equivalent to $h : \mathbb{Z} \to \overline{h(\mathbb{Z})}$, where $\overline{h(\mathbb{Z})}$ is the closure of $h(\mathbb{Z})$ in the semigroup $\mathcal{E}(V)$ (see Definition 4.19).

By way of illustration consider, given an irrational number $\alpha \in \mathbb{R}$, the minimal distal dynamical $\mathbb{Z}$-system on the two torus $(\mathbb{T}^2, T)$ given by:

$$T(x, y) = (x + \alpha, y + x) \qquad (\mathrm{mod}\ 1).$$

Since this system is not equicontinuous, Theorem 8.27 shows that it is neither tame nor injective.

The fact that tame systems are injective also yields the result that metric tame minimal $\mathbb{Z}$-systems have zero topological entropy [51, Corollary 1.8]. But, see Theorem 8.28.2 below for a much stronger statement.

In [60] we present an example of a Toeplitz minimal $\mathbb{Z}$-subshift which is not injective.

8.8 Minimal tame dynamical systems

The following theorem was proved (independently) by Huang [72], Kerr and Li [82], and Glasner [52].

Theorem 8.28 *(A structure theorem for minimal tame dynamical systems) Let (G, X) be a tame minimal metrizable dynamical system with G abelian. Then:*

1. *(G, X) is an almost 1-1 extension $\pi : X \to Y$ of a minimal equicontinuous system (G, Y).*

2. *(G, X) is uniquely ergodic and the factor map π is, measure theoretically, an isomorphism of the corresponding measure preserving system on X with the Haar measure on the equicontinuous factor Y.*

8.9 Some examples of WAP, HNS and Tame dynamical systems

We begin with an explicit construction of a WAP $\mathbb{Z}$-system (X, T) (actually an $\mathbb{R}$-flow). This example is a member of a large family of similar constructions which go back to Nemyckiĭ then Katznelson-Weiss [81], and finally to Akin-Auslander-Berg [3].

Example 8.29 • *The tent example: Let $\beta : \mathbb{R} \to [0, 1]$ be a continuous function of period 2, with $\beta(-1) = \beta(1) = 1$, $\beta(0) = 0$ and $\beta(t) < 1$ for every $t \neq 0$ in $[-1, 1]$.*

The tent function β

- For a rapidly increasing sequence p_n with $p_0 = 1$ and $p_{n+1} | p_n$, set $\beta_n(t) = \beta(\frac{t}{p_n})$, and let

$$\beta_\infty(t) = \sup\{\beta_n(t) : n = 1, 2, \dots\}.$$

  Set $X = \overline{\mathcal{O}}_T(\beta_\infty) \subset \mathbb{R}^{\mathbb{Z}}$, where $Tx(n) = x(n + 1)$.

- For a detailed description of the enveloping semigroup of this example see [64].

In order to use a similar method to create an example of an HNS system which is not WAP a much more intricate construction is needed (See [64] and [56]). In particular a one dimensional picture seems to be inadequate here and we use instead a plane picture.
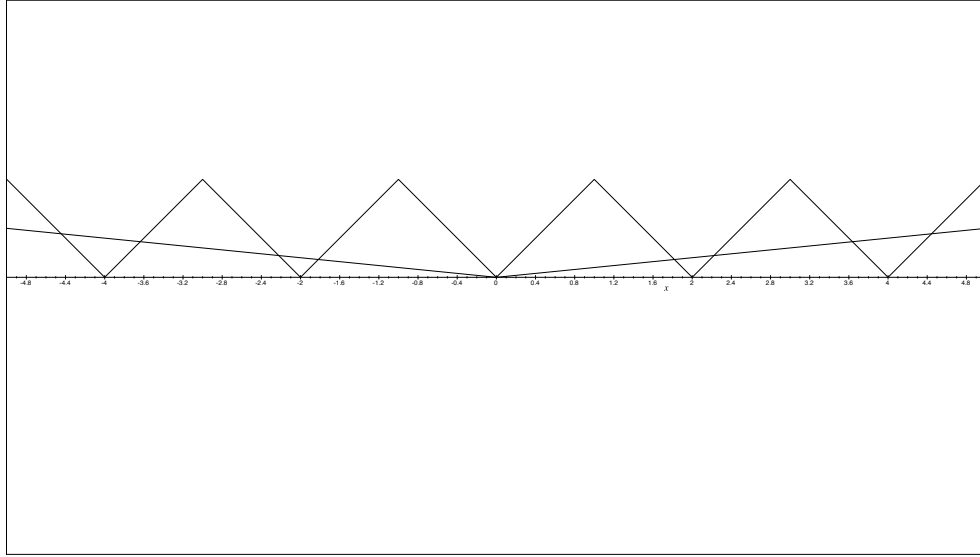
Figure 1: The tent function β

Example 8.30 *The kite example: Again, starting with the picture of the "kite" (Figure 2) and choosing an appropriate sequence of rescalings one defines a limiting picture whose orbit closure (under horizontal $\mathbb{R}$-translations) is the desired $\mathbb{R}$-flow $(\mathbb{R}, X)$.*
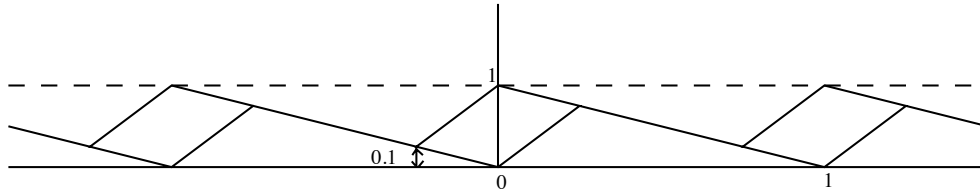

Figure 2: The kite function α

The kite function α

Examples 8.31 *1. As a simple illustration of Proposition 7.7 note that the two-point semigroup compactifications of $\mathbb{Z}$ and $\mathbb{R}$ are obviously metrizable. So the characteristic function $\xi_{\mathbb{N}} : \mathbb{Z} \to \mathbb{R}$ and $arctg : \mathbb{R} \to \mathbb{R}$ are both Asplund. Grothendieck's double limit criterion shows that these functions are not WAP.*

2. (See [56]) Consider an irrational rotation $(\mathbb{T}, R_\alpha)$. Choose $x_0 \in \mathbb{T}$ and split each point of the orbit $x_n = x_0 + n\alpha$ into two points $x_n^{\pm}$. This procedure results in a Sturmian *almost automorphic dynamical system (X, T) which is a minimal almost 1-1 extension of $(\mathbb{T}, R_\alpha)$. Then $E(X, T) \setminus \{T^n\}_{n \in \mathbb{Z}}$ is homeomorphic to the two arrows space, a basic example of a non-metrizable Rosenthal compactum. It follows that $E(X, T)$ is also a Rosenthal compactum. Hence, (X, T) is tame but not HNS.*

3. A more explicit example of this kind is given in [61] where we show that $\varphi_D(n) = sgn \cos(2\pi n\alpha)$ is a tame function on $\mathbb{Z}$ which is not Asplund.

