WAP SYSTEMS AND LABELED SUBSHIFTS

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Abstract. The main object of this work is to present a powerful method of construction of subshifts which we use chiefly to construct WAP systems with various properties. Among many other applications of this so called labeled subshifts, we obtain examples of null as well as non-null WAP subshifts, WAP subshifts of arbitrary countable (Birkhoff) height, and completely scrambled WAP systems of arbitrary countable height. We also construct LE but not HAE subshifts, and recurrent non-tame subshifts

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Introduction

The notion of weakly almost periodic (WAP) functions on a LCA group G was introduced by Eberlein [10], generalizing Bohr's notion of almost periodic (AP) functions. As the theory of AP functions was eventually reduced to the study of the largest topological group compactification of G, so the theory of WAP functions can be reduced to the study of the largest semi-topological semigroup compactification of G. Following Eberlein's work there evolved a general theory of WAP functions on a general topological group G, or even more generally, on various type of semigroups. From the very beginning it was realized that a dual approach, via topological dynamics, is a very fruitful tool as well as an end in itself. Thus in the more recent literature on the subject one is usually concerned with WAP dynamical systems (X, G). These are defined as continuous actions of the group G on a compact Hausdorff space X such that, for every $f \in C(X)$, the weak closure of the orbit $\{f \circ g : g \in G\}$ is weakly compact. The turning point toward this view point is the paper of Ellis and Nerurkar [11], which used the famous double limit criterion of Grothendiek to reformulate the definition of WAP dynamical systems as those (X, G) whose enveloping semigroup $E(X, G)$ consists of continuous maps (and is thus a semi-topological semigroup).

In the last two decades the theory of WAP dynamical systems was put into the broader context of hereditarily almost equicontinuous (HAE) and tame dynamical systems. The starting point for this direction was the proof, in the work [5] of Akin Auslander and Berg, that WAP systems are HAE. For later development along these lines see e.g. [17].

Most of the extensive literature on the subject of WAP functions and WAP dynamical systems has a very abstract flavor. The research in these works is mostly concerned with related questions in harmonic analysis, Banach space theory, and the topology and the algebraic structure of the universal WAP semigroup compactification. Very few papers deal with presentations and constructions of concrete WAP dynamical systems. As a few exceptions let us point out the works of Katznelson-Weiss [28], Akin-Auslander-Berg [4], Downarowicz [9] and Glasner-Weiss [20, Example 1, page 349]. Even in these few works the attention is usually directed toward examples of **recurrent** WAP topologically transitive systems. These are the (usually metric) WAP dynamical systems (X, G) which admit a recurrent transitive point.

A point x in a metric dynamical system (X, G) is a point of equicontinuity if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(gx, gx') < \epsilon$ for every $x' \in B_\delta(x)$ and every $g \in G$. The system is called *almost equicontinuous* (AE) if it has a dense (necessarily G_δ) subset of equicontinuity points. It is *hereditarily almost equicontinuous* (HAE) if every subsystem (i.e. non-empty closed invariant subset) is AE.

As our work deals almost exclusively with cascades (i.e. $\mathbb{Z}$-dynamical systems), in the sequel we will consider dynamical systems of the form (X, T) where $T : X \to X$ is a surjective map, usually a homeomorphism. A large and important class of cascades is the class of symbolic systems or subshifts. We will deal only with subsystems of the *Bernoulli* dynamical system $(\{0, 1\}^{\mathbb{Z}}, S)$, where S is the shift transformation defined by

$$(Sx)_n = x_{n+1} \ (x \in \{0, 1\}^{\mathbb{Z}},\ n \in \mathbb{Z}). \tag{0.1}$$

We will call such dynamical systems *subshifts*. It was first observed in [17] that a subshift is HAE iff it is countable (see Proposition 2.2 below). In particular it follows that WAP subshifts are countable. Since a dynamical system which admits a recurrent non-periodic point is necessarily uncountable, it follows that in a WAP subshift the only recurrent points are the periodic points. These considerations immediately raise the question which countable subshifts are WAP, and how rich is this class ? This question was the starting point of our investigation.

As we proceeded with our study of that problem we were able to construct several simple examples of both WAP and non WAP topologically transitive countable subshifts, but particular constructions of WAP subshifts turned out to be quite complicated. After many trials we finally discovered the beautiful world of *labels*. We begin with $FIN(\mathbb{N})$, the additive semigroup of nonnegative integer-valued functions with finite support defined on $\mathbb{N}$, the set of positive integers. A label is a subset $\mathcal{M}$ of $FIN(\mathbb{N})$ which is hereditary in the sense that $\mathbf{0} \leq \mathbf{m}_1 \leq \mathbf{m}$ and $\mathbf{m} \in \mathcal{M}$ imply $\mathbf{m}_1 \in \mathcal{M}$. The space $\mathcal{LAB}$ of labels has a natural compact metric space structure. An *expansion system* consists of a choice of an *expanding function* $k : \mathbb{Z} \to \mathbb{Z}$ such that $k(-n) = -k(n)$ and $k(n + 1) > 3\Sigma_{i=0}^{n}\, k(i)$ for $n \geq 0$, together with an infinite partition $\{D_\ell : \ell \in \mathbb{N}\}$ of $\mathbb{N}$ by infinite subsets. The set $IP(k) \subset \mathbb{Z}$ consists of the sums of finite subsets of the image of k. Each $t \in IP(k)$, which we call an *expanding time*, has a unique expansion $t = k(j_1) + \cdots + k(j_r)$ with $|j_1| > \cdots > |j_r|$. The *length vector* $\mathbf{r}(t) \in FIN(\mathbb{N})$ counts the number of occurrences of each member of the partition in the expansion. That is, $\mathbf{r}(t)_\ell = \#\{i : j_i \in D_\ell\}$. For a label $\mathcal{M}$ the set $A[\mathcal{M}]$ is the set of $t \in IP(k)$ such that $\mathbf{r}(t) \in \mathcal{M}$. For example, $\emptyset$ and $0 = \{\mathbf{0}\}$ are labels with $A[\emptyset] = \emptyset$ and $A[0] = \{0\}$.

Once a system of expansion is fixed, there is a canonical injective, continuous map from the space of labels $\mathcal{LAB}$ into $\{0,1\}^{\mathbb{Z}}$, $\mathcal{M} \mapsto x[\mathcal{M}]$ where $x[\mathcal{M}]$ is the characteristic function of the set $A[\mathcal{M}]$. Thus to a label $\mathcal{M}$ there is assigned a subshift $X(\mathcal{M})$, the orbit closure of $x[\mathcal{M}]$ under S.

We show in Theorem 5.7 that the set $IP(k)$ of all expanding times has upper Banach density zero. This, in turn, implies that for every label $\mathcal{M}$ the corresponding subshift $(X(\mathcal{M}), S)$ is uniquely ergodic with the point measure at $e = x[\emptyset]$ the unique invariant probability measure. It follows that each such system has zero topological entropy.

The space $\mathcal{LAB}$ is naturally equipped with an action of the discrete semigroup $FIN(\mathbb{N})$,

$$(\mathbf{r}, \mathcal{M}) \mapsto \mathcal{M} - \mathbf{r} = \{\mathbf{w} \in FIN(\mathbb{N}) : \mathbf{w} + \mathbf{r} \in \mathcal{M}\}.$$

We denote the compact orbit closure of a label $\mathcal{M}$ under this action by $\Theta(\mathcal{M})$.

The key lemma which connects the two actions (the $FIN(\mathbb{N})$ action on labels and the shift S on subshifts) is Lemma 5.22. Let $\{t^i\}$ be any sequence of expanding times such that the sequence of smallest terms $\{|j_r(t^i)|\}$ tends to infinity and let $\{\mathbf{r}(t^i)\}$ be the corresponding sequence of length vectors. Then for any sequence of labels $\{\mathcal{M}^i\}$, the sequences $\{S^{t^i}(x[\mathcal{M}^i])\}$ and $\{x[\mathcal{M}^i - \mathbf{r}(t^i)]\}$ are asymptotic in $\{0,1\}^{\mathbb{Z}}$.

We show that for a $FIN(\mathbb{N})$-recurrent label the corresponding $x[\mathcal{M}]$ is an S-recurrent point. At the other extreme we have the *labels of finite type*. For such a label $\mathcal{M}$, $e = x[\emptyset]$ is the only recurrent point in $X(\mathcal{M})$. These labels are particularly amenable to our analysis, which leads to a complete picture of the resulting subshift. In fact for a label $\mathcal{M}$ of finite type

$$X(\mathcal{M}) = \{S^k x[\mathcal{N}] : k \in \mathbb{Z},\ \mathcal{N} \in \Theta(\mathcal{M})\} = \bigcup_{k \in \mathbb{Z}} S^k x[\Theta(X(\mathcal{M}))].$$

Two useful subcollections of the collection of finite type labels are the classes of the *finitary labels* and of the *simple labels*. For each label $\mathcal{M}$ in either one of these special classes, the corresponding subshift $X(\mathcal{M})$ is a countable WAP system whose enveloping semigroup structure is encoded in the structure of the label $\mathcal{M}$. This fact enables us to produce WAP subshifts with various dynamical properties by tinkering with their labels.

The recurrent labels are far less transparent and for these labels the image $x[\Theta(X(\mathcal{M}))]$, which in this case is a Cantor set, forms only a meagre subset of the subshift $X(\mathcal{M})$. Nonetheless it seems that this image forms a kind of nucleolus which encapsulates the dynamical properties of $X(\mathcal{M})$.

The table of contents will now give the reader a rough notion of the structure of our work. In the first section we deal with abstract WAP systems, their enveloping semigroups, and, for an arbitrary separable metric system the hierarchies z_{NW} and z_{LIM} of non-wandering and $\alpha \cup \omega$ limiting procedures, which lead by transfinite induction to the Birkhoff center of the dynamical system. We call the ordinal at which the limiting $\alpha \cup \omega$ transfinite sequence stabilizes, the *height* of the system. We also consider various simple examples of some WAP and non-WAP systems. In the second section we study HAE systems and show, among other considerations, that topologically transitive WAP systems are coalescent and that a general WAP system is E-coalescent. The third section, describes some general constructions like the discrete suspension, and the spin construction.

The space of labels is introduced and studied in section 4. The expanding function systems and the associated subshifts are introduced and studied in section 5. A rather technical subsection 5.4 is dedicated to the fine structure of a general system of the form $X(\mathcal{M})$. We hope it will be useful in further investigations of these enigmatic systems. On a first reading the reader may choose to skip this detailed analysis as it is not used in the sequel. Subsection 5.5 is dedicated to WAP systems that arise as $X(\mathcal{M})$ for labels $\mathcal{M}$ which are either finitary or simple. Finally, in sections 6 and 7 these tools are applied to obtain many interesting and subtle constructions of subshifts. Let us mention just a few. On the finite type side we obtain examples of null as well as non-null WAP subshifts, Example 6.21 (answering a question of Downarowicz); WAP subshifts of arbitrary (countable) height, Theorem 6.31; topologically transitive subshifts which are LE but not HAE, Example 5.36 and Remark 6.22 (these seem to be the first such examples); and completely scrambled WAP systems (although not subshifts) of arbitrary countable height, Example 7.14 (answering a question which is left open in Huang and Ye's work [27]). On the recurrent side we construct various examples of non-tame subshifts. Of course many questions are left open, especially when labels which are not of finite type are considered, and we present some of these throughout the work at the relevant places.

1. WAP Systems

A compact dynamical system (X, T) is a homeomorphism T on a compact space X.

We follow some of the notation of [1] concerning relations on a space. In particular, we will use the the *orbit relation*

$$O_T \ = \ \{ \ (x, T^n(x)) \ : \ x \in X, n \in \mathbb{Z} \ \}$$

and the associated limit relation:

$R_T = \omega T \cup \alpha T$,

where

$$\omega T \ = \ \{ \ (x, x') \ : \ x \in X, \ x' = \lim_{i \to \infty} T^{n_i} x \text{ with } n_i \nearrow \infty \ \},$$

and

$$\alpha T \ = \ \{ \ (x, x') \ : \ x \in X, \ x' = \lim_{i \to \infty} T^{-n_i} x \text{ with } n_i \nearrow \infty \ \}.$$

R_T is a pointwise closed relation (each $R_T(x)$ is closed) but not usually a closed relation (i.e. R_T is usually not closed in $X \times X$).

The dynamical system (X, T) is called *topologically transitive* if for every two non-empty open sets U, V in X there is an $n \in \mathbb{Z}$ with $T^{-n}U \cap V \neq \emptyset$. When X is metrizable this is equivalent to the requirement that X_{tr}, the set of points with dense orbit, is a dense G_δ subset of X. The points of X_{tr} are the *transitive points* of X. The system (X, T) is called *weakly mixing* when the product system $(X \times X, T \times T)$ is topologically transitive. We also recall the definitions of ϵ-chains and chain transitivity. Given $\epsilon > 0$ an ϵ-chain from x to y *ϵ-chain from x to y* is a finite sequence $x = x_0, x_1, \dots, x_n = y$ such that $n > 0$ and $d(T(x_i), x_{i+1}) < \epsilon$ for $i = 0, \dots, n-1$. The system (X, T) is *chain transitive* if for any nonempty open sets U, V and any $\epsilon > 0$ there is an ϵ-chain going from a point in U to a point in V. An *asymptotic chain* is an infinite sequence $\{x_i : i \in \mathbb{Z}_+\}$ or $\{x_i : i \in \mathbb{Z}\}$ such that $\lim_{|i| \to \infty} \ d(T(x_i), x_{i+1}) \ = \ 0$. It is a *dense asymptotic chain* if for every $N \in \mathbb{N}$ $\{x^i : i \geq N\}$ is dense in X. If (X, T) is chain transitive and $x \in X$ then there exists a dense asymptotic chain $\{x^i : i \in \mathbb{Z}\}$ with $x = x_0$.

The following construction is due to Takens (see, e. g. [1, Chapter 4, Exercise 29].

Example 1.1. If (X,T) is a chain transitive metric system and $\{x^i : i \in \mathbb{Z}\}$ is a dense asymptotic chain then let

$$z^i \quad = \quad \begin{cases} (x^i, (2i+1)^{-1}) & \text{for } i \geq 0, \\ (x^i, (2|i|)^{-1} & \text{for } i < 0. \end{cases} \tag{1.1}$$

Let $X^* = X \times \{0\} \cup \{z^i : i \in \mathbb{Z}\}, x^* = z^0$. Extend $T = T \times id_0$ on $X \times \{0\}$ identified with X, by $T(z^i) = z^{i+1}$ for $i \in \mathbb{Z}$. Then (X^*, T) is a topologically transitive metrizable system with transitive point x^* and $X^* = X \cup O_T(x^*)$ and $X = \omega T(x^*)$.

More generally given a monoid (= a semigroup with an identity element) Γ, a Γ-dynamical system is a pair (X, Γ) where X is a compact Hausdorff space and Γ acts on X via a homomorphism of Γ into the semigroup $C(X,X)$ of continuous maps from X to itself, mapping id_Γ to id_X.

The *enveloping semigroup* $E = E(X,\Gamma)$ of the dynamical system (X,Γ) is defined as the closure in X^X (with its compact, usually non-metrizable, pointwise convergence topology) of the image of Γ in $C(X,X)$ considered as a subset of X^X.

It follows directly from the definitions that, under composition of maps, E forms a compact semigroup in which the operations

$$p \mapsto pq \qquad \text{and} \qquad p \mapsto \gamma p$$

for $p, q \in E,\ \gamma \in \Gamma$, are continuous. Notice that this makes Γ act on E by left multiplication, so that (E, Γ) is a Γ-system (though usually non-metrizable). It is easy to see that the subset $A(X,\Gamma) \subset E(X,\Gamma)$, consisting of the limit points in $E(X,\Gamma)$, forms a closed left ideal, called the *adherence semigroup* of (X,Γ). Technically, $A(X,\Gamma)$ is the intersection of $\overline{E(X,\Gamma) \setminus K}$ as K varies over compact (and so finite in the discrete case) subsets of Γ.

The elements of E may behave very badly as maps of X into itself; usually they are not even Borel measurable. However our main interest in E lies in its algebraic structure and its dynamical significance. A key lemma in the study of this algebraic structure is the following:

Lemma 1.2 (Ellis-Numakura)**.** *Let L be a compact Hausdorff semigroup in which all maps $p \mapsto pq$ are continuous. Then L contains an* idempotent *; i.e., an element v with $v^2 = v$.*

Given two Γ dynamical systems, say (X,Γ) and (Y,Γ), a continuous surjective map $\pi : X \to Y$ is a *homomorphism* or an *action map* if it intertwines the Γ actions, i.e. $\gamma\pi(x) = \pi(\gamma x)$ for every $x \in X$ and

$\gamma \in \Gamma$. An action map $\pi : X \to Y$ induces a surjective semigroup homomorphism (and an action map) $\pi_* : E(X, \Gamma) \to E(Y, \Gamma)$.

For more details see e.g. [13, Chapter 1, Section 4] and [6].

In our cascade case, from a dynamical system (X, T), we let $(E(X, T), A(X, T))$ denote the enveloping semigroup and the ideal which is the adherence semigroup (= the limit points of $\{T^n\}$ as $n \to \pm\infty$). Let T_* on $E(X, T)$ be the homeomorphism given by $T_*(p) = Tp = pT$. Thus, $y \in R_T(x)$ iff $y = px$ for some $p \in A(X, T)$ and $A(X, T) = R_{T_*}(id_X)$.

A point $x \in X$ is *recurrent* when $x \in R_T(x)$ and so when there exists an idempotent $u \in A(X, T)$ such that $ux = x$. On the other hand, x is non-recurrent iff the orbit $\mathcal{O}_T(x)$ is disjoint from the limit set $R_T(x)$. The closure of the set of recurrent points is the *Birkhoff Center*. We will denote by $Cent_T$ the Birkhoff center for (X, T).

A point x^* is a *transitive point* when the orbit $O_T(x^*)$ is dense in X. In that case, the evaluation map ev_{x^*} defined by $p \to px^*$ is a surjective action map from $(E(X, T), T_*)$ to (X, T).

Lemma 1.3. *If $p \in E(X, T)$ is continuous at $x \in X$, then for any $q \in E(X, T)$ and $y \in X$ if $qy = x$ then $pqy = qpy$.*

Proof: If $T^{n_i} \to q$ pointwise then $T^{n_i}y \to qy = x$ and so by continuity of p at x, $pT^{n_i}y \to pqy$. But $pT^{n_i}y = T^{n_i}py \to qpy$.
□

Proposition 1.4. *If (X, T) is a compact system with a transitive point x^* and $p \in E(X, T)$ then the following are equivalent.*

(i) *p is continuous on X.*
(ii) *For all $q \in E(X, T)$ $pq = qp$ on X.*
(iii) *For all $q \in E(X, T)$ $pqx^* = qpx^*$.*

Proof: (i) $\Rightarrow$ (ii): This follows from Lemma 1.3.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): Suppose $x_i \to x$. To show that then $px_i \to px$, it suffices to show that every convergent subnet has limit px. So we can assume that $\lim px_i$ exists. There are $r_i \in E(X, T)$ with $x_i = r_i x^*$. Let $r_{i'} \to r$ be a convergent subnet. Then, necessarily $rx^* = x$ and $\lim px_i = \lim px_{i'} = \lim pr_{i'}x^* = \lim r_{i'}px^* = rpx^* = prx^* = px$.
□

(X, T) is called WAP when the elements of $E(X, T)$ are all continuous functions on X.

Corollary 1.5. *If (X,T) is a compact system with a transitive point x^* then the following are equivalent.*

(i) *(X,T) is WAP*
(ii) *$E(X,T)$ is abelian.*
(iii) *For every $p,q \in E(X,T)$ $pqx^* = qpx^*$.*

When these conditions hold $ev_{x^*} : (E(X,T),T_*) \to (X,T)$ is an isomorphism and there is a unique minimal subset of X.

Proof: The equivalence of (i), (ii) and (iii) follows from Proposition 1.4

If (X,T) is WAP and $px^* = qx^*$ then $p = q$ on $O(x^*)$ which is dense and so $p = q$ by continuity. That is, ev_{x^*} is injective and so is an isomorphism.

If $x_i \in M_i$ for minimal sets M_1, M_2 there exist $p_1, p_2 \in E(X,T)$ s.t. $p_i(x^*) = x_i$ and so $p_2(x_1) = p_2(p_1(x^*)) \in M_1$ while $p_1(x_2) = p_1(p_2(x^*)) \in M_2$. Since $E(X,T)$ is abelian, $M_1 \cap M_2 \neq \emptyset$ and so $M_1 = M_2$.
□

Proposition 1.6. *If $ev_{x^*} : E(X,T) \to X$ is an homeomorphism (e.g. if (X,T) is WAP with transitive point x^*) and $\{T^{n_i}(x^*)\}$ is a net converging to a point $x \in X$ then $\{T^{n_i}(z)\}$ is a net converging in X for every $z \in X$. In fact, $\{T^{n_i}\}$ converges pointwise to the unique $p \in E(X,T)$ such that $p(x^*) = x$.*

In general, if $p \in E(X,T)$ and $\{q_i\}$ is a net in $E(X,T)$ such that $\{q_i(x^*)\} \to p(x^*)$ in X then $\{q_i\} \to p$ in $E(X,T)$.

Proof: Obvious by inverting the homeomorphism ev_{x^*}.
□

Example 1.7. The surjection ev_{x^*} can be a homeomorphism in non-WAP cases.

Let X be a compact, connected metric space and $T = id_X$, the identity map. So $E(X,T) = \{id_X\}$. Let $\{x_i : i \in \mathbb{Z}\}$ be a sequence of distinct points in X such that $Lim_{|i| \to \infty} d(x_i, x_{i+1}) = 0$ and so that the positive and negative tails are dense in X. Thus, $\{x_i\}$ is a dense asymptotic chain for (X,T). Following Example 1.1 we embed (X,T) as a subsystem of (X^*,T) with $X^* = X \cup O_T(x^*)$ and $\{T^i(x^*)\}$ asymptotic to $\{x_i\}$.

Now assume that $y \in X$ and T^{n_k} converges to y (and so with $|n_k| \to \infty$). Then $T^{n_k+N}(x^*)$ converges to $T^N(y) = y$. Furthermore, for any $z \in X_0$, $T^{n_k}(z) = z$ converges to z. Thus, T^{n_k} converges pointwise to the function which is the identity on X and which is constantly y on the orbit of x^*. In particular, $ev_{x^*} : (E(X,T), T_*) \to (X,T)$ is an isomorphism. On the other hand, $E(X,T)$ is not abelian and none of the elements of $E(X,T)$ are continuous except for the iterates T^n.

□

Example 1.8. Abelian enveloping semigroup does not imply WAP in general. (a) Let the circle be $\mathbb{R}/\mathbb{Z}$. Let $X = \mathbb{R}/\mathbb{Z} \times \mathbb{Z}^*$ where $\mathbb{Z}^*$ is the one-point compactification of Z. Define T to be the identity on $\mathbb{R}/\mathbb{Z} \times \{\infty\}$ and by $(t,n) \mapsto (t + 3^{-(|n|+1)}, n)$. On each circle the map is just a rotation and so is WAP. Hence the enveloping semigroup is abelian. Consider the sequence $\{T^{\Sigma_{i=0}^k 3^i}\}$. On $\mathbb{R}/\mathbb{Z} \times \{n\}$ this is eventually constant at the rotation $t \mapsto t + \Sigma_{i=0}^{|n|} 3^{-(|n|-i+1)}$. As $|n| \to \infty$ this approaches the rotation $t \mapsto t + \frac{1}{2}$. But on the circle $\mathbb{R}/\mathbb{Z} \times \{\infty\}$ the identity is the only element of the enveloping semigroup.

(b) A countable example is the *spin* (Z,T) of the identity on $\mathbb{Z}^*$. Z is a subset of $\mathbb{Z}^* \times \mathbb{Z}^*$

$$Z \quad = \quad (\mathbb{Z}^* \times \{\infty\}) \ \cup \ \bigcup_{n \in \mathbb{Z}} \{[-|n|, +|n| \ ] \times \{n\}\},$$

$$f(x) \quad = \quad \begin{cases} (\infty,\infty) & \text{for} \quad x \ = \ (\infty,\infty) \\ (t+1,\infty) & \text{for} \quad x \ = \ (t,\infty) \\ (t+1,n) & \text{for} \quad x \ = \ (t,n) \text{ with } t < |n|, \\ (-|n|,n) & \text{for} \quad x \ = \ (|n|,n). \end{cases} \tag{1.2}$$

□

The following describes the equivalences to what might be called local WAP.

Proposition 1.9. *For a system (X,T) the following are equivalent.*

(i) *Multiplication for the enveloping semigroup is continuous in each variable.*
(ii) *Every element of the enveloping semigroup has a continuous restriction on the orbit closure of each element of X.*
(iii) *The enveloping semigroup is abelian.*
(iv) *Each orbit closure in X is a WAP system.*

Proof: Since each orbit closure is invariant for the enveloping semigroup and since the topology of the latter is pointwise convergence, each of these conditions holds (X,T) iff it holds for the restriction to each orbit closure. This restricts to the topologically transitive case for which (iii) implies (ii) by Proposition 1.4. Because of pointwise convergence, (ii) implies (i) is obvious. If p, q are in the enveloping semigroup and T^{n_j} is a net converging to q then $pT^{n_j} = T^{n_j}p$ and two-sided continuity at p imply $pq = qp$.

□

Let (X, T) be an arbitrary separable metric system. Define $z_{CAN}(X)$ to be the complement of the set of isolated points in X. Let $z_{NW}(X)$ be the complement of the union of all wandering open sets. Note that if a point is isolated and non-periodic then it is wandering. Thus, if there are no isolated periodic points, then $z_{NW}(X) \subset z_{CAN}(X)$.

Let $z_{LIM}(X) = \overline{R_T(X)}$. If U meets $\overline{R_T(X)}$ then there exist $x, y \in X$ such that $y \in U \cap R_T(x)$ and so for infinitely many $n \in \mathbb{Z}$ $T^n(x) \in U$. It follows that $z_{LIM}(X) \subset z_{NW}(X)$.

For each of these operators we define the descending transfinite sequence of closed sets by

$$z_0(X) = X, \quad z_{\alpha+1}(X) = z(z_\alpha(X)), \quad z_\beta = \bigcap_{\alpha<\beta} z_\alpha(X), \tag{1.3}$$

for β a limit ordinal. We say that the sequence stabilizes at β when $z_\beta(X) = z_{\beta+1}(X)$ in which case it is constant from then on. The first β at which stabilization occurs for the $CAN/NW/LIM$ sequence is called the *$CAN/NW/LIM$ level*. Since X is a separable metric space, each level is a countable ordinal (because $\{X \setminus z_\alpha : \alpha \leq \beta\}$ is an increasing open cover of the Lindelöf space $X \setminus z_\beta$).

We let $z_\infty(X) = z_\beta(X)$ when the sequence stabilizes at β. Clearly $z_{LIM,\alpha}(X) \subset z_{NW,\alpha}(X)$ for all α. Recall that when (X, T) is non-wandering, i.e. $z_{NW}(X) = X$ then the recurrent points are dense. Since all recurrent points are contained in $z_{LIM,\infty}(X)$ it follows that $z_{LIM,\infty}(X) = z_{NW,\infty}(X)$ is the closure of the set of recurrent points, i.e. the Birkhoff Center.

If X is Polish then a nonempty G_δ subset without isolated points contains a Cantor Set. Hence, if X is Polish, $z_{CAN,\infty} = \{x :$ every neighborhood of x is uncountable $\}$. In particular, if X is Polish and countable then $z_{CAN,\infty} = \emptyset$ and the isolated points are dense in X. Since the intersection of the decreasing family of nonempty closed sets has a nonempty intersection, $\beta_{CAN}(X)$ is not a limit ordinal if X is compact and countable.

Definition 1.10. We will call the ordinal $\beta_{LIM}(X)$ at which the z_{LIM} sequence stabilizes, the *height* of (X, T).

Call (X,T) *semi-trivial* (hereafter ST) if $R_T = X \times \{e\}$ for a point, a fixed point, $e \in X$. That is, for every $x \in X$, $R_T(x) = \{e\}$. Call (X,T) *center periodic* (hereafter CP) if the only recurrent points are periodic. Call (X,T) *center trivial* (hereafter CT) if there is a unique recurrent point e, necessarily a fixed point, and so the Birkhoff center is $\{e\}$. Call (X,T) *min center trivial* (hereafter minCT) if there is a unique minimal point e, necessarily a fixed point.

Clearly, ST implies CT and CT implies CP and minCT. A nontrivial system is ST iff it is a CT system of height 1. For a minCT system we will denote by u the retraction to the fixed point e. If (X,T) is minCT then u is the only minimal element of $E(X,T)$ and so $(E(X,T),T_*)$ is minCT.

In a CP system every point is isolated in its orbit closure. If (X,T) is CP and non-wandering then the recurrent points and so the periodic points form a dense G_δ and so for some finite positive n $\{x : T^n(x) = x\}$ has nonempty interior. In fact the union of such interiors is dense in X. If there are only countably many periodic points then this open dense set is countable and Polish and so the isolated points are dense in X. If X is non-wandering then every isolated point must be periodic.

The identity on any compact space defines a CP system and the finite product of CP's is CP (not the infinite product since the product of periodic orbits can contain an adding machine). Any subsystem and factor of CP is CP (since any recurrent point in the factor lifts to some recurrent point in the top). Inverse limit does not work. Again, an adding machine is the inverse limit of periodic orbits.

Remark 1.11. A nontrivial CP system (X,T) can never be weak mixing, i.e. $(X \times X, T \times T)$ is never topologically transitive. If x^* is a transitive point for a CP system (X,T) then it is isolated in X. If x^* is an isolated, transitive point for a nontrivial system (X,T) then $T(x^*) \neq x^*$ and so $U = \{x^*\}$ and $V = \{T(x^*)\}$ are nonempty open subsets of X, but $N(U \times U, U \times V) = \emptyset$.

□

The CT condition is closed under arbitrary products and subsystems. In particular, the enveloping semigroup of a CT system is CT. The retraction u to the fixed point $e \in X$ is the unique fixed pointf in $E(X,T)$ (the system is minCT). Also, it is the unique idempotent in $A(X,T)$. If $\pi : (X,T) \to (Y,S)$ is an action map and X is CT then Y is. In general, (X,T) is CT iff (Y,S) is CT and $\pi^{-1}(e)$ is a CT subsystem of X. If Y is a metrizable CT then since it is chain transitive we can attach a single orbit of isolated points and obtain a metrizable CT which is topologically transitive, see Example 1.1.

Mapping (X,T) to the factor system on $X/Cent_T$ defines a functor from compact systems to CT systems. An action map $X \to Y$ with Y CT factors through the projection from X to $X/Cent_T$ and so the functor is adjoint to the inclusion functor.

If (X,T) is a countable CT system, then e is not an isolated point in any invariant closed subset, of X except $\{e\}$ itself. Thus, if the Cantor sequence stabilizes at $\beta+1$ then $z_{CAN,\beta} = \{e\}$ and conversely. In that case, for any $\alpha \leq \beta$ $z_{LIM,\alpha} \subset z_{NW,\alpha} \subset z_{CAN,\alpha}$. That is, up to $\alpha = \beta$ the isolated points are all non-wandering.

A CT system (X,T) has height 0 iff $X = \{e\}$, i.e. the system is trivial, and has height 1 iff it is ST.

Proposition 1.12. *(a) If (X,T) is an ST system then it is WAP.*

(b) If (X,T) is a CT system with height at most 2 then $E(X,T)$ is abelian. If, in addition, (X,T) is topologically transitive then it is WAP.

Proof: (a) $E(X,T) = \{T^n : n \in \mathbb{Z}\} \cup \{u\}$ if (X,T) is ST.

(b)If $p,q \in A(X,T)$ then $pq = u = qp$ where u is the retraction onto e. Hence, the semigroup is abelian. So if (X,T) is topologically transitive, then it is WAP by Proposition 1.4.

□

Example 1.13. In his work [33] Shapovalov shows that within the class of countable subshifts one can find, for any countable ordinal α, a subshift $X_\alpha \subset \{0,1\}^{\mathbb{Z}}$ whose Birkhoff degree, i.e. its NW level, is $\alpha+1$. Now it is easy to verify that all of these subshifts X_α constructed by Shapovalov are in fact ST and therefore also WAP. One can make them topologically transitive by attaching a single orbit. Thus we conclude that *the class of WAP, topologically transitive subshifts is rich enough to present every countable Birkhoff degree.* Note however that being semi-trivial Shapovalov's original examples all are of height 2 and they become of height 3 when an orbit is attached to make them topologically transitive. As we will show later (Theorem 6.31) the class of WAP, topologically transitive subshifts is also rich enough to present every countable height.

Let S denote the shift homeomorphism on $\{0,1\}^{\mathbb{Z}}$.

Example 1.14. Various non-WAP examples.

Let $e = \bar{0}, x[0] = 0^\infty \dot{1} 0^\infty$. Let $X(0)$ be the ST subshift generated by $x[0]$. Thus, $(X(0), S)$ is isomorphic to translation on the one point compactification $\mathbb{Z}_*$ of $\mathbb{Z}$.

(a) For $k = 1, 2, \dots$ let $b^k_j = 1$ for $j = 10^{nk}, n \in \mathbb{Z}$ and $= 0$ otherwise. Let (X, S) be the generated subshift. $R_S(X) = X(0)$ and so (X, S) has height 2. $b^k \to x[0]$ as $k \to \infty$. The sequence $S^{10^{n!}} \to p$ in $A(X, T)$ with $p(b^k) = x[0]$ for all k and $p(x[0]) = e$. So p is not continuous at $x[0]$, despite the fact that all of the points of $X \setminus X(0)$ are isolated. That is, the assumption of topological transitivity in Proposition 1.12 (b) is necessary.

(b) A topologically transitive system of height 1 with minimal set not a fixed point need not be WAP. Let c be given by $c_i = 1$ for $i = 2n, -1 - 2n$ for $n \in \mathbb{N}$ and $= 0$ otherwise, i.e. $c = (01)^\infty(10)^\infty$. The orbit closure of c consists of $O_S(c)$ together with the periodic orbit $\{\overline{10}, \overline{10}\}$. $S^{-2k}(c) \to \overline{01}$, $S^{2k}(c) \to \overline{10}$. $S^{-2k} \to p$ and $S^{2k} \to q$ both p, q are identity on the periodic orbit. Hence, $q = pq \neq qp = p$ on c.

(c) For a countable topologically transitive, height 3 CT subshift which is not WAP let $d_i = 1$, for $i = 0, \pm 2^k, (k = 0, 1, \dots)$ and $= 0$ otherwise, X_1 be the orbit closure of d which is the orbit of d together with $X(0)$.

Let $Block_k$ denote the k^{th} block, of d defined to be the word of length $2^{k+1} + 1$ which agrees with $d_{[-2^k, +2^k]}$. Three successive 1's uniquely determine where in the block a subblock is. It follows, that one can build x^* on the positive side as follows

$$Block_1 \qquad N_1 0's \qquad Block_2 \qquad N_2 0's \qquad Block_3 \quad \dots \tag{1.4}$$

with $Block_1 = 111$ centered at position 0. Then reflect to define x^*. Provided the sequence N_k increases much faster than $2^{k+1} + 1$ then it can be chosen arbitrarily and the orbit closure X_2 of x^* will consist of the dense orbit of x^* together with the limit point set which is equal to X_1. Let m_k be the location of the center of $Block_k$ in x^*. Since the N_k's are rather arbitrary we can arrange that m_k is a power of 2 for k even, and 1 plus a power of 2 for k odd. Then $S^{m_k}(x^*)$ converges to d, but $S^{m_{2k}}(d)$ converges to $x[0]$ and $S^{m_{2k+1}}(d)$ converges to $S(x[0])$. Thus, $S^{m_k}(d)$ is not a convergent sequence. From Proposition 1.6 it follows that (X, S) is not WAP.

(d) For (Y, S) any compact metric system, let (X, T) be the one point compactification of $(Y \times \mathbb{Z}, S \times t)$ with t the translation on $\mathbb{Z}$. This is an ST system. If $S = id_Y$ it is easy to build a countable sequence of periodic orbits with limit set (X, T). The expanded system is CP with an uncountable center although there are only countably many periodic orbits.

(e) Call x *selective* if for any n the word 10^n1 occurs at most once in x. Let X be the set of all selective x. Clearly, if x is selective then $R_S(x) \subset X(0)$. Note that if $A \subset \mathbb{Z}$ is such that all the nonzero differences $a_i - a_j$ are distinct then the characteristic function $\chi(A)$ of A in $\{0,1\}^{\mathbb{Z}}$, is a selective element. (X,S) is an uncountable CT subshift with height 2.

(f) Let (Y,S) be any CP subshift. Let $\{w_i\}$ count the finite words in Y. Then $Y \cup \bigcup \{\overline{w_i}\}$ is a CP subshift with dense periodic points.

□

Any CT metric WAP (X,T) is chain transitive and so there exists (X^*,T) topologically transitive so that of X^* the disjoint union of X and the dense orbit of isolated points $O_T(x^*)$. See Example 1.1.

Example 1.15. It may happen that we cannot choose the extension so that (X^*,T) is WAP.

Let (X,T) be a CT WAP with fixed point e and which is not semi-trivial. That is, there exists p in the $A(X,T)$ with $p \neq u$ and so $p(X) \setminus \{e\}$ is nonempty. Let $\bar{X} = X_1 \vee X_2$, two copies of X with the fixed points identified. For any map g on X which fixes e, let $\bar{g}$ on $\bar{X}$ be copies of g on each term. The system $(\bar{X},\bar{T})$ is clearly WAP and $p \mapsto \bar{p}$ is an isomorphism from $E(X,T)$ onto $E(\bar{X},\bar{T})$. Notice the $\bar{p}(\bar{X}) \setminus X_i$ is nonempty for $i = 1,2$.

Now let $(\hat{X},\hat{T})$ contain $(\bar{X},\bar{T})$ and with $\hat{X} \setminus \bar{X}$ consisting of a single dense orbit $O_{\hat{T}}(x^*)$.

Let q be an element of the enveloping semigroup of $E(\hat{X},\hat{T})$ with $q(x^*) \in X_1$. Then q maps the whole orbit of x^* into X_1 and if q is continuous then $q(\hat{X}) \subset X_1$. Thus, every continuous element of the enveloping semigroup $E(\hat{X},\hat{T})$ maps all of $\hat{X}$ either into X_1 or into X_2. Every element of the enveloping semigroup of $(\bar{X},\bar{T})$ extends to some element of the enveloping semigroup of $(\hat{X},\hat{T})$. Thus, if $\hat{p}$ extends $\bar{p}$ it cannot be continuous and so $(\hat{X},\hat{T})$ is not WAP.

□

We will say that $A(X,T)$ *distinguishes points* when $p(x_1) = p(x_2)$ for all $p \in A(X,T)$ implies $x_1 = x_2$. It suffices that some $p \in A(X,T)$ be injective. If X has any non-trivial, but semi-trivial subspace then $A(X,T)$ does not distinguish points.

Let T_* be composition with T on $E(X,T)$. Clearly id_X is a transitive point for T_*. If (X,T) is not weakly rigid, i.e. id_X is not a recurrent point for T_*, then id_X is an isolated transitive point for T_* and $A(X,T) = R_{T_*}(id_X)$ is a proper subset of $E(X,T)$.

Assume that x^* is a transitive point for (X^*,T). Then $ev_{x^*} : (E(X^*,T), T_*) \to (X^*,T)$ is a factor map sending $A(X^*,T)$ to $X = R_T(x^*)$. If (X^*,T) is WAP then the map is an isomorphism by Proposition 1.4.

Now assume that the subspace (X,T) is WAP. As is true for any subsystem the restriction map $\rho : A(X^*,T) \to A(X,T)$ is surjective.

Proposition 1.16. *Assume that (X^*,T) is topologically transitive with an isolated transitive point x^* such that the subsystem (X,T) with $X = R_T(x^*)$ is WAP. The map ρ is injective, and so is an isomorphism, iff (X^*,T) is WAP and, in addition, $A(X,T)$ distinguishes points of X.*

Proof: Since X is WAP, $A(X,T)$ is abelian. If ρ is injective then $A(X^*,T)$ is abelian and so (X^*,T) is WAP by Proposition 1.4.

Now assume (X^*,T) is WAP. We show that ρ is injective iff $A(X,T)$ distinguishes the points of X.

Let $p_1, p_2 \in A(X^*,T)$. Since $A(X^*,T)$ is abelian, $p_1(q(x^*)) = p_2(q(x^*))$ for all $q \in A(X^*,T)$ iff $q(p_1(x^*)) = q(p_2(x^*))$ for all $q \in A(X^*,T)$. The first says $\rho(p_1) = \rho(p_2)$ and the second says $p_1(x^*), p_2(x^*) \in X_1$ are not distinguished by $A(X,T)$. ρ is injective says that the first implies $p_1 = p_2$ while $A(X,T)$ distinguishes points says that the second implies $p_1(x^*) = p_2(x^*)$ and so by continuity $p_1 = p_2$. This proves the equivalence.

□

Corollary 1.17. *Assume (X,T) is WAP, is not weakly rigid and is such that $A(X,T)$ distinguishes points. If there exists (X^*,T) topologically transitive with an isolated transitive point x^* such that (X,T) is the subsystem with $X = R_T(x^*)$ then (X^*,X,T) is isomorphic to $(E(X,T), A(X,T), T_*)$.*

Proof: Since X is not weakly rigid, ρ is an isomorphism from $(E(X^*,T), A(X^*,T), T_*)$ onto $(E(X,T), A(X,T), T_*)$. On the other hand, ev_{x^*} is an isomorphism from $(E(X^*,T), A(X^*,T), T_*)$ onto (X^*,X,T).

□

More generally, if we define $E = \{(x_1,x_2) \in X \times X : q(x_1) = q(x_2)$ for all $q \in A(X,T)\}$ then E is an ICER and the factor $(X/E, X_1/E, T)$ is isomorphic to $(E(X,T), A(X,T), T_*)$, because $\rho(p_1) = \rho(p_2)$ iff $(p_1(x^*), p_2(x^*)) \in E$.

2. Coalescence, LE, HAE and CT-WAP systems

Given a metric dynamical system (X,T), a point $x \in X$ is an *equicontinuity point* if for every $\epsilon > 0$ there is $\delta > 0$ such that $d(x',x) < \delta$ implies $d(T^n x', T^n x) < \epsilon$ for every $n \in \mathbb{Z}$. The system (X,T) is called *equicontinuous* if every point in X is an equicontinuity point (and then it is already *uniformly equicontinuous* meaning that the δ in the above definition does not depend on x). It is called *almost equicontinuous*, hereafter AE, when there is a dense set of points in X at which $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous. Following [16] we will call (X,T) *hereditarily almost equicontinuous*, hereafter HAE, when every subsystem (i.e. closed invariant subset) is again an AE system. As was shown in [5] every WAP is HAE (see also [13, Chapter 1, Sections 8 and 9]).

An isolated point is an equicontinuity point and so if the isolated points are dense then the system is AE. Any countable Polish space has isolated points and so applying this to any nonempty open subset we see that the isolated points are dense. Hence, if (X,T) is countable then every subsystem is AE i.e. it is HAE.

A compact, metric system (X,T) is *expansive* if there exists $\epsilon > 0$ such that for every $x_1 \neq x_2$ there exists $n \in \mathbb{Z}$ such that $d(T^n(x_1), T^n(x_2)) > \epsilon$. Any subshift is expansive. The following is obvious.

Lemma 2.1. *If (X,T) is expansive then $x \in X$ is an equicontinuity point iff it is isolated.*

□

From this follows the result from [17] that a subshift is HAE iff it is countable.

Proposition 2.2. *An expansive, compact, metric dynamical system (X,T) is HAE iff X is countable. In particular, a subshift is HAE iff it is countable.*

Proof: It was observed above that a countable system is HAE.

Now assume that X is uncountable and so contains a Cantor set C. If X_1 is the closure of $\bigcup_n \{T^n(C)\}$, then the subsystem (X_1,T) is expansive and contains no isolated points. So by Lemma 2.1 it has no equicontinuity points. Thus, (X,T) is not HAE.

□

Thus:

Proposition 2.3. *A WAP subshift is countable.*

□

Following [20] we call (X,T) *locally equicontinuous* (hereafter LE) if each point x is an equicontinuity point in its orbit closure or, equivalently, if each orbit closure is an almost equicontinuous subsystem. The equivalence follows from the Auslander-Yorke Dichotomy Theorem, [8], which says that in a topologically transitive system the set of equicontinuity points either coincides with the set of transitive points or else it is empty.

