This is a correction to the proof of Theorem 3.3 in [1]. On page 124, line 8, from above replace the part of the proof starting with the words "Consider the $\mathbb{Z}$-system $(\mathbb{T}^k, R_\alpha)$ ..." till the end of the proof, with the following:

Consider the $\mathbb{Z}$-system $(\mathbb{T}^k, R_\alpha)$, that is the rotation by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ on the k-torus $\mathbb{T}^k$, and let $K \subset \mathbb{T}^k$ be given by

$$K = \{\kappa : \exists n_i \nearrow \infty, \quad T^{n_i} \to \mathrm{Id}, \quad \& \quad n_i\alpha \to \kappa\}.$$

Clearly K is a closed sub-semi-group of $\mathbb{T}^k$ and therefore also a closed subgroup.

If K is a proper subgroup of $\mathbb{T}^k$ then, as is easy to check, the map $T^n \mapsto n\alpha K$ is a continuous group homomorphism which can be extended to a continuous homomorphism $\phi : G \to \mathbb{T}^k/K$. However, since we assume that G is minimally almost periodic, such homomorphism can not exist. Thus we are led to the conclusion that $K = \mathbb{T}^k$. In turn this implies that the subgroup

$$L = \mathrm{closure}\{(T^n, n\alpha) : n \in \mathbb{Z}\} \subset G \times \mathbb{T}^k$$

coincides with $G \times \mathbb{T}^k$.

In particular it is possible to find a sequence $\{n_j\} \subset \mathbb{Z}$ such that $(T^{n_j}, n_j\alpha) \to (T, 0)$. For all sufficiently large j we have

$$n_j \in B \subset S - S = N(U, U),$$

and we conclude that also $TU \cap U \neq \emptyset$, contradicting the way U was chosen.

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[1] E. Glasner, *On minimal actions of Polish groups*, Topology and its applications **85**, (1998), 119-125.

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TOPOLOGY AND ITS APPLICATIONS

On minimal actions of Polish groups

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Abstract

We show the existence of an infinite monothetic Polish topological group G with the fixed point on compacta property. Such a group provides a positive answer to a question of Mitchell who asked whether such groups exist, and a negative answer to a problem of R. Ellis on the isomorphism of $L(G)$, the universal point transitive G-system (for discrete G this is the same as βG the Stone–Čech compactification of G) and $E(M,G)$, the enveloping semigroup of the universal minimal G-system (M,G). For G with the fixed point on compacta property M is trivial while $L(G)$ is not. Our next result is that even for $\mathbb{Z}$ with the discrete topology, $L(\mathbb{Z}) = \beta\mathbb{Z}$ is not isomorphic to $E(M,\mathbb{Z})$. Finally we show that the existence of a minimally almost periodic monothetic Polish topological group which does not have the fixed point property will provide a negative answer to an old problem in combinatorial number theory. © 1998 Elsevier Science B.V.

Keywords: Fixed point property; Minimal actions; Stone–Čech compactification

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0. Introduction

During the 8th Prague Topological Symposium, August 1996, a recent paper of Pestov [14], was brought to my attention in which he deals with several problems concerning actions of certain Polish topological groups and related to a problem of R. Ellis. Reading the paper I found that to some of the questions he mentions I have ready answers. Curiously enough in the same week, talking with J. Lawson who also participated in the Symposium, I learned that he and N. Hindman were also interested in some of these questions. I therefore concluded that the time to write down these results has come and that the Symposium's proceedings is the most appropriate place for it. I would like to thank the organizers for the invitation to speak at the Symposium and for their warm hospitality. I would like to thank H. Furstenberg and B. Weiss for many disscusions on

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these subjects and Vladimir Pestov for supplying information regarding the history of these problems.

1. A monothetic group with the fixed point property

Let G be a topological group. We say that G has the *fixed point on compacta property* (*f.p.c.*), if for every (jointly continuous) G action on a compact space X, there is a fixed point; i.e., there exists $x \in X$ with $gx = x$ for all $g \in G$. (When G acts on a compact space X we say that (X, G) is a compact G-system.) The following theorem answers a question of Mitchell [11].