4. (Huang and Ye [73]) Every null *dynamical system is Tame. (Recall that a system is null if it has zero sequence topological entropy with respect to every subsequence $n_i \nearrow \infty$.)*

5. (Huang [72]) An almost 1-1 extension $\pi : X \to Y$ of a minimal equicontinuous system Y with $X \setminus X_0$ countable, where $X_0 = \{x \in X : |\pi^{-1}\pi(x)| = 1\}$, is tame.

6. *In his paper [35] Ellis, following Furstenberg's classical work [43], investigates the projective action of $GL(n, \mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$. It follows from his results that the corresponding enveloping semigroup is not first countable. In a later work [2], Akin studies the action of $G = GL(n, \mathbb{R})$ on the sphere $\mathbb{S}^{n-1}$ and shows that here the enveloping semigroup is first countable (but not metrizable). The dynamical systems $D_1 = (G, \mathbb{P}^{n-1})$ and $D_2 = (G, \mathbb{S}^{n-1})$ are tame but not HNS. Note that $E(D_1)$ is Fréchet, being a continuous image of a first countable compact space, namely $E(D_2)$.*

7. *Evidently, the Bernoulli system $\Omega = \{0, 1\}^G$ in Example 12.1, as well as all the mixing subshifts of finite type 12.2 are not tame.*

8. *As first noted by Downarowicz [31] (see also [54, Theorem 1.48]), when the acting group G is abelian, a point transitive WAP system is always isomorphic to its enveloping semigroup, which in this case is a commutative semitopological semigroup. Thus for such G the class of all metric, point transitive, WAP systems coincides with the class of all metrizable, commutative, semitopological semigroup compactifications of G. In [31] one can find many interesting examples of WAP but not equicontinuous $\mathbb{Z}$-systems. These arise as the orbit closures of Fourier-Stieltjes transforms of Dirichlet measures on the circle.*

8.10   Subshifts

In this subsection we let G be a countable discrete group. The first three theorems here are from [56] and the last one from a work in progress [61].

Theorem 8.32 *Every* scattered *(e.g., countable) compact G-space X is HNS (see also [97]).*

A metric G-space (X, d) is called *expansive* if there exists a constant $c > 0$ such that $d_G(x, y) := sup_{g \in G} d(gx, gy) > c$ for every distinct $x, y \in X$.

Theorem 8.33 *An expansive compact metric G-space (X, d) is HNS iff X is countable.*

Proof. If X is HNS then by Theorem 7.3, (X, d_G) is separable. On the other hand, (X, d_G) is discrete for every expansive system (X, d). Thus, X is countable. $\square$

For a countable discrete group G and a finite alphabet $\mathcal{L} = \{1, 2, \ldots, l\}$, the compact space $\Omega = \mathcal{L}^G$ is a G-space under left translations $(g\omega)(h) = \omega(g^{-1}h)$, $\omega \in \Omega$, $g, h \in G$. A closed invariant subset $X \subset \mathcal{L}^G$ defines a subsystem (G, X). Such systems are called *subshifts* or *symbolic dynamical systems*. For a nonempty $L \subseteq G$ define the natural projection $\pi_L : \mathcal{L}^G \to \mathcal{L}^L$. In particular for any $g \in G$ the map $\pi_g : \Omega \to \mathcal{L}$ is the usual coordinate projection.

Theorem 8.34 *For a countable discrete group G and a finite alphabet $\mathcal{L}$ let $X \subset \mathcal{L}^G$ be a subshift. The following properties are equivalent:*

1. *X is HNS.*

2. *X is countable.*

Moreover if $X \subset \mathcal{L}^G$ is a HNS subshift and $x \in X$ is a recurrent point then it is periodic (i.e. Gx is a finite set).

Theorem 8.35 *Let X be a subshift of Ω. The following conditions are equivalent:*

1. *(G, X) is a tame system.*

2. *For every infinite subset $L \subseteq G$ there exists an infinite subset $K \subseteq L$ and a **countable** subset $Y \subseteq X$ such that*

$$\pi_K(X) = \pi_K(Y).$$

That is,

$$\forall x \in X \quad \exists y \in Y \quad x_k = y_k \quad \forall k \in K.$$

(This is equivalent to saying that $\pi_K(X)$ is a countable subset of $\mathcal{L}^K$.)

9 General tame, HNS and WAP systems

Our goal in this section is to extend the theory of tame systems to general (not necessarily metric) compact systems. We also review, in this light, the general theory of WAP and HNS systems. The results of this section come mainly from [97, 56, 58, 60]. Our starting point is Theorem 8.5.4 (taking into account Remark 3.2.2).

Definition 9.1 *A compact separately continuous S-system X is said to be* tame *if the translation $\lambda_a : X \to X$, $x \mapsto ax$ is a fragmented map for every element $a \in E(X)$ of the enveloping semigroup.*

According to Remark 4.6, define, for every S-space X, the S-subalgebras $\mathrm{Tame}(X)$ and $\mathrm{Tame_c}(X)$ of $C(X)$. Recall that in several natural cases we have $\mathcal{P}_c(X) = \mathcal{P}(X)$ (see Lemma 4.7).

Lemma 9.2 *Every WAP system is HNS and every HNS system is tame. Therefore, for every semitopological semigroup S and every S-space X (in particular, for $X := S$) we have*

$$\mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{Tame}(X) \quad \mathrm{WAP_c}(X) \subset \mathrm{Asp_c}(X) \subset \mathrm{Tame_c}(X).$$

Proof. If (S, X) is WAP then $E(X) \times X \to X$ is separately continuous. By Lemma 3.3.1 we obtain that E is a fragmented family of maps from X to X. In particular, its subfamily of translations $\{\tilde{s} : X \to X\}_{s \in S}$ is fragmented. Hence, (S, X) is HNS. Directly from the definitions we conclude that every HNS is tame. $\square$

By [61], a compact metrizable S-system X is tame iff S is *eventually fragmented* on X, that is, for every infinite (countable) subset $C \subset G$ there exists an infinite subset $K \subset C$ such that K is a fragmented family of maps $X \to X$. If a compact S-space X is not necessarily metrizable then (S, X) is tame iff for every ε from the uniformity on X and every infinite subset $C \subset S$ there exists an infinite subset $K \subset C$ (depending on (ε, C)) such that K is an ε-fragmented family of maps $X \to X$.

A crucial part of the following theorem from [60] relies on the characterization of Rosenthal Banach spaces due to E. Saab and P. Saab, Theorem 3.7.2.