□

Remark 2.4. From the latter condition, it follows that an HAE system is LE. Any CP system is LE since each point is isolated in its orbit closure. From Proposition 2.2 it follows that any uncountable CP subshift is LE but not HAE.

A system (X,T) is *coalescent* when any surjective action map π on (X,T) is an isomorphism.

Proposition 2.5. *A topologically transitive system which is WAP is coalescent.*

Proof: There exists p in the enveloping semigroup with $p(x^*) = \pi(x^*)$. Because p is continuous it is an action map and so since p and π agree on the dense orbit of x^*, $p = \pi$. Since p is surjective, $p(x^*)$ is a transitive point and so there exists q such that $qp(x^*) = x^*$ and so $qp = id$. Hence, p is injective with inverse q.

□

Example 2.6. In general a WAP system need not be coalescent.

If (X,T) is WAP then the countable product $(X^{\mathbb{N}}, T^{\mathbb{N}})$ is WAP and the shift map is a surjective action map which is not injective. If X is CT with fixed point e then the *infinite wedge* which is $\{x \in X^{\mathbb{N}} : x_i \neq e$ for at most one $i\}$ is a closed invariant set which is shift invariant as well. This is also WAP and not coalescent. In addition, it is countable if X is.

□

Lemma 2.7. *If a dynamical system (X,T) contains an increasing net of topologically transitive subsystems $\{X^i\}$ with $\bigcup_i \{X^i\}$ dense in X, then X is also topologically transitive.*

Proof: Let $U, V \subset X$ be two nonempty open subsets. For some i

$$U \cap X^i \neq \emptyset \quad \text{and} \quad V \cap X^i \neq \emptyset.$$

As X^i is topologically transitive, there exists $k \in \mathbb{Z}$ with $T^k(U \cap X^i) \cap (V \cap X^i) \neq \emptyset$ and, a fortiori, also $T^k(U) \cap V \neq \emptyset$.
□

Proposition 2.8. *Every dynamical system is a union of maximal topologically transitive subsystems.*

Proof: Let (X,T) be a dynamical system and consider the family $\mathcal{T}$ of topologically transitive subsystems of X. Using Lemma 2.7 it is easy to check that this family is inductive. Hence, by Zorns lemma, every topologically transitive subsystem of X is contained in a maximal element of $\mathcal{T}$. In particular, for $x \in X$, the orbit closure of X is contained in a maximal element of $\mathcal{T}$.
□

We then obtain the following results on *E-Coalescence* (i.e. the property that every continuous surjective element of $E(X,T)$ is injective).

Recall that a dynamical system (X,T) is called (i) *weakly rigid* if there is a net $\{T^{n_i}\}$ with $|n_i| \to \infty$ and $\{T^{n_i}(x)\} \to x$ for every $x \in X$, or, equivalently, if $id_X \in A(X,T)$. (ii) *rigid* if the net can be chosen to be a sequence, and (iii) *uniformly rigid* if the convergence can be taken to be uniform (see [15]). Recall that if X is a metrizable, topologically transitive AE system, and a fortiori a metrizable, topologically transitive WAP, then it is uniformly rigid (see [17], [18] and [4]).

Theorem 2.9. *Let (X,T) be an AE compact, metrizable system. Assume that $p \in A(X,T)$ is continuous and surjective.*

(i) *If (X,T) is topologically transitive then p is injective and so is an isomorphism. If T^{n_j} is a net converging pointwise to p then it converges uniformly to p and T^{-n_j} converges uniformly to p^{-1}. Thus, $p^{-1}, id_X \in E(X,T) = A(X,T)$.*
(ii) *If X_1 is a maximal topologically transitive subset of X then $p(X_1) = X_1$.*
(iii) *If (X,T) is HAE then p is an isomorphism and if T^{n_j} is a net converging pointwise to p then T^{-n_j} converges pointwise to p^{-1}.*

Thus, $p^{-1}, id_X \in E(X,T) = A(X,T)$ and the system is weakly rigid.

Proof: (i) Let x^* be a transitive point for X. Since p is surjective and continuous, px^* is a transitive point and so there exists a sequence T^{n_i} with $T^{n_i}px^*$ converging to x^*. Let $\epsilon > 0$. Since x^* is an equicontinuity point eventually $T^{n_i}pT^kx^* = T^kT^{n_i}px^*$ is within ϵ of T^kx^* for all k. Since the orbit of x^* is dense it follows that $T^{n_i}p$ converges uniformly to id_X. Hence, p is injective and so is an isomorphism by compactness. If q is any limit point of T^{n_i} in $E(X,T)$ then pq which is the limit of $pT^{n_i} = T^{n_i}p$ is the identity and so $q = p^{-1}$. Thus, $p^{-1} \in E(X,T)$. Hence, $id_X = p^{-1}p \in A(X,T)$ and so (X,T) is weakly rigid and $E(X,T) = A(X,T)$. Since $pT^{n_i} = T^{n_i}p$ converges uniformly to pp^{-1}, uniform continuity of p^{-1} implies T^{n_i} converges uniformly to p^{-1}. Hence, $Lim_{i,j\to\infty}T^{n_i-n_j} = id_X$ and so the system is uniformly rigid. Finally, if a net T^{m_i} converges to p pointwise then $p^{-1}T^{m_i}x^*$ is eventually close to x^* and so as above $p^{-1}T^{m_i}$ converges to id_X uniformly and so T^{m_i} converges to p uniformly and T^{-m_i} converges uniformly to p^{-1}.

(ii) Let x^* be a transitive point for X_1. Since p is surjective, there exists $x_1 \in X$ such that $px_1 = x^*$. The orbit closure of x_1 is a topologically transitive subset of X and it contains $px_1 = x^*$ and so contains X_1, which is the orbit closure of x^*. Hence, by maximality, $X_1 = \overline{O_T(x_1)}$. Since p is a continuous action map, $p(X_1) = p(\overline{O_T(x_1)}) = \overline{O_T(x^*)} = X_1$.

(iii) Each point is contained in a maximal topologically transitive subset of X, which is necessarily closed.

Now if for some points $x_1, x_2 \in X$ we have $z = px_1 = px_2$, then let X_i be a maximal topologically transitive subset of X which contains x_i. Since they are closed, $z \in X_1 \cap X_2$. By (ii) p is surjective on each X_i and so by (i) there exist $q_i \in A(X,T)$ such that on X_i q_ip is the identity. In particular, $q_iz = x_i$. Hence, $q_2px_1 = x_2$ and so x_2 as well as x_1 is in X_1. Since q_1p is the identity on X_1 it follows that $x_1 = x_2$. Thus, p is an isomorphism. Now let T^{m_i} be a net converging to p pointwise. By part (i) T^{-m_i} converges to p^{-1} uniformly on each orbit closure and so pointwise on X. Hence, p^{-1} and $id_X = p^{-1}p$ are in $A(X,T)$. Hence, $E(X,T) = A(X,T)$ and (X,T) is weakly rigid.

□

Corollary 2.10. *Every WAP dynamical system is E-coalescent.*

Proof: If (X,T) is a WAP dynamical system then (i) it is is HAE, and (ii) every $p \in E(X,T)$ is continuous. Now apply Theorem 2.9.

□

Corollary 2.11. *Let (X,T) be a WAP system and $p \in A(X,T)$.*

The restriction of p to the subsystem $Z = \bigcap_{n\in\mathbb{N}} p^n X$ is an automorphism of Z. In particular the system (Z,T) is weakly rigid.

If, in addition, (X,T) is CP, then every point of Z is periodic. If moreover (X,T) is topologically transitive then Z consists of a single periodic orbit.

Proof: Assume first that X is metrizable. Hence, (X,T) is HAE because it is WAP.

Clearly Z is a (nonempty) subsystem and $pZ = Z$. Thus, by Theorem 2.9 $p|Z$ is an automorphism of Z. Since $p \in A(X,T)$ it follows that Z is weakly rigid and so every point of Z is recurrent. The last assertion is now easily deduced.

Since the group $\mathbb{Z}$ is countable, (X,T) is an inverse limit of a net $\{(X^i,T^i)\}$ metrizable systems which are WAP since the latter property is preserved by factors. The set Z projects onto the corresponding set Z^i. Thus, the restriction of p to Z is the inverse limit of isomorphisms and so is an isomorphism. The inverse limit of weakly rigid systems is weakly rigid.

□

Questions 2.12.

(1) Is there a metric WAP system with every point non-wandering (and so it is its own Birkhoff center) but which is not rigid?
(2) If a homeomorphism for X is in $A(X,T)$ then is its inverse also in $A(X,T)$ (and so it is weakly rigid)? For WAP or even for HAE the answer is yes, by Theorem 2.9 above.

Lemma 2.13. *Assume (X,T) is a CP system. If there is an infinite sequence $\{x_i : i \in \mathbb{N}\}$ in X such that $x_i \in R_T(x_{i+1})$ for all i, then x_1 is a periodic point and all x_i's are in the orbit of x_1.*

Proof: First, assume that X is metrizable.

If x_i is periodic then all x_j's with $j < i$ are in the orbit of x_i. Hence, if infinitely many of the x_i's are periodic then they all are and all lie in the same periodic orbit. If $x_{i+1} \in R_T(x_{i+1})$ then it is positively or negatively recurrent. Since X is CP the only recurrent points are periodic. Hence, if x_{i+1} is not periodic then x_{i+1} is not in the orbit closure of x_i.

We show that the alternative to finitely many periodic points cannot happen. As usual we can use inverse limits to reduce to the metric case. If instead there are only finitely many periodic points in the sequence then by omitting finitely many initial terms and re-numbering we can assume that none are periodic. Let A_i be the orbit closure of x_i. Since $x_i \in A_{i+1}$ we have $A_i \subset A_{i+1}$. Since $x_{i+1} \in A_{i+1} \setminus A_i$, each inclusion is strict. Each $\overline{A_i}$ is topologically transitive. Therefore the closure of the union $A = \overline{\bigcup A_i}$ is topologically transitive. Let z be a transitive point for A. It is isolated in $A = \overline{\bigcup A_i}$ and so must lie in some A_j, but since the $\{A_i\}$ sequence is a strictly increasing sequence of closed invariant sets, it cannot be in any of them.

For the general case, we can assume that some x_i is not periodic. There is a metrizable factor for which the image of x_i is not periodic and this contradicts the metric result above.

□

Proposition 2.14. *Assume that (X, T) is a CP-WAP. If $\{p_i : i \in \mathbb{N}\}$ is a sequence in $A(X, T)$ then $\bigcap_{n=1}^{\infty} p_n p_{n-1} \cdots p_1 X$ is a closed subset consisting of periodic points.*

Proof: For $i = 1, 2, \dots$ and $n \geq i$ let $X_{i,n} = p_i p_{i+1} \cdots p_n X$. $X_i = \bigcap_{n=i}^{\infty} X_{i,n}$. Since each p_i is a continuous, each $X_{i,n}$ is a closed, invariant subspace and the sequence is decreasing in n and $p_i(X_{i+1,n}) = X_{i,n}$ when $n \geq i + 1$. Hence, continuity and compactness imply that $p_i(X_{i+1}) = X_i$. Since the semigroup is abelian, $X_{i,n} = p_n p_{n-1} \cdots p_i X$. Let $x_1 \in X_1$. By induction we can build a sequence $x_i \in X_i$ such that $p_i(x_{i+1}) = x_i$ and so $x_i \in R_T(x_{i+1})$. From Lemma 2.13 it follows that x_1 is periodic.

□

Corollary 2.15. *Assume that (X, T) is a CP-WAP. If $\{x_i : i \in \mathbb{N}\}$ is a sequence in X such that $x_{i+1} \in R_T(x_i)$ for all i, then $\bigcap_i \overline{O(x_i)}$ (the orbit closures of the x_i's) is a closed subset consisting of periodic points.*

Proof: The sequence $\overline{O(x_i)}$ is decreasing and so we can restrict to the case $X = \overline{O(x_1)}$. As this is a transitive subspace and the system is WAP the elements of the semigroup are continuous on it. By assumption, there exists $p_i \in A(X, T)$ such that $p_i(x_i) = x_{i+1}$. In the notation of the proof of Proposition 2.14, $x_{n+1} \in X_{1,n}$ for all $n \geq 1$. Hence, $\bigcup_n \overline{O(x_n)} \subset \bigcup_n X_{1,n}$ and the latter consists of periodic points by Proposition 2.14.

□

Remark 2.16. A closed invariant set $K \subset X$ is an *isolated invariant set* if there is an open set U containing K such that K is the maximum invariant subset of U, i.e. $K = \bigcap_{i \in \mathbb{Z}} T^i(U)$. If $\{K_n\}$ is a decreasing sequence of closed invariant sets with intersection K isolated, then $K_n \subset U$ implies $K_n = K$ and so eventually $K_n = K$. In an expansive system, like a subshift, any fixed point is an isolated invariant set. It thus follows from Proposition 2.14 that if X be a countable CT-WAP subshift with $\{e\}$ the unique minimal subset of X, then for any infinite sequence $p_1, p_2, \dots$ of elements of $A(X,T)$ there there exists an n with $p_1 p_2 \cdots p_n(X) = \{e\}$.

Now we restrict to the case where (X,T) is a metrizable CT-WAP with a single minimal set consisting of a fixed point e.

For $\epsilon > 0$ let A_ϵ denote the maximal invariant subset of X which is contained in the closed neighborhood $\bar{V}_\epsilon(e)$, i.e. $A_\epsilon = \bigcap_{i \in \mathbb{Z}} T^i(\bar{V}_\epsilon(e))$. If $V_\epsilon(e)$ is clopen, A_ϵ is an isolated invariant set. If (X,T) is expansive then we can choose $\epsilon > 0$ so that $A_\epsilon = \{e\}$.

Proposition 2.17. *Let A be any invariant set which contains A_ϵ. If $x \notin A$ then there exists $q \in E(X,T)$ such that $qx \notin A \cup V_\epsilon(e)$ but for all $p \in A(X,T)$, $pqx \in A$, i.e. $R_T(qx) = q(R_T(x)) \subset A$.*

Proof: Clearly,

$$A \quad \subset \quad \bigcap_{\ell \in \mathbb{Z}} T^\ell(A \cup V_\epsilon(e)) \quad \subset \quad A \cup A_\epsilon \quad = \quad A \tag{2.1}$$

So for some $i \in \mathbb{Z}$, $T^i(x) \notin A \cup V_\epsilon(e)$. If $R_T(T^i(x)) \subset A \cup V_\epsilon(e)$ then let $q = T^i$. Otherwise, there exists q_1 such that $q_1(T^i(x)) \notin A \cup V_\epsilon(e)$. Continue inductively defining q_n such that $q_n \cdots q_1(T^i(x)) \notin A \cup V_\epsilon(e)$ whenever $R_T(q_{n-1} \cdots q_1(T^i(x))$ is not contained in $A \cup V_\epsilon(e)$. This process must terminate because for any infinite sequence $\{q_n\}$ in $A(X,T)$ the sequence $\{R_T(q_n \dots q_1(T^i(x))\}$ is a decreasing sequence of invariant sets which must eventually equal e.

If $q_n \cdots q_1(T^i(x)) \notin A \cup V_\epsilon(e)$ but $R_T(q_n \cdots q_1(T^i(x)) \subset A \cup V_\epsilon(e)$, then let $q = q_n \cdots q_1 T^i$. Since $R_T(qx) \subset A \cup V_\epsilon(e)$ and $R_T(qx)$ is an invariant set, equation 2.1 implies $R_T(qx) \subset A$.

□

Remark: Notice that A need not be closed.

Recall that for the relation R_T on X and $A \subset X$ we let $R_T^*(A) = \{x : R_T(x) \subset A\}$. Let $R_T^{*n}(A) = R_T^*(R_T^{*(n-1)}(A)) = \{x : R_T^n(x) \subset A\}$. Observe that $R_T^*(A) = \bigcap\{p^{-1}(A) : p \in A(X,f)\}$. Hence, in the case with all members of $A(X,T)$ continuous, $R_T^*(A)$ is closed when A is. Proposition 2.17 says exactly

Proposition 2.18. *Let A be any invariant set which contains A_ϵ. If $x \notin A$ then there exists $q \in E(X,T)$ such that $qx \in R_T^*(A) \setminus (A \cup V_\epsilon(e))$.*

□

If A is not closed then $R_T^*(A)$ need not be a closed invariant set, but it is always invariant and satisfies the following weakening of closure.

Lemma 2.19. *Let $A \subset X$.*

$$x \in R_T^*(A) \quad \Rightarrow \quad \overline{O(x)} \subset R_T^*(A) \tag{2.2}$$

Proof: Since $R_T(T^i(x)) = R_T(x)$ for all $i \in \mathbb{Z}$ it follows that $R_T^*(A)$ is invariant. Since $R_T \circ R_T \subset R_T$ it follows that $x \in R_T^*(A)$ implies $R_T(x) \subset R_T^*(A)$.
□

Inductively define $A_0 = A_\epsilon$ and for an ordinal α, $A_{\alpha+1} = R_T^*(A_\alpha)$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for α a limit ordinal. At every stage, $x \in A_\alpha \Rightarrow \overline{O(x)} \subset A_\alpha$ although it is not clear that A_α is closed when α is infinite.

Proposition 2.20. *For any ordinal α, if $x \notin A_\alpha$ then there exists $q \in E(X,f)$ such that*

$$\begin{aligned} qx \ \notin \ A_\alpha \cup V_\epsilon(e) \qquad &\textit{and} \qquad R_T(qx) \ \subset \ A_\alpha \\ \textit{i.e.} \qquad qx \in A_{\alpha+1} &\setminus (A_\alpha \cup V_\epsilon(e)). \end{aligned} \tag{2.3}$$

For any such q, $R_T(qx)$ meets $A_{\beta+1} \setminus (A_\beta \cup V_\epsilon(e))$ for every $\beta < \alpha$.

Proof: We repeatedly apply Proposition 2.17. First we obtain $q \in E(X,T)$ which satisfies (2.3).
For any $\beta < \alpha$ there exists $q_1 \in E(X,T)$ such that $q_1qx \in A_{\beta+1} \setminus (A_\beta \cup V_\epsilon(e))$. Since $\beta + 1 \leq \alpha$, $qx \notin A_{\beta+1}$ and so $T^i(qx) \notin A_{\beta+1}$ for any $i \in \mathbb{Z}$. Hence, $q_1 \in A(X,T)$ and so $q_1(qx) \in R_T(qx)$.
□

Corollary 2.21. *If for any ordinal α, A_α is a proper subset of X then $A_{\alpha+1} \setminus A_\alpha$ is nonempty. So the transfinite sequence $\{A_\alpha\}$ is strictly increasing until the first ordinal α^* such that $A_{\alpha^*} = X$. If (X,T)*

is topologically transitive with transitive point x^* then α^* is the first ordinal such that $x^* \in A_{\alpha^*}$ and α^* is a non-limit ordinal.

Proof: That the sequence is strictly increasing until $A_\alpha = X$ is clear from Proposition 2.20. It is also clear that if x^* is a transitive point then $x^* \in A_\alpha$ implies $X = \overline{O(x^*)} \subset A_\alpha$ by (2.2) and so $A_\alpha = X$ iff $x^* \in A_\alpha$. If $x^* \in \bigcup_{\beta<\alpha} A_\beta$ then $x^* \in A_\beta$ for some $\beta < \alpha$ and so α^* cannot then be a limit ordinal.
□

For the case when (X, T) is a metrizable CT-WAP with the fixed point e isolated (as a closed invariant subset), e.g. a CT-WAP subshift, we choose $\epsilon > 0$ so that $A_\epsilon = A_0 = \{e\}$. We then call the ordinal at which the A_α sequence stabilizes the *height** of (X, T). Because X is then countable it follows that the ordinal α^* is countable.

3. Discrete suspensions and spin constructions

For any (X, T) and positive integer N we define on $X \times [0, N-1]$ the homeomorphism $\tilde{T}$ by

$$\tilde{T}(x,i) = \begin{cases} (x, i+1) & \text{for } i < N-1, \\ (T(x), 0) & \text{for } i = N-1. \end{cases} \qquad (3.1)$$

$$\text{so that} \quad \tilde{T}^N = T \times 1_{[0,N-1]}.$$

$(X \times [0, N-1], \tilde{T})$ is called the *discrete N step suspension*. It is countable if X is, it is CP if (X, T) is CT. It is WAP if (X, T) is. Apply the following

Lemma 3.1. *A point $x \in X$ is an equicontinuity point for T on X iff it is an equicontinuity point for T^N on X. Hence, (X, T) is AE or HAE iff (X, T^N) is. In general, (X, T) is WAP iff (X, T^N) is.*

Proof: Since $\{T^{Ni} : i \in \mathbb{Z}\} \subset \{T^i : i \in \mathbb{Z}\}$, an equicontinuity point for T is one for T^N, $E(X, T^N) \subset E(X, T)$ and so each of the conditions for T implies the corresponding condition for T^N. In fact,

$$\begin{aligned} \{T^i : i \in \mathbb{Z}\} &= \{T^{Ni} \circ T^k : i \in \mathbb{Z}, k = 0, \dots, N-1\} \\ &= \bigcup_{k=0}^{N-1} T^k \circ \{T^{Ni} : i \in \mathbb{Z}\}. \end{aligned} \qquad (3.2)$$

It follows that if x is an equicontinuity point for T^N then it is for T. Also, we obtain $E(X,T) = \bigcup_{k=0}^{N-1} T^k \circ E(X,T^N)$ and so the elements of $E(X,T)$ are continuous when those of $E(X,T^N)$ are.

□

Theorem 3.2. *If (X,T) is a CP WAP system with a unique minimal set, a periodic orbit of period N, then (X,T) is isomorphic to the discrete N step suspension of a CT WAP.*

Proof: Let $\{x_0,\dots,x_{N-1}\}$ be the periodic orbit in X. By Lemma spinlem1 (X,T^N) is a WAP with N minimal fixed points $x_0,\dots,x_{N-1}$. For any $x \in X$ the restriction of T^N to the T^N orbit closure of x is a point transitive WAP and so with a unique minimal set, necessarily one of the x_i's. Let $X_i = \{x \in X : x_i \in \overline{\mathcal{O}_{T^N}(x)}\} = R^*_{T^N}(\{x_i\})$. Clearly, $T(X_i) = X_{i+1}$ (addition mod N) and the X_i's are pairwise disjoint. Each is T^N invariant. Let u be a minimal element of $E(X,T^N)$. If $x \in X_i$ then $u(x)$ is a minimal element of the T^N orbit closure of x and so $u(x) = x_i$. That is, u retracts X_i to x_i. Because (X,T^N) is WAP, u is continuous and so each $X_i = u^{-1}(x_i)$ is closed.

Define $H : X_0 \times \{0,\dots,N-l\} \to X$ by $H(x,i) = T^i(x)$. This is continuous and surjective with inverse, $x \mapsto (T^{-i}x, i)$ for $x \in X_i$ and so H is bijective. Furthermore, $H(x,i+1) = T(H(x,i))$ for $i < N-1$ and $H(T^N(x),0) = T^N(x) = T(T^{N-1}(x)) = T(H(x,N-1))$. Thus, H is an isomorphism from the discrete suspension of height N of (X_0,T^N) onto (X,T).

□

Recall that (X,T) minCT when there is a fixed point which is the unique minimal subset, i.e. the mincenter is a single point.

Lemma 3.3. *Let (X,T) be a nontrivial, metric minCT system with fixed point e and let $\epsilon > 0$. There exists an ϵ-dense sequence of distinct points $\{e = x_0,\dots,x_{N-1}\}$ in X such that with $x_N = e$, $\{x_0,\dots,x_N\}$ is an ϵ chain for (X,T), i.e. $d(T(x_i),x_{i+1}) < \epsilon$ for $i = 0,\dots,N-1$.*

Proof: Since X is separable we can choose a finite or infinite sequence $\{a_1,a_2,\dots\}$ of points of $X \setminus \{e\}$ with pairwise distinct orbits and such that the union of the orbits is dense in $X \setminus \{e\}$. Since this set is nonempty the sequence contains at least one point. Since e is the only minimal point, $e \in \alpha_T(x) \cap \omega_T(x)$ for every $x \in X$. Now truncate so that the union of the orbits of the finite sequence $\{a_1,\dots,a_K\}$ is $\epsilon/2$ dense in $X \setminus \{e\}$. For each a_i we can choose a finite piece of the orbit $\{y_{0,i},\dots,y_{K_i+1,i}\}$ which begins and ends $\epsilon/2$ close to e and which is $\epsilon/2$ dense in the orbit of a_i. We concatenate to obtain the sequence $\{x_1, \dots, x_{N-1}\}$. Then let $x_0 = e$.

□

On the one-point compactification $\mathbb{Z}^*$ with e the point at infinity and $T(t) = t + 1$, define the ultra-metric d by

$$(3.3) \qquad d(i,j) \quad = \quad \begin{cases} 0 & \text{if} \quad i = j, \\ max(1/(|i|+1), 1/(|j|+1)) & \text{if} \quad i \neq j, \end{cases}$$

where $1/(|i|+1) = 0$ if $i = e$. If $(X,T) = (\mathbb{Z}^*, T)$ then with $K > 1/\epsilon$ and $N = 2K + 2$ we can use the sequence $\{e, -K, -K+1, \dots, K\}$.

When (X,T) is a nontrivial metric minCT system we define a *preparation* for (X,T) to be a choice for each $i = 0, 1, \dots$ of a sequence $\{e = x^i_0, \dots, x^i_{N_i-1}\}$ which is an 2^{-i} dense sequence of distinct elements of X so that $\{e = x^i_0, \dots, x^i_{N_i-1}, e\}$ is an 2^{-i} chain. For $i = 0$ we let $N_0 = 1$ so that with $i = 0$ the sequence is $\{e\}$.

An *ultrametric minCT system* is a minCT system (X,T) with $d \leq 1$ an ultra-metric on X (and so X is zero-dimensional).

Let $(X_1,T_1), (X_2,T_2)$ be nontrivial ultrametric minCT systems with fixed points e_1, e_2. Assume that (X_2,T_2) is given a preparation.

Let $1 \geq \epsilon > 0$. We will define the ϵ *spin of* (X_2,T_2) *into* (X_1,T_1) to be the ultrametric system (X,T) where X is the closed subset of $X_1 \times X_2$ described below. On $X_1 \times X_2$ we will use the ultrametric $max(\pi_1^* d_1, \epsilon \pi_2^* d_2)$ so that π_1 has Lipschitz constant 1 and for any $\delta > 0$

$$(3.4) \qquad V_\delta(e_1, e_2) \quad \subset \quad \pi_1^{-1}(V_\delta(e_1))$$

with equality if $\delta \geq \epsilon$.

In X_1 we define the sequence of pairwise disjoint clopen sets: $A_0 = X_1 \setminus V_\epsilon(e_1)$ and for $i = 1, 2, \dots, A_i = V_{\epsilon 2^{-i+1}}(e_1) \setminus V_{\epsilon 2^{-i}}(e_1)$. So $X_1 = \{e_1\} \cup \bigcup_{i=0}^\infty A_i$.

(3.5)

$$X = (\{e_1\} \times X_2) \ \cup \ (\bigcup_{i=0}^\infty A_i \times \{ \ x^i_0, \dots, x^i_{N_i-1}\},$$

$$T(x,y) \quad = \quad \begin{cases} (e_1, T_2(y)) & \text{when} \quad x = e_1, \\ (x, x^i_{k+1}) & \text{when} \quad x \in A_i, \quad y = x^i_k \quad \text{with } k < N_i - 1, \\ (T_1(x), e_2) & \text{when} \quad x \in A_i, \quad y = x^i_{N_i-1}. \end{cases}$$

It is obvious that X is a closed subset of $X_1 \times X_2$ and easy to check that T is invertible with $T^{-1}(x,e) = (T_1^{-1}(x), x^i_{N_i-1})$ when $T_1^{-1}(x) \in A_i$.

Continuity of T is clear on each $A_i \times \{ x_0^i, \dots, x_{N_i - 1}^i\}$. Continuity at the points of $\{e_1\} \times X_2$ follows from the following estimate.

Lemma 3.4. *(a) Let $\delta \geq \epsilon$. If $(x,y) \in X$ with $x \in V_\delta(e_1) \cap T_1^{-1}(V_\delta(e_1))$ then $(x,y) \in V_\delta(e_1,e_2) \cap T^{-1}(V_\delta(e_1,e_2))$.*

(b) For $i \geq 1$, let δ with $2^{-i} > \delta > 0$ be a 2^{-i} modulus of uniform continuity for T_2 on X_2. For $(x,y) \in X, \tilde{y} \in X_2$

(3.6)

$$x \in V_{\epsilon 2^{-i+1}}(e_1) \cap T_1^{-1}(V_{2^{-i+1}}(e_1)), \ y \in V_\delta(\tilde{y}) \quad \Longrightarrow \quad T(x,y) \in V_{2^{-i+1}}(e_1, T_2(\tilde{y})).$$

Proof: (a) follows from (3.4) applied to x and to $T_1(x)$.

(b) If $x = e_1$ then $T(x,y) = (e_1, T_2(y))$ and $d(T_2(y), T_2(\tilde{y})) \leq 2^{-i}$ if $y \in V_\delta(\tilde{y})$.

$x \in V_{\epsilon 2^{-i+1}}(e_1) \setminus \{e_1\}$ then the fiber in X over x is a 2^{-i} chain for T_2 and so if $y \in V_\delta(\tilde{y})$ then the second coordinate of $T(x,y)$ is within $2^{-i+1} = 2^{-i} + 2^{-i}$ of $T(\tilde{y})$.

□

So we obtain the ultrametric system (X,T).

From (3.4) it follows that the restriction

(3.7)

$$\begin{aligned} \pi_1 : X \setminus V_\epsilon(e_1,e_2) \ &= \ (\pi_1)^{-1}(X_1 \setminus V_\epsilon(e_1)) \ \to \ X_1 \setminus V_\epsilon(e_1) \quad \text{is bijective,} \\ \text{and on it} \qquad \pi_1 \circ T \ &= \ T_1 \circ \pi_1, \\ x \in X_1 \setminus V_\epsilon(e_1) \quad \Longrightarrow \quad (\pi_1)^{-1}(x) \ &= \ \{(x,e_2)\}, \quad T(x,e_2) \ = \ (T_1(x), e_2). \end{aligned}$$

On the rest of the space the map $\pi_1 : X \to X_1$ does not define an action map, but we obviously have for $x \in X_1 \setminus \{e_1\}$:

(3.8)

$$\begin{aligned} \pi_1^{-1}(\{T_1^i(x) : i = 0,1,\dots\}) \quad &= \quad \{T^i(x,e_2) : i = 0,1,\dots\}, \\ \pi_1^{-1}(\{T_1^{-i}(x) : i = 1,2,\dots\}) \quad &= \quad \{T^{-i}(x,e_2) : i = 1,2,\dots\}. \end{aligned}$$

Proposition 3.5. *If $x \in X_1 \setminus \{e_1\}$, then*

(3.9)

$$\pi_1^{-1}(\omega T_1(x)) \quad = \quad \omega T(x,e_2), \qquad \pi_1^{-1}(\alpha T_1(x)) \quad = \quad \alpha T(x,e_2).$$

If A is a T_1 invariant subset of X_1 then $\pi_1^{-1}(A)$ is a T invariant subset of X.

If B is a T invariant subset of X then $\pi_1(B)$ is a T invariant subset of X_1.

Proof: The equations (3.8) clearly imply that π_1 maps the limit point set $\omega T(x,e_2)$ onto $\omega T_1(x)$. Then they imply that if $z \in X_1 \setminus \{e_1\}$ then all the points of $\pi_1^{-1}(z)$ all lie in the same orbit. Finally, for

$z \in V_{2^{-i}}(e_1) \setminus \{e_1\}$ the set $\pi_2(\pi_1^{-1}(z))$ is $\epsilon 2^{-i}$ dense in X_2. This proves the result for $\omega T(x)$ and the result for $\alpha T(x)$ is similar.

The invariant set results are obvious from (3.8).

□

Corollary 3.6. *(X,T) is a minCT system.*

If (X_1,T_1) and (X_2,T_2) are CT systems then (X,T) is a CT system. Furthermore, if the Birkhoff center sequences for (X_1,T_1) and (X_2,T_2) stabilize at ordinals ω_1 and ω_2 respectively, then the Birkhoff center sequence for (X,T) stabilizes at $\omega_1+\omega_2$.

Proof: If M is a minimal subset of X then by Proposition 3.5 π_1 is a minimal subset of X_1 and so $M \subset \{e_1\} \times X_2$ where T is isomorphic to T_2 and so the $M = \{(e_1,e_2)\}$.

Now assume (X_1,T_1) and (X_2,T_2) are CT systems. If (x,y) is a recurrent point for T then x_1 is a recurrent point for X_1 by (3.9). Hence, $x = e_1$ and y is a recurrent point for T_2. So $y = e_2$. Hence, (X,T) is CT. If A is a closed T_1 invariant subset of X_1 then the limit point set $R_T(\pi_1^{-1}(A))$ is the limit point set $\pi_1^{-1}(R_{T_1}(A))$. So exactly at ω_1 the Birkhoff center sequence for X arrives at $\{e_1\} \times X_2$. It then stabilizes at (e_1,e_2) after ω_2 more steps.

□

Notice that by replacing the metric d on X by the equivalent metric $min(1, max_{n=0}^{\infty} 2^{-n}(T^n)^*d)$ we can assume that the metric is bounded by 1 and that T has Lipschitz constant at most 2. The new metric is an ultrametric if d was.

Let $\{(X_n,T_n) : n = 1,2,\dots\}$ be a sequence of ultrametric minCT systems with ultrametric $d_n \leq 1$ on X_n and with each T_n having Lipschitz constant at most 2. Assume that for $n > 1$ each (X_n,T_n) is given a preparation. Let $(Z_1,U_1) = (X_1,T_1)$, let (Z_2,U_2) be the 2^{-1} spin of (X_2,T_2) into (Z_1,U_1) with $\xi_2 : Z_2 \to Z_1$ be the first coordinate projection. Thus, $Z_2 \subset X_1 \times X_2$. Inductively, let (Z_{n+1},U_{n+1}) be the 2^{-n} spin of (X_{n+1},T_{n+1}) into (Z_n,U_n) which we can regard as a subset of the product $\Pi_n = X_1 \times \cdots \times X_n \times X_{n+1}$ equipped with the ultrametric $max(\pi_1^*d_1, 2^{-1}\pi_2^*d_2,\dots,2^{-n-1}\pi_{n+1}^*d_{n+1})$. Let $\xi_{n+1} : Z_{n+1} \to Z_n$ be the restriction of the coordinate projection from $\Pi_{n+1} \to \Pi_n$ which has Lipschitz constant 1. Note again that the ξ_n's are not action maps, but by (3.7)the restriction $\xi_n : (\xi_n)^{-1}(Z_n \setminus V_{2^{-n}}(e_1,\cdots,e_n)) \to Z_n \setminus V_{2^{-n}}(e_1,\dots,e_n)$ is injective and on it $\xi_n \circ U_{n+1} = U_n \circ \xi_n$.

Let Z_∞ denote the inverse limit, regarded as a closed subset of $\Pi_\infty = \Pi_{i=1}^{\infty} X_i$ equipped with the ultrametric $max\{2^{-i+1}\pi_i^*d_i : i = 1,2,\dots\}$

which yields the product topology. The space Z_∞ consists of the points z such that $\xi_n(z) = (z_1, \dots, z_n) \in Z_n$ for $n = 1, 2, \dots$. We let $e \in Z_\infty$ denote the point $(e_1, e_2, \dots)$.

Assume that $z \in Z_\infty$ with $z \neq e$ and let n be the smallest value such that $x = \xi_n(z) \neq \xi_n(e) = (e_1, \dots, e_n)$. Let $\delta = \frac{1}{2}d(x, (e_1, \dots, e_n))$. Let $i > n$ be such that $2^{-i} \leq \delta$. Since the projections have Lipschitz constant 1, $(\xi_{n+k-1} \circ \dots \circ \xi_n)^{-1}(V_\delta(x))$ is disjoint from $V_\delta(e_1, \dots, e_{n+k})$ for every positive integer k. Once $n + k \geq i$ it follows that $(\xi_{n+k-1} \circ \xi_n)^{-1}(V_\delta(x))$ is disjoint from $V_{2^{-n-k}}(e_1, \dots, e_{n+k})$. By (3.7) if $\tilde{x} \in Z_{n+k} \setminus V_{2^{-n-k}}(e_1, \dots, e_{n+k})$ and $\tilde{z} = (\tilde{x}, e_{n+k+1}, e_{n+k+2}, \dots) \in Z_\infty$ then $\xi_{n+k}^{-1}(\tilde{x}) = \{\tilde{z}\}$ and by (3.7) $U(\tilde{z})$ is unambiguously defined by $U(\tilde{z}) = (U_{n+k}(\tilde{x}), e_{n+k+1}, e_{n+k+2}, \dots)$. Since each U_{n+k} has Lipschitz constant at most 2, it follows that on each $Z_\infty \setminus \xi_{n+k}^{-1}(V_{2^{-n-k}}(e_1, \dots, e_{n+k}))$ U has Lipschitz constant at most 2. Finally,

$$d(U(\tilde{z}), e) = d(U_{n+k}(\tilde{x}), (e_1, \dots, e_{n+k})) \leq 2d(\tilde{x}, (e_1, \dots, e_{n+k})) = 2d(\tilde{z}, e) \tag{3.10}$$

shows that U has Lipschitz constant at most 2 on all of Z_∞.

Finally, with essentially the same proof as that of Corollary 3.6 we have

Corollary 3.7. *(Z_∞, U) is a minCT system.*

If each (X_n, T_n) is a CT system then (Z_∞, U) is a CT system. Furthermore, if the Birkhoff center sequences for (X_n, T_n) stabilize at the ordinals ω_n, then the Birkhoff center sequence for (Z_∞, U) stabilizes at $Lim_{n\to\infty}\, \omega_1 + \omega_2 + \dots + \omega_n$.

□

4. The space of labels

Let $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ denote the sets of integers, of non-negative integers and of positive integers, respectively. Let $\mathbb{Z}_{+\infty} = \mathbb{Z}_+ \cup \{\infty\} = \mathbb{N} \cup \{0, \infty\}$. On the vector space $\mathbb{R}^\mathbb{N}$ we will use the lattice structure, with $x \geq y, x \leq y, x \vee y, x \wedge y$, the pointwise relations and the pointwise operations of maximum and minimum for vectors $x, y \in \mathbb{R}^\mathbb{N}$. As usual $x > y$ means $x \geq y$ and $x \neq y$ so that the inequality is strict for at least one coordinate. The support of a vector $x \in \mathbb{R}^\mathbb{N}$, denoted *supp* x, is $\{\ell : x_\ell \neq 0\}$.

We will call $\mathbf{m} \in \mathbb{Z}^\mathbb{N}$ an $\mathbb{N}$-vector when it is non-negative and has finite support, that is, when $\mathbf{m} \geq 0$ for all $\ell \in \mathbb{N}$ and *supp* $\mathbf{m} = \{\ell :$

$\mathbf{m}_\ell > 0\}$ is finite. We call $\# supp\ \mathbf{m}$ the *size* of $\mathbf{m}$ and call $|\mathbf{m}| = \Sigma_\ell\ \mathbf{m}_\ell$ the *norm* of $\mathbf{m}$.

If $S \subset \mathbb{N}$ we let $\chi(S)$ be the characteristic function of S with $\chi(\ell) = \chi(\{\ell\})$. Thus, $\chi(S) = \Sigma_{\ell \in S}\ \chi(\ell)$ is an $\mathbb{N}$-vector when S is a finite set.

We denote by $FIN(\mathbb{N})$ the discrete abelian monoid of all $\mathbb{N}$-vectors with identity $\mathbf{0}$. It is also a lattice via the pointwise order relations described above.

For an $\mathbb{N}$-vector $\mathbf{m}$ and a positive integer ℓ^* we define $\mathbf{m} \wedge [1, \ell^*]$ to be the $\mathbb{N}$-vector with

$$(\mathbf{m} \wedge [1, \ell^*])_\ell \quad = \quad \begin{cases} \mathbf{m}_\ell & \text{for} \quad \ell \ \leq \ \ell^*, \\ 0 & \text{for} \quad \ell \ > \ \ell^*. \end{cases} \tag{4.1}$$

Definition 4.1. A set $\mathcal{M}$ of $\mathbb{N}$-vectors is called a *label* when it satisfies the following

[Heredity Condition] $0 \ \leq \ \mathbf{m}^1 \ \leq \ \mathbf{m}$ and $\mathbf{m} \in \mathcal{M}$ imply $\mathbf{m}^1 \in \mathcal{M}$.

Definition 4.2. (a) For $S \subset \mathbb{Z}_+^{\mathbb{N}}$, we define $\langle S \rangle \ = \ \{\mathbf{m} \in FIN(\mathbb{N}) : \mathbf{m} \leq \nu \ \text{ for some } \nu \in S\}$. We call $\langle S \rangle$ the *label generated by* S. We will write $\langle \nu \rangle$ for $\langle S \rangle$ when $S = \{\nu\}$.

(b) For $\rho : \mathbb{N} \to \mathbb{Z}_{+\infty}$ let $\langle \rho \rangle \ = \ \{\mathbf{m} \in FIN(\mathbb{N}) : \mathbf{m} \leq \rho\}$. In particular, $\langle \nu \rangle = \langle S \rangle$ when $S = \{\nu\}$.

Definition 4.3. Given $N \in \mathbb{Z}_+$, let $\mathcal{B}_N = \langle N\chi([1, N]) \rangle$. That is, $\mathbf{m} \in \mathcal{B}_N$ iff $\mathbf{m} \leq N$ and $supp\ \mathbf{m} \subset [1, N]$.

In particular, $\mathcal{B}_0 = \emptyset$. Thus, $\{\mathcal{B}_N\}$ is an increasing sequence of finite labels with union $FIN(\mathbb{N})$, the maximum label.

Definition 4.4. A label $\mathcal{M}$ is *bounded* if it satisfies the following

[Bound Condition] There exists $\mu \in \mathbb{Z}_+^{\mathbb{N}}$ such that $0 \ \leq \ \mathbf{m} \ \leq \ \mu$ for all $\mathbf{m} \in \mathcal{M}$.

A label $\mathcal{M}$ is *of finite type* if it satisfies the following

[Finite Chain Condition] There does not exist an infinite increasing sequence in $\mathcal{M}$, or equivalently, any infinite nondecreasing sequence in $\mathcal{M}$ is eventually constant.

A label $\mathcal{M}$ is *size bounded* if it satisfies the following [Size Bound Condition] There exists $n \in \mathbb{N}$ such that $size(\mathbf{m}) \leq n$ for all $\mathbf{m} \in \mathcal{M}$.

Clearly, a finite label is of finite type.

For example, $\emptyset$ and $0 = \{\mathbf{0}\}$ are finite labels. $\mathcal{M} \neq \emptyset$ iff $\mathbf{0} \in \mathcal{M}$. We call $\mathcal{M}$ a *positive label* when it is neither empty nor 0.

Define the *roof* $\rho(\mathcal{M}) : \mathbb{N} \to \mathbb{Z}_{+\infty}$ of a label $\mathcal{M}$ by

$$\rho(\mathcal{M})_\ell = sup_{\mathbf{m} \in \mathcal{M}} \ \{\mathbf{m}_\ell\} = sup\{r \in \mathbb{Z}_+ \ : \ r\chi(\ell) \in \mathcal{M} \ \} \leq \infty$$

Thus, $\mathcal{M}$ is bounded iff $\rho(\mathcal{M})_\ell < \infty$ for all ℓ in which case the roof is the minimum of the functions $\mu \in \mathbb{Z}_+^{\mathbb{N}}$ which bound the elements of $\mathcal{M}$. Clearly, $\rho(0) = 0$ and by convention we let $\rho(\emptyset) = 0$ as well.

We let $Supp\ \mathcal{M} = \{ \ supp\ \mathbf{m} \ : \ \mathbf{m} \in \mathcal{M} \ \}$.

Lemma 4.5. *If a label* $\mathcal{M}$ *is of finite type then it is bounded. If a label is bounded and size bounded then it is of finite type.*

Proof: If $\rho(\mathcal{M})_\ell = \infty$ then $\{i\chi(\ell) : i \in \mathbb{N}\}$ is an infinite increasing sequence in $\mathcal{M}$.