Theorem 1.1. *There exists an infinite monothetic Polish topological group with the fixed point on compacta property.*

The proof relies on the following definitions and theorems. Let (X_i, d_i, μ_i), $i = 1, 2, 3 \ldots$, be a family of metric spaces (X_i, d_i) with $\operatorname{diam}(X_i) = 1$ and μ_i probability measures. Call such a family a *Levy family* if the following condition is satisfied: If $A_i \subset X_i$ is such that $\liminf \mu_i(A_i) > 0$ then for any $\varepsilon > 0$, $\lim \mu(B_\varepsilon(A_i)) = 1$, where $B_\varepsilon(A)$ is the ε neighborhood of A.

Let G be a topological group and $d(\cdot, \cdot)$ a metric on G with diameter 1. Suppose $G = \bigcup G_i$ where $G_i \subset G_{i+1}$ are compact subgroups. For each i let μ_i be normalized Haar measure on G_i. The group G (with the metric d) is called a *Levy group* if the family (G_i, d, μ_i) is a Levy family. The next theorem is due to Gromov and Milman [10] (the equicontinuity of the action assumed there is not necessary). For completeness we reproduce the short proof.

Theorem 1.2. *Let (X, G) be a continuous action of the Levy group G on a compact space X. Then there exists a point $x \in X$ with $gx = x$ for all $g \in G$; i.e., G has the f.p.c. property*

Proof. Since G is a Polish group it is enough to prove the theorem for compact metric spaces. Fix $x_0 \in X$ and let $\phi: G \to X$ be defined by $\phi(g) = gx_0$. Let $\nu_i = \phi_*(\mu_i)$. Fix $\varepsilon > 0$ and let $B_\varepsilon(x_j)$, $j = 1, 2, \ldots, N$, be a finite cover of X with balls of radius ε. There exists then some $x_j = x(\varepsilon)$ with $\nu_{n_i}(B_\varepsilon(x_j)) \geqslant b = 1/N > 0$ for some infinite sequence n_i. The Levy property implies $\lim_{i\to\infty} \nu_{n_i}(B_{2\varepsilon}(x_j)) = 1$. Now take a convergent sequence $\lim_{k\to\infty} x_j(\varepsilon_k) = x$ with $\varepsilon_k \to 0$. It is easy to see that x is a fixed point for every $g \in G$. In fact if $gx \neq x$ then for some open ball U around x we have $gU \cap U = \emptyset$. Now for large enough i, $g \in G_i$ and $\mu_i(\{h \in G_i\colon hx_0 \in U\}) > 1/2$. Since also $\mu_i(\{h \in G_i\colon hx_0 \in gU\}) > 1/2$ we get a contradiction. □

Theorem 1.3. *Let $(\Omega, \mathcal{B}, m)$ be a nonatomic Lebesgue measure space, and let*

$$G = \{f : \Omega \to \mathbb{C} \mid f \ \text{measurable}, \ |f| = 1\},$$

equipped with the metric

$$d(f,g) = \int |f - g| \, \mathrm{d}m.$$

Then G (with pointwise product as the group operation) is a Polish topological group and it contains a dense Levy subgroup.

Proof. With a subset $A \subset \Omega$, associate the finite group $H(A) \cong \mathbb{T}$ of all two-valued functions f on Ω with $f(x) = \lambda$ for some $\lambda \in \mathbb{T}$ when x is in A and $f(x) = 1$ for $x \in A^c$. With a measurable partition $\mathcal{P} = \{A_1, \ldots, A_N\}$ of Ω associate the group $G(\mathcal{P}) = H(A_1) \times \cdots \times H(A_N) \cong \mathbb{T}^N$. Clearly $G_0 = \bigcup G(\mathcal{P}_n)$, for a suitable sequence of refining partitions $\mathcal{P}_n$, is a dense subgroup of G and it is well known that G_0 is Levy (see, e.g., [13, Section 7.13]). □

Proof of Theorem 1.1. It is clear that up to isomorphism the group G described in Theorem 1.3 does not depend on the particular Lebesgue space. Take Ω to be a Kronecker subset of the circle $\mathbb{T}$ and m any continuous probability measure on Ω. Then from the definition of a Kronecker set (the continuous characters on $\mathbb{T}$, restricted to Ω, are uniformly dense in the set of complex valued continuous functions on Ω of modulus 1) it follows that the subgroup

$$G_0 = \{\chi : \Omega \to \mathbb{C} \mid \chi \text{ continuous character of } \mathbb{T}\}$$

is dense in G. Since the group of continuous characters of $\mathbb{T}$ is isomorphic to $\mathbb{Z}$, it follows that G is monothetic. Theorem 1.1 now follows from Theorems 1.2 and 1.3. □

Remarks.