Theorem 9.3 *Let V be a Banach space. The following conditions are equivalent:*

1. *V is a Rosenthal Banach space.*

2. *(Θ^{op}, B^*) is a tame system.*

3. *$p : B^* \to B^*$ is a fragmented map for each $p \in \mathcal{E}$.*

4. *$\mathcal{E}$ is a tame semigroup.*

For the next result, in order to check that every $p \in E(X)$ is a fragmented map one can apply Lemma 3.3.2 to a suitable evaluation map $F \times X \to \mathbb{R}$ with $p \in F$.

Proposition 9.4 *If the enveloping semigroup $E(X)$ is a Fréchet (e.g., Rosenthal) space as a topological space then (S, X) is a tame system and $E(X)$ is a tame semigroup.*

Theorem 9.5 *Let S be a semitopological semigroup and X a compact S-system with a separately continuous action.*

1. *(S, X) is a tame (continuous) system if and only if (S, X) is weakly (respectively, strongly) Rosenthal-approximable.*

2. *(S, X) is a HNS (continuous) system if and only if (S, X) is weakly (respectively, strongly) Asplund-approximable.*

3. *(S, X) is a WAP (continuous) system if and only if (S, X) is weakly (respectively, strongly) reflexively-approximable.*

If X is metrizable then in (1), (2) and (3) "approximable" can be replaced by "representable".

The proof of Theorem 9.5.1 is based on the following characterization of WRN systems (and in particular of WRN compacta) which, in turn, uses a dynamical modification of the celebrated Davis-Figiel-Johnson-Pełczyński [27] construction.

Theorem 9.6 *Let X be a compact S-space. The following conditions are equivalent:*

1. *(S, X) is WRN (i.e., Rosenthal representable).*

2. *There exists an S-invariant Rosenthal family $F \subset C(X)$ of X which separates the points of X.*

Furthermore, the approach of Theorem 9.6 leads also to the following characterization of tame functions in terms of matrix coefficients of Rosenthal representations.

Theorem 9.7   *1. Let X be a compact S-space. The following conditions are equivalent:*

  (a) *$f \in \mathrm{Tame}(S)$ (respectively, $f \in \mathrm{Tame}_c(S)$).*

  (b) *There exist: a weakly (respectively, strongly) continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v \in V$ such that $f(x) = \langle v, \alpha(x) \rangle \quad \forall\, x \in X$.*

2. *Let S be a semitopological semigroup and $f \in C(S)$. The following conditions are equivalent:*

  (a) *$f \in \mathrm{Tame}(S)$ (respectively, $f \in \mathrm{Tame}_c(S)$).*

  (b) *f is a matrix coefficient of a weakly (respectively, strongly) continuous co-representation of S on a Rosenthal space. That is, there exist: a Rosenthal space V, a weakly (respectively, strongly) continuous co-homomorphism $h : S \to \Theta(V)$, and vectors $v \in V$ and $\psi \in V^*$ such that $f(s) = \psi(vs)$ for every $s \in S$.*

3. *Similar results are valid for*

  (a) *Asplund functions and Asplund Banach spaces;*

  (b) *WAP functions and reflexive Banach spaces.*

If in Theorem 9.7, $S := G$ is a semitopological group then for any monoid co-homomorphism $h : G \to \Theta(V)$ we have $h(G) \subset \mathrm{Iso}\,(V)$. Recall also that $\mathrm{WAP}(G) = \mathrm{WAP}_c(G)$ (Lemma 4.7.4).

Corollary 9.8 *Let $S \times X \to X$ be a separately continuous action. Then:*

1. *$\mathrm{Tame}(X) \subset \mathrm{WRUC}(X)$. In particular, $\mathrm{Tame}(S) \subset \mathrm{WRUC}(S)$.*

2. If X is a compact tame (e.g., HNS or WAP) system then (S, X) is WRUC.

Let X be a continuous compact G-space. Then WAP functions on X come from reflexively representable factors. Similarly, Asplund functions on a compact G-system X are exactly the functions which come from Asplund representable factors. Every HNS is tame. Hence

$$\text{WAP}(X) \subset \text{Asp}(X) \subset \text{Tame}(X).$$

This is another way to prove Lemma 9.2. One more explanation for the above inclusions, for metrizable X, is the following topological hierarchy which has its own interest.

Theorem 9.9 *[58] Let X be a continuous compact metric G-space, $f \in C(X)$ and $\text{cls}_p(fG)$ is the (compact) pointwise closure of fG in $\mathbb{R}^X$. Then*

1. *$\text{cls}_p(fG) \subset C(X)$ if and only if $f \in \text{WAP}(X)$.*

2. *$\text{cls}_p(fG)$ is a metrizable subspace in $\mathbb{R}^X$ iff $f \in \text{Asp}(X)$ iff fG is a fragmented family of functions on X.*

3. *$\text{cls}_p(fG) \subset \mathcal{B}_1(X) = \mathcal{F}(X)$ if and only if $f \in \text{Tame}(X)$.*

9.1 Representation of enveloping semigroups

Another attractive direction here is the study of representations of compact right topological semigroups. This direction is closely related to affine compactifications of dynamical systems, [60].

Let P be a compact right topological monoid which is admissible. P is semitopological (HNS, tame) iff P admits a faithful embedding into the monoid $\mathcal{E}(V)$ where V is reflexive (respectively, Asplund, Rosenthal).

By Theorem 9.3 the semigroup $\mathcal{E}(V)$ is tame for every Rosenthal space V. In the converse direction, every tame (respectively, HNS) semigroup P, or equivalently, every enveloping semigroup of a tame (respectively, HNS) system, admits a faithful representation on a Rosenthal (respectively, Asplund) Banach space V. Theorem 5.9 (for semitopological semigroups and reflexive spaces) is a particular case of the following result.

In the following two theorems we use Haydon's characterization of Rosenthal Banach spaces (Theorem 3.7.3).

Theorem 9.10 *[60] (Enveloping semigroup representation theorem)*

1. *Let P be a tame semigroup. Then there exist a Rosenthal Banach space V and an admissible embedding of P into $\mathcal{E}(V)$.*

2. *If P is a HNS-semigroup then there is an admissible embedding of P into $\mathcal{E}(V)$ with V an Asplund Banach space.*

3. *If P is a semitopological semigroup then there is an admissible embedding of P into $\Theta(V) = \mathcal{E}(V^*)$ with V a reflexive Banach space.*

In the next result the main ingredients are Theorem 9.10 and Glasner's result [49] about minimally generated distal affine G-systems.

Theorem 9.11 *[60] (A generalized Ellis theorem) Every tame compact right topological* **group** *P is a topological group.*

Since every compact semitopological semigroup is tame, Ellis' classical theorem (Theorem 5.8) now follows as a special case of Theorem 9.11.

We also have:

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Corollary 9.12 *Let P be a compact admissible right topological group. Assume that P, as a topological space, is Fréchet. Then P is a topological group. In particular this holds in each of the following cases:*

1. *(Moors and Namioka [105]) P is first countable.*

2. *(Namioka [107] and Ruppert [130]) P is metrizable.*

10 Banach representations of groups (selected topics)

Several results concerning group representations have sources in dynamical systems representations. Here we present some cases of this phenomenon and pose some new questions. This section partially is based on [99].