Now assume that $\mathcal{M}$ is bounded by $\mu \in \mathbb{Z}_+^{\mathbb{N}}$. If $\mathbf{m}^1 < \mathbf{m}^2 < \cdots$ is an infinite sequence of $\mathbb{N}$-vectors then at each step either some already positive entry increases or the size increases. Since the entries in $\mathcal{M}$ are bounded by μ and the size is assumed bounded the sequence must eventually leave $\mathcal{M}$. Hence, the Finite Chain Condition holds.

□

For a label $\mathcal{M}$ and $\ell^* \in \mathbb{N}$ we define

$$\mathcal{M} \wedge [1, \ell^*] \quad = \quad \begin{cases} \emptyset & \text{when } \ell^* = 0, \\ \{ \ \mathbf{m} \wedge [1, \ell^*] \ : \ \mathbf{m} \in \mathcal{M} \ \} & \text{when } \ell^* > 0. \end{cases} \tag{4.2}$$

Thus, $\mathcal{M} \wedge [1, \ell^*] \ = \ \{ \ \mathbf{m} \in \mathcal{M} \ : \ supp\ \mathbf{m} \subset [1, \ell^*] \ \}$.

For a label $\mathcal{M}$ and an $\mathbb{N}$-vector $\mathbf{r}$ we define

$$\mathcal{M} - \mathbf{r} \quad = \quad \{ \ \mathbf{w} \in FIN(\mathbb{N}) \ : \ \ \mathbf{w} + \mathbf{r} \in \mathcal{M} \ \}. \tag{4.3}$$

Thus, $\mathcal{M} - \mathbf{r}$ is the set of all non-negative vectors of the form $\mathbf{m} - \mathbf{r}$ for $\mathbf{m} \in \mathcal{M}$. Clearly, $(\mathcal{M} - \mathbf{r}) - \mathbf{s} = \mathcal{M} - (\mathbf{r} + \mathbf{s}) = (\mathcal{M} - \mathbf{s}) - \mathbf{r}$ (and so we can omit the parentheses) since each is the set of $\mathbb{N}$-vectors $\mathbf{w}$ such that $\mathbf{w} + \mathbf{r} + \mathbf{s} \in \mathcal{M}$.

Let $max\ \mathcal{M}$ be the set of $\mathbb{N}$-vectors which are maximal in $\mathcal{M}$. That is, $\mathbf{n} \in max\ \mathcal{M}$ if

$$\mathbf{m} \geq \mathbf{n} \text{ and } \mathbf{m} \in \mathcal{M} \qquad \Longleftrightarrow \qquad \mathbf{m} = \mathbf{n}. \tag{4.4}$$

Definition 4.6. We will say that $Supp\ \mathcal{M}$ *f-contains* a set $L \subset \mathbb{N}$ when every finite subset of L is a member of $Supp\ \mathcal{M}$. That is, $\mathcal{P}_f L \subset Supp\ \mathcal{M}$ where $\mathcal{P}_f L$ is the set of finite subsets of L. Equivalently, $\mathcal{M} \supset \langle \chi(L) \rangle$.

Proposition 4.7. *Let $\mathcal{M}$ be a label and let $\mathbf{r} > \mathbf{0}$ be an $\mathbb{N}$-vector.*

(a) *If $\mathcal{M}$ is of finite type then for every $\mathbf{m} \in \mathcal{M}$ there exists $\mathbf{n} \in max\mathcal{M}$ such that $\mathbf{m} \leq \mathbf{n}$. Hence, $\mathcal{M} = \langle max\ \mathcal{M} \rangle$. Thus, a label $\mathcal{M}$ of finite type is determined by max $\mathcal{M}$. Also, every $A \in$ Supp $\mathcal{M}$ is contained in some $B \in$ Supp $\mathcal{M}$ which is maximal with respect to inclusion in Supp $\mathcal{M}$.*

(b) *If a label $\mathcal{M}$ f-contains some infinite set $L \subset \mathbb{N}$ then $\mathcal{M}$ is not of finite type. If $\mathcal{M}$ is bounded but not of finite type then it f-contains some infinite set L. Thus a bounded label is of finite type iff it does not f-contain an infinite set.*

(c) *If $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{M}_1$ satisfies the Heredity Condition then $\mathcal{M}_1$ is a label and is bounded/of finite type/size bounded/finite if $\mathcal{M}$ satisfies the corresponding condition.*

(d) *If $\mathcal{M}$ is a bounded label and $\ell^* \in \mathbb{N}$ then $\mathcal{M} \wedge [1, \ell^*]$ is a finite label.*

(e) *$\mathcal{M} - \mathbf{r}$ is a label contained in $\mathcal{M}$ with max $\mathcal{M} \subset \mathcal{M} \setminus (\mathcal{M} - \mathbf{r})$. If $\mathcal{M}$ is nonempty and bounded then $\mathcal{M} - \mathbf{r}$ is a proper subset of $\mathcal{M}$.*

(f) *$\mathcal{M} - \mathbf{r} \neq \emptyset$ iff $\mathbf{r} \in \mathcal{M}$.*

(g) *If Φ is a set of labels then $\bigcup \Phi$ and $\bigcap \Phi$ are labels.*

(h) *If Φ is a finite set of labels then $\bigcup \Phi$ is of finite type iff all of the labels in Φ are.*

Proof: (a): If $\mathbf{m}$ is not maximal then there exists $\mathbf{m}_1 \in \mathcal{M}$ with $\mathbf{m}_1 > \mathbf{m}$. Continue if $\mathbf{m}_1$ is not maximal. This sequence can continue only finitely many steps by the Finite Chain Condition. It terminates at a maximal vector $\mathbf{n}$. Similarly, if $A \in$ *Supp* $\mathcal{M}$ is not contained in a maximal element then there is an increasing sequence in $\{A_0, A_1, \dots\}$ in *Supp* $\mathcal{M}$ with $A = A_0$. Then $\mathbf{m}^k = \chi(A_k)$ is a strictly increasing sequence in $\mathcal{M}$.

(b) If $\mathcal{M}$ f-contains $L = \{\ell_1, \ell_2, \dots\}$ then $\mathbf{n}^k = \Sigma_{i=1}^k \chi(\ell_i)$ is a strictly increasing infinite sequence in $\mathcal{M}$ and so $\mathcal{M}$ is not of finite type. Conversely, that if $\mathbf{n}^k \in \mathcal{M}$ is an infinite increasing sequence in $\mathcal{M}$ then $\mathcal{M}$ f-contains the union L of the increasing sequence $\{supp\ \mathbf{n}^k\}$ of finite sets. If $\mathcal{M}$ is bounded then L must be an infinite set.

(c): Obvious.

(d): For any $\mu \in \mathbb{Z}_+^{\mathbb{N}}$ there are only finitely many $\mathbf{m} \in \mathbb{Z}_+^{\mathbb{N}}$ such that $\mathbf{m} \leq \mu$ and *supp* $\mathbf{m} \subset [1, \ell^*]$.

(e) $\mathcal{M} - \mathbf{r}$ satisfies the Heredity Condition and so is a label contained in $\mathcal{M}$. If $\mathbf{m} \in \mathcal{M} - \mathbf{r}$ then $\mathbf{m} + \mathbf{r}$ is an element of $\mathcal{M}$ with $\mathbf{m} + \mathbf{r} > \mathbf{m}$ since $\mathbf{r} > \mathbf{0}$. Hence, $\mathbf{m}$ is not a maximal element of $\mathcal{M}$.

Now assume $\mathcal{M}$ is nonempty and bounded. Then $\mathbf{0} = 0\mathbf{r} \in \mathcal{M}$. If $\mathbf{r}_\ell > 0$ then for $n \in \mathbb{N}$ such that $n > \rho(\mathcal{M})_\ell / \mathbf{r}_\ell$, $n\mathbf{r} \notin \mathcal{M}$. So there is a maximum $n \geq 0$ such that $n\mathbf{r} \in \mathcal{M}$. It follows that $n\mathbf{r} \in \mathcal{M} \setminus (\mathcal{M} - \mathbf{r})$.

(f) If $\mathbf{r} \in \mathcal{M}$ then $0 \in \mathcal{M} - \mathbf{r}$. If $\mathbf{m} \in \mathcal{M} - \mathbf{r}$ then $\mathbf{m} + \mathbf{r} \in \mathcal{M}$ and so $\mathbf{m} + \mathbf{r} \geq \mathbf{r}$ implies $\mathbf{r} \in \mathcal{M}$.

(h) If Φ is finite collection of labels of finite type and $\{\mathbf{m}^i\}$ is a strictly increasing sequence of N-vectors then it can remain in each member of Φ for only finitely many terms. As Φ is finite, the sequence must eventually leave $\bigcup \Phi$. Hence, the union satisfies the Finite Chain Condition.

If the union is of finite type then each member of Φ is of finite type by (b).

□

Example 4.8. A label $\mathcal{M}$ which is generated by *max* $\mathcal{M}$ need not be of finite type.

(a) If $\mathcal{M} = \langle \{ \ 2\chi(k+1) + \Sigma_{\ell=1}^k \ \chi(\ell) : k \in \mathbb{N} \} \rangle$ then $\mathcal{M}$ is not of finite type although every $\mathbf{m} \in \mathcal{M}$ is bounded by an element of *max* $\mathcal{M}$.

(b) Let $\mathcal{M} = \langle \{ \ \chi(2k+1) + \Sigma_{\ell=1}^k \ \chi(2\ell) \ : \ k \in \mathbb{N} \} \rangle$. Clearly $\mathcal{M}$ f-contains L the infinite set of even numbers, but every $A \in$ *Supp* $\mathcal{M}$ is contained in some $B \in$ *Supp* $\mathcal{M}$ which is maximal with respect to inclusion in *Supp* $\mathcal{M}$.

□

A set Φ (or a sequence $\{\mathcal{M}^i\}$) of labels is said to be *uniformly bounded* when the union is a bounded label, or, equivalently, there exists a bounded label $\mathcal{N}$ such that $\mathcal{M} \subset \mathcal{N}$ for all $\mathcal{M} \in \Phi$ (resp. $\mathcal{M} = \mathcal{M}^i$ for all i). In that case we will call $\mathcal{N}$ a *bound* for the set or sequence or say that the set or sequence is bounded by $\mathcal{N}$.

For any label $\mathcal{N}$ the set $[[\mathcal{N}]]$ of all labels which are contained in $\mathcal{N}$ is a uniformly bounded set of labels if $\mathcal{N}$ is bounded. Thus a set or sequence is uniformly bounded iff it is contained in $[[\mathcal{N}]]$ for some bounded label $\mathcal{N}$.

If $\mathcal{N}$ is of finite type then all the members of $[[\mathcal{N}]]$ are labels of finite type.

We denote by $\mathcal{LAB}$ the space of labels. On $\mathcal{LAB}$ we define an ultrametric by

$$(4.5) \quad d(\mathcal{M}_1, \mathcal{M}_2) \ = \ inf \ \{ \ 2^{-N} \ : \ N \in Z_+ \text{ and } \mathcal{M}_1 \cap \mathcal{B}_N = \mathcal{M}_2 \cap \mathcal{B}_N \ \}.$$

Notice that since $\mathcal{B}_0 = \emptyset$, $\mathcal{M}_1 \cap \mathcal{B}_0 = \mathcal{M}_2 \cap \mathcal{B}_0$ is always true.

Lemma 4.9. *(a)* $d(\mathcal{M}_1, \mathcal{M}_2) = 0$ *iff* $\mathcal{M}_1 = \mathcal{M}_2$.

(b) The label $\emptyset$ *is an isolated point of* $\mathcal{LAB}$ *with* $d(\emptyset, \mathcal{M}) = 1$ *for all* $\mathcal{M} \neq \emptyset$.

(c) If $\mathcal{N}_1 \subset \mathcal{N}$ *are finite labels and* $\mathbf{m} \in FIN(\mathbb{N})$ *then each of the following is a clopen subset of* $\mathcal{LAB}$*:*

$$\{\mathcal{M} : \mathcal{M} \cap \mathcal{N} = \mathcal{N}_1\}, \quad \{\mathcal{M} : \mathcal{M} \cap \mathcal{N} = 0\}, \quad \{\mathcal{M} : \mathcal{N} \subset \mathcal{M}\},$$
$$\{\mathcal{M} : \mathbf{m} \in \mathcal{M}\}, \quad \{\mathcal{M} : \mathbf{m} \not\in \mathcal{M}\}.$$

The set $\{(\mathcal{M}, \mathcal{M}_1) : \mathcal{M} \cap \mathcal{N} = \mathcal{M}_1 \cap \mathcal{N}\}$ *is a clopen subset of* $\mathcal{LAB} \times \mathcal{LAB}$.

(d) For any label $\mathcal{M}$ *the set* $[[\mathcal{M}]]$ *of labels contained in* $\mathcal{M}$ *is a closed subset of* $\mathcal{LAB}$. *The set*

$$INC = \{(\mathcal{M}_1, \mathcal{M}_2) : \mathcal{M}_1 \subset \mathcal{M}_2\}$$

is a closed subset of $\mathcal{LAB} \times \mathcal{LAB}$.

(e) The set of finite labels is a countable dense subset of $\mathcal{LAB}$.

(f) The set of bounded labels is a dense G_δ *subset of* $\mathcal{LAB}$.

Proof: (a) Every $\mathbf{m} \in \mathcal{B}_N$ for some N.

(b) If $\mathcal{M} \neq \emptyset$ then $\mathbf{0} \in \mathcal{M} \cap \mathcal{N}_1$.

(c) If $\mathcal{N} \subset \mathcal{B}_N$ then the 2^{-N} ball around $\mathcal{M}$ is either contained in or disjoint from $\{\mathcal{M} : \mathcal{M} \cap \mathcal{N} = \mathcal{N}_1\}$. So $\{\mathcal{M} : \mathcal{M} \cap \mathcal{N} = \mathcal{N}_1\}$ is clopen. With $\mathcal{N}_1 = 0$ or $\mathcal{N}$ these become $\{\mathcal{M} : \mathcal{M} \cap \mathcal{N} = 0\}$. and $\{\mathcal{M} : \mathcal{N} \subset \mathcal{M}\}$. Finally, let $\mathcal{N} = \langle \mathbf{m} \rangle$.

The set of pairs such that $\mathcal{M} \cap \mathcal{N} = \mathcal{M}_1 \cap \mathcal{N}$ is the union of the set of pairs such that $\mathcal{M} \cap \mathcal{N} = \mathcal{N}_1 = \mathcal{M}_1 \cap \mathcal{N}$ taken of the finite set of labels $\mathcal{N}_1 \subset \mathcal{N}$.

(d) The complement of $[[\mathcal{M}]]$ is the union of $\{\mathcal{M}_1 : \mathbf{m} \in \mathcal{M}_1\}$ as $\mathbf{m}$ varies over $FIN(\mathbb{N}) \setminus \mathcal{M}$. The complement of INC is the union of $\{\mathcal{M}_1 : \mathbf{m} \in \mathcal{M}_1\} \times \{\mathcal{M}_2 : \mathbf{m} \not\in \mathcal{M}_2\}$ as $\mathbf{m}$ varies over $FIN(\mathbb{N})$.

(e) $\mathcal{M} \cap \mathcal{B}_N$ is a finite label in the 2^{-N} ball about $\mathcal{M}$. The set of finite labels is countable since $FIN(\mathbb{N})$ is countable.

(f) For each $\ell, \{\mathcal{M} : \rho(\mathcal{M})_\ell = \infty\} = \bigcap_k \{\mathcal{M} : k\chi(\ell) \in \mathcal{M}\}$ is a closed set. So the set of bounded labels is G_δ. It is dense because it contains the set of finite labels.

□

Let $\mathcal{M}^i$ be a sequence of labels. Define the labels

$$\begin{aligned} LIMSUP \; \{\mathcal{M}^i\} \quad &= \quad \bigcap_i [\bigcup_{j \geq i} \{\mathcal{M}^j\} \;], \\ LIMINF \; \{\mathcal{M}^i\} \quad &= \quad \bigcup_i [\bigcap_{j \geq i} \{\mathcal{M}^j\} \;]. \end{aligned} \tag{4.6}$$

Clearly, $\mathbf{m} \in LIMSUP$ iff frequently $\mathbf{m} \in \mathcal{M}^i$ and $\mathbf{m} \in LIMINF$ iff eventually $\mathbf{m} \in \mathcal{M}^i$ and so $LIMINF \subset LIMSUP$.

As usual, if we go to a subsequence $\{\mathcal{M}^{i'}\}$ with $LIMSUP'$ and $LIMINF'$ then $LIMINF \subset LIMINF' \subset LIMSUP' \subset LIMSUP$.

Both $LIMSUP$ and $LIMINF$ are bounded labels if $\bigcup_i \{\mathcal{M}^i\}$ satisfies the Bound Condition. That is, if the sequence is bounded. Conversely, if $LIMSUP$ is a bounded label with roof ρ then for every ℓ there exists i_ℓ such that $(\rho(\ell)+1)\chi(\ell) \not\in \bigcup_{j \geq i_\ell} \{\mathcal{M}^j\}$. Hence, $\{\rho(\mathcal{M}^i)(\ell)\}$ is bounded for each ℓ. That is, $\bigcup_i \{\mathcal{M}^i\}$ is bounded. Thus, $LIMSUP$ is a label iff $\bigcup_i \{\mathcal{M}^i\}$ satisfies the Bound Condition, i. e. iff the sequence is uniformly bounded.

If $\mathcal{M}^i \subset \mathcal{M}$ for all i then both $LIMINF \subset LIMSUP \subset \mathcal{M}$ and both $LIMSUP$ and $LIMINF$ are labels of finite type if $\mathcal{M}$ is of finite type.

Proposition 4.10. *Let $\{\mathcal{M}^i\}$ be a sequence of labels.*

(a) *The following are equivalent.*
   (1) *The sequence $\{\mathcal{M}^i\}$ is a Cauchy sequence.*
   (2) *For every finite label $\mathcal{N}$, the sequence $\{\mathcal{M}^i \cap \mathcal{N}\}$ of finite labels is eventually constant.*
   (3) *$LIMSUP = LIMINF$.*

   The common value $LIMSUP = LIMINF$ is then the limit, and is then denoted $LIM\ \{\mathcal{M}^i\}$.

(b) *If $M^i \subset M^{i+1}$ then $LIM\{\mathcal{M}^i\} = \bigcup\{\mathcal{M}^i\}$. If $M^i \supset M^{i+1}$ then $LIM\{\mathcal{M}^i\} = \bigcap\{\mathcal{M}^i\}$.*

Proof: (a) (1) $\Leftrightarrow$ (2): Since $\mathcal{N} \subset \mathcal{B}_N$ for some N this is obvious from the definition of the ultrametric.

(2) $\Rightarrow$ (3): If $\mathbf{m} \in \mathcal{B}_N$ then since $\mathcal{M}^i \cap \mathcal{B}_N$ is eventually constant, either eventually $\mathbf{m} \in \mathcal{M}^i$ or eventually $\mathbf{m} \not\in \mathcal{M}^i$. This means $LIMSUP = LIMINF$.

(3) $\Rightarrow$ (2): Assume that $\mathcal{M} = LIMSUP = LIMINF$. Let $\mathcal{N}_1 = \mathcal{M} \cap \mathcal{N}$. If $\mathbf{m} \in \mathcal{N}_1$ then eventually $\mathbf{m} \in \mathcal{M}^i$ and if $\mathbf{m} \in \mathcal{N} \setminus \mathcal{N}_1$ then eventually $\mathbf{m} \not\in \mathcal{M}^i$. Since $\mathcal{N}$ is a finite set it follows that eventually $\mathcal{M}^i \cap \mathcal{N} = \mathcal{N}_1$.

(b): For an increasing sequence the $LIMSUP = LIMINF$ is the union and for a decreasing sequence $LIMSUP = LIMINF$ is the intersection.

□

Proposition 4.11. *Let $\{\mathcal{M}^i\}$ be a sequence of labels.*

$\mathbf{m} \in LIMSUP\{\mathcal{M}^i\}$ *iff there is a subsequence* $\{\mathcal{M}^{i'}\}$ *which is convergent with* $\mathbf{m} \in LIM\ \{\mathcal{M}^{i'}\}$.

$\mathbf{m} \notin LIMINF\{\mathcal{M}^i\}$ *iff there is a subsequence* $\{\mathcal{M}^{i'}\}$ *which is convergent with* $\mathbf{m} \notin LIM\ \{\mathcal{M}^{i'}\}$.

Proof: The $LIMSUP$ of a subsequence is contained in the $LIMSUP$ of the original sequence and the $LIMINF$ of a subsequence contains the $LIMINF$ of the original sequence Thus, sufficiency is clear in each case.

Let $\{\mathbf{m}_1, \mathbf{m}_2, \dots\}$ be a numbering of the countable set $FIN(\mathbb{N})$ with $\mathbf{m}_1 = \mathbf{m}$. Since $\mathbf{m} \in \mathcal{M}^i$ frequently, we can choose SEQ_1 an infinite subset of $\mathbb{N}$ so that $\mathbf{m}_1 \in \mathcal{M}^i$ for all $i \in SEQ_1$. If eventually $\mathbf{m}_2 \in \mathcal{M}^i$ for $i \in SEQ_1$ let these values of i define $SEQ_2 \subset SEQ_1$. Otherwise, let SEQ_2 be the $i \in SEQ_1$ such that $\mathbf{m}_2 \notin \mathcal{M}^i$. Inductively we define a decreasing sequence SEQ_n of infinite subsets of $\mathbb{N}$ such that $p \leq n$ implies either $\mathbf{m}_p \in \mathcal{M}^i$ for all $i \in SEQ_n$ or for no $i \in SEQ_n$. Diagonalizing, we obtain a convergent subsequence whose limit contains $\mathbf{m}$. That is, if i_n be the n^{th} element of SEQ_n, then $\{M^{i_n}\}$ is convergent and the limit contains $\mathbf{m}$.

Alternatively, if $\mathbf{m} \notin LIMINF$ we begin by choosing SEQ_1 so that $\mathbf{m}_1 \notin \mathcal{M}^i$ for all $i \in SEQ_1$ and continue as before.

□

Corollary 4.12. *$\mathcal{LAB}$ is a compact, separable, zero-dimensional metric space with $\emptyset$ the only isolated point.*

Proof: $\mathcal{LAB}$ is compact by Proposition 4.11. It is separable because the countable set of finite labels is dense. It is zero-dimensional because it has an ultrametric. If $\mathcal{N}$ is a finite, nonempty label then $\mathcal{N} \cup \{\chi(\ell)\}$ is a sequence of finite labels which converges to $\mathcal{N}$ as $\ell \to \infty$. Hence, no nonempty label is isolated.

□

Lemma 4.13. *Let Φ be a compact subset of $\mathcal{LAB}$. If $\{\mathbf{m}^i : i \in \mathbb{N}\}$ is a nondecreasing sequence in $\bigcup\ \Phi$ then there exists $\mathcal{M} \in \Phi$ such that $\mathbf{m}^i \in \mathcal{M}$ for all i.*

Proof: Assume $\mathbf{m}^i \in \mathcal{M}^i \in \Phi$ for all i. By compactness some subsequence $\{\mathcal{M}^{i'}\}$ converges to $\mathcal{M} \in \Phi$. For each k, $i' > k$ implies $\mathbf{m}^k \leq \mathbf{m}^{i'} \in \mathcal{M}^{i'}$. That is, each $\mathbf{m}^k$ is eventually in $\mathcal{M}^{i'}$ as $i' \to \infty$. Hence, $\mathbf{m}^k \in \mathcal{M}$ for all k.

□

Proposition 4.14. *Let $\mathcal{L} \subset \mathcal{LAB}$ be either the set of bounded labels or the set of labels of finite type. A subset $\Phi \subset \mathcal{L}$ is compact iff Φ is closed in the relative topology of $\mathcal{L}$ and $\bigcup \ \Phi \in \mathcal{L}$. In particular, a compact set of bounded labels is uniformly bounded.*

Proof: If $\bigcup \ \Phi \in \mathcal{L}$ then the compact set $[[\bigcup \ \Phi]]$ is contained in $\mathcal{L}$ and since $\Phi \subset [[\bigcup \ \Phi]]$ is closed relative to L, it is closed relative to $[[\bigcup \ \Phi]]$ and so is itself compact.

Now assume that Φ is compact.

If $\bigcup \ \Phi$ is not bounded then for some $\ell \in \mathbb{N}$ the strictly increasing sequence $\{\mathbf{m}^i = i\chi(\ell)\}$ is in $\bigcup \ \Phi$ and so Lemma 4.13 implies that the sequence is contained in some $\mathcal{M} \in \Phi$ and so $\mathcal{M}$ is unbounded.

Similarly, if $\bigcup \ \Phi$ is not of finite type then there exists a strictly increasing sequence $\{\mathbf{m}^i\}$ in $\bigcup \ \Phi$ and so again Lemma 4.13 implies that the sequence is contained in some $\mathcal{M} \in \Phi$ and so $\mathcal{M}$ is not of finite type.

In each case the contrapositive says that $\Phi \subset \mathcal{L}$ implies $\bigcup \ \Phi \in \mathcal{L}$.
$\square$

Given an $\mathbb{N}$-vector $\mathbf{r}$ we define the map $P_{\mathbf{r}}$ on $\mathcal{LAB}$, by $P_{\mathbf{r}}(\mathcal{M}) = \mathcal{M} - \mathbf{r}$.

Proposition 4.15. *The function $P_{\mathbf{r}}$ is continuous. In particular, if $\{\mathcal{M}^i\}$ is a convergent sequence of labels then $\{\mathcal{M}^i - \mathbf{r}\}$ is convergent with $LIM\{\mathcal{M}^i - \mathbf{r}\} = LIM\{\mathcal{M}^i\} - \mathbf{r}$.*

Proof: Let $N(\mathbf{r})$ be the minimum value such that $\mathbf{r} \in \mathcal{B}_{N(\mathbf{r})}$. Notice that $N(\mathbf{m}+\mathbf{r}) \leq N(\mathbf{m}) + N(\mathbf{r})$ because $\mathbf{m}+\mathbf{r} \leq N(\mathbf{m}) + N(\mathbf{r})$ and $supp\ \mathbf{m}+\mathbf{r} = (supp\ \mathbf{m}) \cup (supp\ \mathbf{r}) \subset [1, max(N(\mathbf{m}), N(\mathbf{r}))]$.

It follows that for labels $\mathcal{M}_1, \mathcal{M}_2$

$$\mathcal{M}_1 \cap \mathcal{B}_{N+N(\mathbf{r})} \ = \ \mathcal{M}_2 \cap \mathcal{B}_{N+N(\mathbf{r})} \quad \Longrightarrow \quad (\mathcal{M}_1 - \mathbf{r}) \cap \mathcal{B}_N \ = \ (\mathcal{M}_2 - \mathbf{r}) \cap \mathcal{B}_N. \tag{4.7}$$

For if $\mathbf{m} \in (\mathcal{M}_1 - \mathbf{r}) \cap \mathcal{B}_N$ then $\mathbf{m}+\mathbf{r} \in \mathcal{M}_1 \cap \mathcal{B}_{N+N(\mathbf{r})} = \mathcal{M}_2 \cap \mathcal{B}_{N+N(\mathbf{r})}$. Hence, $\mathbf{m} \in \mathcal{M}_2 - \mathbf{r}$. Since $\mathbf{m} \in \mathcal{B}_N$, it follows that $\mathbf{m} \in (\mathcal{M}_2 - \mathbf{r}) \cap \mathcal{B}_N$. Symmetrically for reverse inclusion.

From (4.7) it follows that $d(\mathcal{M}_1, \mathcal{M}_2) < 2^{-N-N(\mathbf{r})}$ implies $d(P_{\mathbf{r}}\mathcal{M}_1, P_{\mathbf{r}}\mathcal{M}_2) < 2^{-N}$. This shows that $P_{\mathbf{r}}$ is Lipschitz with Lipschitz constant at most $2^{N(\mathbf{r})}$.
$\square$

Corollary 4.16. *The map* $P : FIN(\mathbb{N}) \times \mathcal{LAB} \to \mathcal{LAB}$ *given by* $(\mathbf{r}, \mathcal{M}) = \mathcal{M} - \mathbf{r} = P_{\mathbf{r}}(\mathcal{M})$ *is a continuous monoid action of* $FIN(\mathbb{N})$ *on* $\mathcal{LAB}$.

The action is faithful *i.e. if* $P_{\mathbf{r}}(\mathcal{M}) = P_{\mathbf{s}}(\mathcal{M})$ *for all* $\mathcal{M}$ *then* $\mathbf{r} = \mathbf{s}$.

Proof: It is an action since $(\mathcal{M} - \mathbf{r}_1) - \mathbf{r}_2 = \mathcal{M} - (\mathbf{r}_1 + \mathbf{r}_2)$ for $\mathbb{N}$-vectors $\mathbf{r}_1, \mathbf{r}_2$ and $\mathcal{M} - \mathbf{0} = \mathcal{M}$. It is a continuous action by Proposition 4.15.

For a label $\mathbf{r}$ let $\mathcal{M}$ be the finite label $\langle \mathbf{r} \rangle$. Since $\{\mathbf{r}\} = max\ \mathcal{M}$, $P_{\mathbf{r}}(\mathcal{M}) = 0$. If $P_{\mathbf{s}}(\mathcal{M}) = 0$ then $\mathbf{s} \in \mathcal{M}$ and so $\mathbf{s} \leq \mathbf{r}$. Clearly, $\mathbf{r} - \mathbf{s} \in P_{\mathbf{s}}(\mathcal{M})$ and so $\mathbf{r} - \mathbf{s} = \mathbf{0}$.

□

Notice that $FIN(\mathbb{N})$ is the free abelian monoid generated by $\{ \chi(\ell) : \ell \in \mathbb{N}\}$ and it is a submonoid of the free abelian group consisting of the members of $\mathbb{Z}^{\mathbb{N}}$ with finite support. In particular, it is a cancellation semigroup: $\mathbf{r}_1 + \mathbf{s} = \mathbf{r}_2 + \mathbf{s}$ implies $\mathbf{r}_1 = \mathbf{r}_2$. In particular, $\mathbf{r} + \mathbf{r} = \mathbf{r}$ only when $\mathbf{r} = \mathbf{0}$. Also, $\mathbf{r} + \mathbf{s} = \mathbf{0}$ iff $\mathbf{r} = \mathbf{s} = \mathbf{0}$.

Giving $FIN(\mathbb{N})$ the discrete topology, we obtain on the Stone-Čech compactification $\beta FIN(\mathbb{N})$ the structure of an Ellis semigroup with product which extends the addition on $FIN(\mathbb{N})$ and is such that $Q \mapsto QR$ is continuous for any $R \in \beta FIN(\mathbb{N})$. Let $\beta^* FIN(\mathbb{N}) = \beta FIN(\mathbb{N}) \setminus \{0\}$ and $\beta^{**} FIN(\mathbb{N}) = \beta FIN(\mathbb{N}) \setminus FIN(\mathbb{N})$. Notice that since $FIN(\mathbb{N})$ is discrete, it is the set of isolated points in $\beta FIN(\mathbb{N})$. Since the elements of $FIN(\mathbb{N})$ commute with all elements of $\beta FIN(\mathbb{N})$, the submonoid $FIN(\mathbb{N})$ acts continuously on $\beta FIN(\mathbb{N})$.

The action of $FIN(\mathbb{N})$ extends to an Ellis action of $\beta FIN(\mathbb{N})$ on $\mathcal{LAB}$.

Theorem 4.17. *(a) For any label* $\mathcal{M}$ *the map* $\mathbf{r} \mapsto P_{\mathbf{r}}(\mathcal{M})$ *extends to a continuous map from* $\beta FIN(\mathbb{N})$ *to* $[[\mathcal{M}]]$ *and this defines an Ellis action* $\beta P : \beta FIN(\mathbb{N}) \times \mathcal{LAB} \to \mathcal{LAB}$.

(b) If $\mathcal{N} \subset \mathcal{M}$ *then* $Q(\mathcal{N}) \subset Q(\mathcal{M})$ *for all* $Q \in \beta FIN(\mathbb{N})$.

(c) The sets $\beta^* FIN(\mathbb{N})$ *and* $\beta^{**} FIN(\mathbb{N})$ *are closed, invariant subsets of* $\beta FIN(\mathbb{N})$ *and so are ideals in the Ellis semigroup.*

(d) Every nonempty, closed sub-semigroup of $\beta FIN(\mathbb{N})$ *contains an idempotent and all the idempotents of* $\beta^* FIN(\mathbb{N})$ *lie in* $\beta^{**} FIN(\mathbb{N})$.

Proof: (a) The extension to $\beta FIN(\mathbb{N})$ of the map to the compact space $\mathcal{LAB}$ is a standard property of the Stone-Čech compactification. It thus defines a function $\beta FIN(\mathbb{N}) \times \mathcal{LAB} \to \mathcal{LAB}$. As usual the equation $(QR)(\mathcal{M}) = Q(R(\mathcal{M}))$ for $Q, R \in \beta FIN(\mathbb{N})$ holds when $Q, R \in FIN(\mathbb{N})$. With Q fixed in $FIN(\mathbb{N})$, continuity of Q implies that it holds for all $R \in \beta FIN(\mathbb{N})$. Then with $R \in \beta FIN(\mathbb{N})$ fixed it then extends to all $Q \in \beta FIN(\mathbb{N})$.

(b) Each $P_{\mathbf{r}}$ preserves inclusions. Thus, the closed set $INC \subset \mathcal{LAB} \times \mathcal{LAB}$ is invariant with respect to the $FIN(\mathbb{N})$ product action. It follows that it is invariant with respect to the $\beta FIN(\mathbb{N})$ product action as well.

(c) Suppose that $\mathbf{r}Q = Q\mathbf{r} = \mathbf{s}$ for some $\mathbf{r}, \mathbf{s} \in FIN(\mathbb{N})$. Assume that $\{\mathbf{r}^i\}$ is a net in $FIN(\mathbb{N})$ converging to Q. Since $\mathbf{s}$ is isolated and $\{\mathbf{r}^i + \mathbf{r}\}$ converges to $Q\mathbf{r}$ it follows that eventually $\mathbf{r}^i + \mathbf{r} = \mathbf{s}$. By cancelation, eventually $\{\mathbf{r}^i\}$ is constant and so the limit Q is in $FIN(\mathbb{N})$. Contrapositively, $Q \in \beta^{**}FIN(\mathbb{N})$ implies $P_{\mathbf{r}}Q \in \beta^{**}FIN(\mathbb{N})$. Hence, $\beta^{**}FIN(\mathbb{N})$ is invariant. As the complement of the set of isolated points, it is closed. If $\mathbf{r}Q = \mathbf{0}$ then $Q = \mathbf{s}$ for some $\mathbf{s} \in FIN(\mathbb{N})$ and $\mathbf{r} + \mathbf{s} = \mathbf{0}$. So $\mathbf{r} = \mathbf{s} = \mathbf{0}$. Hence, $\beta^*FIN(\mathbb{N})$ is invariant and since $\mathbf{0}$ is isolated, it is closed. A closed, $FIN(\mathbb{N})$ invariant subset of $\beta FIN(\mathbb{N})$ is an ideal.

(d) The existence of idempotents is the Ellis-Namakura Lemma. We saw above that $\mathbf{0}$ is the only idempotent in $FIN(\mathbb{N})$ and so there are no idempotents in $\beta^*FIN(\mathbb{N}) \setminus \beta^{**}FIN(\mathbb{N})$.

□

Define $\beta^0 FIN(\mathbb{N}) = \{\ Q \in \beta FIN(\mathbb{N})\ :\ Q(\mathcal{M}) = \emptyset$ for all $\mathcal{M} \in \mathcal{LAB}\}$. Let $\gamma FIN(\mathbb{N})$ be the quotient space of $\beta FIN(\mathbb{N})$ obtained by collapsing $\beta^0 FIN(\mathbb{N})$ to a point which we will denote U.

Proposition 4.18. *$\beta^0 FIN(\mathbb{N})$ is a closed, two-sided ideal in the semigroup $\beta FIN(\mathbb{N})$. The projection map $\beta FIN(\mathbb{N}) \to \gamma FIN(\mathbb{N})$ induces an Ellis semigroup structure so that the projection becomes a continuous, surjective homomorphism. The images $\gamma^* FIN(\mathbb{N})$ and $\gamma^{**}FIN(\mathbb{N})$ of $\beta^*FIN(\mathbb{N})$ and $\beta^{**}FIN(\mathbb{N})$ are closed ideals in $\gamma FIN(\mathbb{N})$. The action of $\beta FIN(\mathbb{N})$ on $\mathcal{LAB}$ factors to define an Ellis action of $\gamma FIN(\mathbb{N})$ on $\mathcal{LAB}$. If $\mathbf{r} > \mathbf{0}$ then any idempotent in the closed ideal $(\beta FIN(\mathbb{N}))P_{\mathbf{r}}$ maps to U in $\gamma FIN(\mathbb{N})$.*

Proof: By definition of an Ellis action $Q \mapsto Q(\mathcal{M})$ is continuous and so $Q(\mathcal{M}) = \emptyset$ is a closed condition. If $Q \in \beta^0 FIN(\mathbb{N})$ and $Q_1 \in \beta FIN(\mathbb{N})$ then $Q_1(\emptyset) = \emptyset$ implies $Q_1 Q \in \beta^0 FIN(\mathbb{N})$ and $Q_1(\mathcal{M}) \in \mathcal{LAB}$ implies $QQ_1 \in \beta^0 FIN(\mathbb{N})$. So multiplication is well-defined on $\gamma FIN(\mathbb{N})$ so that the projection is a homomorphism and $Q_1 \mapsto Q_1 Q_2$ and $Q_1 \mapsto Q_1(\mathcal{M})$ are continuous by definition of the quotient topology. Finally, if $Q = Q_1 P_{\mathbf{r}}$ is an idempotent in $\beta FIN(\mathbb{N})$ then since $P_{\mathbf{r}}$ commutes with Q_1, we have $Q = Q^n = Q_1^n P_{n\mathbf{r}}$ for all positive integers n. For any $\mathcal{M}$, $P_{n\mathbf{r}}(\mathcal{M}) = \emptyset$ for n sufficiently large. Hence, $Q \in \beta^0 FIN(\mathbb{N})$ and so maps to U.

□

Let $\Theta(\mathcal{M})$ be the closure in the space of labels of the set $\{ \mathcal{M} - \mathbf{r} \ : \ \mathbf{r}$ an $\mathbb{N}$-vector $\}$. That is, $\Theta(\mathcal{M})$ is the orbit closure of $\mathcal{M}$ with respect to the $FIN(\mathbb{N})$ action or, equivalently, $\Theta(\mathcal{M}) = \beta FIN(\mathbb{N})(\mathcal{M})$. Since $[[\mathcal{M}]]$ is closed and invariant, $\Theta(\mathcal{M}) \subset [[\mathcal{M}]]$. Even in the finite case, it can happen that the inclusion is proper.

Example 4.19. Set $\mathcal{M} = \langle \chi(1) + \chi(2), 2\chi(2) + \chi(3) \rangle$, and let $\mathcal{N} = \langle \chi(1) + \chi(2), \chi(2) + \chi(3) \rangle$. It is easy to check that $\mathcal{N} \in [[\mathcal{M}]] \setminus \Theta\mathcal{M}$.

If $\mathcal{M} = FIN(\mathbb{N})$, the maximum label then $P_{\mathbf{r}}(\mathcal{M}) = \mathcal{M}$ for all $\mathbf{r} \in FIN(\mathbb{N})$ and so $\Theta(FIN(\mathbb{N})) = \{FIN(\mathbb{N})\}$.

Lemma 4.20. *For any label $\mathcal{M} \neq FIN(\mathbb{N})$, $\emptyset \in \Theta(\mathcal{M})$. If $\mathcal{M}$ is nonempty and bounded then $0 \in \Theta(\mathcal{M})$.*

Proof: If $\mathbf{r} \not\in \mathcal{M}$ then $\mathcal{M} - \mathbf{r} = \emptyset$ and so $\emptyset \in \Theta(\mathcal{M})$. If $\mathbf{r} \in max\ \mathcal{M}$ then $\mathcal{M} - \mathbf{r} = 0$.

Now assume that $\mathcal{M}$ is bounded so that each $\mathcal{M} \wedge [1, i]$ is a finite label. Let $\mathbf{r}^i$ be maximal element of $\mathcal{M} \wedge [1, i]$. in the finite label . Clearly $\mathbf{0} \in LIMINF\{\mathcal{M} - \mathbf{r}^i\}$. On the other hand if $\mathbf{w} \in LIMSUP\{\mathcal{M} - \mathbf{r}^i\}$ then for some j we have *supp* $\mathbf{w} \subset [1, j]$. Frequently $\mathbf{w} + \mathbf{r}^i \in \mathcal{M}$, and so there exists $i \geq j$, $\mathbf{w} + \mathbf{r}^i \in \mathcal{M} \wedge [1, i]$. Maximality implies $\mathbf{w} = \mathbf{0}$. That is, $0 = LIM\{\mathcal{M} - \mathbf{r}^i\}$.

□

Remark: Notice that $\mathcal{M} - \mathbf{r} = 0$ iff $\mathbf{r} \in max\ \mathcal{M}$ and so if $max\ \mathcal{M} = \emptyset$ then $\mathcal{M} - \mathbf{r} \neq 0$ for any $\mathbf{r} \in FIN(\mathbb{N})$.

Let $\Theta'(\mathcal{M})$ be the closure in the space of labels of the set $\{ \mathcal{M} - \mathbf{r} \ : \ \mathbf{r} \in FIN(\mathbb{N}) \ \text{ with } \mathbf{r} > \mathbf{0}\}$. Thus, $\Theta(\mathcal{M}) = \Theta'(\mathcal{M}) \cup \{\mathcal{M}\}$ and $\Theta'(\mathcal{M}) = \beta^* FIN(\mathbb{N})\mathcal{M}$.

If $\mathbf{m} \in max\ \mathcal{M}$ then $\{\mathcal{N} : \mathbf{m} \in \mathcal{N}\}$ is a clopen subset of $\mathcal{LAB}$ which is disjoint from $\Theta'(\mathcal{M})$. In particular, if $\mathcal{M}$ is of finite type and nonempty then $\mathcal{M} \not\in \Theta'(\mathcal{M})$.

Definition 4.21. In general, call $\mathcal{M}$ a *recurrent label* if $\mathcal{M} \in \Theta'(\mathcal{M})$. So $\mathcal{M}$ is recurrent if there exists a sequence $\{\mathbf{r}^i > \mathbf{0}\}$ such that $\mathcal{M} = LIM\{\mathcal{M} - \mathbf{r}^i\}$ and so for all $\mathbf{m} \in \mathcal{M}$ eventually $\mathbf{m} + \mathbf{r}^i \in \mathcal{M}$.

Clearly, if $\mathcal{M}$ is a recurrent label then $max\ \mathcal{M} = \emptyset$.

For example $\mathcal{M} = FIN(\mathbb{N})$ is a recurrent label since then $\mathcal{M} = P_{\mathbf{r}}(\mathcal{M})$ for all $\mathbf{r} \in \mathcal{M}$. By Proposition 4.7 (e) $P_{\mathbf{r}}(\mathcal{M})$ is a proper subset of $\mathcal{M}$

when $\mathcal{M}$ is bounded and so then cannot equal $\mathcal{M}$. Nonetheless, there are bounded recurrent labels.

If $\mathcal{M}$ is bounded then

$$\mathcal{M} \ = \ LIM\{\mathcal{M} - \mathbf{r}^i\} \qquad \Longrightarrow \qquad LIMSUP\{supp\ \mathbf{r}^i\} = \emptyset. \tag{4.8}$$

This is because $\ell \in supp\ \mathbf{r}^i$ implies $\rho(\mathcal{M})_\ell \cdot \chi(\ell) \in \mathcal{M} \setminus (\mathcal{M} - \mathbf{r}^i)$. So it cannot happen that $\ell \in supp\ \mathbf{r}^i$ infinitely often.

Define

$$ISO(\mathcal{M}) \ = \ \{ \ Q \in \beta FIN(\mathbb{N}) \ : \ Q(\mathcal{M}) = \mathcal{M} \ \} \tag{4.9}$$

Clearly, $\mathbf{0} \in ISO(\mathcal{M})$ and $\mathcal{M}$ is recurrent iff there exists $Q \in \beta^* FIN(\mathbb{N})$ such that $Q(\mathcal{M}) = \mathcal{M}$ and so iff $ISO(\mathcal{M}) \cap \beta^* FIN(\mathbb{N}) \neq \emptyset$.