(1) One gets a faithful unitary representation of the group G of Theorem 1.3 on $L^2(\mathbb{T})$ by considering the elements of G as multiplication operators. The associated Gaussian process will yield nontrivial ergodic measure preserving actions. Thus although G admits no nontrivial minimal actions on a compact space it does have nontrivial unitary representations and nontrivial ergodic measure preserving actions.

(2) Independently of the work of Gromov and Milman and at about the same time, H. Furstenberg and B. Weiss proved Theorem 1.1 (private communication) but this work was never published.

(3) In [14] Pestov shows that, e.g., the Polish group $\mathrm{Aut}(\mathbb{Q})$, of order automorphisms of the rational numbers $\mathbb{Q}$ with the topology of pointwise convergence, has the f.p.c. property by a clever use of Ramsey's theorem.

2. Ellis' problem

The enveloping semigroup $E(X,G)$ of a G-system (X,G) is by definition the closure of G in X^X. Introduced by Ellis in [3], this object became central in topological dynamics. It is well known that the enveloping semigroup of the Bernoulli system

$(\Omega, \mathbb{Z})$—where $\Omega = \{0,1\}^{\mathbb{Z}}$ and the $\mathbb{Z}$ action is by translation—is isomorphic to $\beta\mathbb{Z}$ the Stone–Čech compactification of $\mathbb{Z}$. A question, due apparently to Ellis (see also [2, p. 120] and [16, note IV.7.4.13]), is whether the enveloping semigroup $E(M)$ of the universal minimal dynamical system (M, G) is isomorphic to the universal point transitive G-system $L(G)$. The latter system can be identified with the Gelfand space of the Banach algebra $\mathcal{L}$ consisting of bounded left uniformly continuous real valued functions on G. Thus when G is discrete we have $L(G) = \beta G$, where βG is the Stone–Čech compactification of G. As was observed by Pestov, for a nontrivial group G with the f.p.c. property, M is trivial while $L(G)$, which contains a copy of G, is not. Thus every nontrivial f.p.c. group provides a negative answer to Ellis' question. In particular if we let τ denote the topology induced on $\mathbb{Z}$ as a dense subgroup of the monothetic f.p.c. group G described in Section 1, then $(\mathbb{Z}, \tau)$ has the f.p.c. property and we get a negative answer to Ellis's problem with acting group $(\mathbb{Z}, \tau)$. However, using the results of the 1983 paper [9], we can show that even for $\mathbb{Z}$ with the discrete topology $E(M)$ is not isomorphic to $\beta\mathbb{Z}$. (Recall that a system (X, G) is called *point transitive* if there exists a point $x \in X$ with dense orbit.)

Theorem 2.1. *For the group of integers $\mathbb{Z}$ with the discrete topology, the enveloping semigroup $E(M, \mathbb{Z})$ of the universal minimal system $(M, \mathbb{Z})$ is a proper factor of the universal point transitive $\mathbb{Z}$-system $(\beta\mathbb{Z}, \mathbb{Z})$. Hence these two systems are not isomorphic.*

Proof (*Sketch*). Briefly the argument is as follows. Let

$$(W, w_0) = \bigvee \{(X, x_0)\colon\ (X, x_0) \text{ a pointed minimal system}\}$$

be the join of all minimal pointed systems. It is not hard to check that the $\mathbb{Z}$-system W is isomorphic to the enveloping semigroup $E(M)$ and we identify $(E(M), \mathrm{id})$ with (W, w_0). Next we show that the natural homomorphism $\psi : (\beta\mathbb{Z}, 0) \to (E(M), \mathrm{id})$ is not 1–1. Let