Let $\mathcal{K}$ be a class of Banach spaces. We say that a topological group G is $\mathcal{K}$-*representable* if there exists a strongly continuous representation [2] $h\colon G \to \mathrm{Iso}\,(V)$ for some $V \in \mathcal{K}$ such that h is a topological embedding; notation: $G \in \mathcal{K}_r$. In the opposite direction, we say that G is $\mathcal{K}$-trivial if every continuous $\mathcal{K}$-representation of G is trivial. Of course, $TopGr = Ban_r \supset Ros_r \supset Asp_r \supset Ref_r \supset Hilb_r$. As we already mentioned (Remark 4.10.1) by Teleman's representation every topological group G is representable on $V := \mathrm{RUC}(G)$. So, any G is *Banach representable, $TopGr = Ban_r$.*

Even for Polish groups very little is known about their representabilty on well behaved Banach spaces.

Euclidean-approximable groups are exactly the (pre)compact groups. This follows easily by Peter-Weyl theorem. Every locally (pre)compact group is Hilbert representable. Indeed, if σ is the Haar measure on a locally compact G then the regular representation of G on $V := L_2(G, \sigma)$, yields the embedding $i : G \hookrightarrow \mathrm{Iso}\,(V)$. There are many known examples of Polish groups which are not Hilbert representable (see Banasczyk [15]). So, $TopGr \neq Hilb_r$.

Moreover, there are examples of Hilbert trivial groups, the so-called *exotic* groups (Herer–Christensen and Banasczyk [15]). An arbitrary Banach space V, as a topological group, cannot be exotic because the group V, in the weak topology, is Hilbert representable. However $C[0,1], c_0 \notin Hilb_r$ (see Theorem 10.8 below).

The classical kernel construction (CSG) implies that a group is Hilbert representable iff the positive definite functions separate the closed subsets and the neutral element. By results of Shoenberg the function $f(v) = e^{-\|x\|^p}$ is positive definite on the $L_p(\mu)$ spaces for every $1 \leq p \leq 2$. Every non-Archimedean group G is Hilbert representable.

Theorem 10.1 *([134] and [95]) The following conditions are equivalent:*

1. *A topological group G is (strongly) reflexively representable;*

2. *The algebra $\mathrm{WAP}(G)$ determines the topology.*

This result, replacing 'strong' by 'weak', appears in Shtern [134]. Recall that by Corollary 3.13 the weak and the strong operator topologies coincide on $\mathrm{Iso}\,(V)$ for every Banach space V with PCP (e.g., reflexive).

The following result can be obtained using Theorem 9.7.

Theorem 10.2 *Let G be a topological group such that $Tame_c(G)$ (respectively, $Asp_c(G)$) separates points and closed subsets. Then there exists a Rosenthal (respectively, Asplund,*

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[2] For every topological group G the involution $inv : g \mapsto g^{-1}$ defines a topological isomorphism between G and its opposite group G^{op}. So it is equivalent to showing that there exists a topological group embedding $h : G \to \mathrm{Iso}\,(V)^{op}$.

reflexive) Banach space V and a topological group embedding $h : G \hookrightarrow \mathrm{Iso}\,(V)$ with respect to the strong topology. Furthermore, if G is second countable then we can suppose in addition that V is separable.

The question if WAP(G) determines the topology of every Hausdorff topological group G was raised by Ruppert [131]. This question was negatively answered in [95] by showing that the topological group $G := H_+[0, 1]$ has only constant WAP functions (and that every representation on a reflexive Banach space is trivial). The WAP triviality of $G := H_+[0, 1]$ was conjectured by Pestov. Recall also (see Remark 4.10) that for the group $G := H_+[0, 1]$ every Asplund (hence also every WAP) function is constant and every continuous representation $G \to \mathrm{Iso}\,(V)$ on an Asplund (hence also reflexive) space V must be trivial. In contrast one may show that G is Rosenthal representable.

Theorem 10.3 *[60]* $G := H_+[0, 1]$ *is representable on a (separable) Rosenthal space.*

Proof. (A sketch) Consider the natural action of G on the closed interval $X := [0, 1]$ and the corresponding enveloping semigroup $E = E(G, X)$. Every element of G is a (strictly) increasing self-homeomorphism of $[0, 1]$. Hence every element $p \in E$ is a nondecreasing function. It follows that E is naturally homeomorphic to a subspace of the Helly compact space (of all nondecreasing selfmaps of $[0, 1]$ in the pointwise topology). Hence E is a Rosenthal compactum. So by the dynamical BFT dichotomy, Theorem 8.4, the G-system X is tame. By Theorem 9.5 we have a faithful representation (h, α) of (G, X) on a separable Rosenthal space V. Therefore we obtain a G-embedding $\alpha : X \hookrightarrow (V^*, w^*)$. Then the strongly continuous homomorphism $h : G \to \mathrm{Iso}\,(V)^{op}$ is injective. Since $h(G) \times \alpha(X) \to \alpha(X)$ is continuous (and we may identify X with $\alpha(X)$) it follows, by the minimality properties of the compact open topology, that h is an embedding. Thus $h \circ \mathrm{inv} : \mathrm{G} \to \mathrm{Iso}\,(\mathrm{V})$ is the required topological group embedding. $\square$

Remark 10.4   *1. Recall that by Theorem 3.18 continuous group representations on Asplund spaces have the* adjoint continuity property. *In contrast this is not true in general for Rosenthal spaces. Indeed, assuming the contrary we would have, from Theorem 10.3, that the dual action of the group $H_+[0, 1]$ on V^* is continuous, but this is impossible by the following fact [57, Theorem 10.3] (proved also by Uspenskij (private communication)): every adjoint continuous (co)representation of $H_+[0, 1]$ on a Banach space is trivial.*

   2. There exists a semigroup compactification $\nu : G = H_+[0, 1] \to P$ into a tame semigroup P such that ν is an embedding. In fact, the associated enveloping semigroup compactification $j : G \to E$ of the tame system $(G, [0, 1])$ is tame and topologically the compactum E is a hereditarily separable Rosenthal compactum. Observe that j is a topological embedding because the compact open topology on $j(G) \subset \mathrm{Homeo}\,([0, 1])$ coincides with the pointwise topology.

Question 10.5 *Is every Polish topological group G Rosenthal representable ? Equivalently, is this true for the universal Polish groups $G = \mathrm{Homeo}\,([0, 1]^{\mathbb{N}})$ or $G = \mathrm{Iso}\,(\mathbb{U})$ (the isometry group of the Urysohn space $\mathbb{U}$) ? By Theorem 10.2 a closely related question is: whether the algebra $\mathrm{Tame}(G)$ separates points and closed subsets.*

Glasner and Megrelishvili asked [99] if there is an *abelian* group which is not reflexively representable ? Equivalently: is it true that the algebra WAP(G) on an abelian group G separates the identity from closed subsets? The following result of Ferri and Galindo solves this and also some related questions posed in [99] (the second volume of 'Open Problems in Topology').