Definition 4.22. Call $\mathcal{M}$ a *strongly recurrent label* if $\mathcal{M}$ is bounded and infinite and for every $\mathbf{m} \in \mathcal{M}$, there is a finite set $F(\mathbf{m}) \subset \mathbb{N}$ such that $\mathcal{M} - \mathbf{m} \supset \{ \ \mathbf{w} \in \mathcal{M} \ : \ (supp\ \mathbf{w}) \ \cap \ F(\mathbf{m}) = \emptyset \}$.

Call a label $\mathcal{N}$ a *strongly recurrent set* for a bounded label $\mathcal{M}$ if $\mathcal{N}$ is infinite, $\mathcal{N} \subset \mathcal{M}$ and for every $\mathbf{m} \in \mathcal{M}$, there is a finite set $F(\mathbf{m}) \subset \mathbb{N}$ such that $\mathcal{M} - \mathbf{m} \supset \{ \ \mathbf{w} \in \mathcal{N} \ : \ (supp\ \mathbf{w}) \ \cap \ F(\mathbf{m}) = \emptyset \}$ and if $\mathbf{m} \in \mathcal{N}$, $\mathcal{N} - \mathbf{m} \supset \{ \ \mathbf{w} \in \mathcal{N} \ : \ (supp\ \mathbf{w}) \ \cap \ F(\mathbf{m}) = \emptyset \}$. Thus, the label $\mathcal{N}$ is strongly recurrent and $\mathcal{M}$ is strongly recurrent iff $\mathcal{M}$ itself is a strongly recurrent set for $\mathcal{M}$.

Proposition 4.23. *(a) If $\{\mathbf{m}^i\}$ is a strictly increasing infinite sequence in $\mathcal{M}$ then $\rho \in \mathbb{Z}_{+\infty}^{\mathbb{N}}$ defined by $\rho_\ell = sup_i\{\mathbf{m}^i(\ell)\}$ satisfies $\rho \ \leq \ \rho(\mathcal{M})$ and $\mathcal{N} \ = \ \langle \rho \rangle \ = \ \bigcup_i \{\langle \mathbf{m}^i \rangle\}$ is a recurrent label with $\mathcal{N} \subset \mathcal{M}$ and $\mathbf{m}^i \in \mathcal{N}$ for all i. If $\mathcal{M}$ is bounded then $\mathcal{N}$ is strongly recurrent.*

(b) For any label $\mathcal{M}$ $ISO(\mathcal{M})$ is a closed submonoid of $\beta FIN(\mathbb{N})$ such that for $Q_1, Q_2 \in \beta FIN(\mathbb{N})$ the product $Q_1 Q_2$ is in $ISO(\mathcal{M})$ iff both Q_1 and Q_2 are in $ISO(\mathcal{M})$.

(c) A label $\mathcal{N}$ is recurrent iff there exists an idempotent $Q \in \beta^{**} FIN(\mathbb{N})$ such that $Q(\mathcal{N}) = \mathcal{N}$. If $\mathcal{M}$ is any label and Q is an idempotent in $\beta^{**} FIN(\mathbb{N})$ then $Q(\mathcal{M})$ is a recurrent element of $\Theta'(\mathcal{M})$. In particular, if $\mathcal{M}$ is of finite type then $Q(\mathcal{M}) = \emptyset$.

(d) If $\mathcal{N}$ is any nonempty recurrent label with $\mathcal{N} \subset \mathcal{M}$, then there is a recurrent label $\mathcal{M}_\infty \in \Theta(\mathcal{M})$ such that $\mathcal{N} \subset \mathcal{M}_\infty$.

(e) If $\mathcal{M}$ is a recurrent label then $\mathcal{M} - \mathbf{r}$ is a recurrent label for any $\mathbf{r} \in FIN(\mathbb{N})$.

(f) If $\mathcal{N}$ is a strongly recurrent set for a bounded label $\mathcal{M}$ then for every net $\{\mathbf{r}^i\}$ of elements of $\mathcal{N}$ such that $LIMSUP\{supp\ \mathbf{r}^i\} = \emptyset$, $\mathcal{M} = LIM \ \{\mathcal{M} - \mathbf{r}^i\}$ and $\mathcal{N} = LIM \ \{\mathcal{N} - \mathbf{r}^i\}$. In particular, $\mathcal{M}$ is recurrent if it has a strongly recurrent set.

(g) A bounded, infinite label $\mathcal{N}$ *is strongly recurrent iff for every sequence* $\{\mathbf{r}^i\}$ *of elements of* $\mathcal{N}$ *such that* $LIMSUP\{supp\ \mathbf{r}^i\} = \emptyset$, $\mathcal{N} = LIM\ \{\mathcal{N} - \mathbf{r}^i\}$.

(h) If label $\mathcal{M}$ *is bounded and recurrent then there exists an infinite set* $L \subset \mathbb{N}$ *such that* $\langle \chi(L) \rangle$ *is a strongly recurrent set for* $\mathcal{M}$.

(i) For any nonempty label $\mathcal{M}$ *the following conditions are equivalent.*

(1) *If* $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}$ *with disjoint supports then* $\mathbf{m}_1 + \mathbf{m}_2 \in \mathcal{M}$.
(2) $\mathcal{M}$ *is a sublattice of* $FIN(\mathbb{N})$, *i.e. if* $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}$ *then* $\mathbf{m}_1 \vee \mathbf{m}_2 \in \mathcal{M}$.
(3) $\mathcal{M} = \langle \rho \rangle$ *for some* $\rho \in \mathbb{Z}_{+\infty}^{\mathbb{N}}$.
(4) $\mathcal{M} = \langle \rho(\mathcal{M}) \rangle$.

When these conditions hold, Supp $\mathcal{M}$ *f-contains supp* $\rho(\mathcal{M})$. *If, in addition,* $\bigcup$ *Supp* $\mathcal{M}$ *is an infinite set, then* $\mathcal{M}$ *is strongly recurrent. For example, if* $\mathcal{M}$ *is bounded and infinite then* $\bigcup$ *Supp* $\mathcal{M}$ *is an infinite set.*

Proof: (a) Let $\mathbf{r}^i = \mathbf{m}^{i+1} - \mathbf{m}^i$. If $\mathbf{m} \leq \mathbf{m}^j$ and $i \geq j$ then $\mathbf{m} + \mathbf{r}^i \in \mathcal{N}$. That is, $\mathcal{N} = LIM\{\mathcal{N} - \mathbf{r}^i\}$. Thus, $\mathcal{N}$ is recurrent. When $\mathcal{M}$ is bounded, it is strongly recurrent by (i), proved below.

(b) It is clear that $ISO(\mathcal{M})$ is a closed subsemigroup. For $Q, P \in \beta FIN(\mathbb{N})$ if $Q(\mathcal{M}) \neq \mathcal{M}$ then $\mathcal{N} = Q(\mathcal{M})$ is a proper subset of $\mathcal{M}$ and so $PQ(\mathcal{M}) = P(\mathcal{N}) \in [[\mathcal{N}]]$ and so it,too, is a proper subset of $\mathcal{M}$. Also, $P(\mathcal{M}) \subset \mathcal{M}$ and so $QP(\mathcal{M}) \subset Q(\mathcal{M}) = \mathcal{N}$, also a proper subset of $\mathcal{M}$. Thus, $\{Q \in \beta FIN(\mathbb{N}) : Q(\mathcal{M}) \neq \mathcal{M}\}$ is a two-sided ideal (though it is not closed when $\mathcal{M}$ is recurrent). It follows that $Q_1Q_2(\mathcal{M}) = \mathcal{M}$ implies $Q_1(\mathcal{M}) = \mathcal{M}$ and $Q_2(\mathcal{M}) = \mathcal{M}$. The converse is true because $ISO(\mathcal{M})$ is a semigroup.

(c) $ISO(\mathcal{M}) \cap \beta^* FIN(\mathbb{N})$ is a closed subsemigroup of $\beta^* FIN(\mathbb{N})$ which is nonempty iff $\mathcal{N}$ is recurrent. In that case, the subsemigroup contains an idempotent which must lie in $\beta^{**} FIN(\mathbb{N})$. If $Q \in \beta^* FIN(\mathbb{N})$ is an idempotent then $Q(Q(\mathcal{M})) = Q(\mathcal{M})$ is recurrent and lies in $\Theta'(\mathcal{M}) = \beta^* FIN(\mathbb{N})(\mathcal{M})$. If $\mathcal{M}$ is of finite type and $\mathcal{N} \in [[\mathcal{M}]]$ is nonempty then $\mathcal{N}$ is of finite type and so *max* $\mathcal{N}$ is nonempty. Thus, $\emptyset$ is the only recurrent label in $[[\mathcal{M}]]$.

(d) Since $\mathcal{N}$ is recurrent, there exists Q an idempotent in $\beta^{**} FIN(\mathbb{N})$ such that $Q(\mathcal{N}) = \mathcal{N}$. Since $\mathcal{N} \subset \mathcal{M}$, $\mathcal{N} = Q(\mathcal{N}) \subset Q(\mathcal{M})$ which is recurrent by (c).

(e) If $Q(\mathcal{M}) = \mathcal{M}$, then $Q(P_{\mathbf{r}}(\mathcal{M})) = P_{\mathbf{r}}(Q(\mathcal{M})) = P_{\mathbf{r}}(\mathcal{M})$ and so $P_{\mathbf{r}}(\mathcal{M})$ is recurrent.

(f) For any $\mathbf{m} \in \mathcal{M}$, $\mathbf{m} \in \mathcal{M} - \mathbf{r}^i$ as soon as $F(\mathbf{m}) \cap$ *supp* $\mathbf{r}^i = \emptyset$ which happens eventually.

(g) If $\mathcal{M}$ is bounded and infinite but not strongly recurrent then there exists $\mathbf{m} \in \mathcal{M}$ and for every F finite subset of $\mathbb{N}$ there exists $\mathbf{n} \in \mathcal{M}$ with $supp\ \mathbf{n} \ \cap \ F = \emptyset$ but with $\mathbf{m} + \mathbf{n} \notin \mathcal{M}$. Note that this implies $\mathbf{n} > \mathbf{0}$.

Let $F_1 = supp\ \mathbf{m}$ and choose a positive $\mathbf{r}_1 \in \mathcal{M}$ with support disjoint from F_1 and is such that $\mathbf{m} + \mathbf{r}^1 \notin \mathcal{M}$. Let $F_2 = F_1 \cup supp\ \mathbf{r}^1$. Inductively, we build an increasing sequence of finite sets $\{F^i\}$ and positive elements $\mathbf{r}^i \in \mathcal{M}$ such that $supp\ \mathbf{r}^i \subset F^{i+1} \setminus F^i$ and $\mathbf{m} + \mathbf{r}^i \notin \mathcal{M}$. Since the supports are disjoint, $LIMSUP\{supp\ \mathbf{r}^i\} = \emptyset$. Because $\mathbf{m} \notin LIM\ \{\mathcal{M} - \mathbf{r}^i\}$, the limit is not $\mathcal{M}$.

The converse follows from (f).

(h) Since $\mathcal{M}$ is recurrent there exists a sequence $\{\mathbf{r}^i > \mathbf{0}\}$ be such that $\mathcal{M} = LIM\{\mathcal{M} - \mathbf{r}^i\}$. Since $\mathcal{M}$ is bounded, it follows from (4.8) that $\bigcup_i supp\ \mathbf{r}^i$ is infinite. Choose $\ell_1 \in supp\ \mathbf{r}^1$ and let $\mathcal{N}_1 = \{\chi(\ell_1)\} \cup \mathcal{M} \wedge [1,1]$. There exists $\mathbf{r}^{i_2}$ with $\mathbf{r}^{i_2}_{\ell_2} > 0$ and ℓ_2 not in the support of a member of $\mathcal{N}_1$ and is such that $\mathbf{m} + \mathbf{r}^{i_2} \in \mathcal{M}$ for all $\mathbf{m} \in \mathcal{N}_1$. Let $\mathcal{N}_2 = \{\mathbf{m} + \chi(\ell_2) : \mathbf{m} \in \mathcal{N}_1\} \cup \mathcal{M} \wedge [1,2]$. Inductively, we can choose ℓ_{k+1} such that $\mathbf{m} + \chi(\ell_{k+1}) \in \mathcal{M}$ for all $\mathbf{m} \in \mathcal{N}_k$ and with ℓ_{k+1} not in the support of any member of $\mathcal{N}_k$. Let $\mathcal{N}_{k+1} = \{\mathbf{m} + \chi(\ell_{k+1}) : \mathbf{m} \in \mathcal{N}_k\} \cup \mathcal{M} \wedge [1, k+1]$. By the inductive construction if $\mathbf{m} \in \mathcal{N}_{k-1}$ then $\mathbf{m} + \Sigma_{i=k}^{j} \chi(\ell_i) \in \mathcal{M}$ for all $j \geq k$. With $\mathbf{m} = 0$ this says that $\Sigma_{i=1}^{j} \chi(\ell_i) \in \mathcal{M}$ for all j and so $Supp\ \mathcal{M}$ f-contains $L = \{\ell_k\}$ and $\langle \chi(L) \rangle$ is a strongly recurrent set for $\mathcal{M}$.

(i) (4) $\Rightarrow$ (3) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): If $\mathbf{m}_1$ and $\mathbf{m}_2$ have disjoint supports then $\mathbf{m}_1 \vee \mathbf{m}_2 = \mathbf{m}_1 + \mathbf{m}_2$.

(1) $\Rightarrow$ (4): For any $\mathcal{M}$, $\rho(\mathcal{M})_\ell \chi(\ell) \in \mathcal{M}$. So if $\mathbf{m} \leq \rho(\mathcal{M})$ then (1) (and induction) implies that $\mathbf{m} \leq \Sigma\ \{\ \rho(\mathcal{M})_\ell \chi(\ell)\ :\ \ell \in supp\ \mathbf{m}\ \}$ is in $\mathcal{M}$.

If $L = supp\ \rho(\mathcal{M})$ then $\chi(L) \leq \rho(\mathcal{M})$ and so $\langle \chi(L) \rangle \subset \langle \rho(\mathcal{M}) \rangle$. This implies that $Supp\ \langle \rho(\mathcal{M}) \rangle$ f-contains L.

Now assume that $\bigcup\ Supp\ \mathcal{M}$ is infinite and that $\mathcal{M}$ is bounded. For any $\mathbf{m} \in \mathcal{M}$, let $F(\mathbf{m}) = supp\ \mathbf{m}$ and apply (i) to get the strong recurrence condition.

□

Remark 4.24. Let $FIN^\ell = \{\ \mathbf{m} \in FIN(\mathbb{N})\ :\ supp\, \mathbf{m} \cap [1, \ell] = \emptyset\ \}$. It is clear from Proposition 4.23 (f) that, when $\mathcal{N}$ is a strongly recurrent set for a bounded label $\mathcal{M}$, $\bigcap_\ell \overline{\mathcal{N} \cap FIN^\ell} \subset ISO(\mathcal{M}) \cap ISO(\mathcal{N})$, where the closure is taken in $\beta FIN(\mathbb{N})$. One can prove, by using (4.8), that the intersection is equal to $ISO(\mathcal{N})$.

Corollary 4.25. *(a) A label $\mathcal{M}$ is of finite type iff $\emptyset$ is the only recurrent label contained in $\mathcal{M}$, iff $\emptyset$ is the only recurrent label in $\Theta(\mathcal{M})$.*

(b) For a label $\mathcal{M}$, max $\mathcal{M} = \emptyset$ iff $\mathcal{M}$ is the union of the recurrent labels contained in $\mathcal{M}$. Thus:

$$\{\mathcal{M} : \textit{finite type}\} \subset \{\mathcal{M} : \textit{max } \mathcal{M} \neq \emptyset\} =$$
$$\{\mathcal{M} : \textit{not a union of recurrent labels}\} \subset \{\mathcal{M} : \textit{not recurrent}\}.$$

Proof: (a): If $\mathcal{N}$ is a nonempty recurrent label then it is not of finite type and so if $\mathcal{N} \subset \mathcal{M}$ then $\mathcal{M}$ is not of finite type. If $\mathcal{M}$ is not of finite type then it contains a strictly increasing infinite sequence and so by Proposition 4.23 (a) it contains a recurrent label. For the second claim apply Proposition 4.23.(d).

(b) If $\mathbf{m} \in \max\ \mathcal{M}$ and $\mathbf{m} \in \mathcal{N} \subset \mathcal{M}$ then $\mathbf{m} \in max\ \mathcal{N}$ and so $\mathcal{N}$ is not recurrent. That is, *max* $\mathcal{M}$ is disjoint from all nonempty recurrent labels contained in $\mathcal{M}$.

If *max* $\mathcal{M} = \emptyset$ and $\mathbf{m} \in \mathcal{M}$ then inductively we can define a strictly increasing sequence $\{\mathbf{m}^i\}$ in $\mathcal{M}$ with $\mathbf{m} = \mathbf{m}^1$. By Proposition 4.23 there is a recurrent label $\mathcal{N} \subset \mathcal{M}$ with $\mathbf{m} \in \mathcal{N}$.

□

Corollary 4.26. *Assume $\mathcal{M}$ is a nonempty, bounded label. If $\mathcal{M}$ is a recurrent label then $\Theta(\mathcal{M})$ is a Cantor set. If $\mathcal{M}$ is a label not of finite type, then $\Theta(\mathcal{M})$ is uncountable.*

Proof: If $\mathcal{M}$ is a nonempty recurrent label then for some sequence $\{\mathbf{r}^i\}$ of positive elements of $\mathcal{M}$, $\{\mathcal{M} - \mathbf{r}^i\}$ converges to $\mathcal{M}$. Since $\mathcal{M}$ is bounded, each of the $\mathcal{M} - \mathbf{r}^i$ is a proper subset of $\mathcal{M}$ and each lies in $\Theta(\mathcal{M})$. It follows that $\mathcal{M}$ is not an isolated point of $\Theta(\mathcal{M})$. For each $\mathbf{r} \in \mathcal{M}$, the label $\mathcal{M} - \mathbf{r}$ is nonempty and bounded and it is recurrent by Proposition 4.23(e). Hence, $\mathcal{M} - \mathbf{r}$ is not isolated in $\Theta(\mathcal{M} - \mathbf{r}) \subset \Theta(\mathcal{M})$. It follows that $\{ \mathcal{M} - \mathbf{r} \ : \ \mathbf{r} \in \mathcal{M} \}$ is a dense subset of $\Theta(\mathcal{M})$ no point of which is isolated and so no point of $\Theta(\mathcal{M})$ is isolated. On the other hand, $\Theta(\mathcal{M})$ is a nonempty, compact, ultra-metric space and so it is a Cantor set.

If $\mathcal{M}$ is not of finite type then there exists a nonempty recurrent $\mathcal{N} \in \Theta(\mathcal{M})$ by Corollary 4.25(a). Hence, $\Theta(\mathcal{M})$ contains the Cantor set $\Theta(\mathcal{N})$.

□

Corollary 4.27. *For any label $\mathcal{M}$, $\mathcal{M}$ is the only $FIN(\mathbb{N})$-transitive point in $\Theta(\mathcal{M})$.*

Proof: Suppose $\mathcal{N} \in \Theta(\mathcal{M})$ is a transitive point. Then there are $P, Q \in \beta FIN(\mathbb{N})$ with $P(\mathcal{M}) = \mathcal{N}$ and $Q(\mathcal{N}) = \mathcal{M}$. Thus $QP \in ISO(\mathcal{M})$ and it follows, by Proposition 4.23(b), that $P \in ISO(\mathcal{M})$ and so $\mathcal{N} = \mathcal{M}$.
□

Remark 4.28. Note the sharp contrast with the case where the acting semigroup is a group. In fact, for a dynamical system (X, G), where X is a compact metric space and G is a group, the existence of one dense orbit implies that the set X_{tr} of transitive points forms a dense G_δ subset of X.

Example 4.29. It can happen that $\Theta(\mathcal{M})$ is uncountable even with $\mathcal{M}$ of finite type.

Define a bijection $w \mapsto \ell_w$, from the set of finite words on the alphabet $\{0, 1\}$ onto $\mathbb{N}$. For $x \in \{0, 1\}^{\mathbb{N}}$ let $w_k(x)$ be the initial word of length k in x, that is, $w_k(x) = x_1 x_2 \dots x_k$, for $k \in \mathbb{N}$. Let

(4.10)
$$\begin{aligned} M_x &= \{ \ \ell_{w_k(x)} \ : \ k \in \mathbb{N} \ \} \\ \mathcal{M}_x &= \{\mathbf{0}\} \cup \{ \ \chi(\ell) \ : \ \ell \in M_x \ \}, \\ \mathcal{M}_x^{(2)} &= \mathcal{M}_x \oplus \mathcal{M}_x \ = \ \langle \{ \ \chi(\ell^1) + \chi(\ell^2) \ : \ \ell^1, \ell^2 \in M_x \ \} \rangle, \\ \mathcal{M} &= \bigcup \{ \mathcal{M}_x^{(2)} \ : \ x \in \{0,1\}^{\mathbb{N}} \ \}. \end{aligned}$$

Since $\rho(\mathcal{M}) \leq 2$ and the size of the elements of $\mathcal{M}$ are bounded by 2, it follows from Lemma 4.5 that $\mathcal{M}$ is a label of finite type. Notice that if $x \neq y$ in $\{0, 1\}^{\mathbb{N}}$ then $M_x \ \cap \ M_y$ is finite.

It is easy to see that for each $x \in \{0, 1\}^{\mathbb{N}}$, $LIM \ \{\mathcal{M} - \chi(\ell_{w_i(x)})\} = \mathcal{M}_x$ and it follows that $\Theta(\mathcal{M})$ is uncountable.

If Φ is a compact, invariant subset of $\mathcal{LAB}$ then the action of $FIN(\mathbb{N})$ restricts to an action on Φ. Since Φ^Φ is an Ellis semigroup with an Ellis action on Φ, the closure in Φ^Φ of $\{P_{\mathbf{r}} : \mathbf{r} \in FIN(\mathbb{N})\}$, denoted $\mathcal{E}(\Phi)$, is an Ellis semigroup with an Ellis action on Φ. We call $\mathcal{E}(\Phi)$ the *enveloping semigroup* of Φ. A map Q on Φ is an element of $\mathcal{E}(\Phi)$ iff for every finite sequence $\{\mathcal{M}^i\}$ in Φ and any $N \in \mathbb{N}$ there exists $\mathbf{r} \in FIN(\mathbb{N})$ such that $P_{\mathbf{r}}(\mathcal{M}^i) \cap \mathcal{B}_N = Q(\mathcal{M}^i) \cap \mathcal{B}_N$ for all i. It follows that for $\mathcal{N} \in \Phi$, $\mathcal{N}_1 = Q(\mathcal{N})$ for some some $Q \in \mathcal{E}(\Phi)$ iff $\mathcal{N}_1$ is the limit of some sequence $\{\mathcal{N} - \mathbf{r}^i\}$. Notice that this does not necessarily imply that Q is a pointwise limit of some sequence $\{P_{\mathbf{r}^i}\}$.

The *adherence semigroup* $\mathcal{A}(\Phi)$ is the closure in Φ^{Φ} of $\{P_{\mathbf{r}} : \mathbf{r} > \mathbf{0}\}$. It follows that
(4.11)

$$\mathcal{E}(\Phi) = \mathcal{A}(\Phi) \cup \{P_{\mathbf{0}} = id_{\Phi}\}, \qquad \Theta(\mathcal{M}) = \mathcal{E}(\Phi)\mathcal{M}, \qquad \Theta'(\mathcal{M}) = \mathcal{A}(\Phi)\mathcal{M},$$

for $\mathcal{M} \in \Phi$.

If $\Phi_1 \subset \Phi$ is also closed and invariant then the restriction map defines a continuous, surjective homomorphism from $\mathcal{E}(\Phi)$ to $\mathcal{E}(\Phi_1)$.

The homomorphism from $FIN(\mathbb{N})$ into $\mathcal{E}(\Phi)$ extends to an Ellis semigroup homomorphism $\beta FIN(\mathbb{N}) \to \mathcal{E}(\Phi)$ with $\beta^* FIN(\mathbb{N})$ mapping onto $\mathcal{A}(\Phi)$.

The sequences $\{\mathcal{M}^i\}$ that we will find most useful will be given by $\mathcal{M}^i = \mathcal{M} - \mathbf{r}^i = P_{\mathbf{r}^i}(\mathcal{M})$ for a sequence $\{\mathbf{r}^i\}$ of $\mathbb{N}$-vectors. Such a sequence is bounded by $\mathcal{M}$ and $\mathbf{m} \in LIMSUP$ iff frequently $\mathbf{m} + \mathbf{r}^i \in \mathcal{M}$ and $\mathbf{m} \in LIMINF$ iff eventually $\mathbf{m} + \mathbf{r}^i \in \mathcal{M}$.

If $\mathbf{r}^i > \mathbf{0}$ for all i then $max\ \mathcal{M}$ is disjoint from $LIMSUP$.

If $\mathbf{r}^i \in \mathcal{M}$ for all i then $\mathbf{0} \in LIMINF$ and so $LIMINF \neq \emptyset$.

Lemma 4.30. *Assume that $\{\mathcal{M} - \mathbf{r}^i\}$ is convergent.*

(a) If $max\ \mathcal{M} \neq \emptyset$, e.g. if $\mathcal{M}$ is of finite type, then either eventually $\mathbf{r}^i = \mathbf{0}$ and $LIM\ \{\mathcal{M} - \mathbf{r}^i\} = \mathcal{M} = \mathcal{M} - \mathbf{0}$ or eventually $\mathbf{r}^i > \mathbf{0}$ and $(max\ \mathcal{M}) \cap LIM\ \{\mathcal{M} - \mathbf{r}^i\} = \emptyset$.

(b) Either eventually $\mathbf{r}^i \in \mathcal{M}$ and $\mathbf{0} \in LIM\ \{\mathcal{M} - \mathbf{r}^i\}$ or eventually $\mathbf{r}^i \notin \mathcal{M}$ and $LIM\ \{\mathcal{M} - \mathbf{r}^i\} = \emptyset$.

(c) If $\mathcal{M} - \mathbf{m}_1 = \mathcal{M} - \mathbf{m}_2$ then $LIM\ \{\mathcal{M} - \mathbf{r}^i\} - \mathbf{m}_1 = LIM\ \{\mathcal{M} - \mathbf{r}^i\} - \mathbf{m}_2$.

(d) If there exists $\mathbf{r}$ such that for infinitely many i, $\mathcal{M} - \mathbf{r}^i = \mathcal{M} - \mathbf{r}$ then the limit is $\mathcal{M} - \mathbf{r}$.

Proof: (a) Since $\mathbf{r} > \mathbf{0}$ implies $max\ \mathcal{M} \cap (\mathcal{M} - \mathbf{r}) = \emptyset$, if $\mathbf{r}^i = \mathbf{0}$ infinitely often then convergence implies that the limit is $\mathcal{M}$ and so an element of $max\ \mathcal{M}$ is eventually in $\mathcal{M} - \mathbf{r}^i$ which can only happen when $\mathbf{r}^i$ is eventually 0.

(b) Since $\mathbf{0} \in \mathcal{M} - \mathbf{r}$ iff $\mathbf{r} \in \mathcal{M}$ we see that if $\mathbf{r}^i \in \mathcal{M}$ infinitely often then convergence implies that eventually $\mathbf{0} \in \mathcal{M} - \mathbf{r}^i$ and so eventually $\mathbf{r}^i \in \mathcal{M}$.

(c) Since $P_{\mathbf{m}_1}$ and $P_{\mathbf{m}_2}$ are continuous,

$$\begin{aligned} LIM\ \{\mathcal{M} - \mathbf{r}^i\} - \mathbf{m}_1 \ = \ LIM\ \{\mathcal{M} - \mathbf{m}_1 - \mathbf{r}^i\} \ = \\ LIM\ \{\mathcal{M} - \mathbf{m}_2 - \mathbf{r}^i\} \ = \ LIM\ \{\mathcal{M} - \mathbf{r}^i\} - \mathbf{m}_2. \end{aligned} \tag{4.12}$$

(d) By assumption there is a subsequence $\mathbf{r}^{i'}$ such that $\mathcal{M} - \mathbf{r}^{i'}$ is constant at $\mathcal{M} - \mathbf{r}$ and so converges to $\mathcal{M} - \mathbf{r}$. By the assumption of convergence the limit of the original sequence is $\mathcal{M} - \mathbf{r}$.

□

Remark: (c) has the following interpretation: If $\mathcal{M} - \mathbf{m}_1 = \mathcal{M} - \mathbf{m}_2$ then as elements of $\mathcal{E}(\Theta(\mathcal{M}))$, $P_{\mathbf{m}_1} = P_{\mathbf{m}_2}$ because the set of $\mathcal{N} \in \Theta(\mathcal{M})$ on which they agree is closed and invariant and includes $\mathcal{M}$.

Corollary 4.31. *Assume that $\mathcal{M}$ is a bounded label and $\{\mathbf{r}^i\}$ is a sequence of $\mathbb{N}$-vectors such that $\{\mathcal{M} - \mathbf{r}^i\}$ is convergent. If $\bigcup_i$ supp $\mathbf{r}^i$ is finite then the sequence $\{\mathcal{M} - \mathbf{r}^i\}$ is eventually constant. That is, there exists $I \in \mathbb{N}$ such that for $i, j \geq I$:*

$$\mathcal{M} - \mathbf{r}^i \quad = \quad \mathcal{M} - \mathbf{r}^j, \tag{4.13}$$

and this common value is the limit.

Proof: If eventually $\mathbf{r}^i \notin \mathcal{M}$ then there exists $I \in \mathbb{N}$ such that $i \geq I$ implies $\mathcal{M} - \mathbf{r}^i = \emptyset$ and $\emptyset$ is the limit. Lemma 4.30 (b) implies that otherwise eventually $\mathbf{r}^i \in \mathcal{M}$. So we may assume $\mathbf{r}^i \in \mathcal{M}$ for all i. Since $\mathcal{M}$ is bounded $\bigcup_i$ *supp* $\mathbf{r}^i$ finite implies that the set of vectors $\{\mathbf{r}^i\}$ is finite. There exists I such that for $i \geq I$ each $\mathbf{r}^i$ occurs infinitely often in the sequence. It follows from Lemma 4.30 (d) the limit is $\mathcal{M} - \mathbf{r}^i$ for each $i \geq I$. As the limit is unique, $\mathcal{M} - \mathbf{r}^i = \mathcal{M} - \mathbf{r}^j$ for all $i, j \geq I$.

□

Lemma 4.32. *Let $\mathcal{M}$ be a label. Assume that $\{\mathbf{r}^i\}$ and $\{\mathbf{s}^j\}$ are sequences of $\mathbb{N}$-vectors such that $\{\mathcal{M} - \mathbf{r}^i\}$ and $\{\mathcal{M} - \mathbf{s}^j\}$ are convergent with $LIM\{\mathcal{M} - \mathbf{r}^i\} = \mathcal{M} - \mathbf{r}$ for some $\mathbb{N}$-vector $\mathbf{r}$ and $LIM\{\mathcal{M} - \mathbf{s}^j\}$ denoted LIM_s. If either $\{\mathcal{M} - \mathbf{r}^i\}$ is eventually constant at $\mathcal{M} - \mathbf{r}$ or $LIM_s = \mathcal{M} - \mathbf{s}$ for some $\mathbb{N}$-vector $\mathbf{s}$, then*

$$\begin{aligned} LIM_{i\to\infty} LIM_{j\to\infty}\{\mathcal{M} - \mathbf{r}^i - \mathbf{s}^j\} \quad &= \quad LIM_s - \mathbf{r} \\ &= \quad LIM_{j\to\infty} LIM_{i\to\infty}\{\mathcal{M} - \mathbf{r}^i - \mathbf{s}^j\}. \end{aligned} \tag{4.14}$$

Proof: By Proposition 4.15 applied twice $LIM_{i\to\infty}\{\mathcal{M} - \mathbf{r}^i - \mathbf{s}^j\} = \mathcal{M} - \mathbf{r} - \mathbf{s}^j$ and this sequence converges to $LIM_s - \mathbf{r}$. If $LIM_s = \mathcal{M} - \mathbf{s}$ so that $LIM_s - \mathbf{r} = \mathcal{M} - \mathbf{s} - \mathbf{r}$, we similarly, $LIM_i LIM_j = \mathcal{M} - \mathbf{r} - \mathbf{s}$. On the other hand, if eventually $M - \mathbf{r}^i = \mathcal{M} - \mathbf{r}$ we can omit terms and assume this is true for all i. Then by 4.12 $LIM_{j\to\infty}\{\mathcal{M} - \mathbf{r}^i - \mathbf{s}^j\} = LIM_s - \mathbf{r}^i = LIM_s - \mathbf{r}$. That is the sequence $\{LIM_s - \mathbf{r}^i\}$ is eventually constant at $LIM_s - \mathbf{r}$. Hence, in both cases $LIM_i LIM_j = LIM_s - \mathbf{r} = LIM_j LIM_i$.

□

4.1. **Finitary and simple labels.**

Definition 4.33. A label $\mathcal{M}$ is *finitary* if it is bounded and satisfies the following

[Finitary condition] Whenever $\{S_i\}$ is a sequence of finite subsets of $\mathbb{N}$ with $\bigcup_i S_i$ infinite, there are only finitely many subsets S of $\mathbb{N}$ such that eventually $S \cup S_i \in Supp\ \mathcal{M}$.

Clearly, if $\mathcal{M}_1 \subset \mathcal{M}$ is a label then $\mathcal{M}_1$ is finitary if $\mathcal{M}$ is.

The unbounded label $\mathcal{M} = \langle\{k\chi(1) : k \in \mathbb{N}\}\rangle$ shows that the above finitary condition alone does not imply that $\mathcal{M}$ is bounded.

For a label $\mathcal{M}$ and $\mathcal{N}$ a nonempty set of $\mathbb{N}$-vectors we define

$$(4.15)\quad \mathcal{M} - \mathcal{N} \ = \ \{\ \mathbf{m} : \mathbf{m} + \mathbf{r} \in \mathcal{M} \quad \text{for all } \mathbf{r} \in \mathcal{N}\ \} \ = \ \bigcap_{\mathbf{r} \in \mathcal{N}} \mathcal{M} - \mathbf{r}.$$

Proposition 4.34. *Let $\mathcal{M}$ be a bounded label.*

(a) The following conditions are equivalent.

(i) *$\mathcal{M}$ is finitary.*

(ii) *If $\{\mathbf{r}^i\}$ is a sequence of $\mathbb{N}$-vectors with $\bigcup_i$ supp $\mathbf{r}^i$ infinite then $LIMINF$ $\{\mathcal{M} - \mathbf{r}^i\}$ is finite.*

(iii) *If $\mathbf{r}^i$ is a sequence of $\mathbb{N}$-vectors with $\bigcup_i$ supp $\mathbf{r}^i$ infinite and $\{\mathcal{M} - \mathbf{r}^i\}$ convergent then then LIM $\{\mathcal{M} - \mathbf{r}^i\}$ is finite.*

(iv) *If $\mathcal{N}$ is infinite then $\mathcal{M} - \mathcal{N}$ is finite, and there is no strictly increasing sequence of members of $\{\mathcal{M} - \mathcal{N} : \mathcal{N}$ infinite $\}$.*

(b) If $\mathcal{M}$ is finitary then it is of finite type.

(c) If $\mathcal{M}$ is finitary, then the following conditions on a finite subset $\mathcal{F}$ of $\mathcal{M}$ are equivalent.

(i) *There exists $\mathbf{r}^i$ a sequence of finite vectors with $\bigcup_i$ supp $\mathbf{r}^i$ infinite such that LIM $\{\mathcal{M} - \mathbf{r}^i\} \ = \ \mathcal{F}$.*

(ii) *There exists $\mathbf{r}^i$ a sequence of distinct $\mathbb{N}$-vectors such that LIM $\{\mathcal{M} - \mathbf{r}^i\} \ = \ \mathcal{F}$.*

(iii) *There is an infinite set $\mathcal{N}$ such that $\mathcal{F} = \mathcal{M} - \mathcal{N}_1$ for every infinite subset $\mathcal{N}_1$ of $\mathcal{N}$.*

Proof: If $\mathcal{N} \subset \mathcal{M}$ then by the Bound Condition $\mathcal{N}$ is infinite iff $\bigcup$ *Supp* $\mathcal{N}$ is infinite. Hence, if $\{\mathbf{r}^i\}$ is a sequence with $\bigcup_i$ $\{supp\ \mathbf{r}^i\}$ infinite then we can choose a subsequence of distinct vectors.

(a) (i) $\Rightarrow$ (ii): If $\mathbf{m} \in LIMINF$ then eventually $supp\ \mathbf{m} \cup supp\ \mathbf{r}^i \in Supp\ M$. Because $\mathcal{M}$ is finitary there are only finitely many such sets $supp\ \mathbf{m}$ and by the Bound Condition there are only finitely many $\mathbf{m} \in \mathcal{M}$ with such supports.

(ii) $\Leftrightarrow$ (iii): If $\{\mathbf{r}^i\}$ is a sequence with $\bigcup_i \{supp\ \mathbf{r}^i\}$ infinite then we can choose a subsequence of distinct vectors and then go to a further subsequence $\{\mathbf{r}^{i'}\}$ which is convergent. Assuming (iii) $LIM\ \{\mathcal{M}-\mathbf{r}^{i'}\} = LIMINF\ \{\mathcal{M} - \mathbf{r}^{i'}\} \supset LIMINF\ \{\mathcal{M} - \mathbf{r}^i\}$ is finite. This shows that (iii) implies (ii). The converse is obvious.

(ii) $\Rightarrow$ (iv): Suppose that $\{\mathcal{F}^i\}$ is a nondecreasing sequence of subsets of $\mathcal{M}$ with each $\mathcal{M}-\mathcal{F}^i$ infinite. Inductively, choose $\mathbf{r}^i \in \mathcal{M}-\mathcal{F}^i$ distinct from the $\mathbf{r}^j$'s with $j\langle i$. Since the sequence $\{\mathcal{F}^i\}$ is a nondecreasing, $\mathcal{F}^j \subset \mathcal{M} - \mathbf{r}^i$ for $j \leq i$. Hence, $\bigcup_i \mathcal{F}^i \subset LIMINF\ \{\mathcal{M} - \mathbf{r}^i\}$ and so it is finite by (ii). Thus, the sequence $\{\mathcal{F}^i\}$ is eventually constant.

(iv) $\Rightarrow$ (iii): By going to a subsequence we can assume that $\{\mathbf{r}^i\}$ is a sequence of distinct elements. Then $\{\bigcap_{j\geq i}\{\mathcal{M}-\mathbf{r}^j\}\}$ is a nondecreasing sequence in $\{\mathcal{M} - \mathcal{N} : \mathcal{N} \text{ infinite }\}$. Hence, its union, which is the limit, is finite by (iv).

(b) Assume that $\{\mathbf{m}^i\}$ is a strictly increasing sequence in $\mathcal{M}$ and that $\mathbf{r}^i = \mathbf{m}^{i+1} - \mathbf{m}^i$. Since $\mathcal{M} \wedge [1, \ell]$ is finite and so is of finite type, it follows that $\bigcup_i \{supp\ \mathbf{m}^i\}$ is infinite and hence so is $\bigcup_i \{supp\ \mathbf{r}^i\}$. Since each $\mathbf{m}^j \in LIMINF\{\mathcal{M} - \mathbf{r}^i\}$ it follows that $\mathcal{M}$ is not finitary.

(c) (i) $\Leftrightarrow$ (ii): This is obvious from our initial remarks.

(ii) $\Rightarrow$ (iii): Assume $\mathbf{r}^i$ a sequence of $\mathbb{N}$-vectors with $LIM\ \{\mathcal{M}-\mathbf{r}^i\} = \mathcal{F}$. By discarding finitely many terms $\mathbf{r}^i$ we can assume that the finite set $\mathcal{F}$ equals $\mathcal{M} - \mathcal{N}$ with $\mathcal{N} = \{\mathbf{r}^i\}$. If $\mathcal{N}_1$ is any infinite subset of $\mathcal{N}$ then $\mathcal{F} = \mathcal{M} - \mathcal{N} \subset \mathcal{M} - \mathcal{N}_1$. On the other hand, if $\mathbf{m} \in \mathcal{M} - \mathcal{N}_1$ then $\mathbf{m} \in \mathcal{M}-\mathbf{r}^i$ for infinitely many i and so by convergence $\mathbf{m} \in LIM = \mathcal{F}$.

(iii) If $\mathcal{N} = \{\mathbf{r}^1, \mathbf{r}^2, \dots\}$ is the infinite set given by (iii) then $\{\mathbf{r}^i\}$ is a sequence of distinct elements with $\mathcal{F} = LIM\{\mathcal{M} - \mathbf{r}^i\}$.

□

We will call $\mathcal{F}$ an *external limit set* (or an *external label*) for a finitary label $\mathcal{M}$ when it is a limit of a sequence $\{\mathcal{M} - \mathbf{r}^i\}$ for some sequence of distinct vectors $\{\mathbf{r}^i\}$ in $\mathcal{M}$ (and so $\mathbf{0} \in \mathcal{F}$).

The value of the assumption that a label is finitary will come from the following result.

Lemma 4.35. *Assume that $\{\mathbf{r}^i\}$ and $\{\mathbf{s}^j\}$ are sequences in $FIN(\mathbb{N})$ such that $\{\mathcal{M}-\mathbf{r}^i\}$ and $\{\mathcal{M}-\mathbf{s}^j\}$ are convergent and with $LIM\{\mathcal{M}-\mathbf{r}^i\}$ and $LIM\{\mathcal{M} - \mathbf{s}^j\}$ are both finite. If $\bigcup_j$ supp $\mathbf{s}^j$ is infinite then for sufficiently large i, $(LIM\{\mathcal{M} - \mathbf{s}^j\}) - \mathbf{r}^i = \emptyset$.*

Proof: Suppose for some $\mathbf{w} \in LIM_s = LIM\{\mathcal{M} - \mathbf{s}^j\}$ that $\mathbf{w} - \mathbf{r}^k \geq \mathbf{0}$ for k in some infinite subset SEQ of $\mathbb{N}$. For $k \in SEQ$, $\mathbf{0} \leq \mathbf{r}^k \leq \mathbf{w}$ and so each such $\mathbf{r}^k \in LIM_s$. Since LIM_s is finite there must be a $\mathbf{w}_1 \in LIM_s$ such that $\mathbf{r}^k = \mathbf{w}_1$ for $k \in SEQ_1$ an infinite subset of SEQ. There exists J such that $j \geq J$ implies $\mathbf{w}_1 + \mathbf{s}^j \in \mathcal{M}$. Thus, for each $j \geq J$ $\mathbf{s}^j + \mathbf{r}^k \in \mathcal{M}$ for all $k \in SEQ_1$. By convergence of $\{\mathcal{M} - \mathbf{r}^i\}$, this means $\mathbf{s}^j \in LIM_r = LIM\{\mathcal{M} - \mathbf{r}^i\}$ for all $j \geq J$. Since LIM_r is also finite, this contradicts the assumption that $\bigcup_j$ *supp* $\mathbf{s}^j$ is infinite. Thus, for each $\mathbf{w} \in LIM_s$ it follows that eventually $\mathbf{w} - \mathbf{r}^k$ is not nonnegative.

As there are only finitely many $\mathbf{w} \in LIM_s$, it follows that eventually $LIM_s - \mathbf{r}^i$ is empty.

□

This immediately yields

Corollary 4.36. *Assume that $\{\mathbf{r}^i\}$ and $\{\mathbf{s}^j\}$ are sequences of $\mathbb{N}$-vectors such that for a label $\mathcal{M}$ both $\{\mathcal{M} - \mathbf{r}^i\}$ and $\{\mathcal{M} - \mathbf{s}^j\}$ are convergent and both $LIM\{\mathcal{M} - \mathbf{r}^i\}$ and $LIM\{\mathcal{M} - \mathbf{s}^j\}$ are finite. If $\bigcup_i$ supp $\mathbf{s}^j$ and $\bigcup_i$ supp $\mathbf{r}^i$ are both infinite then*

(4.16)
$$LIM_{i\to\infty}LIM_{j\to\infty}\{\mathcal{M} - \mathbf{r}^i - \mathbf{s}^j\} = LIM_{j\to\infty}LIM_{i\to\infty}\{\mathcal{M} - \mathbf{r}^i - \mathbf{s}^j\} = \emptyset.$$

□

Definition 4.37. A label $\mathcal{M}$ is *simple* if it is bounded and satisfies the following

[Convergence Condition] If $\{\mathbf{r}^i\}$ is a sequence of vectors in $\mathcal{M}$ such that the sequence $\{\mathcal{M} - \mathbf{r}^i\}$ is convergent then $LIM\ \{\mathcal{M} - \mathbf{r}^i\} = \mathcal{M} - \mathbf{r}$ for some $\mathbf{r} \in FIN(\mathbb{N})$.