$$\psi^* : C(W) \to C_b(\mathbb{Z}) \cong C(\beta\mathbb{Z}) \cong l^\infty(\mathbb{Z}),$$

be the adjoint map from $C(W)$, the algebra of real valued continuous functions on W, into $l^\infty(\mathbb{Z})$. Let $\mathcal{A} = \psi^* C(W)$ be its image in $l^\infty(\mathbb{Z})$. By [9], $\mathcal{A}$-interpolation sets are "small" subsets of $\mathbb{Z}$ while every subset of $\mathbb{Z}$ is an $l^\infty(\mathbb{Z})$-interpolation set. (A subset L of $\mathbb{Z}$ is *small* if for every n the set of starting points of intervals of length at least n contained in L^c is syndetic (i.e., has bounded gaps).) Thus $\mathcal{A} \neq l^\infty(\mathbb{Z})$, hence ψ is not 1–1. The universal property of $\beta\mathbb{Z}$ now shows that no isomorphism can exist between $\beta\mathbb{Z}$ and $W = E(M)$. □

3. An open problem in combinatorial number theory

A subset B of $\mathbb{Z}$ is called a *Bohr-neighborhood of zero* if there exist $\varepsilon > 0$ and a finite set of real numbers $\{\alpha_1, \ldots, \alpha_k\}$ such that

$$\{n \in \mathbb{Z}\colon\ \forall j\ |1 - \exp(2n\pi i\alpha_j)| < \varepsilon\} \subset B.$$

Let $b\mathbb{Z}$ denote the Bohr compactification of $\mathbb{Z}$. It is a compact topological group containing an isomorphic dense copy of $\mathbb{Z}$. In fact $b\mathbb{Z}$ is characterized up to topological isomorphism as the largest compact group with this property. A subset B of $\mathbb{Z}$ is a Bohr neighborhood of zero iff there exists a neighborhood V of zero in $b\mathbb{Z}$ such that $B = V \cap \mathbb{Z}$. Notice that $\mathbb{Z}$ acts on $b\mathbb{Z}$ in a natural way and that this action is minimal. (Recall that a compact system (X, G) is *minimal* if X and $\emptyset$ are the only closed invariant subsets of X.)

A subset S of $\mathbb{Z}$ is called *syndetic* if there exists a finite subset F of $\mathbb{Z}$ such that $\mathbb{Z} = S + F$. The following characterization of minimality is well known and easy to prove.

Lemma 3.1. *Let (X, T) be a compact $\mathbb{Z}$-system. Then the orbit closure of a point $x \in X$ is minimal iff for every neighborhood U of x the set $N(x, U) = \{n \in \mathbb{Z}\colon\ T^n x \in U\}$ is syndetic.*

The proof of the following lemma is an easy exercise.

Lemma 3.2. *Let (X, T) be a minimal $\mathbb{Z}$-system, $U \subset X$ a nonempty open subset and x_0 an arbitrary point in U. Let $N(U, U) = \{n \in \mathbb{Z}\colon\ T^n U \cap U \neq \emptyset\}$, then*

$$N(U, U) = N(x_0, U) - N(x_0, U).$$

Let B be a Bohr neighborhood of zero in $\mathbb{Z}$. Then, $\mathbb{Z}$ with the Bohr topology being a topological group, we can find a Bohr neighborhood of zero S with $S - S \subset B$. There exists a Bohr neighborhood of zero in $b\mathbb{Z}$ say V such that $V \cap \mathbb{Z} = S$ and since the action of $\mathbb{Z}$ on $b\mathbb{Z}$ is minimal we conclude that $S = \{n\colon\ 0 + n \in V\} = N(0, V)$ is syndetic.

Conversely, in [15] it was shown that if S is a syndetic subset of $\mathbb{Z}$ then there exists a Bohr neighborhood of zero B such that $(S - S) \setminus B$ is a set of upper Banach density zero. It is an open question whether one can actually always find a Bohr neighborhood $B \subset S - S$.

A monothetic topological group G is called *maximally almost periodic* if the continuous characters separate points on G. It is called *minimally almost periodic* if the only continuous character on G is the constant function 1. Thus $\mathbb{Z}$ with the discrete topology is maximally almost periodic while clearly every monothetic group with the f.p.c. property (like the one described in Theorem 1.3) is minimally almost periodic. A wealth of Polish monothetic minimally almost periodic topological groups is constructed in [1].