Theorem 10.6 (Ferri-Galindo [41]) *The additive group $G := c_0$ is not reflexively representable.*

However, there exists a continuous injective representation of the group c_0 on a reflexive space. This fact led to the following natural question:

Question 10.7 (Ferri-Galindo [41]) *Does every abelian topological group admit a continuous injective representation on a reflexive Banach space?*

For a partial answer, at least for 2-step nilpotent groups, see [29].

Theorem 10.8 ([92, 94]) *Let G be a (separable) metrizable group and let $\mathcal{U}_L$ denote its left uniform structure. If G is reflexively representable, than $(G, \mathcal{U}_L)$ as a uniform space is embedded into a (separable) reflexive space V. Moreover, if G is Hilbert representable then G is uniformly embedded into a (separable) Hilbert space.*

As a corollary it follows that the uniformly universal group c_0 is not Hilbert representable because this group cannot be uniformly embedded into a Hilbert space. Moreover, c_0 is not even uniformly embedded into reflexive spaces as was shown by Kalton [80]. Thus, this argument provides another proof that c_0 is not reflexively representable. Similarly, as was mentioned in [16], the well known quasi-reflexive James space J cannot be uniformly embedded into a reflexive space. Hence the group J is not reflexively representable.

Chaatit [25] proved that every separable *stable* (in the sense of Krivine and Maurey) Banach space (e.g., the $L_p(\mu)$ spaces $(1 \leq p < \infty)$) is reflexively representable. In [16] I. Ben Yaakov, A. Berenstein and S. Ferri proved the following result: a metrizable group is reflexively representable iff its left invariant metric is uniformly equivalent to a stable metric. This result yields one more proof of Theorem 10.6 that c_0 is not reflexively representable (because it is known that the metric on c_0 is not uniformly equivalent to a stable metric). Moreover, Tsirelson's reflexive space, as a topological group also is not reflexively representable.

Shtern [134] conjectured the coincidence of Ref_r and $Hilb_r$. The first example which distinguishes the Hilbert and the reflexive cases is due to Megrelishvili [94]: the additive group of the Banach space $L_4[0,1]$ is reflexively but not Hilbert representable. Glasner and Weiss [66] proved that there is a Polish *monothetic* group P which is reflexively but not Hilbert representable (see also section 6.6 above). More specifically, P is a Banaczyk group of the form $l_4(\mathbb{N})/\Gamma$, where Γ is a certain discrete subgroup of $l_4(\mathbb{N})$ (see Banaczyk [15]). Also they proved that if a Polish monothetic group P is Hilbert-representable and if $K \leq P$ is a compact subgroup, then the quotient group P/K is Hilbert-representable.

Ferri and Galindo [41] proved that Scwartz locally convex spaces, as topological groups, are reflexively representable. Such groups are not in general Hilbert representable.

By [1], if a metrizable abelian (in fact, *metrizable amenable*, is enough.) group, as a uniform space, is embedded into a Hilbert space then the positive definite functions separate the identity and closed subsets. Combining this with Theorem 10.8 we can conclude: a metric abelian group is Hilbert representable if and only if it can be uniformly embedded into a Hilbert space. The same observation (for second countable abelian groups) is mentioned by Galindo [44].

In the same paper Galindo has shown that for every compact space X the free abelian topological group $A(X)$ is Hilbert representable. Uspenskij found [143] that, in fact, this is true for every Tykhonov space X. The case of a general $F(X)$ is open.

Question 10.9 *Let X be a Tykhonov (or, even a compact) space.*

1. Is the free topological group $F(X)$ reflexively representable?

2. (See also Pestov [119]) *Is $F(X)$ Hilbert representable?*

Every abelian Polish group is a factor-group of a Hilbert representable Polish group (Gao and Pestov [45]).

Question 10.10 (Kechris) *Is every Polish (nonabelian) topological group a topological factor-group of a subgroup of $U(\ell_2)$ with the strong operator topology?*

A natural test case is the group $H_+[0,1]$. Since it is reflexively trivial, every bigger group $G \supset H_+[0,1]$ is not reflexively representable. As observed by Pestov [120], if G, in addition, is topologically simple then it is reflexively trivial. For instance, the Polish group $\mathrm{Iso}\,(\mathbb{U}_1)$ of all isometries of $\mathbb{U}_1$ (the sphere of radius $1/2$ in the Urysohn space $\mathbb{U}$) is reflexively trivial. It follows that every Polish group is a subgroup of a reflexively trivial Polish group.

Question 10.11 *Is the group $H(I^{\aleph_0})$ reflexively trivial?*

It is enough to show that the group $H(I^{\aleph_0})$ is topologically simple.
It is still an open problem to distinguish Asplund and reflexive representability.

Question 10.12 *[57] Does there exist a Polish group G such that $G \in Asp_r$ and $G \notin Ref_r$. What about $G := c_0$?*

By a result of Rosendal and Solecki [126] every homomorphism of $H_+[0,1]$ into a separable group is continuous. Hence every representation (of the discrete group) $H_+[0,1]$ on a separable reflexive space is trivial.

Question 10.13 *[57] Find a Polish group G which is reflexively (Asplund) trivial but the discrete group G_d admits a nontrivial representation on a separable reflexive (Asplund) space.*

For every topological group G, every left and right uniformly continuous bounded function f in $LUC(G) \cap RUC(G)$ on G comes as a matrix coefficient of a representation of G on a bilinear map $E \times F \to \mathbb{R}$ with Banach spaces E and F. The proof of the latter result is based on representations of dynamical systems and has applications in minimal topological groups theory. It was one of the crucial steps (together with generalized Heisenberg groups) used in [101] to resolve some long-standing problems posed by Pestov and Arhangelskii. Namely, representing every topological group as a group retract of a minimal topological group. For more details see [101, 29].

11   Fixed point theorems

Let (S, X) be a dynamical system. If in addition $X = Q$ is a convex and compact subset of a locally convex vector space and each $\lambda_s : Q \to Q$ is an affine map, then the S-system (S, Q) is called an *affine dynamical system*. We recall the following well known fixed point theorem of Ryll-Nardzewski.

Theorem 11.1 (Ryll-Nardzewski) *[132] Let V be a locally convex vector space equipped with its uniform structure ξ. Let Q be an affine compact S-system such that*

1. *Q is a weakly compact subset in V.*

2. *S is ξ-distal on Q.*

Then Q contains a fixed point.

In the special case where Q is compact already in the ξ-topology, we get a version of Hahn's fixed point theorem [48]. There are several geometric proofs of Theorem 11.1, see Namioka and Asplund [113], Namioka [106, 107, 109, 112], Glasner [47, 48], Veech [148], and Hansel-Troallic [69]. The subject is treated in several books, see for example [48], Berglund-Junghenn-Milnes [19], and Granas-Dugundji. A crucial step in these proofs is the lifting of distality on Q from ξ to the original compact topology.