Again $\mathcal{M} = \langle\{k\chi(1) : k \in \mathbb{N}\}\rangle$ shows that the above convergence condition alone does not imply that $\mathcal{M}$ is bounded.

It follows from Corollary 4.31 that any finite label is simple as well as finitary.

Proposition 4.38. *Let $\mathcal{M}$ be a label.*

(a) *If $\mathcal{M}$ is simple then $\mathcal{M} - \mathbf{r}$ is simple for all $\mathbb{N}$-vectors $\mathbf{r}$.*
(b) *If $\mathcal{M}$ is simple then it is of finite type.*
(c) *$\mathcal{M}$ is simple iff $\Theta(\mathcal{M}) = \{\ \mathcal{M} - \mathbf{r}\ :\ \mathbf{r} \in FIN(\mathbb{N})\ \}$.*
(d) *If $\mathcal{M}$ is simple then $\mathcal{E}(\Theta(\mathcal{M})) = \{\ P_{\mathbf{r}}\ :\ \mathbf{r} \in FIN(\mathbb{N})\ \}$ or, more precisely, the restrictions of these maps to $\Theta(\mathcal{M})$. Thus,*

$\mathcal{E}(\Theta(\mathcal{M}))$ is an abelian semigroup whose members act continuously on $\Theta(\mathcal{M})$.

Proof: (a): If $\{\mathcal{M} - \mathbf{r} - \mathbf{s}^j\}$ is a convergent sequence, then we can choose a convergent subsequence $\{\mathcal{M} - \mathbf{s}^{j'}\}$ with limit $LIM_{s'}$. Because $\mathcal{M}$ is simple, $LIM_{s'} = \mathcal{M} - \mathbf{s}$ for some $\mathbf{s}$. By Lemma 4.32 the subsequence $\{\mathcal{M} - \mathbf{r} - \mathbf{s}^{j'}\}$ converges to $\mathcal{M} - \mathbf{r} - \mathbf{s}$ and so this is the limit of the full convergent sequence $\{\mathcal{M} - \mathbf{r} - \mathbf{s}^j\}$.

(c): Since $\Theta(\mathcal{M})$ consists exactly of the limits of convergent sequences $\{\mathcal{M} - \mathbf{r}^i\}$, the equivalence is clear.

(b) If $\mathcal{M}$ is not of finite type then by Corollary 4.25 there exists $\mathcal{N} \in \Theta(\mathcal{M})$ which is recurrent. Since $\mathcal{M}$ is recurrent, $max\ \mathcal{N} = \emptyset$. By Lemma 4.20 and the Remark thereafter $0 \in \Theta(\mathcal{N})$ but $0 \neq \mathcal{N} - \mathbf{r}$ for any $\mathbb{N}$-vector $\mathbf{r}$. Hence, $\mathcal{N}$ is not simple. Since $\mathcal{N} \in \Theta(\mathcal{M})$, (a) and (c) imply that $\mathcal{M}$ is not simple.

(d) We observe first that if $\{\mathcal{M}^i\}$ is a net converging to some $\mathcal{N}$, then there is a sequence of elements $\{\mathcal{M}^{i'}\}$ which converges to $\mathcal{N}$, although not necessarily a subnet. Now let $Q \in \mathcal{E}(\Theta(\mathcal{M}))$ with $\mathcal{M}$ of finite type. There is a net $\{P_{\mathbf{r}^i}\}$ which converges pointwise to Q. For any $\mathbb{N}$-vector $\mathbf{r}$ we can choose a sequence $\{i'\}$ so that $\{P_{\mathbf{r}^{i'}}(\mathcal{M})\}$ converges to $Q(\mathcal{M})$ and $\{P_{\mathbf{r}^{i'}}(\mathcal{M} - \mathbf{r})\}$ converges to $Q(\mathcal{M} - \mathbf{r})$. Because $\mathcal{M}$ is simple, there exists $\mathbf{s}$ such that $\mathcal{M} - \mathbf{s} = LIM\ \{\mathcal{M} - \mathbf{r}^{i'}\} = Q(\mathcal{M})$. Then Lemma 4.32 implies that $\mathcal{M} - \mathbf{r} - \mathbf{s} = LIM\ \{\mathcal{M} - \mathbf{r} - \mathbf{r}^{i'}\} = Q(\mathcal{M} - \mathbf{r})$. That is, $Q = P_{\mathbf{s}}$ on $\Theta(\mathcal{M})$.

From Corollary 4.16 it follows that $\mathcal{E}(\Theta(\mathcal{M}))$ is abelian and acts continuously on $\Theta(\mathcal{M})$. Notice that $\mathbf{r} \mapsto P_{\mathbf{r}}$ is a homomorphism from the discrete monoid $FIN(\mathbb{N})$ onto the compact monoid $\mathcal{E}(\Theta(\mathcal{M}))$.

□

If the sequence $\{\mathcal{M} - \mathbf{r}^i\}$ is eventually constant then, of course, the limit is of the form $\mathcal{M} - \mathbf{r}$. However, it can happen that $\{\mathbf{r}^i\}$ is a sequence of distinct vectors in $\mathcal{M} \setminus max\ \mathcal{M}$ such that $0 = LIM\ \{\mathcal{M} - \mathbf{r}^i\}$. In that case, the sequence is not eventually constant but the limit is $\mathcal{M} - \mathbf{r}$ for $\mathbf{r} \in max\ \mathcal{M}$. Notice that because $\mathcal{M}$ is of finite type by (b), $max\ \mathcal{M} \neq \emptyset$.

For labels $\mathcal{M}_1$ and $\mathcal{M}_2$ define $\mathcal{M}_1 \oplus \mathcal{M}_2 = \{\mathbf{m}_1 + \mathbf{m}_2 : \mathbf{m}_1 \in \mathcal{M}_1, \mathbf{m}_2 \in \mathcal{M}_2\}$. Clearly, this is $\emptyset$ if either is $\emptyset$. If neither term is empty then

$$\rho(\mathcal{M}_1 \oplus \mathcal{M}_2) \quad = \quad \rho(\mathcal{M}_1) + \rho(\mathcal{M}_2). \tag{4.17}$$

and so $\mathcal{M}_1 \oplus \mathcal{M}_2$ is bounded iff both $\mathcal{M}_1$ and $\mathcal{M}_2$ are bounded.

Observe that for any $\ell \in \mathbb{N}$, we have $(\mathcal{M}_1 \oplus \mathcal{M}_2) \wedge [1, \ell] = (\mathcal{M}_1 \wedge [1, \ell]) \oplus (\mathcal{M}_2 \wedge [1, \ell])$. It follows that for any $N \in \mathbb{N}$, $(\mathcal{M}_1 \oplus \mathcal{M}_2) \cap \mathcal{B}_N$

is determined by $\mathcal{M}_1 \cap \mathcal{B}_N$ and $\mathcal{M}_2 \cap \mathcal{B}_N$. Hence, the map $\oplus : \mathcal{LAB} \times \mathcal{LAB} \to \mathcal{LAB}$ is continuous.

Lemma 4.39. *(a) If $\mathcal{M}$ is a recurrent label, then $\mathcal{M}_1 \oplus \mathcal{M}$ is a recurrent label for any label $\mathcal{M}_1$. If $\mathcal{M}$ is a strongly recurrent label, then $\mathcal{M}_1 \oplus \mathcal{M}$ is a strongly recurrent label for any finite label $\mathcal{M}_1$.*

(b) If $\mathcal{M}$ is any label and $\ell \in \mathbb{N}$ then there exists a recurrent label $\mathcal{M}_1$ with $\mathcal{M} \subset \mathcal{M}_1$ and $\mathcal{M} \wedge [1,\ell] = \mathcal{M}_1 \wedge [1,\ell]$. There exists a strongly recurrent label $\mathcal{M}_2$ with $\mathcal{M} \wedge [1,\ell] = \mathcal{M}_2 \wedge [1,\ell]$.

Proof: (a) Clearly $\mathcal{M}_1 \oplus (\mathcal{M} - \mathbf{r}) \subset (\mathcal{M}_1 \oplus \mathcal{M}) - \mathbf{r}$ for any $\mathbf{r} \in \mathcal{M}$. Hence, if $\{\mathcal{M} - \mathbf{r}^i\} \to \mathcal{M}$ then $\{(\mathcal{M}_1 \oplus \mathcal{M}) - \mathbf{r}^i\} \to \mathcal{M}_1 \oplus \mathcal{M}$ and so $\mathcal{M}_1 \oplus \mathcal{M}$ is recurrent.

If $\mathbf{m}_1 \in \mathcal{M}_1$ and $\mathbf{m} \in \mathcal{M}$ we let $F(\mathbf{m}_1 + \mathbf{m}) = \bigcup Supp\ \mathcal{M}_1 \cup F(\mathbf{m})$ (see Definition 4.22). If $\mathbf{r}_1 + \mathbf{r}$ has support disjoint from this set then $\mathbf{r}_1 = 0$ and $\mathbf{r} + \mathbf{m} \in \mathcal{M}$. Hence, $\mathbf{r}_1 + \mathbf{r} + \mathbf{m} \in \mathcal{M}_1 \oplus \mathcal{M}$.

(b) Build a recurrent label $\mathcal{N}_1$ with $\mathcal{N}_1 \wedge [1,\ell] = 0$. Then $\mathcal{M}_1 = \mathcal{M} \oplus \mathcal{N}_1$ is a recurrent label with $\mathcal{M} \wedge [1,\ell] = \mathcal{M}_1 \wedge [1,\ell]$. If one starts with a strongly recurrent label $\mathcal{N}_2$ with $\mathcal{N}_2 \wedge [1,\ell] = 0$ then $\mathcal{M}_2 = (\mathcal{M} \wedge [1,\ell]) \oplus \mathcal{N}_2$ is a strongly recurrent label.

□

Example 4.40. If L is an infinite set disjoint from $\bigcup Supp\ \mathcal{M}$. Let $\mathcal{M}_1 = \{\emptyset\} \cup \{\ \chi(\ell)\ :\ \ell \in L\ \}$. If $\mathcal{M}$ is strongly recurrent then $\mathcal{M}_1 \oplus \mathcal{M}$ is recurrent by (a) but it is not strongly recurrent.

□

Proposition 4.41. *The set $RECUR$ of recurrent labels is a dense, G_δ subset of $\mathcal{LAB}$.*

Proof: For any $N \in \mathbb{N}$ and $\mathbf{r} \in FIN(\mathbb{N})$, the set $\{\ \mathcal{M}\ :\ \mathcal{M} \cap \mathcal{B}_N = P_{\mathbf{r}}(\mathcal{M}) \cap \mathcal{B}_N\ \}$ is a clopen subset of $\mathcal{LAB}$ by Lemma 4.9 (c) and continuity of the map $P_{\mathbf{r}}$.

Clearly,

$$RECUR = \bigcap_{N \in \mathbb{N}} \bigcup_{\mathbf{r} \in FIN} \{\ \mathcal{M}\ :\ \mathcal{M} \cap \mathcal{B}_N = P_{\mathbf{r}}(\mathcal{M}) \cap \mathcal{B}_N\ \} \tag{4.18}$$

and so $RECUR$ is a G_δ subset of $\mathcal{LAB}$. It is dense by Lemma 4.39 (b).

□

Thus, the labels of finite type, upon which we focus most of our attention, comprise a subset of first category. Since the finite labels are dense, the labels of finite type are dense. Thus, the set of recurrent labels has empty interior. As the intersection of dense G_δ sets the set of bounded recurrent labels is a dense G_δ set.

We will need a combinatorial lemma. Let M be a collection of finite subsets of $\mathbb{N}$. Call it *hereditary* if $A \subset B$ and $B \in M$ implies $A \in M$. We say that M f-contains $L \subset \mathbb{N}$ if every finite subset of L is in M, i.e. $\mathcal{P}_f L \subset M$. For two hereditary collections M_1, M_2 define $M_1 \oplus M_2 = \{A_1 \cup A_2 : A_1 \in M_1, A_2 \subset M_2\}$.

Lemma 4.42. *Let M_1 and M_2 be hereditary collections of finite subsets of $\mathbb{N}$. If $M_1 \oplus M_2$ f-contains an infinite set then either M_1 or M_2 f-contains an infinite set.*

Proof: Let $L = \{\ell_1, \ell_2, \dots\}$ be a counting for an infinite set f-contained in $M_1 \oplus M_2$. Define the directed binary tree with vertices at level $n = 0, 1, \dots$ consisting of the 2^n ordered partitions (A, B) of $\{\ell_1, \dots, \ell_n\}$. Connect (A, B) to the $n+1$ level vertices $(A \cup \{\ell_{n+1}\}, B)$ and $(A, B \cup \{\ell_{n+1}\})$. The set of paths to infinity form a Cantor set. Call a path *good at level* n if for the partition (A_n, B_n) at level n, $A_n \in M_1$ and $B_n \in M_2$. Since $M_1 \oplus M_2$ f-contains L, the set G_n of paths good at level n is nonempty. Each G_n is closed and $G_{n+1} \subset G_n$. So the intersection contains a path $\{(A^i, B^i) : i = 0, 1, \dots\}$. Let $A^\infty = \bigcup A^i, B^\infty = \bigcup B^i$. Clearly, $\{A^i\}$ is a nondecreasing sequence of finite sets in M_1 with union A^∞ and so A^∞ is f-contained in M_1. Similarly, B^∞ is f-contained in M_2. Since $A^\infty \cup B^\infty = L$, at least one of them is infinite.

$\Box$

For a label $\mathcal{M}$, if $\mathbf{m} \in \mathcal{M}$ then $\mathbf{0} \leq \chi(\mathit{supp}\ \mathbf{m}) \leq \mathbf{m}$ and so $\chi(\mathit{supp}\ \mathbf{m}) \in \mathcal{M}$. Hence, $\mathit{Supp}\ \mathcal{M} = \{ A \subset \mathbb{N} : \chi(A) \in \mathcal{M}\}$. Thus, $\mathit{Supp}\ \mathcal{M}$ is a hereditary collection of finite subsets of $\mathbb{N}$ with $\mathit{Supp}\ \mathcal{M} = \emptyset$ iff $\mathcal{M} = \emptyset$. Thus, Proposition 4.7(b) says that a bounded label $\mathcal{M}$ is not of finite type iff $\mathit{Supp}\ \mathcal{M}$ f-contains some infinite subset.

Theorem 4.43. *Let $\mathcal{N}$ be a nonempty label.*

(a) *Assume $\mathcal{M}$ is a nonempty label. The nonempty label $\mathcal{N} \oplus \mathcal{M}$ is of finite type iff $\mathcal{N}$ and $\mathcal{M}$ are labels of finite type.*

If $\mathcal{N} \cap \mathcal{M} = 0$ and both $\mathcal{N}$ and $\mathcal{M}$ are simple, then $\mathcal{N} \oplus \mathcal{M}$ is simple.

If $\mathcal{N}$ is finite and $\mathcal{M}$ is finitary, then $\mathcal{N} \oplus \mathcal{M}$ is finitary.

(b) *If $\{\mathcal{M}_a\}$ is a finite or infinite sequence of labels such that $\mathcal{M}_a \cap \mathcal{M}_b \subset \mathcal{N}$ when $a \neq b$, then $\mathcal{M} = \bigcup_a \mathcal{M}_a$ is a label which is* *bounded if all the $\mathcal{M}_a$'s and $\mathcal{N}$ are bounded, and of finite type if all the $\mathcal{M}_a$'s and $\mathcal{N}$ are of finite type.*

If $\mathcal{N}$ is finite and the $\mathcal{M}_a$'s are all finitary then $\bigcup_a \mathcal{M}_a$ is finitary.

If $\mathcal{N} = 0$ and the $\mathcal{M}_a$'s are all simple then $\bigcup_a \mathcal{M}_a$ is simple.

Proof: Notice first that for labels $\mathcal{M}_1, \mathcal{M}_2$

$$\bigcup Supp(\mathcal{M}_1 \cap \mathcal{M}_2) = (\bigcup Supp\, \mathcal{M}_1) \cap (\bigcup Supp \mathcal{M}_2). \tag{4.19}$$

For if $\ell \in (\bigcup Supp\, \mathcal{M}_1) \cap (\bigcup Supp \mathcal{M}_2)$ then $\chi(\ell) \in \mathcal{M}_1 \cap \mathcal{M}_2$.

With $\mathcal{M}$ and $\mathcal{N}$ labels with $\mathcal{N}$ finite, we let $\mathcal{N}^+$ be the finite label consisting of the $\mathbb{N}$-vectors $\mathbf{m}$ with support contained in $\bigcup Supp\, \mathcal{N}$ and with $\mathbf{m} \leq \rho(\mathcal{N}) + \rho(\mathcal{M})$. Let $\mathcal{M} \ominus \mathcal{N} = \{\mathbf{m} \in \mathcal{M}$ with $supp\ \mathbf{m} \cap \bigcup Supp\, \mathcal{N} = \emptyset\}$. Any finite vector $\mathbf{m} \in \mathcal{N} \oplus \mathcal{M}$ can be written uniquely as $\mathbf{m}_{\mathcal{N}} + \mathbf{m}_{\mathcal{M} \ominus \mathcal{N}}$ where $\mathbf{m}_{\mathcal{N}} \in \mathcal{N}^+$ and $\mathbf{m}_{\mathcal{M} \ominus \mathcal{N}} \in \mathcal{M} \ominus \mathcal{N}$.

(a) Assume that $\mathcal{M}$ and $\mathcal{N}$ are nonempty labels of finite type and so $\mathcal{N} \oplus \mathcal{M}$ is bounded. If $\mathcal{N} \oplus \mathcal{M}$ is not of finite type then $Supp\ (\mathcal{N} \oplus \mathcal{M})$ f-contains an infinite set by Proposition 4.7(c).

Since $Supp\ (\mathcal{N} \oplus \mathcal{M}) = (Supp\, \mathcal{N}) \oplus (Supp\, \mathcal{M})$, by Lemma 4.42, either $Supp\, \mathcal{N}$ or $Supp\, \mathcal{M}$ f-contains an infinite set. Thus, $\mathcal{N}$ and $\mathcal{M}$ of finite type implies $\mathcal{N} \oplus \mathcal{M}$ is of finite type. The converse is obvious since $\mathcal{N} \cup \mathcal{M} \subset \mathcal{N} \oplus \mathcal{M}$.

Now define the disjoint sets $L_0 = (\bigcup Supp\, \mathcal{N}) \setminus (\bigcup Supp\, \mathcal{M}), L_1 = (\bigcup Supp\, \mathcal{M}) \setminus (\bigcup Supp\, \mathcal{N})$ and $L_{01} = (\bigcup Supp\, \mathcal{N}) \cap (\bigcup Supp\, \mathcal{M})$. If $supp\ \mathbf{m} \subset (\bigcup Supp\, \mathcal{N}) \cup (\bigcup Supp\, \mathcal{M})$ then we can decompose $\mathbf{m} = \mathbf{m}_0 + \mathbf{m}_1 + \mathbf{m}_{01}$ with $\mathbf{m}_\epsilon = \mathbf{m} \wedge L_\epsilon$ for $\epsilon = 0, 1$ or 01.

Assume $\mathcal{N}$ and $\mathcal{M}$ are simple and $\mathcal{N} \cap \mathcal{M} = 0$ and so $L_{01} = \emptyset$. We have $\mathbf{w} \in (\mathcal{N} \oplus \mathcal{M}) - \mathbf{r}$ for $\mathbf{r} \in \mathcal{N} \oplus \mathcal{M}$, iff $\mathbf{w}_0 \in \mathcal{N} - \mathbf{r}_0$ and $\mathbf{w}_1 \in \mathcal{M} - \mathbf{r}_1$. Thus, if $\{(\mathcal{N} \oplus \mathcal{M}) - \mathbf{r}^i\}$ is convergent with a nonempty limit then the limit is $(LIM\ \{\mathcal{N} - \mathbf{r}_0^i\}) \oplus (LIM\ \{\mathcal{N} - \mathbf{r}_1^i\})$. Because $\mathcal{N}$ and $\mathcal{M}$ are simple this is $(\mathcal{N} - \mathbf{n}) \oplus (\mathcal{M} - \mathbf{m}) = (\mathcal{N} \oplus \mathcal{M}) - (\mathbf{n} + \mathbf{m})$. Hence, $\mathcal{N} \oplus \mathcal{M}$ is simple. The empty limit is $(\mathcal{N} \oplus \mathcal{M}) - \mathbf{r}$ with $\mathbf{r} \notin (\mathcal{N} \oplus \mathcal{M})$.

Now assume $\mathcal{M}$ is finitary and $\mathcal{N}$ is finite so that $L_0 \cup L_{01}$ is finite. Let $\{\mathbf{r}^i\}$ be a sequence in $\mathcal{N} \oplus \mathcal{M}$ with $\bigcup_i \{supp\ \mathbf{r}^i\}$ infinite. Then $\bigcup_i \{supp\ \mathbf{r}_1^i\}$ is infinite since $\bigcup Supp\, \mathcal{N} = L_0 \cup L_{01}$ is finite. If $\mathbf{m}_0 + \mathbf{m}_1 + \mathbf{m}_{01} + \mathbf{r}^i \in \mathcal{N} \oplus \mathcal{M}$ then $\mathbf{m}_1 + \mathbf{r}_1^i \in (\mathcal{N} \oplus \mathcal{M}) \wedge L_1 \subset \mathcal{M}$. Because $\mathcal{M}$ is finitary, there are only finitely many $\mathbf{m}_1$ such that eventually $\mathbf{m}_1 + \mathbf{r}_1^i \in \mathcal{M}$. Since there are only finitely many $\mathbf{m}_0$'s and $\mathbf{m}_{01}$'s it follows that $LIMINF\{\mathcal{N} \oplus \mathcal{M} - \mathbf{r}_1^i\}$ is finite. Since this contains $LIMINF\{\mathcal{N} \oplus \mathcal{M} - \mathbf{r}^i\}$ it follows that $\mathcal{N} \oplus \mathcal{M}$ is finitary.

(b) Let $a \neq b$. If $\rho(\mathcal{M}_a)_\ell > \rho(\mathcal{M}_b)_\ell$ then $\rho(\mathcal{M}_b)_\ell \chi(\ell) \in \mathcal{M}_a \cap \mathcal{M}_b \subset \mathcal{N}$. Thus, for each ℓ there is at most one index a with $\rho(\mathcal{M}_a)_\ell > \rho(\mathcal{N})_\ell$.

Hence, $\rho(\mathcal{M})_\ell \;=\; max_a \; \rho(\mathcal{M}_a)_\ell \;<\; \infty$ for all ℓ and so the Bound Condition is satisfied if all the $\mathcal{M}_a$'s and $\mathcal{N}$ are bounded.

If $\mathbf{m}_a \in \mathcal{M}_a, \; \mathbf{m}_b \in \mathcal{M}_b$ with $\mathbf{m}_a \geq \mathbf{m}_b$ then $\mathbf{m}_b \in \mathcal{M}_a \cap \mathcal{M}_b \subset \mathcal{N}$. Thus, an increasing sequence in $\mathcal{M}$ which is not contained in $\mathcal{N}$ is contained in some $\mathcal{M}_a$. Thus, the Finite Chain Condition for $\mathcal{M}$ follows from the condition for each $\mathcal{M}_a$.

Let $L_{\mathcal{M}\ominus\mathcal{N}} = (\bigcup \; Supp \; \mathcal{M}) \setminus (\bigcup \; Supp \; \mathcal{N}) = \bigcup_a \; \{L_a\}$ and $L_{\mathcal{N}} = \bigcup \; Supp \; \mathcal{N}$. For $\mathbf{m} \in \mathcal{M}$ let $\mathbf{m}_{\mathcal{N}} = \mathbf{m} \wedge L_{\mathcal{N}}$ and $\mathbf{m}_{\mathcal{M}\ominus\mathcal{N}} = \mathbf{m} \wedge L_{\mathcal{M}\ominus\mathcal{N}}$. If $\mathbf{m}_{\mathcal{M}\ominus\mathcal{N}} > \mathbf{0}$ then $\mathbf{m} \in \mathcal{M}_a$ for a unique a.

Assume that $\mathcal{N} = 0$ and the $\mathcal{M}_a$'s are simple. We have $\mathcal{M} = \mathcal{M} - 0, 0 = \mathcal{M} - \mathbf{r}$ with $\mathbf{r} \in max \; \mathcal{M}$ and $\emptyset = \mathcal{M} - \mathbf{r}$ with $\mathbf{r} \not\in \mathcal{M}$. Now note that $\mathcal{M}_a \ni \mathbf{r} > 0$ implies $\mathcal{M}_b - \mathbf{r} = \emptyset$ if $b \neq a$ and so $\mathcal{M} - \mathbf{r} = \mathcal{M}_a - \mathbf{r}$. Thus, if $\{\mathcal{M} - \mathbf{r}^i\}$ is convergent with limit not equal to $\mathcal{M}, 0$ or $\emptyset$, there is then a unique a such that eventually $\mathbf{r}^i \in \mathcal{M}_a$ and $\{\mathcal{M}_a - \mathbf{r}^i\}$ is convergent and with the same limit. Since $\mathcal{M}_a$ is simple, this limit is $\mathcal{M}_a - \mathbf{r}$ for some $\mathbf{r} \in \mathcal{M}_a$ and $\mathbf{r} > \mathbf{0}$. It follows that $\mathcal{M} - \mathbf{r} = \mathcal{M}_a - \mathbf{r}$ is the limit. Hence, $\mathcal{M}$ is simple.

Now assume that $\mathcal{N}$ is finite and that the $\mathcal{M}_a$'s are finitary. Let $\{\mathbf{r}^i \in \mathcal{M}\}$ with $\bigcup_i \{supp \; \mathbf{r}^i\}$ infinite. Again $\bigcup_i \{supp \; \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}}\}$ is infinite. If $\mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} > \mathbf{0}$ then it is contained in a unique $\mathcal{M}_{a(i)}$ and so if $\mathbf{m} + \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} \in \mathcal{M}$ then $\mathbf{m} + \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} \in M_{a(i)}$ and so $\mathbf{m} \in \mathcal{M}_{a(i)}$.

Case (i) - There exists a such that eventually $\mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} > \mathbf{0}$ implies $a(i) = a$. In that case, if $\mathbf{m} + \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} \in \mathcal{M}$ eventually then $\mathbf{m} + \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} \in \mathcal{M}_a$ eventually and so $LIMINF\{\mathcal{M} - \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}}\} = LIMINF\{\mathcal{M}_a - \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}}\}$ and the latter is finite because $\mathcal{M}_a$ is finitary.

Case (ii) - For every I there exist $i_1, i_2 \geq I$ with $\mathbf{r}^{i_1}_{\mathcal{M}\ominus\mathcal{N}}, \mathbf{r}^{i_2}_{\mathcal{M}\ominus\mathcal{N}} > \mathbf{0}$ and $a(i_1) \neq a(i_2)$. In that case, if $\mathbf{m} + \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}} \in \mathcal{M}$ eventually then $\mathbf{m} \in \mathcal{M}_{a(i_1)} \cap \mathcal{M}_{a(i_2)}$ with $a(i_1) \neq a(i_2)$ and so $\mathbf{m} \in \mathcal{N}$. Hence, $LIMINF\{\mathcal{M} - \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}}\} \subset \mathcal{N}$ which is finite.

Thus, in either case $LIMINF\{\mathcal{M} - \mathbf{r}^i_{\mathcal{M}\ominus\mathcal{N}}\}$ is finite. As before this contains $LIMINF\{\mathcal{M} - \mathbf{r}^i\}$ and so $\mathcal{M}$ is finitary.

□

Remark 4.44. (a) Since any label $\mathcal{M}$ is the union of the finite labels $\mathcal{M} \cap \mathbb{N}_N$ it follows that some condition like that in (b) above is needed to get the finite type or finitary conditions.

(b) If $\mathcal{N} \oplus \mathcal{M}$ is finitary and $\mathcal{M}$ is infinite then $\mathcal{N}$ must be finite since if $\{\mathbf{r}^i\}$ is an infinite sequence of distinct positive vectors in $\mathcal{M}$ then $\mathcal{N} \subset LIMINF \; \{(\mathcal{N} \oplus \mathcal{M}) - \mathbf{r}^i\}$.

□

Example 4.45. (a) If $\mathcal{M}$ is defined by $\mathcal{M} = \langle\{\chi(1) + \chi(\ell) : \ell > 1\} \cup \{\chi(\ell) + \chi(\ell+1) : \ell > 1\}\rangle$, then $\mathcal{M}$ is size bounded and **finitary, but not simple**.

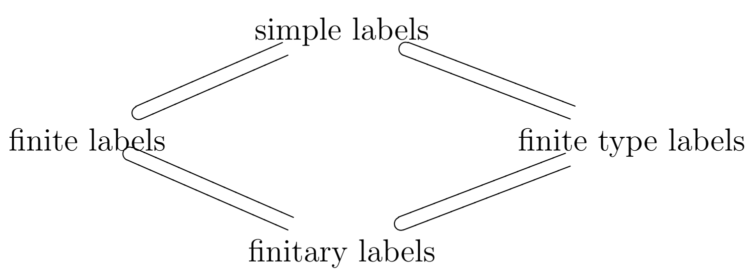
$$\begin{aligned} \mathcal{M} - \chi(1) &= \{0\} \cup \{\chi(\ell) : \ell > 1\}, \\ \mathcal{M} - \chi(\ell) &= \{0\} \cup \{\chi(1), \chi(\ell - 1), \chi(\ell + 1)\}, \quad \text{for } \ell > 1, \\ \mathcal{F} = \{0, \chi(1)\} &= LIM_{\ell\to\infty}\{\mathcal{M} - \chi(\ell)\}. \end{aligned} \tag{4.20}$$

Notice that $\mathcal{F} \neq \mathcal{M} - \mathbf{r}$ for any $\mathbf{r} \in \mathcal{M}$.

(b) If $\mathcal{M}$ be defined by $\mathcal{M} = \langle\{\chi(2a - 1) + \chi(2b) : a, b \geq 1\}\rangle$, then $\mathcal{M}$ is size bounded and is simple but is not finitary. In general, if $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_1, \mathcal{M}_2$ infinite simple labels with $\mathcal{M}_1 \cap \mathcal{M}_2 = 0$ then $\mathcal{M}$ is **simple but not finitary**.

□

Summarizing we have the following inclusions:



5. Labeled subshifts

5.1. Expanding functions.

Call h a function on $\mathbb{Z}$ when $h : \mathbb{Z} \to \mathbb{Z}$. A function h on $\mathbb{Z}$ is *odd* when $h(-j) = -h(j)$ for all $j \in \mathbb{Z}$ and so, in particular, $h(0) = 0$. We call h *increasing* when $h(j + 1) > h(j)$ for all $j \in \mathbb{Z}$. When h is odd, then it is increasing when its restriction to $\mathbb{Z}_+$ is increasing. Then, of course, h is positive on $\mathbb{N}$. For an odd increasing function $|h(j)| = h(|j|)$ for all $j \in \mathbb{Z}$.

On $\mathbb{Z}$ let ϵ be the *signum function* so that

$$\epsilon(N) \quad = \quad \begin{cases} +1 & N > 0, \\ 0 & N = 0 \\ -1 & N < 0. \end{cases} \tag{5.1}$$

For integers a, b we will denote by $[\ a \pm b\ ]$ the interval $[\ a-|b|, a+|b|\ ]$ in $\mathbb{Z}$. When $a = 0$ we will write $[\ \pm b\ ]$ for the interval $[\ -|b|, +|b|\ ]$.

Let b be a positive integer. Call a function $k : \mathbb{Z} \to \mathbb{Z}$ a *b-expanding function* when it satisfies

- $k(-n) \ = \ -k(n)$ for all $n \in \mathbb{Z}$.
- $k(n+1) \ > \ b \cdot \Sigma_{i=0}^{n} \ k(i)$ and, in particular, $k(n) > 0$, for all $n \geq 0$.

Thus, k is an odd, increasing function and so $k(|n|) = |k(n)|$ for all $n \in \mathbb{Z}$.

Define $k_0(n) = \epsilon(n)(1+b)^{|n|-1}$.

Define for k the function k^+ by

$$k^+(n) \quad = \quad \epsilon(n)k(|n|+1). \tag{5.2}$$

When k is b-expanding, $k^+(n) > bk(n)$ for all $n > 0$.

Define for k the function $sk : \mathbb{N} \to \mathbb{N}$ by

$$sk(n) \quad = \quad \begin{cases} 1 & \text{for } n = 1, \\ \Sigma_{i=1}^{n-1} \ k(i) & \text{for } n > 1. \end{cases} \tag{5.3}$$

Lemma 5.1. (i) *If k is b-expanding and C is a positive integer then $C \cdot k$ is b-expanding. If h, k are b-expanding then $h + k$ is b-expanding.*

(ii) *k_0 is b-expanding and if k is b-expanding then $k(n) \geq k(1)k_0(n)$ for all $n \geq 0$.*

(iii) *Assume that $h : \mathbb{Z} \to \mathbb{Z}$ is an increasing, odd function. If k is b-expanding then $k \circ h$ is b-expanding, and so, if h, k are both b-expanding then $k \circ h$ is b-expanding.*

(iv) *If k is b-expanding, then k^+ is b-expanding.*

(v) *If k is b-expanding, then $n \geq 0$ implies*

$$k(n+2) - k(n+1) \quad > \quad (b-1)(k(n+1) - k(n)) \tag{5.4}$$

and so the sequence of successive differences $\{\ k(n+1) - k(n) : n \in \mathbb{Z}_+\}$ is a strictly increasing sequence of positive integers with $k(n+1) - k(n) \ \geq \ k(1)(b-1)^n$.

(vi) *If k is b-expanding and $k(1) \geq b+1$ then $k(n) > bsk(n) \geq bn$ for all $n \in \mathbb{N}$ and $\{\ [\ k(n) \ \pm \ \frac{b}{2}sk(|n|)\ ] \ : \ n \in \mathbb{Z}\ \}$ is a pairwise* *disjoint sequence of intervals in* $\mathbb{Z}$ *(for* $n = 0$ *the empty sum is zero and the interval is* $[0]$*).*

Proof: (i) Obvious.

(ii) By summing the geometric series we see that for $n \geq 0, k_0(n+1) = 1 + b\Sigma_{i=0}^{n} k_0(i)$. So k_0 is b-expanding. If k is b-expanding then $k(1) = k(1)k_0(1)$ and so by induction $k(n+1) > b\Sigma_{j=0}^{n+1} k(j) \geq k(1)b\Sigma_{j=0}^{n-1} k_0(j) = k(1)(k_0(n) - 1) > k(1)k_0(n)$.

(iii) Assume $n \geq 0$. Since h is a strictly increasing function,

$$\{h(0), h(1), \dots, h(n)\} \subset \{0, 1, \dots, h(n+1) - 1\}.$$

Hence,

$$k(h(n+1)) \; > \; b\Sigma_{i=0}^{h(n+1)-1} k(i) \; \geq \; b\Sigma_{j=0}^{n} k(h(j)). \tag{5.5}$$

Thus, $k \circ h$ is b-expanding.

(iv) Define $h(n) = \epsilon(n)(|n| + 1)$. From (iii) it follows that $k^+ = k \circ h$ is b-expanding.

(v) $k(n+2) > bk(n+1)$ and so $k(n+2) - k(n+1) \; > \; (b-1)k(n+1) \; \geq (b-1)(k(n+1) - k(n))$. So the sequence of differences is increasing and, by induction, $k(n+1) - k(n) \; \geq \; (b-1)^n k(1)$.

(vi) For $n = 1$, $k(1) \geq 1 + b > b = bsk(1) = b \cdot 1$. For $n > 1$, $k(n) > bsk(n)$ is the definition of a k expanding function. $sk(2) = k(1) \geq 1 + b > 2$. Then $sk(n+1) > sk(n)$ implies $sk(n) \geq n$. Finally, we observe that

$$\begin{aligned} k(|n+1|) - \frac{b}{2}sk(n+1) \; &> \; \frac{b}{2}sk(n+1) \\ = \; \frac{b}{2} \, (k(|n|) + sk(n)) \; &> \; k(|n|) + \frac{b}{2}sk(n). \end{aligned} \tag{5.6}$$

Since $b \geq 3$, $\frac{b}{2} > 1$.

□

With $b \geq 3$ we now fix a b-expanding function k with $k(1) > b + 1$ so that $k(n) \geq (1+b)^n$ for all $n \in \mathbb{N}$, e.g. $k(n) = \epsilon(n)4^{|n|}$.

Let $IP(k)$ denote the set of sums of finite subsets of the image set $k(\mathbb{Z}) \subset \mathbb{Z}$ and let $IP_+(k)$ denote the set of sums of finite subsets of the image set $k(\mathbb{Z}_+) \subset \mathbb{Z}_+$.

Definition 5.2. (a) An *expansion of length* $r \geq 0$ for $t \in \mathbb{Z}$ is a finite sequence $j_1, \dots, j_r \in \mathbb{Z}$ such that $|j_i| > |j_{i+1}|)$ for $i = 1, \dots, r-1$ and with

$$t \; = \; k(j_1) + k(j_2) + \cdots + k(j_r).$$

$0 \in \mathbb{Z}$ has the empty expansion with length 0.

(b) For $0 \leq \tilde{r} \leq r$ the $\tilde{r}$ *truncation* is the element $\tilde{t} \in \mathbb{Z}$ with expansion $j_1, \dots, j_{\tilde{r}}$ so that

$$\tilde{t} \ = \ k(j_1) + k(j_2) + \cdots + k(j_{\tilde{r}}).$$

The sequence $j_{\tilde{r}+1}, \dots, j_r$ is an expansion for the $\tilde{r}$ *residual* $t - \tilde{t}$ with length $r - \tilde{r}$. We call $\tilde{t}$ a *truncation* of t and t an *extension* of $\tilde{t}$. When $\tilde{r} = 0$ the truncation $\tilde{t} = 0$ with the empty expansion.

When there is an expansion for t we will say that t is *expanding* or that t is an *expanding time*. When there is an expansion for t with $j_1 > \cdots > j_r > 0$ we will say that t is *positive expanding* or a *positive expanding time*.

Using properties of b-expanding functions we obtain various useful estimates for expansions and their truncations. The key idea is that the expanding functions grow so rapidly that the expansion of t is dominated by its leading term $k(j_1)$.

Lemma 5.3. *If t has an expansion $j_1, \dots, j_r$ with $r \geq 1$ then $t \neq 0$ and t has the same sign as $k(j_1)$. Furthermore,*

$$\begin{aligned} |t| \quad &\geq \quad (b-1) \cdot \Sigma_{i=2}^{r} \ |k(j_i)|, \\ \frac{b+1}{b}|k(j_1)| \quad \geq \quad |t| \quad &\geq \quad \frac{b-1}{b}|k(j_1)|, \end{aligned} \tag{5.7}$$

If $\tilde{t}$ is the $\tilde{r}$ truncation of t then

$$\frac{b+1}{b}k(|j_{\tilde{r}+1}|) \ > \ |t - \tilde{t}| \ > \ \frac{b-1}{b}k(|j_{\tilde{r}+1}|). \tag{5.8}$$

Proof: Clearly,

$$|t \ - \ k(j_1)| \quad \leq \quad \Sigma_{i=2}^{r} \ |k(j_i)| \quad \leq \quad sk(|j_1|). \tag{5.9}$$

The sequence $\{ \ |j_i| \ : i = 2, \dots, r \ \}$ consists of distinct positive integers all less than $|j_1|$. So we obtain

$$|k(j_1)| \quad \geq \quad b \cdot sk(|j_1|) \quad \geq \quad b \cdot \Sigma_{i=2}^{r} \ |k(j_i)|. \tag{5.10}$$

From(5.9) and (5.10) we see that

$$|t \ - \ k(j_1)| \quad \leq \quad \frac{1}{b}k(|j_1|). \tag{5.11}$$

It follows that $t \neq 0$ and t has the same sign as $k(j_1)$. The estimates of (5.7) follow as well.

Now we apply (5.10) to the expansion of $t - \tilde{t}$ to get

$$\frac{1}{b}k(|j_{\tilde{r}+1}|) \ \geq \ \Sigma_{\tilde{r}+1 < i \leq r} \ |k(j_i)|. \tag{5.12}$$

Thus, (5.8) follows from

$$k(|j_{\tilde{r}+1}|) + \Sigma \ \geq \ |t - \tilde{t}| \ \geq \ k(|j_{\tilde{r}+1}|) - \Sigma \tag{5.13}$$

where $\Sigma = \Sigma_{\tilde{r}+1 < i \leq r} \ |k(j_i)|$.
□

Now suppose that $j_1(t), \ldots, j_{r(t)}(t)$ for t and $j_1(s), \ldots, j_{r(s)}(s)$ for s are different expansions. By convention, let $j_{r(t)+1}(t) = j_{r(s)+1}(s) = 0$. Let $\tilde{r}$ be the smallest positive integer such that $j_{\tilde{r}}(t) \neq j_{\tilde{r}}(s)$ so that $1 \leq \tilde{r} \leq min(r(s), r(t)) + 1$.

Lemma 5.4.

$$|t - s| \ > \ \frac{b-2}{b} \cdot max(\ |k(j_{\tilde{r}}(t))|, |k(j_{\tilde{r}}(s))| \ ).$$

Proof: Let u be the common $\tilde{r} - 1$ truncation of t and s. If $\tilde{r} = 1$, then $u = 0$.

We replace t and s by their $\tilde{r} - 1$ residuals $t - u$ and $s - u$, and observe that $|t - s| = |(t - u) - (s - u)|$.

Thus, we reduce to the case that $\tilde{r} = 1$ and prove that $|t - s| > \frac{b-2}{b} \cdot max(\ |k(j_1(t))|, |k(j_1(s))|)$. Note that we include the case when $r(t) = 0$ and so $t = j_1(t) = 0$ or the similar possibility for s. Since the $j_1(t) \neq j_1(s)$ these two cannot both happen.

Case 1 - $j_1(t) = -j_1(s)$: By Lemma 5.3 t and s are nonzero and have opposite signs. Hence, $|t-s| = |t|+|s| > \frac{b-1}{b} \cdot max(\ k(|j_1(t)|, k(|j_1(s)|) \ )$ by (5.7).

Case 2 - $|j_1(t)| \neq |j_1(s)|$: By symmetry we can assume that $|j_1(t)| > |j_1(s)|$. Then $|t| \geq k(|j_1(t)|) - sk(|j_1(t)|)$ and $|s| \leq sk(|j_1(t)|)$. Furthermore, $sk(|j_1(t)|) < \frac{1}{b} k(|j_1(t)|)$. Hence,

$$|t - s| \quad \geq \quad k(|j_1(t)|) - 2sk(|j_1(t)|) \quad > \quad \frac{b-2}{b} k(|j_1(t)|). \tag{5.14}$$

In this case, $k(|j_1(t)|) = max(\ k(|j_1(t)|), k(|j_1(s)|) \ )$.
□

Proposition 5.5. *An integer $t \in \mathbb{Z}$ is expanding iff $t \in IP(k)$, in which case its expansion is unique. An integer $t \in \mathbb{Z}_+$ is in $IP_+(k)$ iff it is expanding with $j_1 > \cdots > j_r > 0$.*

If t is expanding with expansion length r and s is expanding then s is an extension of t iff $|s - t| \leq \frac{b-2}{b} k(|j_r(t)|)$.

Proof: It is clear that t admits an expansion iff $t \in IP(k)$ and is in $IP_+(k)$ iff the terms of the expansion are positive.

Lemma 5.4 implies that if expansions of s and t differ then $|s-t| > 0$. Hence, the expansion of each $t \in IP(k)$ is unique.

If $t = \tilde{s}$ then $r = \tilde{r}(s)$ and so by (5.8) applied to s

$$|s-t| \ \leq \ \frac{b+1}{b}k(|j_{r+1}(s)|). \tag{5.15}$$

Because k is b-expanding, $\frac{1}{b}k(|j_r(t)|) \ = \ \frac{1}{b}k(|j_r(s)|) \geq k(|j_{r+1}(s)|)$. Since $b > 4$, $\frac{b+1}{b^2} < \frac{b-2}{b}$.

If s is not an extension of t then in Lemma 5.4 $\tilde{r} \leq r$. Hence, the lemma implies that $|s-t| > \frac{b-2}{b}k(|j_r(t)|)$.