Theorem 3.3. *If a Polish monothetic topological group G exists which is minimally almost periodic yet does not have the fixed point on compacta property, then there exists a syndetic subset S of $\mathbb{Z}$ for which $S - S$ does not contain a Bohr neighborhood of zero.*

Proof. Since by assumption G does not have the f.p.c. property there exists a nontrivial minimal metrizable G-system (X, G). We can assume that G is a subgroup of the

Polish group Homeo(X) of all homeomorphisms of X with the topology of uniform convergence of homeomorphisms and their inverses. Let T be the homeomorphism which represents the element $1 \in \mathbb{Z} \subset G$. Let x_0 be an arbitrary point in X and let U be an open neighborhood of x_0 such that $U \cap TU = \emptyset$. Set $S = N(x_0, U) = \{n \in \mathbb{Z}\colon T^n x_0 \in U\}$. Then S is a syndetic subset of $\mathbb{Z}$ and we now assume that for some $\varepsilon > 0$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ the Bohr neighborhood

$$B = \{n \in \mathbb{Z}\colon \forall j\ |1 - \exp(2n\pi \mathrm{i}\alpha_j)| < \varepsilon\}$$

is contained in $S - S = N(U, U) = \{n \in \mathbb{Z}\colon T^n U \cap U \neq \emptyset\}$ (Lemma 3.2). Consider the $\mathbb{Z}$-system $(\mathbb{T}^k, R_\alpha)$ which is the rotation by $\alpha = (\alpha_1, \ldots, \alpha_k)$ on the k-torus $\mathbb{T}^k$ and let $L \subset G \times \mathbb{T}^k$ be given by

$$L = \mathrm{closure}\{(T^n, R_\alpha^n 0)\colon\ n \in \mathbb{Z}\}.$$

Clearly L is a closed subgroup of $G \times \mathbb{T}^k$ which projects onto all of G in the first component and on some closed subgroup K of $\mathbb{T}^k$ in the second component. We claim that $L = G \times K$. Otherwise the set $H = \{\beta \in K\colon (\mathrm{id}, \beta) \in L\}$ is a closed proper subgroup of K and it is easy to check that the map $g \mapsto \beta + H$, where β is any element of K with $(g, \beta) \in L$, is well defined and is a continuous homomorphism from G onto K/H. This, however, contradicts our assumption that G is minimally almost periodic. Thus $L = G \times K$ and in particular it is possible to find a sequence $n_j \in \mathbb{Z}$ such that $(T^{n_j}, R_\alpha^{n_j}) \to (T, 0)$. For all large j we have

$$n_j \in B \subset (S - S) = N(U, U),$$

and we conclude that also $TU \cap U \neq \emptyset$ contradicting the way U was chosen. □

Remarks.

(1) One can think of two approaches to the problem of finding a monothetic group of the type described in Theorem 3.3. The first is to try and show that some of the known examples of monothetic minimally almost periodic groups (like those constructed in [1]) do not have the f.p.c. property. Perhaps this can be done using a modification of the probabilistic method developed in [1]. The other is to construct a minimal $\mathbb{Z}$-system (X, T) with the following properties:
   (i) the action is uniformly rigid (see [7]), and
   (ii) the topology induced from Homeo(X) on $\mathbb{Z} = \{T^n\colon\ n \in \mathbb{Z}\}$ makes it a minimally almost periodic group.

   In fact, when (X, T) is uniformly rigid, the uniform closure, say G, of $\mathbb{Z}$ in Homeo(X) forms a perfect Polish monothetic group which by construction admits a nontrivial minimal action. Notice that such a system (X, T) is necessarily weakly mixing. (Otherwise any character of the nontrivial Kronecker factor of (X, T) can be lifted to obtain a continuous character on G.) A way of constructing minimal weakly mixing uniformly rigid $\mathbb{Z}$-systems is described in [7,8]; however, we do not know how to control the characters of the monothetic groups obtained this way and in particular how to make them minimally almost periodic.

(2) See [17] for a related discussion of this problem. Similar problems are treated in [5,12].

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