In [59] we present a short proof of a fixed point theorem which covers several known generalizations of Theorem 11.1.

Since every weakly compact subset in a locally convex space is fragmented, the following result is indeed a generalization of Ryll-Nardzewski's fixed point theorem.

Theorem 11.2 *[59] Let τ_1 and τ_2 be two locally convex topologies on a vector space V with their uniform structures ξ_1 and ξ_2 respectively. Assume that $S \times Q \to Q$ is a semigroup action such that Q is an affine τ_1-compact S-system. Let X be an S-invariant τ_1-closed subset of Q such that:*

1. *X is (τ_1, ξ_2)-fragmented.*

2. *the S-action is ξ_2-distal on X.*

Then Q contains an S-fixed point.

We apply our results to weak* compact affine dynamical systems in a large class NP (see Remark 3.20) of locally convex spaces including the duals of Asplund Banach spaces.

Fragmentability (or the concept of nonsensitivity) allows us to simplify and strengthen the methods of Veech and Hansel-Troallic for lifting the distality property. As in the proofs of Namioka [107] and Veech [148], the strategy is to reduce the problem at hand to the situation where the existence of an invariant measure follows from the following fundamental theorem of Furstenberg [42].

Theorem 11.3 (Furstenberg) *Every distal compact dynamical system admits an invariant probability measure.*

11.1 Amenable affine compactifications

Let G be a topological group and X a G-space. Let us say that an affine G-compactification $\alpha : X \to Y$ is *amenable* if Y has a G-fixed point. We say that a closed unital linear subspace $\mathcal{A} \subset \mathrm{WRUC}(X)$ is (left) *amenable* if the corresponding affine G-compactification is amenable. By Ryll-Nardzewski's classical theorem $\mathrm{WAP}(G)$ is amenable. For $f \in \mathrm{RUC}(G)$ let $\pi_f : X \to Q_f$ be the corresponding *cyclic affine G-compactification* in the sense of [60]. In [59] we show that even the larger algebra $\mathrm{Asp}_c(G)$ is amenable and that for every $f \in \mathrm{Asp}_c(G)$ there exists a G-fixed point (a *G-average* of f) in Q_f. Note however that in contrast to the $\mathrm{WAP}(G)$ case, where the invariant mean is unique, for some groups (including the integers) there are uncountably many invariant means on $\mathrm{Asp}(G)$.

This result together with Proposition 7.7 yield the following:

Corollary 11.4 *[60] Let G be a topological group and $\mathcal{A}$ a (left) m-introverted closed subalgebra of $\mathrm{RUC}(G)$. If $\mathcal{A}$ is separable then $\mathcal{A}$ is amenable.*

The still larger algebra $\mathrm{Tame}(G)$, of tame functions on G, is not, in general, amenable. Equivalently, tame dynamical systems need not admit an invariant probability measure.

A topological group G is said to be amenable if $\mathrm{RUC}(G)$ is amenable. By a classical result of von Neumann, the free discrete group $\mathbb{F}_2$ on two symbols is not amenable. So, $\mathrm{RUC}(\mathbb{F}_2) = l_\infty(\mathbb{F}_2)$ is not amenable. By [59], neither is $\mathrm{Tame}(\mathbb{F}_2)$ amenable.

It would be interesting to study for which non-amenable groups G the algebra $\mathrm{Tame}_c(G)$ is amenable. For more information on amenable and extremely amenable groups see [118].

12 Some concrete examples of enveloping semigroups

Example 12.1 (Bernoulli shifts) *(See [54, Lemma 4.1]) Let G be a discrete group. We form the product space $\Omega = \{0,1\}^G$ and let G act on Ω by translations: $(g\omega)(h) = \omega(g^{-1}h)$, $\omega \in \Omega$, $g, h \in G$. The corresponding G-dynamical system (Ω, G) is called the Bernoulli G-system. The enveloping semigroup of the Bernoulli system (Ω, G) is isomorphic to the Stone-Čech compactification βG (as a G-system but also as a semigroup, when the semigroup structure on βG is as defined e.g. in [34]).*

To see this recall that the collection $\{\bar{A} : A \subset G\}$ is a basis for the topology of βG consisting of clopen sets. Next identify $\Omega = \{0,1\}^G$ with the collection of subsets of G in the obvious way: $A \longleftrightarrow \mathbf{1}_A$. Now define an "action" of βG on Ω by:

$$p * A = \{g \in G : g^{-1}p \in \overline{A^{-1}}\}.$$

It is easy to check that this action extends the action of G on Ω and defines an isomorphism of βG onto $E(\Omega, G)$.

Example 12.2 *A mixing subshift of finite type $(\mathbb{Z}, X)$ has a Cartesian product with the full 2-shift as a factor (see [22]). Since $E(\mathbb{Z}, X) = E(\mathbb{Z}, X^k)$ for every $k \in \mathbb{N}$, it follows that the enveloping semigroup $E(\mathbb{Z}, X)$ is the Stone-Čech compactification $\beta\mathbb{Z}$.*

Example 12.3 (Kronecker systems) *(See e.g. [54]) Let (X, G) be a point transitive system. Then the action of G on X is equicontinuous if and only if $K = E(X, G)$ is a compact topological group whose action on X is jointly continuous and transitive. It then follows that the system (X, G) is isomorphic to the homogeneous system $(K/H, G)$, where H is a closed subgroup of K and G embeds in K as a dense subgroup. When G is abelian $H = \{e\}$ is trivial, and $E(X, G) = K$. In particular, for $G = \mathbb{Z}$ the collection of Kronecker (= minimal equicontinuous) systems coincides with the collection of compact Hausdorff monothetic topological groups.*

Example 12.4 (WAP functions on semi-simple Lie groups) *(See [149] and [57] for an enhanced version.) Let G be a semisimple analytic group with finite center and without compact factors. For simplicity suppose further that G is a direct product of simple groups. In his paper [149] Veech shows that the algebra $\mathrm{WAP}(G)$, of bounded, right uniformly continuous, weakly almost periodic real valued functions on G, coincides with the algebra $\mathcal{W}^*$ of continuous functions on G which extend continuously to the product of the one-point compactification of the simple components of G ([149, Theorem 1.2]). In particular we have:*

Theorem 12.5 *For a simple Lie group G with finite center (e.g., $SL_n(\mathbb{R})$) $\mathrm{WAP}(G) = \mathcal{W}^*$. The corresponding universal WAP compactification is equivalent to the one point compactification $X = G^*$ of G. Thus $E(X, G) = X$.*

A similar but a bit more interesting situation occurs in the following example.