□

Corollary 5.6. *If $t \in IP(k)$ has expansion $j_1, \dots, j_r$ with $r > 0$ and $1 \geq i^* \geq r$ then $t^* = t - k(j_{i^*})$ is the unique member of $IP(k)$ with $|t - t^*| = |k(j_{i^*})|$, i.e. $t + k(j_{i^*}) \notin IP(k)$.*

Proof: Clearly, t^* has expansion of length $r-1$ obtained by omitting j_{i^*} and so $t^* \in IP(k)$ with $|t-t^*| = |k(j_{i^*})|$. Now let $s = t + k(j_{i^*})$. We assume that $s \in IP(k)$ and so has an expansion $j_1(s), \dots, j_{r(s)}(s)$ which differs from that of t since $s \neq t$. As above let $\tilde{r}$ be the smallest positive integer at which the expansions of t and s disagree. By Lemma 5.4 if $\tilde{r} > i^*$ then $|t-s| \ > \ \frac{b-2}{b} \cdot \ |k(j_{\tilde{r}}(t))| > (b-2)|k(j_{i^*})|$ would contradict $|t-s| = |k(j_{i^*})|$. So t and s have a common i^*-1 truncation u. By subtracting u and replacing t and s by $t-u$ and $s-u$ we reduce to the case with $i^* = 1$. Thus, we derive the contradiction by showing that with $t \neq 0, \ s = t + j_1(t) \notin IP(k)$.

By Lemma 5.1 (vi) the intervals $\{[\ k(j) \pm sk(|j|)\ ] : j \in \mathbb{Z}\}$ are disjoint and by (5.9) $t \in [\ k(j) \pm sk(|j|)\ ]$. On the other hand, $s \in [\ 2k(j) \pm sk(|j|)\ ]$ with $j = j_1(t) \neq 0$.

If $n > |j|$ then $|s| < 2sk(|j|+1) \leq 2sk(n) < k(n) - sk(n)$ because k is b-expanding with $b \geq 3$ (and $k(1) \geq 1+b$). If $n \leq |j|$ then $|s| \geq 2k(|j|) - sk(|j|) > k(|j|) + sk(|j|) \geq k(n) + sk(n)$. Hence, $s \notin \bigcup \{[\ k(j) \pm sk(|j|)\ ] : j \in \mathbb{Z}\}$ and so is not in $IP(k)$.

□

For a subset A of $\mathbb{Z}$ the *upper Banach density* of A is

$$\limsup_{\#I \to \infty} \frac{\#(I \cap A)}{\#I} \tag{5.16}$$

as I varies over finite intervals in $\mathbb{Z}$. When the limsup is zero, or, equivalently, when the limit exists and equals zero, we say that A has *Banach density zero*.

Thanks to a helpful discussion with Benjamin Weiss, we obtain the following

Theorem 5.7. *The subset $IP(k)$ of $\mathbb{Z}$ has Banach density zero.*

Proof: Given an interval $I \subset \mathbb{Z}$ in with $N = \#I$, if t, s are distinct points in I then $0 < |t - s| < N$. Now suppose that $t, s \in IP(k)$ with expansions $j_1(t), \dots, j_{r(t)}(t)$ for t and $j_1(s), \dots, j_{r(s)}(s)$ for s. Since $t \neq s$ the expansions differ. Let $\tilde{r}$ be the smallest positive integer such that $j_{\tilde{r}}(t) \neq j_{\tilde{r}}(s)$. Since $b \geq 3$, $\frac{b-2}{b} \geq \frac{1}{3}$ and so Lemma 5.4 implies that $3N > max(\ k(|j_{\tilde{r}}(t)|), k(|j_{\tilde{r}}(s)|))$. Thus, the terms in the expansions of t and s agree except for terms $k(j)$ with $|k(j)| < 3N$.

Since $|k(j)| \geq (1+b)^{|j|}$ and $b \geq 3$, we have $4^{|j|} < 3N$. So the number of possible $|j|$'s is bounded by $\log_4\,(3N)$. For each such j, $k(|j|)$ can occur with coefficient $-1, 0$ or $+1$. Consequently, $\#(I \cap IP(k))$ is bounded by $3^{\log_4\,(3N)} = (3N)^{\log_4(3)}$. Since $0 < \log_4\,(3) < 1$, it follows that $\frac{(3N)^{\log_4(3)}}{N} \to 0$ as $N \to \infty$.

Hence,

$$\lim_{\#I \to \infty} \frac{\#(I \cap IP(k))}{\#I} = 0.$$

□

Remark 5.8. The above proof works for any b-expanding function k with $b \geq 3$. On the other hand, if $k(n) = \epsilon(n)3^{|n|-1}$, which is b-expanding with $b = 2$, then every integer has a unique expansion. That is, $IP(k) = \mathbb{Z}$.

A similar proof works for $IP_+(k)$ with $b \geq 2$. In that case, we have that the number of j's is bounded by $\log_3\,(3N)$ and for each such j, $k(j)$ can occur with coefficient 0 or +1. Thus, $\#(I \cap IP_+(k))$ is bounded by $(3N)^{\log_3(2)}$.

5.2. **Labeled integers.**

We now fix a partition of $\mathbb{N}$ by an infinite sequence

$$\mathcal{D} \quad = \quad \{D_\ell : \ell \in \mathbb{N}\}$$

with $\#D_\ell = \infty$ for every $\ell \in \mathbb{N}$, numbered so that $minD_\ell < minD_{\ell+1}$. Hence, $minD_1 = 1$ and $minD_\ell \geq \ell$. We then let $Q(\ell, i)$ be the i^{th} smallest member of D_ℓ. Thus, $Q : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection such that $\ell \mapsto Q(\ell, 1)$ is increasing and for each $\ell \in \mathbb{N}$, $i \mapsto Q(\ell, i)$ is increasing with image D_ℓ.

The *support map* $n \mapsto \ell(n)$ associates to each $n \in \mathbb{N}$ the member of $\mathcal{D}$ which contains it, so that $n \in D_{\ell(n)}$. The map is the composition of Q^{-1} with the projection onto the first coordinate.

Definition 5.9. (a) If $j_1, \dots, j_r \in \mathbb{Z}$ is an expansion for $t \in IP(k)$,

$$t \ = \ k(j_1) + k(j_2) + \cdots + k(j_r),$$

then the *length vector* for the expansion is the $\mathbb{N}$-vector

$$\mathbf{r} \ = \ \mathbf{r}(t) \ = \ \chi(\ell(|j_1|)) + \chi(\ell(|j_2|)) + \cdots + \chi(\ell(|j_r|))$$

so that $|\mathbf{r}|$ is the length $r(t)$.

$0 \in IP_+(k) \subset IP(k)$ has the empty expansion with length 0 and length vector $\mathbf{0}$.

Thus, $\mathbf{r}(t)_\ell \ = \ \#\{i : j_i \in D_\ell\}$ in the expansion $j_1, \dots, j_r$ for t.

If $0 \leq \tilde{r} \leq r$, then the $\tilde{r}$ truncation, $\tilde{t}$, has length vector $\tilde{\mathbf{r}} \ = \ \mathbf{r}(\tilde{t}) \ = \ \chi(\ell(j_1)) + \chi(\ell(j_2)) + \cdots + \chi(\ell(j_{\tilde{r}}))$, so that $|\tilde{\mathbf{r}}| = \tilde{r}$. The residual $t - \tilde{t}$ has length vector $\mathbf{r} - \tilde{\mathbf{r}} \geq \mathbf{0}$.

Definition 5.10. For a label $\mathcal{M}$ define $A[\mathcal{M}] \subset \mathbb{Z}$ to consist of those $t \in IP(k)$ which have length vector $\mathbf{r}(t) \in \mathcal{M}$. Define $A_+[\mathcal{M}] = A[\mathcal{M}] \cap IP_+(k)$.

If $\mathcal{M} = \emptyset$ then $A_+[\mathcal{M}] = A[\mathcal{M}] = \emptyset$. Otherwise $0 \in A_+[\mathcal{M}] \subset A[\mathcal{M}]$ since $\mathbf{0} \in \mathcal{M}$.

Proposition 5.11. *Assume $t \in A[\mathcal{M}]$ with expansion length r. For any positive integer N,*

(5.17)
$$2N \ \leq \ |j_r(t)| \quad \Longrightarrow \quad [\, t \pm N] \ \cap \ A[\mathcal{M}] \ = \ t + ([\, \pm N\,] \ \cap \ A[\mathcal{M} - \mathbf{r}(t)]),$$

and the elements of $[\, t \ \pm \ N] \ \cap \ A[\mathcal{M}]$ are extensions of t in $A[\mathcal{M}]$.

If, in addition, $t \in A_+[\mathcal{M}]$ then

$$[\, t \ \pm \ N] \ \cap \ A_+[\mathcal{M}] \ = \ t \ + \ ([\, \pm N\,] \ \cap \ A_+[\mathcal{M} - \mathbf{r}(t)]) \tag{5.18}$$

Proof: If $s \in [\, t \ \pm \ \frac{b-2}{b} k(|j_r(t)|)\,] \ \cap \ A[\mathcal{M}]$ then by Proposition 5.5 s is an extension of t and $\mathbf{r}(s) \in \mathcal{M}$. Because t is a truncation of s with residual $s - t$ we have $\mathbf{r}(s) = \mathbf{r}(t) + \mathbf{r}(s - t)$ so $s - t \in A[\mathcal{M} - \mathbf{r}(t)]$. If $N \leq |j_r(t)|$ then $N \leq (b-2)|j_r(t)| < \frac{b-2}{b} k(|j_r(t)|)$, since $k(n) > bn$ for $n \in \mathbb{N}$.

On the other hand, if $u = k(j_1(u)) + \cdots + k(j_p(u)) \in A[\mathcal{M} - \mathbf{r}(t)]$ with length p then $\mathbf{r}(u) + \mathbf{r}(t) \in \mathcal{M}$.

By (5.7) applied to u, $|u| \leq N$ implies $|k(j_1(u))| \leq 2|u| \leq 2N \leq |j_r(t)|$. Hence, $|j_1(u)| < k(|j_1(u)|) \leq |j_r(t)|$. It follows that

$$(j_1(t), \dots, j_r(t), j_1(u), \dots, j_p(u))$$

is an expansion for $t+u$ with length $r+p$ and length vector $\mathbf{r}(t)+\mathbf{r}(u)$. Hence, $t+u \in [\, t \ \pm \ N] \ \cap \ A[\mathcal{M}]$.

The proof of (5.18) is exactly the same when $t \in A_+[\mathcal{M}]$.

□

Lemma 5.12. *If t is expanding and $|t| \leq (b-1)minD_\ell$ then $\mathbf{r}(t)_\ell = 0$.*

Proof: We may assume $t \neq 0$. If for some $i = 2, \dots, r$, $\ell(|j_i|)(t) = \ell$ then by (5.7) $|t| > (b-1)|k(j_i(t))| \geq (b-1)minD_\ell$. If $\ell(|j_1(t)|) = \ell$ then $|t| \geq \frac{b-1}{b}|k(j_1(t))| > (b-1)|j_1(t)|$. Since $\ell(|j_1(t)|) = \ell$ we have $|j_1(t)| > minD_\ell$.

□

Proposition 5.13. *For any label $\mathcal{M}$ and $N \in \mathbb{N}$*

$$\begin{aligned} [\, \pm N \,] \ \cap \ A[\mathcal{M}] \quad &= \quad [\, \pm N \,] \ \cap \ A[\mathcal{M} \ \cap \mathcal{B}_N], \\ [\, \pm N \,] \ \cap \ A_+[\mathcal{M}] \quad &= \quad [\, \pm N \,] \ \cap \ A_+[\mathcal{M} \ \cap \mathcal{B}_N] \end{aligned} \tag{5.19}$$

Proof: If $N \leq \ell$ then $N \leq minD_\ell$. Hence, $|t| \leq N$ implies $\mathbf{r}(t)_\ell = 0$ by Lemma 5.12. Thus, $supp\ \mathbf{r}(t) \subset [1, N]$.

If $r = |\mathbf{r}(t)| > N$. Then $|j_1| > |j_2| > \cdots > |j_r|$ implies $|j_1| > N$ and so $|t| \geq \frac{b-1}{b}k(|j_1|) > (b-1)N$ since $k(N) \geq bN$. Contrapositively, $|t| \leq N$ implies $|\mathbf{r}(t)| \leq N$.

It follows that $|t| \leq N$ implies $\mathbf{r}(t) \in \mathcal{B}_N$.

□

If $\mathbf{m} = \mathbf{0}$ then $t = 0$ is the unique time with length vector $\mathbf{m}$. For a nonzero $\mathbb{N}$-vector $\mathbf{m}$ there are infinitely many such times.

Lemma 5.14. *(a) If $\mathbf{m}$ is a nonzero $\mathbb{N}$-vector so that $|\mathbf{m}| = r > 0$ then for any positive integer N there exist $t \in IP_+(k)$ such that $\mathbf{r}(t) = \mathbf{m}$. and $j_r(t) > N$.*

(b) For labels $\mathcal{M}_1, \mathcal{M}_2$ if $\mathbf{m} \in \mathcal{M}_1 \setminus \mathcal{M}_2$ and t is expanding with $\mathbf{r}(t) = \mathbf{m}$ then $t \in A[\mathcal{M}_1] \setminus A[\mathcal{M}_2]$ and if In particular, $\mathcal{M}_1 = \mathcal{M}_2$ iff $A[\mathcal{M}_1] = A[\mathcal{M}_2]$ and iff $A_+[\mathcal{M}_1] = A_+[\mathcal{M}_2]$.

(c) For any $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that for all labels $\mathcal{M}_1, \mathcal{M}_2$

$$\begin{aligned} &[\, \pm M \,] \ \cap \ A[\mathcal{M}_1] \ = \ [\, \pm M \,] \ \cap \ A[\mathcal{M}_2] \qquad \Longrightarrow \\ &[\, \pm M \,] \ \cap \ A_+[\mathcal{M}_1] \ = \ [\, \pm M \,] \ \cap \ A_+[\mathcal{M}_2] \\ &\Longrightarrow \qquad \mathcal{M}_1 \cap \mathcal{B}_N \ = \ \mathcal{M}_2 \cap \mathcal{B}_N. \end{aligned} \tag{5.20}$$

Proof: (a) We use the fact that each D_ℓ is an infinite subset of $\mathbb{N}$.

Write $\mathbf{m} = \chi(\ell_1) + \cdots + \chi(\ell_r)$ where $r = |\mathbf{m}|$. Choose $j_r \in D_{\ell_r}$ with $j_r > N$. For $i = 1, ..., r-1$ inductively choose $j_{r+i} \in D_{\ell_{r+i}}$ with $j_{r+i} > j_{r+i-1}$. Then $j_1, \dots, j_r$ is an expansion for $t \in IP(k)$ with $\mathbf{r}(t) = \mathbf{m}$.

(b) If $\mathbf{m} \in \mathcal{M}_1 \setminus \mathcal{M}_2$ then any t with $\mathbf{r}(t) = \mathbf{m}$ then by definition t is in $A[\mathcal{M}_1] \setminus A[\mathcal{M}_2]$. If t is positive expanding then t is in $A_+[\mathcal{M}_1] \setminus A[\mathcal{M}_2]$. Hence, if $\mathcal{M}_1 \neq \mathcal{M}_2$, then $A[\mathcal{M}_1] \neq A[\mathcal{M}_2]$ and $A_+[\mathcal{M}_1] \neq A_+[\mathcal{M}_2]$. The reverse implications are obvious.

(c) The first implication follows by intersecting with $IP_+(k)$.

For each $\mathbf{m} \in \mathcal{B}_N$ choose $t = t(\mathbf{m}) \in IP_+(k)$ with $\mathbf{r}(t) = \mathbf{m}$ and let $M = max\{t(\mathbf{m}) : \mathbf{m} \in \mathcal{B}_N\}$. Now assume $[\pm M] \cap A_+[\mathcal{M}_1] = [\pm M] \cap A_+[\mathcal{M}_2]$.

If $\mathbf{m} \in \mathcal{M}_1 \cap \mathcal{B}_N$ then $0 \leq t(\mathbf{m}) \leq M$. Hence $t(\mathbf{m}) \in [\pm M] \cap A[\mathcal{M}_1] = [\pm M] \cap A[\mathcal{M}_2]$. Hence, $t(\mathbf{m}) \in A[\mathcal{M}_2]$ and so $\mathbf{m} \in \mathcal{M}_2$. That is, $\mathcal{M}_1 \cap \mathcal{B}_N \subset \mathcal{M}_2 \cap \mathcal{B}_N$. Symmetrically for the reverse inclusion.
$\square$

5.3. **Subshifts.**

On $\{0,1\}^{\mathbb{Z}}$ define the ultrametric d by

$$d(x,y) = inf \{ 2^{-N} : N \in Z_+ \text{ and } x_t = y_t \text{ for all } |t| < N\}. \tag{5.21}$$

We denote by S the shift homeomorphism on $\{0,1\}^{\mathbb{Z}}$. That is, $S(x)_t = x_{t+1}$. Hence, for any $k \in \mathbb{Z}$ $S^k(x)_t = x_{t+k}$.

For $A \subset \mathbb{Z}$ we defined $\chi(A) \in \{0,1\}^{\mathbb{Z}}$ by $\chi(A)_t = 1$ iff $t \in A$, so that $\chi(A)$ is the characteristic function A. Thus, $\chi(\emptyset) = e$ with $e_t = 0$ for all t. If $z \in \{0,1\}^{\mathbb{Z}}$ then $z = \chi(A)$ for $A = \{t \in \mathbb{Z} : z_t = 1\}$.

Definition 5.15. Let $X(A)$ be the orbit closure of $\chi(A)$, i.e. the smallest closed invariant subspace which contains $\chi(A)$.

We will say that $x = \chi(A)$ *dominates* $z = \chi(A_1)$ when $A \supset A_1$. Clearly, the set of pairs $\{(x,z)\}$ such that x dominates z is closed.

For $x = \chi(A)$ we will say that x has positive or zero upper Banach density when the subset A has positive or zero upper Banach density, as defined in (5.16) above. The point x has positive upper Banach density iff there exists an invariant probability measure μ on its orbit closure such that the cylinder set $C_0 = \{z \in \{0,1\}^{\mathbb{Z}} : z_0 = 1\}$ has positive μ measure, see [12, Lemma 3.17]. Observe that the union of the translates $S^k(C_0)$ is the complement of $\{e\}$. So for an invariant probability measure μ the set C_0 has positive measure iff μ is not the point measure δ_e where e is the fixed point $\chi(\emptyset)$. Thus, for example, if $X(A)$ contains a minimal subset other than the fixed point e then A has positive upper Banach density.

Definition 5.16. Define $Z(A)$ to be the smallest closed invariant subset of $\{0,1\}^{\mathbb{Z}}$ which contains $\chi(A_1)$ for every $A_1 \subset A$.

Theorem 5.17. *If A has Banach density zero then every point $x \in Z(A)$ has Banach density zero. The dynamical system $(Z(A), S)$ is uniquely ergodic with δ_e, the point mass at e, as the unique invariant probability measure and so $\{e\}$ is the unique minimal subset of $Z(A)$.*

Proof: First observe that any block of a point $x \in X(A)$ is also a block in $\chi(A)$. Thus every such x has Banach density zero. Moreover, every point $y \in Z(A)$ is dominated by a point $x \in X(A)$ and therefore has also Banach density zero. Since, by Birkhoff ergodic theorem, every ergodic measure admits a generic point we can apply Furstenberg characterization [12, Lemma 3.17] to deduce that the only ergodic measure on $Z(A)$ is δ_e. The ergodic decomposition theorem implies that indeed δ_e is the only invariant probability measure on $Z(A)$. The topological entropy of $Z(A)$ is zero by the Variational Principle since the entropy of the unique invariant measure is zero. Finally, as any minimal subset carries an invariant probability measure and distinct minimal subsets are disjoint, it follows that $\{e\}$ is the only minimal subset of $Z(A)$.
□

Definition 5.18. For a label $\mathcal{M}$ let $x[\mathcal{M}] = \chi(A[\mathcal{M}]), x_+[\mathcal{M}] = \chi(A_+[\mathcal{M}])$. Let $X(\mathcal{M})$ and $X_+(\mathcal{M})$ denote the orbit closures $X(A[\mathcal{M}])$ and $X(A_+[\mathcal{M}])$, respectively.

For example, $x_+[\emptyset] = x[\emptyset] = e$ and $x_+[0] = x[0] = (\dots, 0, 0, \dot{1}, 0, 0, \dots)$. In general, the points $x[\mathcal{M}]$ are symmetric in that $x[\mathcal{M}]_{-t} = x[\mathcal{M}]_t$. On the other hand, $x_+[\mathcal{M}]_t = 0$ if $t < 0$. Hence, $x[\mathcal{M}] \neq x_+[\mathcal{M}]$ if $\mathcal{M}$ is a positive label.

Corollary 5.19. *For the subshifts $(Z(IP(k)), S)$ and $(X(\mathcal{M}), S), (X_+(\mathcal{M}), S)$ for labels $\mathcal{M}$, the point measure δ_e is the unique invariant probability measure and so $\{e\}$ is the unique minimal subset. In particular, the systems are minCT and all have entropy zero.*

Proof: By Theorem 5.7 $IP(k)$ has Banach density zero. So Theorem 5.17 δ_e is the unique invariant probability measure for $(Z(IP(k)), S)$ and so the same is true for each subsystem $(X(\mathcal{M}), S)$.
□

Theorem 5.20. *The maps $x[\cdot], x_+[\cdot]$ defined by $\mathcal{M} \mapsto x[\mathcal{M}]$ and $\mathcal{M} \mapsto x_+[\mathcal{M}]$ are homeomorphisms from $\mathcal{LAB}$ onto their images in $\{0,1\}^{\mathbb{Z}}$.*

Proof: Let $\mathcal{M}_1, \mathcal{M}_2$ be labels and N be an arbitrary positive integer.
By Lemma 5.14 (b) $\mathcal{M}_1 \cap \mathcal{B}_N = \mathcal{M}_2 \cap \mathcal{B}_N$ iff $A(\mathcal{M}_1 \cap \mathcal{B}_N) = A(\mathcal{M}_2 \cap \mathcal{B}_N)$. By Proposition 5.13 this implies $[\pm N] \cap A(\mathcal{M}_1) = [\pm N] \cap A(\mathcal{M}_2)$ and so, by intersecting with $IP_+(k)$, that $[\pm N] \cap A_+(\mathcal{M}_1) = [\pm N] \cap A_+(\mathcal{M}_2)$ these are equivalent to $x[\mathcal{M}_1]_t = x[\mathcal{M}_2]_t$ and $x_+[\mathcal{M}_1]_t = x_+[\mathcal{M}_2]_t$ for all $t \in [\pm N]$. Since $d(\emptyset, 0) = 1 = d(x[\emptyset], x[0])$, it follows that $x[\cdot]$ has Lipschitz constant 1.
The map $\mathcal{M} \mapsto A_+(\mathcal{M})$ is injective by Lemma 5.14 (b). Clearly, $A \mapsto \chi(A)$ is injective. So compactness implies that $x[\cdot]$ and $x_+[\cdot]$ are homeomorphisms onto their respective images.
□

Corollary 5.21. *Let $\{\mathcal{M}^i\}$ be a sequence of labels:*
(a) If $\mathcal{M}$ is a nonempty label and $k \in \mathbb{Z}$ then the following are equivalent

(i) *$k = 0$ and $\{\mathcal{M}^i\}$ converges to $\mathcal{M}$.*
(ii) *$\{S^k(x[\mathcal{M}^i])\}$ converges to $x[\mathcal{M}]$.*
(iii) *$\{S^k(x_+[\mathcal{M}^i])\}$ converges to $x_+[\mathcal{M}]$.*

(b) The sequence $\{\mathcal{M}^i\}$ is convergent iff $\{x[\mathcal{M}^i]\}$ is convergent and iff $\{x_+[\mathcal{M}^i]\}$ is convergent

Proof: (a) When $k = 0$ the equivalence follows from Theorem 5.20.
Since $\mathcal{M}$ is nonempty $x[\mathcal{M}]_0 = 1$ and so we can assume $\mathcal{M}^i$ is nonempty for all i.
Now assume $k \neq 0$. The points $x[\mathcal{M}]$ are symmetric about 0 and for every t, $\{(S^k x[\mathcal{M}^i])_t = x[\mathcal{M}^i]_{t+k}\} \to x[\mathcal{M}]_t$. Hence, $\{x[\mathcal{M}^i]_{-t-k} = x[\mathcal{M}^i]_{t+k}\}$ tends to $x[\mathcal{M}]_t$ and to $x[\mathcal{M}]_{-t-2k} = x[\mathcal{M}]_{t+2k}$. This implies that $x[\mathcal{M}]$ is a periodic point with period $2k$. Since $\mathcal{M} \neq \emptyset$, $x[\mathcal{M}] \neq \bar{0}$. Since it has arbitrarily long runs of zeroes it is not periodic. Thus, (ii) implies $k = 0$.
Now assume (iii).
If $k \geq 0$ then $(S^k x_+[\mathcal{M}^i])_{-k} = x_+[\mathcal{M}^i]_0 = 1$ and so in the limit $x_+[\mathcal{M}]_{-k} = 1$ implies $-k \geq 0$ and so $k = 0$.

If $k < 0$ then $(S^k x_+[\mathcal{M}^i])_0 = x_+[\mathcal{M}^i]_k = 0$ and so in the limit $x_+[\mathcal{M}]_0 = 0$ which contradicts the assumption that $\mathcal{M}$ is nonempty.

(b) Obvious from the fact that $x[\cdot]$ and $x_+[\cdot]$ are homeomorphisms between compact sets.

□

Lemma 5.22. *Let $\{t^i\}$ be a sequence of expanding times with r^i the length of t^i. If $\{|j_{r^i}(t^i)|\} \to \infty$ then*

$$(5.22)\qquad \begin{aligned} Lim_{i\to\infty}\ sup_{\mathcal{M}\in\mathcal{LAB}}\ d(S^{t^i}(x[\mathcal{M}])\ ,\ x[\mathcal{M} - \mathbf{r}(t^i)]\ ) \quad &= \quad 0. \\ Lim_{i\to\infty}\ sup_{\mathcal{M}\in\mathcal{LAB}}\ d(S^{t^i}(x_+[\mathcal{M}])\ ,\ x_+[\mathcal{M} - \mathbf{r}(t^i)]\ ) \quad &= \quad 0. \end{aligned}$$

That is, the pairs of sequences

$$(\{S^{t^i}(x[\mathcal{M}])\}, \{x[\mathcal{M} - \mathbf{r}(t^i)]\}) \quad \textit{and} \quad (\{S^{t^i}(x_+[\mathcal{M}])\}, \{x_+[\mathcal{M} - \mathbf{r}(t^i)]\})$$

are uniformly asymptotic in $Z(IP(k))$.

Proof: For any positive integer N, there exists i_N so that $i > i_N$ implies $|j_{r^i}(t^i)| > 2N$. Then Proposition 5.11 implies for any label $\mathcal{M}$ $[\, t^i \pm N] \ \cap \ A[\mathcal{M}] \ = \ t^i + ([\,\pm N\,] \ \cap \ A[\mathcal{M} - \mathbf{r}(t^i)])$ This says that for all t with $|t| \leq N$, $x[\mathcal{M} - \mathbf{r}(t^i)]_t \ = \ x[\mathcal{M}]_{t^i+t}$. Thus, if $i > i_N$, then $x[\mathcal{M} - \mathbf{r}(t^i)]_t = (S^{t^i}(x[\mathcal{M}]))_t$ for all t with $|t| \leq N$. So (5.22) follows from the definition (5.21) of the metric on $\{0,1\}^{\mathbb{Z}}$.

The proof for x_+ is the same.

□

Remark 5.23. For later semigroup applications we note that the proofs of Corollary 5.21 and Lemma 5.22 work just the same for nets instead of sequences.

Lemma 5.24. *For a label $\mathcal{M}$ assume that I is a directed set and $\{t^i : i \in I\}$ is a net in $A[\mathcal{M}]$ with r^i the length of t^i. Assume that $\{|t^i|\} \to \infty$ but $\{|j_{r^i}(t^i)|\}$ is bounded.*

(a) There exists an integer $j^* \neq 0$ and a subnet $\{t^{i'}\}$, defined by restricting to $i' \in I'$ a cofinal subset of I, such that $j_{r^{i'}}(t^{i'}) = j^*$.

(b) If $\mathcal{M}$ is a label of finite type then there exists $\bar{t} \in A[\mathcal{M}]$ with length $\bar{r} > 0$ (and so $\bar{t} \neq 0$) and a subnet $\{t^{i'}\}$, defined by restricting to $i' \in I'$ a cofinal subset of I, such that $t^{i'} - \tilde{t}^{i'} = \bar{t}$ for all i', where $\tilde{t}^{i'}$ is the $\tilde{r}^{i'} = r^{i'} - \bar{r}$ truncation of $t^{i'}$ and, in addition, $\{|j_{\tilde{r}^{i'}}(\tilde{t}^{i'}) = j_{\tilde{r}^{i'}}(t^{i'})|\} \to \infty$.

Proof: The bounded set $\{|j_{r^i}(t^i)|\}$ is finite. It follows that there exists $j^{-1} \in \mathbb{Z} \setminus \{0\}$ such that $j_{r^i}(t^i) = j^{-1}$ for i in a cofinal subset SEQ_{-1} of I. Let $\bar{t}_{-1} = k(j^{-1})$ and let $\tilde{t}^i_{-1}$ be the $\tilde{r}^i_{-1} = r^i - 1$ truncation of t^i for all i in SEQ_{-1}. Since the truncation is proper, $\bar{t}_{-1} \neq 0$. Furthermore, $\mathbf{r}(t^i) = \mathbf{r}(\tilde{t}^i_{-1}) + \chi(\ell(j^{-1})) \in \mathcal{M}$ for all $i \in SEQ_{-1}$.

For (a) we let $j^* = j^{-1}$ and $I' = SEQ_{-1}$. Now assume that $\mathcal{M}$ is of finite type and continue.

If $\{|j_{\tilde{r}^i_{-1}}(\tilde{t}^i_{-1})|\} \to \infty$ as $i \to \infty$ in SEQ_{-1} then let $I' = SEQ_{-1}$ use this as the required subnet. Otherwise, it is bounded on some cofinal subset and we can choose $j^{-2} \in \mathbb{Z} \setminus \{0\}$ such that $j_{\tilde{r}^i_{-1}}(\tilde{t}^i_{-1}) = j^{-2}, \ell^{-2}$ for i in a cofinal subset SEQ_{-2} of SEQ_{-1}. Let $\bar{t}_{-2} = k(j^{-2}) + k(j^{-1})$ and let $\tilde{t}^i_{-2}$ be the $\tilde{r}^i_{-2} = r^i - 2$ truncation of t^i for all i in SEQ_{-2}. Since the truncation is proper, $\bar{t}_{-2} \neq 0$. Furthermore, $\mathbf{r}(t^i) = \mathbf{r}(\tilde{t}^i_{-2}) + \chi(\ell(j^{-1})) + \chi(\ell(j^{-2})) \in \mathcal{M}$ for all $i \in SEQ_{-2}$.

Because $\mathcal{M}$ is of finite type, the Finite Chain Condition implies that this procedure must halt after finitely many steps. That is, for some $k \geq 1$, $\{|j_{\tilde{r}^{i'}_{-k}}(\tilde{t}^{i'}_{-k})|\} \to \infty$ as $i' \to \infty$ in SEQ_{-k}. The subnet is obtained by restricting to $I' = SEQ_{-k}$ and $\bar{t} = \bar{t}_{-k}$ with length $\bar{r} = k$.

Since $\tilde{t}^{i'}$ is the truncation of $t^{i'}$ to $\tilde{r}^{i'}$, we have $|j_{\tilde{r}^{i'}}(\tilde{t}^{i'}) = j_{\tilde{r}^{i'}}(t^{i'})| \to \infty$.

□

Remark 5.25. In (b) if $t^i \in A_+[\mathcal{M}]$ for all i then $\bar{t} \in A_+[\mathcal{M}]$ and the truncations are positive expanding times.

Corollary 5.26. *Assume for a label $\mathcal{M}$, I is a directed set, and the net $\{S^{t^i}(x[\mathcal{M}]) : i \in I\}$ converges to $x[\mathcal{N}]$ (or, alternatively, $\{S^{t^i}(x_+[\mathcal{M}]) : i \in I\}$ converges to $x_+[\mathcal{N}]$) with $\mathcal{N} \neq \emptyset$. If eventually $t^i = 0$ then $\mathcal{N} = \mathcal{M}$. Otherwise, eventually $t^i \in A[\mathcal{M}]$ (and $t^i \in A_+[\mathcal{M}]$ in the x_+ case) with length r_i and length vector $\mathbf{r}(t^i) > 0$. Furthermore, $\{\ |j_{r_i}(t^i)|\} \to \infty$, $\{\ \mathcal{M} - \mathbf{r}(t^i)\ \}$ is convergent and $LIM\ \{\ \mathcal{M} - \mathbf{r}(t^i)\ \}\ =\ \mathcal{N}$. In particular, $\mathcal{N} \in \Theta'(\mathcal{M})$.*

Proof: Since $0 \in A(\mathcal{N})$ it follows that eventually $x[\mathcal{M}]_{t^i} = 1$ and so eventually $t^i \in A[\mathcal{M}]$. So we may assume $t^i \in A[\mathcal{M}]$ for all i. Similarly, in the x_+ case we can assume $t^i \in A_+[\mathcal{M}]$ for all i.

If $\{t^i\}$ is bounded on some cofinal subset of I then there is cofinal subset on which $t^{i'} = k$ for some $k \in \mathbb{Z}$. By Corollary 5.21 (b), $k = 0$ and so the limit is $x[\mathcal{M}]$. Hence, $\mathcal{N} = \mathcal{M}$.

When $\{|t^i|\} \to \infty$ we apply Lemma 5.24 to prove that if $\{S^{t^i}(x[\mathcal{M}]) : i \in I\}$ converges to z (or $\{S^{t^i}(x_+[\mathcal{M}]) : i \in I\}$ converges to z) then $z = x[\mathcal{N}]$ (resp. $z = x_+[\mathcal{N}]$) implies $\{ |j_{r_i}(t^i)|\} \to \infty$ after which the remaining results follow from Lemma 5.22.

If $\{ |j_{r_i}(t^i)|\}$ does not tend to infinity then it remains bounded on some cofinal set and by Lemma 5.24(a) we can restrict further to assume that for some $j^* \neq 0$, $j_{r_i}(t_i) = j^*$.

For the $x[\mathcal{M}]$ case we apply Corollary 5.6. If $s_i = t_i - k(j*)$ then s_i has length vector $\mathbf{r}(t_i) - \chi(j^*) \in \mathcal{M}$ and so $s_i \in A[\mathcal{M}]$. But $u_i = t_i + k(j*) \notin IP(k)$ and so a fortiori is not in $A[\mathcal{M}]$. Hence, for all i we have

$$S^{t^i}(x[\mathcal{M}])_{-k(j^*)} \ = \ 1, \qquad \text{and} \qquad S^{t^i}(x[\mathcal{M}])_{k(j^*)} \ = \ 0.$$

In the limit, this implies $z_{-k(j^*)} = 1$ and $z_{-k(j^*)} = 0$. Since every $x[\mathcal{N}]$ is symmetric about 0 it follows that $z \neq x[\mathcal{N}]$.

Now assume $\{S^{t^i}(x_+[\mathcal{M}]) : i \in I\}$ converges to $x_+[\mathcal{N}]$ so that $j^* > 0$ and $\tilde{t^i} = t^i - k(j^*) \in A_+[\mathcal{M}]$. Hence, for each i, $S^{t^i}(x_+[\mathcal{M}])_{-k(j^*)} = x_+[\mathcal{M}]_{\tilde{t}^i} = 1$. So in the limit, $z_{-k(j^*)} = 1$. Since every $x_+[\mathcal{N}]_t = 0$ for $t < 0$, it follows that $z \neq x_+[\mathcal{N}]$.

□

Remark 5.27. Notice that we can sharpen the first result by merely assuming that the limit z is symmetric and the second by assuming that $z_t = 0$ for all $t < 0$. We still obtain $\{ |j_{r_i}(t^i)|\} \to \infty$, $\{ \mathcal{M} - \mathbf{r}(t^i) \}$ is convergent and $LIM \{ \mathcal{M} - \mathbf{r}(t^i) \} = \mathcal{N}$ with $z = x[\mathcal{N}]$ or $= x_+[\mathcal{N}]$.

N.B. From now on we will omit the proofs of the $A_+(\mathcal{M}), x_+[\mathcal{M}], X_+(\mathcal{M})$ results when they are exactly the same as those of the $A(\mathcal{M}), x[\mathcal{M}], X(\mathcal{M})$ results.

Theorem 5.28. *Let $\mathcal{M}$ be a label. Assume $\{t^i\}$ is a sequence of expanding times with $|t_i| \to \infty$ and $\{u^i\}$ is a sequence in $\mathbb{Z}$ with $\{u^i - t^i\}$ bounded. Let $\mathbf{r}(t^i)$ be the length vector of the expansion for t^i with sum r_i.*

(i) *Assume $|j_{r_i}(t^i)| \to \infty$.*

(a) $\{S^{t^i}(x[\mathcal{M}])\}$ is convergent iff $\{S^{t^i}(x_+[\mathcal{M}])\}$ is convergent iff $\{\mathcal{M} - \mathbf{r}(t^i)\}$ is convergent in which case

$$\begin{aligned} Lim_{i\to\infty}\ S^{t^i}(x[\mathcal{M}]) \quad &= \quad x[LIM\{\mathcal{M} - \mathbf{r}(t^i)\}], \\ Lim_{i\to\infty}\ S^{t^i}(x_+[\mathcal{M}]) \quad &= \quad x_+[LIM\{\mathcal{M} - \mathbf{r}(t^i)\}]. \end{aligned} \tag{5.23}$$

(b) $Lim_{i\to\infty}\ S^{u^i}(x[\mathcal{M}]) = e$ *iff* $Lim_{i\to\infty}\ S^{u^i}(x_+[\mathcal{M}]) = e$ *iff* $LIM\{\mathcal{M} - \mathbf{r}(t^i)\} = \emptyset$.

(c) $\{S^{u^i}(x[\mathcal{M}])\}$ *is convergent with* $Lim_{i\to\infty}\ S^{u^i}(x[\mathcal{M}]) \neq e$ *iff* $\{S^{u^i}(x_+[\mathcal{M}])\}$ *is convergent with* $Lim_{i\to\infty}\ S^{u^i}(x_+[\mathcal{M}]) \neq e$ *iff* $\{\mathcal{M} - \mathbf{r}(t^i)\}$ *is convergent with a nonempty limit and there exists an integer* k *such that eventually* $u^i = t^i + k$. *In that case,*

$$
\begin{aligned}
Lim_{i\to\infty}\ S^{u_i}(x[\mathcal{M}]) \quad &= \quad S^k(x[LIM\ \{\mathcal{M} - \mathbf{r}(t^i)\}]), \\
Lim_{i\to\infty}\ S^{u_i}(x[\mathcal{M}]) \quad &= \quad S^k(x[LIM\ \{\mathcal{M} - \mathbf{r}(t^i)\}]).
\end{aligned} \tag{5.24}
$$

(ii) *If* $\{v^i\}$ *is a sequence in* $\mathbb{Z}$ *such that* $|v^i| \to \infty$ *and* $Lim_{i\to\infty}$ $S^{v^i}(x[\mathcal{M}]) = z \neq e$ *(or* $Lim_{i\to\infty}\ S^{v^i}(x_+[\mathcal{M}]) = z \neq e$*) then there exists* $k \in \mathbb{Z}$ *and a sequence* $s^i \in A[\mathcal{M}]$ *(resp.*$s^i \in A_+[\mathcal{M}]$*) such that eventually* $v_i = s_i + k$ *and* $|s_i| \to \infty$.

(iii) *If* $\{v^i\}$ *is a sequence in* $\mathbb{Z}$ *such that* $|v^i| \to \infty$ *and* $Lim_{i\to\infty}$ $S^{v^i}(x[\mathcal{M}]) = e$ *(or* $Lim_{i\to\infty}\ S^{v^i}(x_+[\mathcal{M}]) = e$*) then* $Lim_{i\to\infty}\ S^{u^i}(x[\mathcal{M}_1]) =$ e *(resp.* $Lim_{i\to\infty}\ S^{u^i}(x_+[\mathcal{M}_1]) = e$*)whenever* $\mathcal{M}_1$ *is a label with* $\mathcal{M}_1 \subset \mathcal{M}$ *and* $\{|u^i - v^i|\}$ *is bounded.*

(iv) *Assume that* $\mathcal{M}$ *is of finite type and eventually* $t^i \in A[\mathcal{M}]$. *If* $\{S^{t^i}(x[\mathcal{M}])\}$ *converges to* z *(or if* $\{S^{t^i}(x[\mathcal{M}])\}$ *converges to* z *and eventually* $t^i \in A_+[\mathcal{M}]$*) then* $z \neq e$ *and there exist* $\tilde{r}_i \leq r_i$ *and* $\bar{t}$ *such that with* $\tilde{t}^i$ *the* $\tilde{r}_i$ *truncation of* t^i *we have* $|j_{\tilde{r}_i}(t^i)| = |j_{\tilde{r}_i}(\tilde{t}^i)| \to \infty$ *and eventually* $t^i - \tilde{t}^i = \bar{t}$. *So if* $\mathbf{r}(\tilde{t}^i)$ *is the length vector of* $\tilde{t}^i$ *then* $\{\mathcal{M} - \mathbf{r}(\tilde{t}^i)\}$ *is convergent and*

$$
\begin{aligned}
z \quad &= \quad S^{\bar{t}}(x[LIM\ \{\mathcal{M} - \mathbf{r}(\tilde{t}_i)\}]), \\
(\textit{resp.} \quad z \quad &= \quad S^{\bar{t}}(x_+[LIM\ \{\mathcal{M} - \mathbf{r}(\tilde{t}_i)\}]).
\end{aligned} \tag{5.25}
$$

Proof: (i): By Lemma 5.22 $\{S^{t^i}(x[\mathcal{M}])\}$ is asymptotic to $\{x[\mathcal{M} - \mathbf{r}(t^i)]\}$ and by Corollary 5.21 (b) the latter converges iff $\{\mathcal{M} - \mathbf{r}(t^i)\}$ converges in which case the common limit is $x[LIM\{\mathcal{M} - \mathbf{r}(t^i)\}]$.

If $\{\mathcal{M} - \mathbf{r}(t^i)\}$ is not convergent. Then then by compactness there exist two convergent subsequences with limits $LIM_1 \neq LIM_2$ such that $\mathbf{m} \in LIM_1 \setminus LIM_2$. Because $LIM_1 \neq LIM_2$ it follows that $x[LIM_1] \neq x[LIM_2]$ by Theorem 5.20. Hence, $\{S^{t^i i}(x[\mathcal{M}])\}$ has subsequences converging to different limits and so is not convergent.

It is clear that if $\{S^{t^i}(x[\mathcal{M}])\}$ is convergent and eventually $u^i = t^i + k$ then $\{S^{u^i}(x[\mathcal{M}])\}$ converges to $S^k(x[LIM])$. If $LIM = \emptyset$ or, equivalently, $LIMSUP = \emptyset$ then $\{u^i\}$ can be partitioned into finitely many subsequences on each of which $u^i - t^i$ is constant and so each of these $\{S^{u^i}(x[\mathcal{M}])\}$ subsequences converges to e.

Finally, if $\{S^{u^i}(x[\mathcal{M}])\}$ converges to $z \neq e$ then the now familiar subsequence argument implies that $\{\mathcal{M} - \mathbf{r}(t^i)\}$ is convergent and eventually $u^i - t^i$ is constant.

(ii) Since $z \neq 0$ there exists $k \in \mathbb{Z}$ such that $z_{-k} = 1$. Hence, $S^{-k}(z)_0 = 1$. Let $s^i = v^i - k$. Since $S^{s^i}(x[\mathcal{M}])$ converges to $S^{-k}(z)$, $x[\mathcal{M}]_{t^i}$ is eventually 1 and so eventually $t^i \in A[\mathcal{M}]$. Since $|v_i| \to \infty$, $|t_i| \to \infty$.

(iii) To say that $S^{v^i}(x[\mathcal{M}]) = e$ says that for all positive integers N, eventually $[v^i \pm N] \ \cap \ A[\mathcal{M}] \ = \ \emptyset$ Since $A[\mathcal{M}_1] \subset A[\mathcal{M}]$ when $\mathcal{M}_1 \subset \mathcal{M}$, it follows that $[v^i \pm N] \ \cap \ A[\mathcal{M}_1] \ = \ \emptyset$.