Example 12.6 (WAP functions on $S(\mathbb{N})$) *(See [57]) Let $G = S(\mathbb{N})$ be the Polish topological group of all permutations of the set $\mathbb{N}$ of natural numbers (equipped with the topology of pointwise convergence). Consider the one point compactification $N^* = \mathbb{N} \cup \{\infty\}$ and the associated natural G action (G, N^*). For any subset $A \subset \mathbb{N}$ and an injection $\alpha : A \to \mathbb{N}$ let p_α be the map in $(N^*)^{N^*}$ defined by*

$$p_\alpha(x) = \begin{cases} \alpha(x) & x \in A \\ \infty & otherwise \end{cases}$$

We have the following simple claim.

Claim. The enveloping semigroup $E = E(N^*, G)$ of the G-system (N^*, G) consists of the maps $\{p_\alpha : \alpha : A \to \mathbb{Z}\}$ as above. Every element of E is a continuous function so that by the Grothendieck-Ellis-Nerurkar theorem [37], the system (N^*, G) is WAP.

In fact, it is shown in [57] that $E = E(N^*, G)$ is isomorphic to the universal WAP compactification G^{WAP} of G; which, in turn, is also the universal $\mathrm{UC}(G)$ compactification G^{UC} of G (where $\mathrm{UC}(G) = \mathrm{RUC}(G) \cap \mathrm{LUC}(G)$ is the algebra of bounded right and left uniformly continuous functions on G).

Example 12.7 *(See [62]) The following is an example of a dynamical system $(X, \mathbb{Z})$ which is distal, HNS, and its enveloping semigroup $E(X)$ is a compact topological group isomorphic to the 2-adic integers. However, $(X, \mathbb{Z})$ is not WAP and a fortiori not equicontinuous.*

Let $\mathbf{S} = \mathbb{R}/\mathbb{Z}$ (reals mod 1) be the circle. Let $X = \mathbf{S} \times (\mathbb{N} \cup \{\infty\})$, where $\mathbb{N} \cup \{\infty\}$ is the one point compactification of the natural numbers. Let $T : X \to X$ be defined by:

$$T(s, n) = (s + 2^{-n}, n), \quad T(s, \infty) = (s, \infty).$$

It is not hard to see that $E(X)$ is isomorphic to the compact topological group $\mathbf{Z}_2$ of 2-adic integers. The fact that X is not WAP can be verified directly by observing that $E(X)$ contains discontinuous maps. Indeed, the map $f_a \in E(X)$ corresponding to the 2-adic integer

$$a = \ldots 10101 = 1 + 4 + 16 + \ldots$$

can be described as follows: $f_a(s, n) = (s + a_n, n)$, where

$$a_{2k} = \frac{2^{2k} - 1}{3 \cdot 2^{2k}} \to \frac{1}{3}, \qquad a_{2k+1} = \frac{2^{2k+2} - 1}{3 \cdot 2^{2k+1}} \to \frac{2}{3}.$$

Geometrically this means that half of the circles are turned by approximately $2\pi/3$, while the other half are turned by approximately the same angle in the opposite direction. The map f_a is discontinuous at the points of the limit circle.

12.1 Nil-systems of class 2

This subsection is taken, almost verbatim, from [53]. For the theory of nil-flows we refer the reader to the book by Auslander, Green and Hahn "Flows on homogeneous spaces" [13], where incidentally a use of Ellis' semigroup theory plays an important role. As we have seen above the enveloping semigroup of a distal system is, in fact, a group. For a special kind of distal systems, namely those that arise from class 2 nil-flows, one can provide an explicit description of the group $E(X, G)$. The first example of such computation was given by Furstenberg in his seminal paper [42].

Example 12.8 *Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-torus and let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $T(z, y) = (z + \alpha, y + z)$, where $\alpha \in \mathbb{R}$ is irrational, and addition is mod 1. Furstenberg shows that $(\mathbb{T}^2, T)$ is a minimal distal but not equicontinuous dynamical system, and exhibits $E(\mathbb{T}^2, T)$ as the collection of all maps $p : \mathbb{T}^2 \to \mathbb{T}^2$ of the form:*

$$p(z, y) = (z + \beta, y + \phi(z)),$$

where $\beta \in \mathbb{T}$ and $\phi : \mathbb{T} \to \mathbb{T}$ is a (not necessarily continuous) group endomorphism.

Now let

$$N = \left\{ \begin{pmatrix} 1 & n & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z}, \ z, y \in \mathbb{T} \right\},$$

so that

$$\begin{pmatrix} 1 & n & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+n' & y+y'+nz' \\ 0 & 1 & z+z' \\ 0 & 0 & 1 \end{pmatrix}.$$

N is a nilpotent group with center $K = \{\left(\begin{smallmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : y \in \mathbb{T}\}$ and $[N,N] \subset K$. Set $a = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{smallmatrix}\right)$, *where $\alpha \in \mathbb{T}$ is irrational and let*

$$\Gamma = \{\left(\begin{smallmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}.$$

Then Γ is a cocompact discrete subgroup of N and the nil-system $(N/\Gamma, a)$, with $a \cdot g\Gamma = (ag)\Gamma$, $g \in G$, is isomorphic to the minimal system $(\mathbb{T}^2, T)$, $T(z, y) = (z + \alpha, y + z)$, described above.

Furstenberg's example and subsequently Namioka's work [110] motivated the work [50] on nil-systems of class 2, where the following theorem is proved. Let X be a compact metric space and $a : X \to X$ a fixed homeomorphism such that the system (X, a) is minimal. Suppose $K \subset \mathrm{Homeo}\,(X)$ is a compact subgroup in the centralizer of a which is topologically isomorphic to a (finite or infinite dimensional) torus. Suppose further that the quotient map $\pi : X \to Z = X/K$ realizes the maximal Kronecker factor of (X, a). Note that under these conditions the system (X, a) is minimal and distal, hence its enveloping semigroup $E = E(X, a)$ is a group.

Theorem 12.9 (Glasner, [50]) *The following conditions on the system (X, a) as above are equivalent.*

1. *The enveloping semigroup E is (algebraically) a nilpotent group.*

2. *There exists a nilpotent class 2 subgroup $N \subset \mathrm{Homeo}\,(X)$ and a closed cocompact subgroup $\Gamma \subset N$ such that: (i) $a \in N$, (ii) $K \subset N$ and K is central in N, (iii) $[N, N] \subset K$, and the nil-system $(N/\Gamma, a)$ is isomorphic to (X, a).*

3. *For every $x_0, x_1 \in X$ the subsystem $\Omega = \bar{\mathcal{O}}_{a \times a}(x_0, x_1)$ of the product $X \times X$ is invariant under the action of the group $\Delta_K = \{(k, k) : k \in K\}$ and the quotient map $\pi_1 : \Omega \to \Omega/\Delta_K = Z_1$ realizes the largest Kronecker factor of the system $(\Omega, a \times a)$.*

When these equivalent conditions hold then Γ is isomorphic to a subgroup of the group $\mathrm{Hom}_c(Z, K)$ of continuous homomorphisms of the compact group Z into K. If, in addition, $\hat{K}$, the dual group of K, is finitely generated, then N is locally compact and σ-compact and Γ is a countable discrete subgroup of N.

Remark 12.10 *The assumption that K is a torus (rather than any central compact subgroup of N) can be removed for a price: The presentation of (X, a) one obtains is now of the form $(W \backslash N/\Gamma, a)$, where W is a compact abelian subgroup of N which commutes with a and satisfies $W \cap K = \{e\}$ ([50, Theorem 2.1*]).*

The easy part of the proof of the theorem consists of yet another concrete computation of an enveloping semigroup:

Example 12.11 *Consider the nil-system (X, a) as described in condition 2 of Theorem 12.9. Thus $X = N/\Gamma$ and we let $x_0 = \Gamma$ be the distinguished point of the system (X, a). Let $\phi_0 : N \to K$ be the group homomorphism defined by $\phi_0(g) = [a, g]$. Let $\mathrm{Hom}\,(N, K)$ be the group of all (not necessarily continuous) homomorphisms from N to K. We endow $\mathrm{Hom}\,(N, K)$ with the (compact) topology of pointwise convergence. Now set*

$$\Phi = \mathrm{cls}\,\{\phi_0^n : n \in \mathbb{Z}\},$$

and

$$\tilde{E} = \mathrm{cls}\,\{(a^n x_0, \phi_0^n) \in X \times \Phi : n \in \mathbb{Z}\}.$$

Proposition 12.12 *The formulas*

$$(g\Gamma, \phi)(h\Gamma, \psi) = (\phi(h)hg\Gamma, \phi\psi)$$
$$(g\Gamma, \phi)^{-1} = (\phi(g)g^{-1}\Gamma, \phi^{-1}),$$

define a group structure on $\tilde{E}$. The resulting group is nilpotent of class 2. Multiplication on the left by $\tilde{a} = (a\Gamma, \phi_0)$ is continuous and $(\tilde{E}, \tilde{a})$ is isomorphic, as a dynamical system and as a group, to (E, a).

Question 12.13 *Extend Theorem 12.9 to other (higher order) nil-flows.*

Example 12.14 *In [121, 122] Pikuła computes enveloping semigroups of affine transformations on the torus of the form $x \mapsto Ax + \alpha$, where A is a unipotent matrix and α is an element of the torus. Among other results he shows that in these cases the enveloping semigroup is a group (i.e. the associated dynamical system is distal) and as such it is nilpotent.*

13 Miscellaneous topics

13.1 Compactifications and representations of semigroups

A topological semigroup S is *compactifiable* if the left action of S on itself is compactifiable in the sense of Definition 4.1. Every Hausdorff topological group is compactifiable. This result cannot be extended to the class of Tykhonov topological monoids. At the same time, several natural constructions lead to compactifiable semigroups and actions, [98]. For example, $\Theta(V)_s^{op}$ is compactifiable for every Banach space V. The semigroup $C(K, K)$ of all continuous selfmaps on the Hilbert cube $K = [0,1]^\omega$ is a universal second countable compactifiable semigroup (*a semigroup version of Uspenskij's theorem*). Moreover, the Hilbert cube K under the action of $C(K, K)$ is universal in the clas of all compactifiable S-flows X with compactifiable S where both X and S are second countable.

Let us say that a semitopological (topological) monoid S is weakly (respectively, strongly) representable if it admits a topological monoid embedding into $\Theta(V)_w^{op}$ (respectively, into $\Theta(V)_s^{op}$) for some Banach space V. S is strongly representable iff S is compactifiable.

Several examples of Tykhonov topological monoids which are not strongly representable appear in [98]. Results of Hindman-Milnes imply that the topological multiplicative semigroup $S := ([0, \infty), \cdot)$ is not even weakly representable. (Indeed, $\mathrm{RMC}(S)$ does not generate the topology, but it does separate the points). It follows that for any right topological semigroup compactification $\alpha : S \to P$, α is not an embedding. On the other hand, $\Theta(V)_w^{op}$ is embedded into $\mathcal{E}(V)$ and for every submonoid S of $\Theta(V)_w^{op}$ its closure $P := \mathrm{cls}\,(S)$ in $\mathcal{E}$ is a compact right topological semigroup.)

It would be interesting to study weak representability of semitopological groups.

Question 13.1 *Which Tykhonov semitopological groups G are weakly representable ? This is equivalent to asking when G can be embedded into $\mathrm{Iso}\,(V)_w^{op}$ for some Banach space V ?*

13.2 Homogeneous compacta

A topological space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism of X onto itself sending x to y. See the survey paper by Arhangelskii and van Mill [10]. Some topics in the theory of homogeneous compact spaces can be linked with topological dynamics. For example one of the famous problems in the theory of homogeneous compact spaces is van Douwen's problem which asks whether there exist compact homogeneous spaces

K whose cellularity $c(K)$ is greater than 2^ω. The existence of Haar measure on a compact topological group K immediately yields the fact that $c(K) \leq \aleph_0$. In fact this is the case for every compact space X for which there is a probability measure μ such that $\mu(U) > 0$ for every nonempty open $U \subset X$. Now, using Furstenberg's structure theorem for distal flows, Milnes and Pym [104] show that any admissible compact right topological group (a CHART) X admits a right invariant probability measure (see also [105]). Thus, we have again $c(X) \leq \aleph_0$. In particular, for any group G, the cellularity of the universal minimal distal G-ambit is at most countable.

As was shown by Kunen [84], see also [10, p.4], under the assumption $\diamond$, a first-countable compact right topological group K need not be metrizable. On the other hand, by Moors and Namioka [105] (see also Corollary 9.12), if a first-countable compact right topological group K is admissible then K is metrizable. In fact, "first-countable" here can be relaxed to "Fréchet"; see Corollary 9.12. This is a remarkable contrast between the admissible and the non-admissible cases.

Question 13.2 *1. Which enveloping semigroups $E(G, X)$ are topologically homogeneous?*

2. Which (familiar) compact spaces are homeomorphic to some $E(X)$, or to the "remainder" space $E(G, X) \setminus \check{G}$?

Of course, for every distal system (G, X), the right topological group $E(G, X)$ is homogeneous. As mentioned by Kunen (see [10, p.4]) the Hilbert cube can not be a right (or left) topological group since it has the fixed-point property.

One of the prototypes for Questions in 13.2 are $\beta\mathbb{Z}$ and $\beta\mathbb{Z} \setminus \mathbb{Z}$ respectively. Recall that $\beta\mathbb{Z}$ can be identified as the enveloping semigroup of the Bernoulli shift $\mathbb{Z}$-system (see Example 12.1). The two arrows (homogeneous) space of Alexandrov and Urysohn is of the form $E(X) \setminus \check{G}$ for some compact metric (tame) G-system X ([56, Example 14.10]).

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