(iv) If $\{|j_{r^i}(t^i)|\} \to \infty$ then by (i), we can choose $\tilde{r}_i = r_i$, $\bar{t} = 0$.

Otherwise, there is a bounded subsequence to which we apply Lemma 5.24 (b). Because $\bar{t} \neq 0$ it follows that there is no subsequence on which $|j_{r^i}(t^i)| \to \infty$. Hence, $\{|j_{r^i}(t^i)|\}$ is bounded. All of the convergent subsequences have the same limit and so $\bar{t}$ is defined independent of the choice of subsequence by Corollary 5.21 (a). Because the length $\bar{r}$ of the expansion of $\bar{t}$ is uniquely determined by $\bar{t}$ it follows that the $\tilde{r}_i = r_i - \bar{r}$ is the truncation used in every subsequence.

□

Definition 5.29. If Y is an invariant subset of $Z(IP(k))$ we let $\Phi(Y) = \{\mathcal{N} \subset \mathcal{M} : x[\mathcal{N}] \in Y\}$ and $\Phi_+(Y) = \{\mathcal{N} \subset \mathcal{M} : x_+[\mathcal{N}] \in Y\}$

That is, $\Phi(Y)$ and $\Phi_+(Y)$ are the preimages of Y by the maps $x[\cdot]$ and $x_+[\cdot]$ respectively. By Theorem 5.20 $\mathcal{N}$ is uniquely determined by $x[\mathcal{N}]$ and by $x_+[\mathcal{N}]$.

Proposition 5.30. *(a) If Y is a closed, shift invariant subset of $Z(IP(k))$ then $\Phi(Y), \Phi_+(Y)$ are closed, $FIN(\mathbb{N})$ invariant subsets of $\mathcal{LAB}$.*

(b) If $\mathcal{N} \in \Theta(\mathcal{M})$, then $x[\mathcal{N}] \in X(\mathcal{M})$ and $x_+[\mathcal{N}] \in X_+(\mathcal{M})$

(c) If $x[\mathcal{N}] \in X(\mathcal{M})$ or $x_+[\mathcal{N}] \in X_+(\mathcal{M})$ with $\mathcal{N}$ nonempty then $\mathcal{N} \in \Theta(\mathcal{M})$. If $\mathcal{M} \neq FIN(\mathbb{N})$ then $\emptyset \in \Theta(\mathcal{M})$.

(d) $\emptyset \in \Phi(X(\mathcal{M})) \cap \Phi_+(X_+(\mathcal{M}))$ and

$$\Theta(\mathcal{M}) \setminus \{\emptyset\} \ = \ \Phi(X(\mathcal{M})) \setminus \{\emptyset\} \ = \ \Phi_+(X_+(\mathcal{M})) \setminus \{\emptyset\}.$$

Proof: (a): Because the map $x[\cdot]$ is continuous, $\Phi(Y)$ is closed when Y is.

If $x[\mathcal{N}] \in Y$ and $\mathbf{r}$ is a nonzero $\mathbb{N}$-vector then we an choose $\{t^i\}$ with length vector $\mathbf{r}$ and with $|j_r(t^i)| \to \infty$. By Theorem 5.28(i)(a) $S^{t_i} x[\mathcal{N}] \to x[\mathcal{N} - \mathbf{r}]$ and since Y is closed, $\mathcal{N} - \mathbf{r} \in \Phi$.

(b): $\mathcal{M} \in \Phi(X(\mathcal{M}))$ and (a) implies that $\Phi(X(\mathcal{M}))$ is closed and invariant. Hence, it contains $\Theta(\mathcal{M})$, the smallest closed, invariant set containing $\mathcal{M}$.

(c): If $x[\mathcal{N}] \in X(\mathcal{M})$ with $\mathcal{N} \neq \emptyset$ then Corollary 5.26 implies that $\mathcal{N} \in \Theta(\mathcal{M})$. If $\mathcal{M} \neq FIN(\mathbb{N})$, then there exists $\mathbf{r} \in FIN(\mathbb{N}) \setminus \mathcal{M}$ and so $\mathcal{M} - \mathbf{r} = \emptyset$.

(d): By Corollary 5.19 $e = x[\emptyset] \in X(\mathcal{M})$. For nonempty $\mathcal{N}$, $\mathcal{N} \in \Phi(X(\mathcal{M}))$ iff $\mathcal{N} \in \Theta(\mathcal{M})$ by (b) and (c).

□

Corollary 5.31. *If $\mathcal{M} \neq FIN(\mathcal{M})$, then*

$$\Theta(\mathcal{M}) = \Phi(X(\mathcal{M})) = \Phi_+(X_+(\mathcal{M})).$$

Example 5.32. If $\mathcal{M} = FIN(\mathbb{N})$, the maximum label, then $\mathcal{M} - \mathbf{r} = \mathcal{M}$ for all $\mathbf{r} \in FIN(\mathbb{N})$ and so $\Theta(\mathcal{M}) = \{\mathcal{M}\}$. Thus, $\emptyset \in \Phi(X(\mathcal{M})) \setminus \Theta(\mathcal{M})$.

Notice that if $t \in IP(k)$ and $n \in \mathbb{N}$ there exists $s \in IP(k)$ such that $|t - s| = k(n)$. If $|j_i(t)| = n$ for some j_i in the expansion of t then let $s = t - k(j_i(t))$, i.e. remove the i^{th} term of the expansion so that s has length $r - 1$. If $|j_i(t)| \neq n$ for any i, e.g. if $t = 0$, let $s = t \pm k(n)$ both with length $r + 1$. It follows that if $z \in X(\mathcal{M})$ and $z_t = 1$ then $z_s = 1$ for $s = t + k(n)$ or $s = t - k(n)$. Hence, $\{t : z_t = 1\}$ is not bounded in $\mathbb{Z}$. In particular, $x[0] \notin X(\mathcal{M})$. More generally, we see that $X(\mathcal{M})$ has no nontrivial but semitrivial subsystem.

Corollary 5.33. *Let $\mathcal{M}$ be a label of finite type with $(X(\mathcal{M}), S)$ and $(X_+(\mathcal{M}), S)$ the associated subshifts, so that $X(\mathcal{M})$ (resp. $X_+(\mathcal{M})$) is the S orbit closure of $x[\mathcal{M}]$ (resp. $x_+[\mathcal{M}]$), and $\Theta(\mathcal{M})$ is the $FIN(\mathbb{N})$ orbit closure of $\mathcal{M}$.*

(a) $X(\mathcal{M}) = \{S^k(x[\mathcal{N}]) : k \in \mathbb{Z},\ \mathcal{N} \in \Theta(\mathcal{M})\}$ *and* $X_+(\mathcal{M}) = \{S^k(x_+[\mathcal{N}]) : k \in \mathbb{Z},\ \mathcal{N} \in \Theta(\mathcal{M})\}$. *Thus,* $\Theta(\mathcal{M}) = \Phi(X(\mathcal{M})) = \Phi_+(X_+(\mathcal{M}))$.

(b) *If $\Phi \subset \Theta(\mathcal{M})$ then $\Phi = \Phi(Y)$ for some closed, invariant subset Y of $X(\mathcal{M})$ iff $\Phi = \Phi_+(Y_+)$ for some closed, invariant subset Y_+ of $X_+(\mathcal{M})$ iff Φ is closed and $FIN(\mathbb{N})$ invariant. In that case,*

$$Y = \{S^k x[\mathcal{N}] : k \in \mathbb{Z}, \mathcal{N} \in \Phi\}, \quad Y_+ = \{S^k x_+[\mathcal{N}] : k \in \mathbb{Z}, \mathcal{N} \in \Phi\}.$$

Proof: From Theorem 5.28 (iv) it follows that when $\mathcal{M}$ is of finite type every element of the orbit closure of $x[\mathcal{M}]$ is on the orbit of some

$x[\mathcal{N}]$ for $\mathcal{N}$ a unique element of $\Theta(\mathcal{M})$. Hence, $\Phi(X(\mathcal{M})) = \Theta(\mathcal{M})$. If Y is an invariant subset of $X(\mathcal{M})$ then Y consists of the orbits of some of these $x[\mathcal{N}]$. That is, $\Phi(Y) \subset \Theta(\mathcal{M})$ and Y consists of the orbits of the points $x[\mathcal{N}]$ for $\mathcal{N} \in \Phi(Y)$. Conversely, if $\Phi \subset \Theta(\mathcal{M})$ then together the orbits of $\{x[\mathcal{N}] : \mathcal{N} \in \Phi\}$ form an invariant set with $\Phi(Y) = \Phi$. Proposition 5.30 implies that $\Phi(Y)$ is closed when Y is. To complete the proof of (b) we must show that Y is closed if $\Phi(Y)$ is closed and $FIN(\mathbb{N})$ invariant.

A sequence in Y is of the form $S^{t^i}x[\mathcal{N}^i]$ with $\mathcal{N}^i \in \Phi(Y)$. Shifting by a finite amount if necessary we can assume the limit point is $x[\mathcal{N}]$ for some $\mathcal{N} \in \Theta(\mathcal{M})$. If there is some subsequence of $\{t^i\}$ which is bounded then by going to a further subsequence we obtain a subsequence with $t^{i'} = k$ and $S^k x[\mathcal{N}^i] \to x[\mathcal{N}]$. Then Corollary 5.21(a) implies that $k = 0$ and $\{\mathcal{N}^{i'}\} \to \mathcal{N}$. Since $\Phi(Y)$ is closed, $\mathcal{N} \in \Phi(Y)$ and so $x[\mathcal{N}] \in Y$, by Lemma 5.14(b).

There remains the case with $\{|t_i|\} \to \infty$. Since $x[\mathcal{N}]_0 = 1$ we can assume that $S^{t_i}x[\mathcal{N}^i]_0 = 1$ for all i and so $t_i \in A[\mathcal{N}^i]$ for all i. By Corollary 5.26 $\{|j_{r(t^i)}|\} \to \infty$ and $\{\mathcal{M}^i - \mathbf{r}(t^i)\}$ converges to $\mathcal{N}$. Since $\{\mathcal{M}^i\}$ is a sequence in $\Phi(Y)$ and $\Phi(Y)$ is closed and invariant, $\mathcal{N} \in \Phi(Y)$ and so $x[\mathcal{N}] \in Y$.

□

Theorem 5.34. *For a label $\mathcal{M}$ the following are equivalent:*

(i) *$\mathcal{M}$ is a recurrent label.*
(ii) *$x[\mathcal{M}]$ is a recurrent point in $(\{0,1\}^{\mathbb{Z}}, S)$.*
(iii) *$x_+[\mathcal{M}]$ is a recurrent point in $(\{0,1\}^{\mathbb{Z}}, S)$.*

Proof: (i) $\Leftrightarrow$ (ii): Let $\{\mathbf{r}^i > 0\}$ be a sequence in $\mathcal{M}$ with $\mathcal{M} = LIM\ \{\mathcal{M} - \mathbf{r}^i\}$. Choose t^i with $\mathbf{r}(t^i) = \mathbf{r}^i$ and with $\{|j_{r^i}(t^i)|\} \to \infty$. By Lemma 5.22 $\{S^{t^i}(x[\mathcal{M}])\}$ is asymptotic to $\{x[\mathcal{M} - \mathbf{r}^i]\}$ which converges to $x[\mathcal{M}]$ by Corollary 5.21(b).

On the other hand, if $\{S^{t^i}(x[\mathcal{M}])\}$ converges to $x[\mathcal{M}]$ then eventually $\mathbf{r}(t^i) \in \mathcal{M}$ and by Corollary 5.26 $\{\mathcal{M} - \mathbf{r}(t^i)\}$ converges to $\mathcal{M}$. Hence, $\mathcal{M}$ is recurrent.

□

Corollary 5.35. *If a label $\mathcal{M}$ is not of finite type then $X(\mathcal{M})$ and $X_+(\mathcal{M})$ each contain non-periodic recurrent points.*

If a label $\mathcal{M}$ is of finite type then e is the only recurrent point of $X(\mathcal{M})$ or $X_+(\mathcal{M})$ and so $(X(\mathcal{M}), S)$ and $(X_+(\mathcal{M}), S)$ are CT systems. In that *case,* $(X(\mathcal{M}), S)$ *and* $(X_+(\mathcal{M}), S)$ *are LE and topologically transitive but not weak mixing.*

Proof: If $\mathcal{M}$ is not of finite type then Proposition 4.23 implies that there is a positive recurrent label $\mathcal{N}$ with $\mathcal{N} \in \Theta(\mathcal{M})$. Hence, $x[\mathcal{N}] \in X(\mathcal{M})$ by Proposition 5.30(b). Theorem 5.34 implies that $x[\mathcal{N}]$ is recurrent. By Corollary 5.19 $e = x[\emptyset]$ is the only periodic point in $Z(IP(k))$.

If $\mathcal{M}$ is of finite type then by Corollary 5.33 every point of $X(\mathcal{M})$ is on the orbit of some $x[\mathcal{M}_1]$ with $\mathcal{M}_1 \subset \mathcal{M}$. These are all labels of finite type and so none are recurrent except for $\mathcal{M}_1 = \emptyset$.

Since $x[\mathcal{M}]$ is always a transitive point for $X(\mathcal{M})$, $(X(\mathcal{M}), S)$ is always topologically transitive. In the finite type case, it is CT and so is LE and not weak mixing (see Remarks 1.11 and 2.4).

□

Example 5.36. With $\mathcal{M}$ as in Example 4.29 the subshift $X = X(\mathcal{M})$ is uncountable, CT and LE. In fact, each point of X is an isolated point in its orbit closure, and e is the unique recurrent point.

Let $SYM = \{ \ x \in \{0,1\}^{\mathbb{Z}} \ : \ x_{-t} = x_t \ \text{ for all } \ t \in \mathbb{Z} \ \}$, and let $ZER = \{ \ x \in \{0,1\}^{\mathbb{Z}} \ : \ x_0 = 1 \text{ and } x_t = 0 \ \text{ for all } \ t < 0 \ \}$.

Lemma 5.37. *SYM is a closed subset of $\{0,1\}^{\mathbb{Z}}$ which contains $x[\mathcal{LAB}]$. ZER is a closed subset of $\{0,1\}^{\mathbb{Z}}$ which contains $x_+[\mathcal{LAB}] \setminus \{e = x_+[\emptyset]\}$. Each non-periodic S orbit in $\{0,1\}^{\mathbb{Z}}$ meets SYM at most once. Each S orbit in $\{0,1\}^{\mathbb{Z}}$ meets ZER at most once*

Proof: Every $A[\mathcal{M}]$ is symmetric about 0 and so $x[\mathcal{LAB}]$ is contained in SYM. SYM is clearly a closed set. Since $\mathcal{LAB}$ is complete and $x[\cdot]$ is a homeomorphism onto its image, $x[\mathcal{LAB}]$ is a G_δ subset although it is not closed.

If $S^{k_1}x, S^{k_2}x \in SYM$ with $k_2 \neq k_1$ then for all $t \in \mathbb{Z}$,

$$x_{t+k_1} \ = \ x_{-t+k_1} \ = \ x_{(-t+k_1-k_2)+k_2} \ = \ x_{(t-k_1+k_2)+k_2} \tag{5.26}$$

Letting $s = t + k_1$ we have that $x_s = x_{s+2(k_2-k_1)}$ for all $s \in \mathbb{Z}$. Since $k_2 \neq k_1$ it follows that x is periodic.

The results for ZER are obvious.

□

Remark 5.38. It follows that if μ is any non-atomic, shift-invariant probability measure on $\{0,1\}^{\mathbb{Z}}$ then $\mu(ZER) = \mu(SYM) = 0$. Observe first that the countable set PER of periodic points in $\{0,1\}^{\mathbb{Z}}$ has measure zero because μ is non-atomic. Since $\{S^k(SYM \setminus PER) : k \in \mathbb{Z}\}$ and $\{S^k(ZER) : k \in \mathbb{Z}\}$ are pairwise disjoint sequences of sets with identical measure the common value must be zero.

Proposition 5.39. *Let x^* be a non-periodic recurrent point of $\{0,1\}^{\mathbb{Z}}$ and let X be its orbit closure, so that (X,S) is the closed subshift generated by x^*. If K is a closed subset of X such that every non-periodic orbit in X meets K at most once, then K is nowhere dense. The set $X \setminus \bigcup_{k\in\mathbb{Z}} \{S^{-k}(K)\}$ is a dense G_δ subset of X.*

In particular, $SYM \cap X$ is nowhere dense in X and the set of points of X whose orbit does not meet SYM is a dense G_δ subset of X. Similarly, $ZER \cap X$ is nowhere dense in X and the set of points of X whose orbit does not meet ZER is a dense G_δ subset of X.

Proof: Since the orbit of x^* is dense in X, it meets any nonempty open subset of X. If the interior of K contained more than one point then there would be two disjoint open sets U_1, U_2 contained in K and so there would exist $k_1, k_2 \in \mathbb{Z}$ such that $S^{k_a}x^* \in U_a \subset K$ for $a = 1,2$. Since U_1 and U_2 are disjoint $k_2 \neq k_1$. This contradicts the assumption on K. Hence, if K has nonempty interior then the interior consists of a single point which is on the orbit of x^*. This implies that x^* is an isolated point and so cannot be recurrent unless it is periodic. Hence, the interior of K is empty.

Since K is nowhere dense, the G_δ set $X \setminus \bigcup_{k\in\mathbb{Z}} \{S^{-k}(K)\}$ is dense by the Baire Category Theorem. A point lies in this set exactly when its orbit does not meet K.

The result applies to $K = SYM \cap X$ by Lemma 5.37.

In the case of ZER the result is clear because the S^{-1} transitive points in the orbit closure of x^* form a dense G_δ set disjoint from ZER.

□

Corollary 5.40. *For any label $\mathcal{M}$, the set $X(\mathcal{M}) \cap SYM = x[\Theta(\mathcal{M})]$ is a compact subset of $X(\mathcal{M})$ which meets each orbit in at most one point. The set $\{e\} \cup X_+(\mathcal{M}) \cap ZER = x_+[\Theta(\mathcal{M})]$ is a compact subset of $X_+(\mathcal{M})$ which meets each orbit in at most one point.*

If $\mathcal{M}$ is of finite type then $x[\Theta(\mathcal{M})]$ meets each orbit in $X(\mathcal{M})$ and $x_+[\Theta(\mathcal{M})]$ meets each orbit in $X_+(\mathcal{M})$.

If $\mathcal{M}$ is not of finite type then $X(\mathcal{M}) \setminus \bigcup\{S^i(SYM)\}$ is non-empty and so $\{S^k x[\mathcal{N}] : k \in \mathbb{Z},\ \mathcal{N} \in \Phi(X(\mathcal{M}))\}$ is a proper subset of $X(\mathcal{M})$.

Furthermore, $X_+(\mathcal{M}) \setminus [\{e\} \ \cup \ \bigcup\{S^i(ZER)]\}$ *is non-empty and so* $\{S^k(x_+[\mathcal{N}]) : k \in \mathbb{Z},\ \mathcal{N} \in \Phi_+(X_+(\mathcal{M}))\}$ *is a proper subset of* $X_+(\mathcal{M})$*.*

Proof: By Lemma 5.37 a non-periodic orbit meets SYM in at most one point. $X(\mathcal{M})$ is compact and SYM is closed and so the intersection is compact. $x[\Theta(\mathcal{M})] \subset X(\mathcal{M})$ and $x[\mathcal{LAB}] \subset SYM$. On the other hand, by Corollary 5.26 and the Remark thereafter, if $z \in X(\mathcal{M}) \cap SYM$ then $z = x[\mathcal{N}]$ for some $\mathcal{N} \in \Theta(\mathcal{M})$.

If $\mathcal{M}$ is of finite type then each orbit of $X(\mathcal{M})$ meets $x[\Theta(\mathcal{M})]$.

If $\mathcal{M}$ is not of finite type then by Corollary 5.35there exists a non-periodic recurrent point $x^* \in X(\mathcal{M})$. If X^* is the orbit closure of x^* then $X^* \subset X(\mathcal{M})$ and by Proposition 5.39 $X^* \setminus \bigcup \{S^i(SYM)\}$ is nonempty.

□

Remark 5.41. Corollaries 5.33 (a) and 5.40 show that for any label $\mathcal{M}$ of finite type the dynamical system $(X(\mathcal{M}), S)$ admits $x[\Theta(\mathcal{M})]$ as a closed *cross-section*; i.e. the orbit of any point $x \in X(\mathcal{M})$ meets $x[\Theta(\mathcal{M})]$ exactly at one point. This is in accordance with the following general theorem (see [22, Section 1.2]).

Theorem 5.42. *For a system* (X, T)*, with* X *a completely metrizable separable space, there exists a Borel cross-section if and only if the only recurrent points are the periodic ones.*

□

Notice, too, that if μ is an invariant probability measure on (X, T) such that the measure of the set of periodic points is zero, then any cross-section is a non-measurable set. The special case of translation by rationals on $\mathbb{R}/\mathbb{Z}$ is used in the usual proof of the existence of a subset of $\mathbb{R}$ which is not Lebesgue measurable.

On the other hand, when $\mathcal{M}$ is recurrent, $x[\Theta(\mathcal{M})]$ is a Cantor subset of $X(\mathcal{M})$ which meets each orbit at most once. This just says that the Cantor set $x[\Theta(\mathcal{M})]$ is *wandering* in $X(\mathcal{M})$, i.e. $S^i(x[\Theta(\mathcal{M})]) \cap S^j(x[\Theta(\mathcal{M})]) = \emptyset$ whenever $i \neq j$ in $\mathbb{Z}$. While this explicit construction may be of interest, in fact any system (X, T) admits wandering Cantor sets when X is perfect and the set of periodic points has empty interior, see [3] Theorem 1.4.

5.4. **The elements of $X(\mathcal{M})$ and $X_+(\mathcal{M})$.**

Any point $z \in \{0,1\}^{\mathbb{Z}}$ is the characteristic function $\chi(A)$ where $A = \{t \in \mathbb{Z} : z_t = 1\}$. Our purpose in this second is to provide an explicit description of the subsets associated with the points in $X(\mathcal{M})$ and $X_+(\mathcal{M})$ for an arbitrary label $\mathcal{M}$.

Let $S^{\mathbb{Z}}x[\Theta(\mathcal{M})] = \{S^t(x[\mathcal{N}]) : t \in \mathbb{Z}$ and $\mathcal{N} \in \Theta(\mathcal{M})\}$ and similarly $S^{\mathbb{Z}}x_+[\Theta(\mathcal{M})] = \{S^t(x_+[\mathcal{N}])\}$. By Proposition 5.30 and Corollary 5.33 these are always subsets of $X(\mathcal{M})$ and $X_+(\mathcal{M})$ respectively with equality when $\mathcal{M}$ is of finite type.

An arbitrary point z of $X(\mathcal{M})$ is the limit of some sequence $\{S^{t^i}(x[\mathcal{M}])\}$. We know that $e = x[\emptyset]$ is in both $X(\mathcal{M})$ and $X_+(\mathcal{M})$ and so, after a finite translation by S, we may restrict attention to points $z = \chi(A)$ with $z_0 = 1$, so with $0 \in A$, and then assume that $t^i \in A[\mathcal{M}]$ for all $i \in \mathbb{N}$.

If $\{t^i\}$ has a bounded subsequence, or if the procedure given in the proof of Lemma 5.24 (b) terminates after a finite number of steps, then the limit z lies in $S^{\mathbb{Z}}x[\Theta(\mathcal{M})]$. The procedure always terminates if $\mathcal{M}$ is of finite type. This is the reason that $X(\mathcal{M}) = S^{\mathbb{Z}}x[\Theta(\mathcal{M})]$ in the finite type case. Now assume the procedure does not terminate. We obtain a decreasing sequence of infinite subsets $\{SEQ_{-n}\}$ of $\mathbb{N}$. We restrict to a subsequence obtained by diagonalizing so that the n^{th} term lies in SEQ_{-n}. We can thus restrict to the following situation where the sequence $\{t^i\}$ is a special sequence built as follows

Definition 5.43. (a) Call $\mathcal{S} = \{a_1, a_2, ...\}$ an *absolute increasing sequence* in $\mathbb{Z}$ when $|a_1| > 0$ and for all $i \in \mathbb{N}$ $|a_{i+1}| > |a_i|$. Call it a *positive increasing sequence* when, in addition, $a_i > 0$ for all $i \in \mathbb{N}$. For an absolute increasing sequence $\{u^i = \Sigma_{p=1}^i \ k(a_p)\}$ is the *associated sequence of expanding times*. Observe that the sequence of lengths, $\{\mathbf{r}(u^i) = \Sigma_{p=1}^i \ \chi(\ell(a_p))\}$, is an increasing sequence in $FIN(\mathbb{N})$

(b) Call $\{t^i \in IP(k)\}$ a *special sequence of times* with associated absolute increasing sequence $\mathcal{S} = \{a_1, a_2, ...\}$ when for each $i \in \mathbb{N}$ there is an r^i truncation $\tilde{t}^i$ of t^i so that the residual $t^i - \tilde{t}^i$ is u^i. Thus, the expansion of t^i is $j_1(\tilde{t}^i), \dots, j_{r^i}(\tilde{t}^i), a_i, \dots, a_1$ and $|j_{r^i}(\tilde{t}^i)| \to \infty$. The sequence is a *positive special sequence of times* when $t^i \in IP_+(k)$ for all i and so $\mathcal{S}$ is a positive increasing sequence.

Define $\rho(\mathcal{S}) \in \mathbb{Z}_{+\infty}^{\mathbb{N}}$ by $\rho(\mathcal{S}) = \Sigma_{i=1}^{\infty} \ \chi(\ell(a_i))$. Since $t^i \in A(\mathcal{M})$ for all i it follows that $\langle \rho(\mathcal{S}) \rangle \subset \mathcal{M}$. As $\rho(\mathcal{S})$ is not a finite vector we see that $\mathcal{M}$ is not of finite type.

Notice that if i_p is an increasing function of $p \in \mathbb{N}$ then we can identify the subsequence $\{t^{i_p}\}$ with the special sequence $\{s^i\}$ associated with the same $\mathcal{S}$. Let $s^i = t^{i_p}$ for all $i \in (i_{p-1}, i_p]$ (by convention $i_0 = 0$). Then $\tilde{s}^i$ has expansion $j_1(\tilde{t}^{n_p}), \dots, j_{r^{n_p}}(\tilde{t}^{n_p}), a_{n_p}, \dots, a_{i+1}$.

Definition 5.44. Let $\mathcal{S} = \{a_1, a_2, ...\}$ be an absolute increasing sequence.

(a) $s \in \mathbb{Z}$ has an $\mathcal{S}$ *adjusted expansion* if $s = s' + \Sigma_{i=1}^{\infty} \epsilon_i k(a_i)$ with $s' \in IP(k)$ with an expansion which includes none of the members of $\mathcal{S}$ and with $\epsilon_i \in \{0, -1, -2\}$ for all i but with $\epsilon_i = 0$ for all but finitely many i. Thus, s has an $\mathcal{S}$ adjusted expansion iff $s + u^i \in IP(k)$ for all $i \in \mathbb{N}$ sufficiently large. For an absolute increasing sequence $\mathcal{S}$ we let $IP(k, \mathcal{S})$ be the set of $s \in \mathbb{Z}$ with an $\mathcal{S}$ adjusted expansion.

For such a time s we define $\rho(\mathcal{S}, s) = \mathbf{r}(s') + \Sigma \ \{\chi(\ell(a_i)) : \epsilon_i \neq -1\}$.

(b) If $\mathcal{S}$ is a positive increasing sequence then a *positive* $\mathcal{S}$ *adjusted expansion* is an S adjusted expansion with $s' \in IP_+(k)$ and $\epsilon_i \in \{0, -1\}$ for all $i \in \mathbb{N}$. Thus, s has a positive $\mathcal{S}$ adjusted expansion iff $s + u^i \in IP_+(k)$ for all $i \in \mathbb{N}$ sufficiently large. For a positive increasing sequence $\mathcal{S}$ we let $IP_+(k, \mathcal{S})$ be the set of $s \in \mathbb{Z}$ with a positive $\mathcal{S}$ adjusted expansion.

Remark 5.45. For $B \subset \mathbb{Z}$ define the *restriction* $k|B : \mathbb{Z} \to \mathbb{Z}$ by

$$(k|B)(t) = \begin{cases} k(t) & \text{for } t \in B \\ 0 & \text{for } t \notin B \end{cases} .$$

For a sequence $\mathcal{S}$ we let $k|\mathcal{S}$ be the restriction to the set of terms of the sequence. For example, $IP_+(k) = IP(k|\mathbb{N})$. It is easy to check that

$$\begin{aligned} IP(k, \mathcal{S}) &= IP(k - k|\mathcal{S}) - IP(k|\mathcal{S}), \\ IP_+(k, \mathcal{S}) &= IP(k|(\mathbb{N} \setminus \mathcal{S})) - IP(k|\mathcal{S}) = IP(k|\mathbb{N} - 2k|\mathcal{S}). \end{aligned} \tag{5.27}$$

For example, 0 has an $\mathcal{S}$ adjusted expansion with $s' = 0$ and $\epsilon_i = 0$ for all i. If $\mathcal{S}$ is positive then this is a positive adjusted expansion.

Lemma 5.46. *Let $\{t^i\}$ be a special sequence of times associated with $\mathcal{S}$ and let $s \in \mathbb{Z}$.*

(a) If $3|s| < |a_i|$ then $s + u^i \in IP(k)$ iff $s + t^i \in IP(k)$. In that case, $\tilde{t}^i$ is the common r^i truncation of t^i and $s + t^i$ and the expansions of u^i and $s + u^i$ each begin with a_i. If, in addition, $\{t^i\}$ is a positive special sequence of times then $s + u^i \in IP_+(k)$ iff $s + t^i \in IP_+(k)$

(b) The following are equivalent:

(i) *$s \in IP(k, \mathcal{S})$.*
(ii) *For all $i \in \mathbb{N}$ if $3|s| < |a_i|$ then $s + u^i \in IP(k)$.*
(iii) *There exists i with $3|s| < |a_i|$ and $s + u^i \in IP(k)$.*
(iv) *For sufficiently large i, $t^i + s \in IP(k)$.*
(v) *There exists i with $3|s| < |a_i|$ and $s + t^i \in IP(k)$.*

(c) If $\{t^i\}$ is a positive special sequence of times then the following are equivalent:

(i) *$s \in IP_+(k, \mathcal{S})$.*
(ii) *For all $i \in \mathbb{N}$ if $3|s| < |a_i|$ then $s + u^i \in IP_+(k)$.*
(iii) *There exists i with $3|s| < |a_i|$ and $s + u^i \in IP_+(k)$.*
(iv) *For sufficiently large i, $t^i + s \in IP_+(k)$.*
(v) *There exists i with $3|s| < |a_i|$ and $s + t^i \in IP_+(k)$.*

Proof: (a) Given s let i be such that $3|s| < |a_i|$.

If $u^i + s \in IP(k)$ with $3|s| < |a_i|$, then by Lemma 5.4 the first term $k(a_i)$ in the expansion of u^i must be the first term in the expansion of $u^i + s$ as well. As $|j_{r^i}(\tilde{t}^i)| > |a_i|$ it follows that $t^i + s = (u^i + s) + \tilde{t}^i \in IP(k)$.

Conversely, if $t^i + s \in IP(k)$ then

$$|t^i - (t^i + s)| \ = |s| \ < \ \frac{1}{3}|a_i| \ < \ \frac{b-2}{b}|j_{r^i}(t^i)|$$

and Lemma 5.4 again imply that the expansions for t^i and $s + t^i$ agree through the r^i term. That is, $\tilde{t}^i$ is a common r^i truncation for t^i and $s + t^i$. Hence, the residual $(t^i + s) - \tilde{t}^i = u^i + s$ is in $IP(k)$.

(b) (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are obvious and (i) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v) follow from (a). We are left with showing (iii) $\Rightarrow$ (ii).

Let i_0 be the smallest positive integer i such that $3|s| < |a_i|$. If $s + u^i \in IP(k)$ for some $i \geq i_0$ then Lemma 5.4 implies that the expansions of $s + u^i$ and u^i agree through a_{i_0} and so have a common truncation with residuals $s + u^{i_0}$ and u^{i_0}. Hence, $s + u^{i_0}$ is an expanding time and the first term in its expansion is a_{i_0}. It follows that $s + u^p = (s + u^{i_0}) + (u^p - u^{i_0}) \in IP(k)$ for all $p > i_0$, proving (ii).

□

In particular, for $\chi(IP(k)) = A[FIN(\mathbb{N})]$ and $\chi(IP_+(k)) = A_+[FIN(\mathbb{N})]$ we immediately have

Corollary 5.47. *If $z_0 = 1$ for $z \in \{0,1\}^{\mathbb{Z}}$, we have*

(5.28)

$$z \in X(\chi(IP(k))) \quad \Longleftrightarrow \quad z = \chi(IP(k - k|B) - IP(k|B)) \text{ for some } \ B \subset \mathbb{Z},$$

$$z \in X(\chi(IP_+(k))) \quad \Longleftrightarrow \quad z = \chi(IP(k|\mathbb{N} - 2k|B)) \text{ for some } \ B \subset \mathbb{N}.$$

Proof: Let $\{t^i\}$ be a special sequence of times associated with $\mathcal{S}$.

$$\lim \{ \ S^{t^i}(\chi(IP(k))) \ \} \ = \ \chi(IP(\mathcal{S},k)). \tag{5.29}$$

If $\{t^i\}$ is a positive special sequence of times then

$$\lim \{ \ S^{t^i}(\chi(IP_+(k))) \ \} \ = \ \chi(IP_+(\mathcal{S},k)). \tag{5.30}$$

Apply (5.27) with B the set of terms of the sequence $\mathcal{S}$.

The remaining cases are $z = S^t(\chi(IP(k))$ with $t \in IP(k)$ and $z = S^t(\chi(IP_+(k)))$ with $t \in IP_+(k)$. If $a_r, \dots, a_1$ is the expansion for t then we let $B = \{a_1, \dots, a_r\}$. If $t+s \in IP(k)$ then $s = s' + \Sigma_{i=1}^r \epsilon_i k(a_i)$ with $s' \in IP(k)$ with an expansion which includes none of the members of B and with $\epsilon_i \in \{0, -1, -2\}$. Proceed similarly for IP_+.

□

Proposition 5.48. *Let $\mathcal{M}$ be a label and $z \in \{0,1\}^{\mathbb{Z}}$ with $z_0 = 1$.*

(a) Assume $z = \chi(A) = \lim\{S^{t^i}(x[\mathcal{M}])\}$ with $\{t^i\}$ a special sequence associated with $\mathcal{S}$ and lengths $\{r^i\}$. Let $\mathbf{s}^i = \mathbf{r}(\tilde{t}^i)$ with length r^i.

(i) *$A \subset IP(k, \mathcal{S})$.*

(ii) *For $s \in IP(k, \mathcal{S})$*

$$\begin{aligned} z_s \ &= \ 1 \quad \Longleftrightarrow \quad \text{eventually} \quad \mathbf{r}(u^i + s) + \mathbf{s}^i \in \mathcal{M}, \\ z_s \ &= \ 0 \quad \Longleftrightarrow \quad \text{eventually} \quad \mathbf{r}(u^i + s) + \mathbf{s}^i \not\in \mathcal{M}. \end{aligned} \tag{5.31}$$

(iii) *If $z_s = 1$ then eventually $\mathbf{r}(u^i + s) \in LIMINF_p\{\mathcal{M} - \mathbf{s}^p\}$. If $z_s = 0$ then eventually $\mathbf{r}(u^i + s) \not\in LIMSUP_p\{\mathcal{M} - \mathbf{s}^p\}$.*

(iv) *$z \in X(\mathcal{M}) \setminus S^{\mathbb{Z}}x[\Theta(\mathcal{M})]$.*

(b)Assume $z = \chi(A) = \lim\{S^{t^i}(x_+[\mathcal{M}])\}$ with $\{t^i\}$ a positive special sequence associated with $\mathcal{S}$ and lengths $\{r^i\}$. Let $\mathbf{s}^i = \mathbf{r}(\tilde{t}^i)$ with length r^i.

(i) *$A \subset IP_+(k, \mathcal{S})$.*

(ii) *For $s \in IP_+(k, \mathcal{S})$*

$$\begin{aligned} z_s \ &= \ 1 \quad \Longleftrightarrow \quad \text{eventually} \quad \mathbf{r}(u^i + s) + \mathbf{s}^i \in \mathcal{M}, \\ z_s \ &= \ 0 \quad \Longleftrightarrow \quad \text{eventually} \quad \mathbf{r}(u^i + s) + \mathbf{s}^i \not\in \mathcal{M}. \end{aligned} \tag{5.32}$$

(iii) *If $z_s = 1$ then eventually $\mathbf{r}(u^i + s) \in LIMINF_p\{\mathcal{M} - \mathbf{s}^p\}$. If $z_s = 0$ then eventually $\mathbf{r}(u^i + s) \not\in LIMSUP_p\{\mathcal{M} - \mathbf{s}^p\}$.*

(iv) *$z \in X_+(\mathcal{M}) \setminus S^{\mathbb{Z}}x_+[\Theta(\mathcal{M})]$.*

Proof: (i): By going to a subsequence we can assume that $S^{t^i}(x[FIN(\mathbb{N})])$ converges as well and so with limit $\chi(IP(\mathcal{S},k))$. Since $\mathcal{M} \subset FIN[\mathbb{N}]$ it follows that $A \subset IP(\mathcal{S},k)$. In particular, if $s \not\in IP(\mathcal{S},k)$ then $z_s = 0$.

(ii): Now fix $N \in \mathbb{N}$. We can choose $i_N \in \mathbb{N}$ so that $3N < |a_{i_N}|$ and for all $s \in \mathbb{Z}$ with $|s| \leq N$ and $i \geq i_N$

$$
\begin{aligned}
z_s &= 1 &\Longleftrightarrow&& t^i + s &= u^i + s + \tilde{t}^i \in A[\mathcal{M}], \\
z_s &= 0 &\Longleftrightarrow&& t^i + s &= u^i + s + \tilde{t}^i \not\in A[\mathcal{M}].
\end{aligned} \tag{5.33}
$$

If $s \in IP(\mathcal{S}, k)$ with $|s| \leq N$ then Lemma 5.46 implies that $u^i + s \in IP(k)$ and $\mathbf{r}(u^i + s) + \mathbf{s}^i = \mathbf{r}(t^i + s)$ for all $i \geq i_N$. So if $z_s = 1$ then $\mathbf{r}(u^i + s) + \mathbf{s}^i \in \mathcal{M}$ while if $z_s = 0$ then $\mathbf{r}(u^i + s) + \mathbf{s}^i \not\in \mathcal{M}$.

(iii): As in the proof above, if $|s| \leq N$ and $z_s = 1$ and if $i_N \leq i \leq p$ then $\mathbf{r}(u^i + s) \leq \mathbf{r}(u^p + s) \in \mathcal{M} - \mathbf{s}^p$.

(iv): For $i \leq p$, the expansion of $t^p - 2u^i$ agrees with that of t^p except that $a_1, \dots, a_i$ are replaced by $-a_1, \dots, -a_i$. So $t^p - 2u^i \in IP(k)$ with $\mathbf{r}(t^p - 2u^i) = \mathbf{r}(t^p) \in \mathcal{M}$. It follows that $-2u^i \in A$ for all $i \in \mathbb{N}$. On the other hand, for any fixed $t \in \mathbb{Z}$ we have eventually $-t - 2u^i \not\in IP(k)$. It follows that $z = x[\chi(A)] \not\in S^{\mathbb{Z}}(X(\mathcal{M}))$.

For the positive case in (b) we observe that $t^p - u^i$ has expansion agreeing with that of t^p except that $a_1, \dots, a_i$ do not appear. Hence, $-u^i \in A$ for all i but for any fixed $t \in \mathbb{Z}$ we have eventually $-t - u^i \not\in IP_+(k)$.

□

Conversely, if we are given an absolute increasing sequence $\mathcal{S}$ and a sequence $\{\mathbf{s}^i \in FIN(\mathbb{N})\}$ and a subset $A \subset IP(\mathcal{S}, k)$ such that

(5.34)

$$
\begin{aligned}
s \in A &\Longleftrightarrow \text{eventually } \mathbf{r}(u^i + s) + \mathbf{s}^i \in \mathcal{M}, \\
s \in IP(\mathcal{S}, k) \setminus A &\Longleftrightarrow \text{eventually } \mathbf{r}(u^i + s) + \mathbf{s}^i \not\in \mathcal{M}.
\end{aligned}
$$

then we can choose $\tilde{t}^i \in IP_+(k)$ with $\mathbf{r}(\tilde{t}^i) = \mathbf{s}^i$ and $j_{r^i}(\tilde{t}^i) > |a_i|$ where r^i is the length of $\mathbf{s}^i$. Then $\{t^i = u^i + \tilde{t}^i\}$ is a special sequence of times and $x[\chi(A)] = Lim\{S^{t^i}(x[\mathcal{M}])\}$. If $\mathcal{S}$ is a positive increasing sequence then $x_+[\chi(A)] = Lim\{S^{t^i}(x_+[\mathcal{M}])\}$.

By going to a subsequence we can assume that $\{\mathcal{M} - \mathbf{s}^i\}$ converges to $\mathcal{N} \in \Theta(\mathcal{M})$. In that case, (iv) becomes, for $s \in IP(\mathcal{S}, k)$, $z_s = 1$ (or $z_s = 0$) iff eventually $\mathbf{r}(u^i + s) \in \mathcal{N}$ (resp. eventually $\mathbf{r}(u^i + s) \not\in \mathcal{N}$.

We can use this to extend - slightly - Corollary 5.47. Recall from Proposition 4.23 (i) that $\mathcal{M}$ is a sublattice of $FIN(\mathbb{N})$ iff $\mathcal{M} = \langle \rho(\mathcal{M}) \rangle$.

Proposition 5.49. *Assume that the infinite label $\mathcal{M}$ is a sublattice.*

(a) $z = \chi(A) \in X(\mathcal{M}) \setminus S^{\mathbb{Z}} x[\Theta(\mathcal{M})]$ iff there exists $\mathcal{N} \in \Theta(\mathcal{M})$ and $\mathcal{S}$ an absolute increasing sequence and a sequence $\{\mathbf{s}^i \in FIN(\mathbb{N})\}$ such that $\mathbf{r}(u^i) + \mathbf{s}^i \in \mathcal{M}$ for all $i \in \mathbb{N}$, $\mathcal{N} = LIM\{\mathcal{M} - \mathbf{s}^i\}$ and $A = \{s \in IP(\mathcal{S}, k) : \langle \rho(\mathcal{S}, s) \rangle \subset \mathcal{N}\}$.

(b) $z = \chi(A) \in X_+(\mathcal{M}) \setminus S^{\mathbb{Z}} x_+[\Theta(\mathcal{M})]$ *iff there exists* $\mathcal{N} \in \Theta(\mathcal{M})$ *and* $\mathcal{S}$ *a positive increasing sequence and a sequence* $\{\mathbf{s}^i \in FIN(\mathbb{N})\}$ *such that* $\mathbf{r}(u^i) + \mathbf{s}^i \in \mathcal{M}$ *for all* $i \in \mathbb{N}$, $\mathcal{N} = LIM\{\mathcal{M} - \mathbf{s}^i\}$ *and* $A = \{s \in IP_+(\mathcal{S}, k) : \langle \rho(\mathcal{S}, s) \rangle \subset \mathcal{N}\}$.

Proof: Notice that $\langle \rho(\mathcal{S}) \rangle \subset \mathcal{N}$ and so $0 \in A$. For $s \in IP(\mathcal{S}, k)$, $\{u^i + s\}$ is increasing in $FIN(\mathbb{N})$ once i is large enough that $3|s| < |a_i|$. So if $p \geq i$, $\mathbf{r}(u^i + s) \leq \mathbf{r}(u^p + s)$ and so $\mathbf{r}(u^p + s) + \mathbf{s}^p \in \mathcal{M}$ eventually implies that eventually $\mathbf{r}(u^i + s) \in \mathcal{N}$ and so $\langle \rho(\mathcal{S}, s) \rangle \subset \mathcal{N}$.

If $\langle \rho(\mathcal{S}, s) \rangle \subset \mathcal{N}$ then $u^i + s \in \mathcal{N}$ for sufficiently large i.

Now assume for some i with $3|s| < |a_i|$ we have $u^i + s \in \mathcal{M}$. There exists a $p_i > i$ such that $u^i + s + \mathbf{s}^p \in \mathcal{M}$ for all $p \geq p_i$. By assumption $u^p + \mathbf{s}^p \in \mathcal{M}$. Because $\mathcal{M}$ is a sublattice, $u^p + s + \mathbf{s}^p = max(u^i + s + \mathbf{s}^p, u^p + \mathbf{s}^p) \in \mathcal{M}$ for $p \geq p_i$.

The remaining possibility is that eventually $u^i + s + \mathbf{s}^i \not\in \mathcal{M}$.

As we saw above this implies that we can choose $\tilde{t}^i$ so that $\{\tau^i = \tilde{t}^i + u^i\}$ is a special sequence of times associated with $\mathcal{S}$ and $Lim\{S^{t^i}(x[\mathcal{M}])\} = \chi(A)$.

□

5.5. **WAP Subshifts.**

Theorem 5.50. *Let* $\mathcal{M}$ *be a label of finite type. Let* $\{t^i\}$ *be a sequence of expanding times with* r_i *the length of* t^i. *If* $|j_{r_i}(t^i)| \to \infty$ *and* $\{\mathcal{M} - \mathbf{r}(t^i)\}$ *is eventually constant at* $\mathcal{M} - \mathbf{r}$, *then* $\{S^{t^i}\}$ *on* $X(\mathcal{M})$ *converges pointwise to* $p_{\mathbf{r}} \in E(X(\mathcal{M}), S)$ *such that* $p_{\mathbf{r}} x[\mathcal{M}] = x[\mathcal{M} - \mathbf{r}] = x[P_{\mathbf{r}}(\mathcal{M})]$ *and* $p_{\mathbf{r}}$ *is continuous on* $X(\mathcal{M})$. *Similarly,* $\{S^{t^i}\}$ *on* $X_+(\mathcal{M})$ *converges pointwise to* $p_{\mathbf{r}} \in E(X_+(\mathcal{M}), S)$ *such that* $p_{\mathbf{r}} x_+[\mathcal{M}] = x_+[\mathcal{M} - \mathbf{r}] = x_+[P_{\mathbf{r}}(\mathcal{M})]$ *and* $p_{\mathbf{r}}$ *is continuous on* $X_+(\mathcal{M})$.

Proof: Discarding the initial values we can assume $\mathcal{M} - \mathbf{r}(t^i) = \mathcal{M} - \mathbf{r}$ for all i.

Let p be any pointwise limit point of $\{S^{t^i}\}$ in $E(X(\mathcal{M}), S)$, i.e. the limit of a convergent subnet. By Theorem 5.28 (i)(a) we have that $px[\mathcal{M}] = x[\mathcal{M} - \mathbf{r}]$. It suffices to prove that p is continuous because it is then determined by its value on the transitive point $x[\mathcal{M}]$ and so the same p is the limit of every convergent subnet of $\{S^{t^i}\}$. By compactness this implies that the sequence $\{S^{t^i}\}$ converges to p.

To prove continuity of p it suffices by Proposition 1.4 to show that $pqx[\mathcal{M}] = qpx[\mathcal{M}]$ for any $q \in E(X(\mathcal{M}), S)$. We assume that q is a limit of a subnet of the sequence $\{S^{s^j}\}$.

By Theorem 5.28 (iii), if $\{S^{s^j}(x[\mathcal{M}])\} \to e$ then $\{S^{s^j}(x[\mathcal{M}-\mathbf{r}])\} \to e$ and so $pqx[\mathcal{M}] = e = qpx[\mathcal{M}]$. We now consider the case when $qx[\mathcal{M}] = z \neq e$.

By Theorem 5.28 again we are reduced to the case when $\{s^j\}$ is a sequence in $A[\mathcal{M}]$, $|j_{r^{s^j}}| \to \infty$ and $\{\mathcal{M}-\mathbf{r}(s^j)\}$ converges to LIM_s. We show that the two double limits of $\{S^{t^i+s^j}(x[\mathcal{M}])\}$ exist and have the same value.

(5.35)

$$Lim_j Lim_i\{S^{t^i+s^j}(x[\mathcal{M}])\} = Lim_j\{S^{s^j}(x[\mathcal{M}-\mathbf{r}])\} = Lim_j x[\mathcal{M}-\mathbf{r}-\mathbf{r}(s^j)]$$

By Proposition 4.15 applied with $\mathcal{M}^j = \mathcal{M} - \mathbf{r}(s^j)$ $\{\mathcal{M} - \mathbf{r} - \mathbf{r}(s^j)\}$ converges to $LIM_s - \mathbf{r}$ and so this limit is $x[LIM_s - \mathbf{r}]$.

(5.36)

$$Lim_i Lim_j\{S^{t^i+s^j}(x[\mathcal{M}])\} = S^{t^i}(x[LIM_s]) = Lim_i x[LIM_s - \mathbf{r}(t^i)].$$

By Lemma 4.30(c), $LIM_s - \mathbf{r}(t^i) = LIM_s - \mathbf{r}$ for all i. So this limit is also $x[LIM_s - \mathbf{r}]$.

So the common value of the double limits of $\{S^{t^i+s^j}(x[\mathcal{M}])\}$ is $x[LIM_s - \mathbf{r}]$. That is $pqx[\mathcal{M}] = qpx[\mathcal{M}] = x[LIM_s - \mathbf{r}]$.

□

In particular, for any $\ell \in \mathbb{N}$ there is a continuous elements p_ℓ of the semigroups with $p_\ell x[\mathcal{M}] = x[\mathcal{M} - \chi(\ell)]$ and $p_\ell x_+[\mathcal{M}] = x_+[\mathcal{M} - \chi(\ell)]$. For $\mathbf{r} \in \mathcal{M}$, $p_\mathbf{r} = \Pi_\ell \ p_\ell^{\mathbf{r}_\ell}$ since these agree on $x[\mathcal{M}]$ and $x_+[\mathcal{M}]$. The set of labels $\mathcal{N}$ in $\Theta(\mathcal{M})$ such that $p_\mathbf{r} x[\mathcal{N}] = x[P_\mathbf{r}\mathcal{N}]$ and $p_\mathbf{r} x_+[\mathcal{N}] = x_+[P_\mathbf{r}\mathcal{N}]$ is closed and invariant since the elements of each set $\{p_\mathbf{s}\}$ and $\{P_\mathbf{s}\}$ commute with one another. Hence, the equations hold for all $\mathcal{N} \in \Theta(\mathcal{M})$. The relations on these elements of the semigroups are given by $p_{\mathbf{r}_1} = p_{\mathbf{r}_2}$ iff $\mathcal{M} - \mathbf{r}_1 = \mathcal{M} - \mathbf{r}_2$. Compare the remark after Lemma 4.30.

Corollary 5.51. *If $\mathcal{M}$ is a label of finite type, then $FIN(\mathbb{N}) \times X(\mathcal{M}) \to X(\mathcal{M})$ and $FIN(\mathbb{N}) \times X_+(\mathcal{M}) \to X_+(\mathcal{M})$ given by $(\mathbf{r}, x) \mapsto p_\mathbf{r} x$ are continuous monoid actions and $x[\cdot] : \Theta(\mathcal{M}) \to X(\mathcal{M})$, $x_+[\cdot] : \Theta(\mathcal{M}) \to X_+(\mathcal{M})$ are injective, continuous action maps. They induce homomorphism $J_\mathcal{M} : \mathcal{E}(\Theta(\mathcal{M})) \to E(X(\mathcal{M}), S)$ and $J_{+\mathcal{M}} : \mathcal{E}(\Theta(\mathcal{M})) \to E(X_+(\mathcal{M}), S)$ each of which is a homeomorphism onto its image. Except for the retraction to e, every $q \in E(X(\mathcal{M}), S)$ can be expressed as $q = S^k J_\mathcal{M}(Q) = J_\mathcal{M}(Q)$ with $k \in \mathbb{Z}$ and $Q \in \mathcal{E}(\Theta(\mathcal{M}))$ uniquely determined by q. In particular, $q \in J_\mathcal{M}(\mathcal{E}(\Theta))$ iff $qx[\mathcal{M}] \in x[\Theta(\mathcal{M})]$. Similarly, except for the retraction to e, every $q \in E(X_+(\mathcal{M}), S)$ can be expressed as $q = S^k J_{+\mathcal{M}}(Q) = J_{+\mathcal{M}}(Q)$ with $k \in \mathbb{Z}$ and $Q \in \mathcal{E}(\Theta(\mathcal{M}))$*

uniquely determined by q. In particular, $q \in J_{+\mathcal{M}}(\mathcal{E}(\Theta))$ *iff* $qx_+[\mathcal{M}] \in x_+[\Theta(\mathcal{M})]$.

Proof: For $\mathbf{r}, \mathbf{s} \in FIN(\mathbb{N})$ we see that $p_\mathbf{r} \circ p_\mathbf{s} = p_{\mathbf{r}+\mathbf{s}}$ on $x[\mathcal{M}]$. As these are continuous action maps of the S actions and $x[\mathcal{M}]$ is a transitive point for $X(\mathcal{M})$ it follows that equality holds on all of $X(\mathcal{M})$. Similarly, $p_\mathbf{0} = id_{X(\mathcal{M})}$. Thus, the map $(\mathbf{r}, x) \mapsto p_\mathbf{r} x$ is a continuous monoid action.

Because the continuous maps $x[\cdot] \circ P_\mathbf{r} = p_\mathbf{r} \circ x[\cdot]$ on every $P_\mathbf{s}(\mathcal{M})$ and these points are dense in $\Theta(\mathcal{M})$, it follows by continuity that the equation holds on all of $\Theta(\mathcal{M})$. Hence, $x[\cdot]$ is an action map.

If a net $\{P_{\mathbf{r}^i}\}$ converges pointwise to $Q \in \mathcal{E}(\Theta(\mathcal{M}))$, then for any limit point q of the net $\{p_{\mathbf{r}^i}\}$ in $E(X(\mathcal{M}), S)$, we have $q(x[\mathcal{N}]) = x[Q(\mathcal{N})]$ for all $\mathcal{N} \in \Theta(\mathcal{M})$. By Corollary 5.33 every point x of $X(\mathcal{M})$ can be expressed uniquely as $x = S^k(x[\mathcal{N}])$ for some $\mathcal{N} \in \Theta(\mathcal{M})$. Hence, $qx = S^k(qx[\mathcal{N}]) = S^k x[Q(\mathcal{N})]$. That is, Q on $x[\Theta(\mathcal{M})]$ has a unique extension q to an element of $E(X(\mathcal{M}), S)$. Since $x[\cdot]$ is an action map, $J_\mathcal{M}$ is a homomorphism and since q is determined by its restriction to $x[\Theta(\mathcal{M})]$, $J_\mathcal{M}$ is injective. If $\{Q^i\}$ is a net converging to Q in $\mathcal{E}(\Theta(\mathcal{M}))$ then the net $\{J_\mathcal{M}(Q^i)\}$ converges to $J_\mathcal{M}(Q)$ pointwise on $x[\Theta(\mathcal{M})]$. Since $X(\mathcal{M}) = \bigcup_k \ \{S^k(x[\Theta(\mathcal{M})])\}$ it follows that $\{J_\mathcal{M}(Q^i)\}$ converges to $J_\mathcal{M}(Q)$ pointwise on all of $X(\mathcal{M})$. Hence, $J_\mathcal{M}$ is an injective, continuous map. Because $\mathcal{E}(\Theta(\mathcal{M}))$ is compact, $J_\mathcal{M}$ is a homeomorphism onto its image.

Now suppose that $\{S^{s^i} : i \in I\}$ is a net converging to $q_1 \in E(X(\mathcal{M}), S)$. By Corollary 5.33 every point x of $X(\mathcal{M})$ can be expressed uniquely as $q_1 X(\mathcal{M}) = S^k(x[\mathcal{N}])$ for some $\mathcal{N} \in \Theta(\mathcal{M})$ and $k \in \mathbb{Z}$. Hence, $\{S^{t^i}(x[\mathcal{M}]) : i \in I\}$ converges to $x[\mathcal{N}]$ with $t_i = s_i - k$. By Corollary 5.26 we may have eventually $t^i = 0$ in which case $q_1 = S^k$. If not then eventually $t^i \in A[\mathcal{M}]$ with length r_i and length vector $\mathbf{r}(t^i) > 0$ and $\{ \ |j_{r_i}(t^i)|\} \to \infty$, $\{ \ \mathcal{M} - \mathbf{r}(t^i) \ \}$ convergent and $LIM \ \{ \ \mathcal{M} - \mathbf{r}(t^i) \ \} \ = \ \mathcal{N}$. By Lemma 5.22 for any $\mathcal{M}_1 \in \Theta(\mathcal{M})$, $\{S^{t^i}(x[\mathcal{M}_1]) : i \in I\}$, which converges to $S^{-k} q_1 x[\mathcal{M}_1]$, is asymptotic to $\{x[\mathcal{M}_1 - \mathbf{r}(t^i)] : i \in I\}$ and so the latter is convergent to $S^{-k} q_1 x[\mathcal{M}_1] \in X(\mathcal{M})$ which is of the form $S^{k_1} x[\mathcal{N}_1]$ for some $\mathcal{N}_1 \in \Theta(\mathcal{M})$ and $k_1 \in \mathbb{Z}$. By Corollary 5.21, $k_1 = 0$ and $\{\mathcal{M}_1 - \mathbf{r}(t^i) : i \in I\}$ converges to $\mathcal{N}_1$. That is, the net $\{P_{\mathbf{r}(t^i)} : i \in I\}$ converges pointwise in $\mathcal{E}(\Theta(\mathcal{M}))$ to $Q \in \mathcal{E}(\Theta(\mathcal{M}))$ with $Q(\mathcal{M}_1) = LIM/\{\mathcal{M}_1 - \mathbf{r}(t^i) : i \in I\}$. Thus, $S^{-k} q_1$ is the image of Q via the injection $J_\mathcal{M} : \mathcal{E}(\Theta(\mathcal{M})) \to E(X(\mathcal{M}), S)$. This shows that $q_1 = S^k J_\mathcal{M}(Q)$. Lemma 5.37 implies that k and hence Q are uniquely determined except when q_1 is the retraction to e in which case $Q = P_\mathbf{r}$

with $\mathbf{r} \notin \mathcal{M}$ but k can be anything in $\mathbb{Z}$. Hence, if $q_1 x[\mathcal{M}] \in x[\Theta(\mathcal{M})]$ and $q_1 x[\mathcal{M}] \neq e$ then $k = 0$ and so $q_1 = J_{\mathcal{M}}(Q)$.
□

If $\mathcal{M}$ is a finitary label then $\Theta(\mathcal{M})$ consists of $\{ \mathcal{M} - \mathbf{r} \ : \ \mathbf{r} \in FIN(\mathbb{N}) \}$ together with the finite external labels for $\mathcal{M}$. As $\mathcal{M}$ is of finite type, $X(\mathcal{M})$ consists of the orbits of the points of the compact set $x[\Theta(\mathcal{M})]$ and except for the fixed point e each orbit intersects $x[\Theta(\mathcal{M})]$ in a single point.

Theorem 5.52. *If $\mathcal{M}$ is a finitary label, then $(X(\mathcal{M}), S)$ and $(X_+(\mathcal{M}), S)$ are countable, WAP subshifts. For $\mathcal{F}$ an external limit set there are unique elements $q_{\mathcal{F}} \in E(X(\mathcal{M}), S)$ with $q_{\mathcal{F}} x[\mathcal{M}] = x[\mathcal{F}]$ and $q_{\mathcal{F}} \in E(X_+(\mathcal{M}), S)$ with $q_{\mathcal{F}} x_+[\mathcal{M}] = x_+[\mathcal{F}]$. There is a unique $Q_{\mathcal{F}} \in \mathcal{E}(\Theta(\mathcal{M}))$ such that $Q_{\mathcal{F}}(\mathcal{M}) = \mathcal{F}$ and $q_{\mathcal{F}}$ in $E(X(\mathcal{M}), S)$ is $J_{\mathcal{M}}(Q_{\mathcal{F}})$ and $q_{\mathcal{F}}$ in $E(X_+(\mathcal{M}), S)$ is $J_{+\mathcal{M}}(Q_{\mathcal{F}})$. The semigroups $\mathcal{E}(\Theta(\mathcal{M})), E(X(\mathcal{M}), S)$ and $E(X_+(\mathcal{M}), S)$ are abelian and act continuously on the spaces $\Theta(\mathcal{M}), X(\mathcal{M})$ and $X_+(\mathcal{M})$, respectively.*

Proof: Since the external labels are finite, $\Theta(\mathcal{M})$ is countable. Since a finitary label is of finite type, every point of $X(\mathcal{M})$ lies on the orbit of an $x[\mathcal{N}]$ with $\mathcal{N} \in \Theta(\mathcal{M})$. It follows that $X(\mathcal{M})$ itself is countable.

To show that the shift is WAP it suffices by Corollary 1.5 to show that $pqx[\mathcal{M}] = qpx[\mathcal{M}]$ for any $p, q \in E(X(\mathcal{M}), S)$.

By Theorem 5.28 (iii), if $S^{t_i} x[\mathcal{M}] \to e$ then $S^{t_i} x[\mathcal{M}_1] = e$ for all $\mathcal{M}_1 \subset \mathcal{M}$. Hence $px[\mathcal{M}] = e$ implies $pqx[\mathcal{M}] = e = qpx[\mathcal{M}]$ for all $q \in E(X(\mathcal{M}), S)$.

We need to show that for sequences, $\{t^i\}, \{s^j\}$ in $\mathbb{N}$, if $Lim_i S^{t^i}(x[\mathcal{M}])$ and $Lim_j S^{s^j}(x[\mathcal{M}])$ exist then the two double limits of $\{S^{t^i + s^j}(x[\mathcal{M}])\}$ exist and have the same value. By Theorem 5.28(iv) we are reduced to the case when $|j_{r_i}(t^i)|, |j_{r_j}(s^j)| \to \infty$ and $\{\mathcal{M} - \mathbf{r}(t^i)\}$ and $\{\mathcal{M} - \mathbf{r}(s^j)\}$ converge in which case $S^{t^i}(x[\mathcal{M}]) \to x[LIM \ \{\mathcal{M} - \mathbf{r}(t^i)\}] = x[LIM_t]$ and $S^{s^j}(x[\mathcal{M}]) \to x[LIM \ \{\mathcal{M} - \mathbf{r}(s^j)\}] = x[LIM_s]$.

If $\{\mathcal{M} - \mathbf{r}(t^i)\}$ is eventually constant at $\mathcal{M} - \mathbf{r}$ then by Lemma 4.32 implies that the two double limits of $\{\mathcal{M} - \mathbf{r}(t^i) - \mathbf{r}(s^j)\}$ both exist and equal $LIM_s - \mathbf{r}$. Applying the continuous map $x[\cdot]$ we see that the double limits of $\{S^{t_i + s_j}(x[\mathcal{M}])\}$ both exist and equal $x[LIM_s - \mathbf{r}]$.

By Corollary 4.31 when neither sequence is eventually constant both $\bigcup_i \ supp \ \mathbf{r}(t^i)$ and $\bigcup_j \ supp \ \mathbf{r}(s^j)$ are infinite. Since $\mathcal{M}$ is finitary both LIM_r and LIM_s are finite by Proposition 4.34(a). Lemma 4.35 then implies that each double limit of $\{\mathcal{M} - \mathbf{r}(t^i) - \mathbf{r}(s^j)\}$ is $\emptyset$ and so, applying $x[\cdot]$ again, each double limit of $\{S^{t_i + s_j}(x[\mathcal{M}])\}$ is e.

Hence, $(X(\mathcal{M}),S)$ is WAP and so each element of $E(X(\mathcal{M}),S)$ is determined by its value on $x[\mathcal{M}]$. Hence, there is a unique $q_{\mathcal{F}} \in E(X(\mathcal{M}),S)$ such that $q_{\mathcal{F}}x[\mathcal{M}] = x[\mathcal{F}]$. By Corollary 5.51 there is a unique $Q_{\mathcal{F}} \in \mathcal{E}(\Theta(\mathcal{M}))$ such that $J_{\mathcal{M}}(Q_{\mathcal{F}}) = q_{\mathcal{F}}$. Since $q_{\mathcal{F}}$ is continuous and $x[\cdot]$ is a homeomorphism, $Q_{\mathcal{F}}$ is continuous.
□

Remark 5.53. In (b) the additional relations are $q_{\mathcal{F}_1}p_{\mathbf{r}_1} = q_{\mathcal{F}_1}p_{\mathbf{r}_2}$ if $\mathcal{F}_1 - \mathbf{r}_1 = \mathcal{F}_2 - \mathbf{r}_2$. Also, $q_{\mathcal{F}_1}q_{\mathcal{F}_2} = u$, the retraction to e for any pair of external limit sets $\mathcal{F}_1, \mathcal{F}_2$.

Corollary 5.54. *If $\mathcal{F} = LIM\ \{\mathcal{M} - \mathbf{r}^i\}$ is an external label for a finitary label $\mathcal{M}$, then $\{p_{\mathbf{r}^i}\}$ converges to $q_{\mathcal{F}}$ in $E(X(\mathcal{M}),S)$ and in $E(X_+(\mathcal{M}),S)$ and $\{P_{\mathbf{r}^i}\}$ converges to $Q_{\mathcal{F}}$ in $\mathcal{E}(\Theta(\mathcal{M}))$*

Proof: $\{p_{\mathbf{r}^i}(x[\mathcal{M}] = x[\mathcal{M} - \mathbf{r}^i]\}$ converges to $x[\mathcal{F}] = q_{\mathcal{F}}(x[\mathcal{M}])$. It follows from Proposition 1.6 that $\{p_{\mathbf{r}^i}\} \to q_{\mathcal{F}}$ pointwise. Since $J_{\mathcal{M}}$ is a homeomorphism onto its image, $\{P_{\mathbf{r}^i}\} \to Q_{\mathcal{F}}$ pointwise.
□

Corollary 5.55. *For a finitary label $\mathcal{M}$, the abelian enveloping semigroup $\mathcal{E}(\Theta(\mathcal{M})) = \{\ P_{\mathbf{r}}\ :\ \mathbf{r} \in FIN(\mathbb{N})\ \} \cup \{\ Q_{\mathcal{F}}\ :\ \mathcal{F}$ an external label $\}$. The relations are given by $P_{\mathbf{r}_1} = P_{\mathbf{r}_2}$ iff $\mathcal{M} - \mathbf{r}_1 = \mathcal{M} - \mathbf{r}_2$ and $P_{\mathbf{r}_1}Q_{\mathcal{F}} = P_{\mathbf{r}_2}Q_{\mathcal{F}}$ iff $\mathcal{F} - \mathbf{r}_1 = \mathcal{F} - \mathbf{r}_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are external labels then $Q_{\mathcal{F}_1}Q_{\mathcal{F}_2}$ is the constant map to $\emptyset$.*

Proof: By continuity there is a unique member of the enveloping semigroup which maps $\mathcal{M}$ to $\mathcal{N} \in \Theta(\mathcal{M})$. The relations follow from this uniqueness.
□

The simple case is easier:

Theorem 5.56. *If $\mathcal{M}$ is a simple label, then $(X(\mathcal{M}),S)$ and $(X_+(\mathcal{M}),S)$ are countable, WAP subshifts. The semigroups $\mathcal{E}(\Theta(\mathcal{M})), E(X(\mathcal{M}),S)$ and $E(X_+(\mathcal{M}),S)$ are abelian and act continuously on the spaces $\Theta(\mathcal{M})$, $X(\mathcal{M})$ and $X_+(\mathcal{M})$, respectively. Using the continuous, injective, homomorphism $J_{\mathcal{M}} : \mathcal{E}(\Theta(\mathcal{M})) \to E(X(\mathcal{M}),S)$ every element of $E(X(\mathcal{M}),S)$ can be expressed in the form $S^k p_{\mathbf{r}} = S^k J_{\mathcal{M}}(P_{\mathbf{r}})$ with $k \in \mathbb{Z}$ and $\mathbf{r} \in FIN(\mathbb{N})$. Similarly, every element of $E(X_+(\mathcal{M}),S)$ can be expressed in the form $S^k p_{\mathbf{r}} = S^k J_{+\mathcal{M}}(P_{\mathbf{r}})$ with $k \in \mathbb{Z}$ and $\mathbf{r} \in FIN(\mathbb{N})$.*

Proof: By Corollary 5.51 $J_{\mathcal{M}} : \mathcal{E}(\Theta(\mathcal{M})) \to E(X(\mathcal{M}), S)$ is a continuous injective homomorphism and every element of $E(X(\mathcal{M}), S)$ is of the form $S^k J_{\mathcal{M}}(Q)$ for some $Q \in \mathcal{E}(\Theta(\mathcal{M}))$. Because $\mathcal{M}$ is simple $\mathcal{E}(\Theta(\mathcal{M})) = \{ \ P_{\mathbf{r}} \ : \ \mathbf{r} \in FIN(\mathbb{N}) \ \}$ by Proposition 4.38 (d). It follows that $E(X(\mathcal{M}), S)$ is abelian and so $(X(\mathcal{M}), S)$ is WAP and consists of the orbits of the points $x[\mathcal{M} - \mathbf{r}]$ for $\mathbf{r} \in FIN(\mathbb{N})$.

□

Example 5.57. (a) If $\mathcal{M}$ is neither finitary nor simple then $X(\mathcal{M})$ need not be WAP even with $\mathcal{M}$ of finite type and $X(\mathcal{M})$ countable. Let $\mathcal{M}$ be defined by $\mathcal{M} = \langle \{\chi(3) + \chi(2a+1) + \chi(2b) : a \geq b \geq 1\} \cup \{\chi(1) + \chi(3) + \chi(2b) : b \geq 1\} \rangle$.

(5.37)

$$\begin{aligned} \mathcal{M} - \chi(1) &= \langle \{\chi(3) + \chi(2b) : b \geq 1\} \rangle, \\ \mathcal{M} - \chi(3) &= \langle \{\chi(2a+1) + \chi(2b) : a \geq b \geq 1\} \cup \{\chi(1) + \chi(2b) : b \geq 1\} \rangle \\ \mathcal{M} - \chi(2\ell+1) &= \langle \{\chi(1) + \chi(2b) : \ell \geq b \geq 1\} \rangle \\ \mathcal{M} - \chi(2\ell) &= \langle \{\chi(1) + \chi(2a+1) : a \geq \ell \geq 1\} \cup \{\chi(1) + \chi(3)\} \rangle. \end{aligned}$$

It follows that

(5.38)

$$\begin{aligned} LIM_{a \to \infty} \{\mathcal{M} - \chi(2a+1)\} &= \langle \{\chi(3) + \chi(2b) : b \geq 1\} \rangle, \\ LIM_{b \to \infty} \{\mathcal{M} - \chi(2b)\} &= \langle \{\chi(1) + \chi(3)\} \rangle, \\ LIM_{b \to \infty} LIM_{a \to \infty} \{\mathcal{M} - \chi(2a+1) - \chi(2b)\} &= \{\chi(3), 0\}, \\ LIM_{a \to \infty} LIM_{b \to \infty} \{\mathcal{M} - \chi(2a+1) - \chi(2b)\} &= \emptyset. \end{aligned}$$

So the enveloping semigroup $E(X(\mathcal{M}), S)$ is not abelian and $(X(\mathcal{M}), S)$ is not WAP. Notice that if $\mathcal{N} \subset \mathcal{M}$ is infinite then $\mathcal{M} - \mathcal{N}$ is finite. Thus, this condition does not suffice to yield $\mathcal{M}$ finitary.

In addition, notice that with $\mathbf{r}^i = \chi(2i+1), \mathbf{r} = \chi(1), \mathbf{s}^j = \chi(2j)$, we have that $LIM \mathcal{M} - \mathbf{s}^j$ is finite, and $LIM \mathcal{M} - \mathbf{r}^i = \mathcal{M} - \mathbf{r}$, but the two double limits disagree. This shows that the conditions given in Lemma 4.32 are needed to get the resulting commutativity.

(b) Let $\mathcal{M}$ be defined by $\mathcal{M} = \langle \{\chi(3a) + \chi(3b+1) + \chi(3(5^a 7^b) + 2) : a, b \geq 1\} \rangle$. It can be shown that the subshift $(X(\mathcal{M}), S)$ is WAP even though $\mathcal{M}$ is not finitary or simple. Notice that if $\ell_1 \neq \ell_2$ then

$$\begin{aligned} (\mathcal{M} - \chi(3\ell_1)) \quad &\cap \quad (\mathcal{M} - \chi(3\ell_2)) = \{\chi(3b+1) : b \geq 1\} \\ (\mathcal{M} - \chi(3\ell_1 + 1)) \quad &\cap \quad (\mathcal{M} - \chi(3\ell_2 + 1)) = \{\chi(3a) : a \geq 1\}. \end{aligned} \tag{5.39}$$

□

6. Dynamical properties of $X(\mathcal{M})$

6.1. Translation finite subsets of $\mathbb{Z}$.

We recall the following combinatorial characterization of WAP subsets of $\mathbb{Z}$ ([32]).

Theorem 6.1 (Ruppert). *For a subset $A \subset \mathbb{Z}$ the following conditions are equivalent:*

1. *The subshift $\overline{\mathcal{O}}(\chi(A)) \subset \{0,1\}^{\mathbb{Z}}$ is WAP.*
2. *For every infinite subset $B \subset \mathbb{Z}$ either:*
   (i) there exists $N \geq 1$ such that

$$\bigcap_{b \in B \cap [-N,N]} A - b \quad \textit{is finite,} \tag{6.1}$$

or:
(ii) there exists $N \geq 1$ and $n \in \mathbb{Z}$ such that

$$A - n \supset B \cap \big(\mathbb{Z} \setminus [-N,N]\big). \tag{6.2}$$

Definition 6.2 (Ruppert). We say that a subset $A \subset \mathbb{Z}$ is *translation finite* (TF hereafter) if for every infinite subset $B \subset \mathbb{Z}$ there exists an $N \geq 1$ such that

$$\bigcap_{b \in B \cap [-N,N]} A - b \;=\; \{\, n \in \mathbb{N} \;:\; A - n \supset B \cap [-N,N] \,\} \quad \text{is finite.} \tag{6.3}$$

Example 6.3. It is easy to check that the set $A = 2\mathbb{N} \cup -(2\mathbb{N}+1)$ (with $c = \chi(A) = (\dots, 1, 0, 1, 0, 1, \dot{1}, 0, 1, 0, \dots)$) does not satisfy Ruppert's condition (and a fortiori is not translation finite), hence $\overline{\mathcal{O}}(\chi(A))$ is not WAP.

(See Example 1.14.(b).)

□

Proposition 6.4. *Let A be a subset of $\mathbb{Z}$. The following conditions are equivalent.*

1. *The subset A is TF.*
2. *Every point in $R_S(\chi(A)) = (\omega_S \cup \alpha_S)(\chi(A))$ has finite support.*
3. *The subshift $\overline{\mathcal{O}}(\chi(A))$ is CT of height at most 2.*

Proof: (1) $\implies$ (2): Suppose first that A is TF and suppose that for some sequence $\{n_i\}_{i=1}^{\infty}$, with $|n_i| \to \infty$, we have $S^{n_i}\chi(A) = x$ with *supp* x an infinite set. Let $B =$ *supp* x and observe that for every $N \geq 1$, eventually,

$$S^{n_i}\chi(A) \wedge [-N, N] = (x_{-N}, \ldots, x_N), \tag{6.4}$$

whence $A - n_i \supset B \cap [-N, N]$. But this contradicts our assumption that A is TF.

(2) $\Rightarrow$ (1): Conversely, suppose A is not TF. Then there exists an infinite $B \subset \mathbb{Z}$ such that for every $N \geq 1$ the intersection

$$\{n \in \mathbb{Z} : A - n \supset B \cap [-N, N]\} \quad \text{is infinite.} \tag{6.5}$$

We can construct a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ with $A - n_i \supset B \wedge [-i, i]$ and so for any limit point $x \in \{0,1\}^Z$ of the sequence $\{S^{n_i}\chi(A) = \chi(A - n_i)\}$ the support *supp* $x \supset B$ and so is infinite.

(2) $\Rightarrow$ (3): is obvious.

(3) $\Rightarrow$ (2): Suppose finally that that $\overline{\mathcal{O}}(\chi(A))$ is CT of height at most 2. Suppose to the contrary that $x \in (\omega_T \cup \alpha_T)(\chi(A))$ has infinite support, say *supp* $x = B$. By compactness there exists a sequence $\{n_i\}_{i=1}^{\infty} \subset B$ such that the sequence $S^{n_i}x$ converges. Let $y = \lim_{i \to \infty} S^{n_i}x$. Then $y \in R_T(x)$ and $y_0 = 1$. Thus $y \neq \mathbf{0}$ and this contradicts our assumption that $\overline{\mathcal{O}}(\chi(A))$ is of height at most 2.

□

We next address the question 'when is $A[\mathcal{M}]$ TF ?'. This turns out to be a rather restrictive condition, because $\emptyset$ and 0 are the only labels $\mathcal{N}$ such that $A[\mathcal{N}]$ has finite support. For $\mathcal{M} = \emptyset$ or 0, $R_S(A[\mathcal{M}]) = \{e\}$ where e is the fixed point $\bar{0} = A[\emptyset]$. Thus, in these cases $x[\mathcal{M}]$ is TF.

Proposition 6.5. *For a positive label* $\mathcal{M}$ *the following conditions are equivalent.*

(i) $\Theta(\mathcal{M}) = \{\mathcal{M}, 0, \emptyset\}$.
(ii) *For all* $\mathbf{r} > \mathbf{0} \in \mathcal{M}$, $\mathcal{M} - \mathbf{r} = 0$.
(iii) *There exists* L *a nonempty subset of* $\mathbb{N}$ *such that* $\mathcal{M} = \langle\{\chi(\ell) : \ell \in L\}\rangle$.
(iv) $A[\mathcal{M}]$ *is TF.*
(v) $(X(\mathcal{M}), S)$ *has height 2.*
(vi) $(X_+(\mathcal{M}), S)$ *has height 2.*

When these conditions hold, $\mathcal{M}$ *is finitary and simple.*

Proof: (iii) $\Rightarrow$ (ii) : Obvious.

(ii) $\Rightarrow$ (i) : From (ii) it is clear that consists of $P_{\mathbf{r}}(\mathcal{M}) = \emptyset$ for $\mathbf{r} \notin \mathcal{M}$ and $P_{\mathbf{r}}(\mathcal{M}) = 0$ for $\mathbf{r} > \mathbf{0} \in \mathcal{M}$ and finally, $P_{\mathbf{0}}(\mathcal{M}) = \mathcal{M}$. Hence, the only limit labels possible in $\Theta(\mathcal{M})$ are $\emptyset, 0$ and $\mathcal{M}$. In passing, we see that $\mathcal{A}(\Theta(\mathcal{M}))$ contains one nontrivial element which maps $\mathcal{M}$ to 0 and maps 0 and $\emptyset$ to $\emptyset$.

(i) $\Rightarrow$ (iii) : If $\mathbf{r} \in \mathcal{M}$ with $|\mathbf{r}| \geq 2$ then there exists $\ell \in \mathbb{N}$ such that $\mathbf{r} - \chi(\ell) > \mathbf{0}$. Hence, $\mathcal{M} - \chi(\ell) \in \Theta(\mathcal{M})$ is neither 0 nor $\emptyset$. Hence, $\mathcal{M} - \chi(\ell) = \mathcal{M}$. That is, $\mathbf{r} + \chi(\ell) \in \mathcal{M}$ for all $\mathbf{r} \in \mathcal{M}$. So $\mathbf{r} \in \mathcal{M}$ implies $\chi(\ell) \in \mathcal{M} - \mathbf{r}$ and thus, $\mathcal{M} - \mathbf{r} = \mathcal{M}$ for all $\mathbf{r} \in \mathcal{M}$. This implies that $0 \notin \Theta(\mathcal{M})$. Contrapositively, (i) implies that $|\mathbf{r}| = 1$ for any nonzero $\mathbf{r}$ in $\mathcal{M}$. That is, each nonzero $\mathbf{r}$ in $\mathcal{M}$ is some $\chi(\ell)$.

(iii) $\Rightarrow$ (v) : $\mathcal{M}$ is bounded and size-bounded and so is of finite type. By Corollary 5.33 the limit points of $x[\mathcal{M}]$ lie on the orbits of $x[\mathcal{N}]$ for some $\mathcal{N} \in \Theta(\mathcal{M})$. Since, $\mathcal{M}$ is not recurrent, the set $R_S(x[\mathcal{M}])$ of limit points consists of the orbits of $x[0]$ and $x[\emptyset]$. These in turn map to the fixed point $x[\emptyset]$ and so $(X(\mathcal{M}), S)$ has height 2.

(v) $\Rightarrow$ (iv) : This follows from Proposition 6.4.

(iv) $\Rightarrow$ (iii) : We prove the contrapositive, assuming, as above, that there exist If $\mathbf{r} \in \mathcal{M}$ and $\ell \in \mathbb{N}$ such that $\mathbf{r} - \chi(\ell) > \mathbf{0}$. Choose an increasing sequence $\{t^i\}$ of expanding times with length $\mathbf{r}(t^i) = \chi(\ell)$ and with $|j_r(t^i)| \to \infty$. By $\{S^{t^i}(x[\mathcal{M}])\}$ converges to $x[\mathcal{M} - \chi(\ell)]$ which does not have finite support since $\mathbf{r} - \chi(\ell) \in \mathcal{M} - \chi(\ell)$. By Proposition 6.4 again $A[\mathcal{M}]$ is not TF.

Finally, it is clear that the labels described in (iii) are finitary and simple.

□

6.2. **Non-null and non-tame labels.**

Definition 6.6. (a) For a subshift (X, S) a subset $K \subset \mathbb{Z}$ is called an *independent set* if the restriction to X of the projection $\pi_K : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^K$ is surjective. The subshift is called *null* if there is a finite bound on the size of the independent sets for (X, S). It is called *tame* if there is no infinite independent set for (X, S).

(b) For a label $\mathcal{M}$ a subset $L \subset \mathcal{M}$ is called an *independent set* if for every $L_1 \subset L$ there exists $\mathcal{N} \in \Theta(\mathcal{M})$ such that $L \cap \mathcal{N} = L_1$.

(c) A label $\mathcal{M}$ is called *non-null* if for every $n \in \mathbb{N}$ there is a finite independent subset $F \subset \mathcal{M}$ with $\#F \geq n$. It is *non-tame* if there is an infinite set $L \subset \mathcal{M}$ such that every finite $F \subset L$ is an independent set.

Notice that an independent set L for a label $\mathcal{M}$ is certainly not a label. In fact, if $\mathbf{m}_1 < \mathbf{m}$ and $\mathbf{m} \in L$ then $\mathbf{m}_1 \notin L$ because if $\mathbf{m} \in \mathcal{N}$

for a label $\mathcal{N}$ then $\mathbf{m}_1 \in \mathcal{N}$. Since there exists a label $\mathcal{N}$ such that $\mathcal{N} \cap L = \{\mathbf{m}\}$ it follows that $\mathbf{m}_1 \notin L$.

Remark 6.7. The concepts 'null' and 'tame' are defined for any dynamical system. The first is defined in terms of sequential topological entropy (see e.g. [24] and the review [23]) and the latter in terms of the dynamical Bourgain-Fremlin-Talagrand dichotomy for enveloping semigroups ([14]). The convenient criteria which we use here for subshifts to be non-null and non-tame, are due basically to Kerr and Li [30] (see [17, Theorem 6.1.(3)]).

Lemma 6.8. *(a) If* $\mathcal{M}$ *is a label and* F *is a finite subset of* $\mathcal{M}$ *then for any* $\mathcal{N} \in \Theta(\mathcal{M})$ *there exists an* $\mathbb{N}$*-vector* $\mathbf{r}$ *such that* $\mathcal{N} \cap F = \mathcal{M} - \mathbf{r} \cap F$. *In particular, if* F *is a finite independent subset of* $\mathcal{M}$ *then for every* $A \subset F$ *there exists* $\mathbf{r}$ *such that* $F \cap \mathcal{M} - \mathbf{r} = A$.

(b) If every finite subset $F \subset L$ *is an independent set for a label* $\mathcal{M}$ *then* L *is an independent set for* $\mathcal{M}$.

(c) If L *is an independent set for a label* $\mathcal{M}$*, and if for every* $\mathbf{m} \in L$*,* $t(\mathbf{m})$ *is a positive expanding time such that* $\mathbf{r}(t(\mathbf{m})) = \mathbf{m}$ *then* $\{ t(\mathbf{m}) : \mathbf{m} \in L\}$ *is an independent set for the subshifts* $(X(\mathcal{M}), S)$ *and* $(X_+(\mathcal{M}), S)$.

Proof: (a) If $\mathcal{B}_N$ contains all the supports of elements of F then $\mathbf{m} \in F$ is in $\mathcal{M}_1 \in \mathcal{LAB}$ iff it is in $\mathcal{M}_1 \cap \mathcal{B}_N$. Hence, for any $A \subset F$ the set $\{ \mathcal{M}_1 : \mathcal{M}_1 \cap F = A \}$ is clopen in $\mathcal{LAB}$. Since $\{\mathcal{M} - \mathbf{r}\}$ is dense in $\Theta(\mathcal{M})$, the result follows.

(b) Let $L_1 \subset L$. Let $\{F^i\}$ be an increasing sequence of finite subsets of L with union L. Because F^i is an independent set, part (a) implies there exists $\mathbf{r}^i$ such that $F^i \cap \mathcal{M} - \mathbf{r}^i = L_1 \cap F^i \cap \mathcal{M} - \mathbf{r}^i$. It follows that if $\mathbf{m} \in L_1$ then eventually $\mathbf{m} \in \mathcal{M} - \mathbf{r}^i$. If $\mathbf{m} \notin L_1$ then eventually $\mathbf{m} \notin \mathcal{M} - \mathbf{r}^i$. By going to a subsequence, we can assume that $\{\mathcal{M} - \mathbf{r}^i\}$ converges to some $\mathcal{N} \in \Theta(\mathcal{M})$. Clearly, $L \cap \mathcal{N} = L_1$.

(c) For any label $\mathcal{N} \in \Theta(\mathcal{M})$, $t \in A[\mathcal{N}]$ iff t is expanding with $\mathbf{r}(t) \in \mathcal{N}$ and so $x[\mathcal{N}]_t = 1$ iff $\mathbf{r}(t) \in \mathcal{N}$.
□

From this we obviously have

Proposition 6.9. *(a) A label* $\mathcal{M}$ *is non-null iff for every* $n \in \mathbb{N}$ *there is a finite subset* $F \subset \mathcal{M}$ *with* $\#F \geq n$ *such that for every* $A \subset F$ *there exists* $\mathbf{r}$ *such that* $F \cap \mathcal{M} - \mathbf{r} = A$.

(b) A label $\mathcal{M}$ *is non-tame if there is an infinite set* $L \subset \mathcal{M}$ *such that for any finite* $A \subset F \subset L$ *there exists* $\mathbf{r}$ *such that* $F \cap \mathcal{M} - \mathbf{r} = A$.

In that case if L_1 is any subset of L then there exists $\mathcal{N} \in \Theta(\mathcal{M})$ such that $L \cap \mathcal{N} = L_1$. In particular, $\Theta(\mathcal{M})$ is uncountable.

□

Remark 6.10. It follows that if $\mathcal{M}$ is a non-tame label then $X(\mathcal{M})$ and $X_+(\mathcal{M})$ are uncountable and so neither $(X(\mathcal{M}), S)$ nor $(X_+(\mathcal{M}), S)$ can be WAP.

Corollary 6.11. *Given a label $\mathcal{M}$, if any label $\mathcal{N} \in \Theta(\mathcal{M})$ is non-null (or non-tame) then the subshifts $(X(\mathcal{M}), S), (X_+(\mathcal{M}), S)$ are not null (resp. not tame).*

Proof: If $\mathcal{N} \in \Theta(\mathcal{M})$ then $X(\mathcal{N}) \subset X(\mathcal{M})$ and so if $X(\mathcal{N})$ projects onto $\{0,1\}^L$ then $X(\mathcal{M})$ does.
□

There are some simple conditions which allow us to find non-tame labels.

Definition 6.12. A label $\mathcal{M}$ with roof $\rho(\mathcal{M})$ is called *flat* over a set $L \subset \mathbb{N}$ if for all $F \in Supp(\mathcal{M})$ with $F \subset L$, $\rho(\mathcal{M})|F \in \mathcal{M}$. In particular, $\rho(\mathcal{M})_\ell < \infty$ for $\ell \in L$. Equivalently, if $\mathbf{m} \in \mathcal{M}$ with $supp\ \mathbf{m} \subset L$ then $\rho(\mathcal{M})|(supp\ \mathbf{m}) \in \mathcal{M}$. The label is called *flat* when it is flat over $\mathbb{N}$. So a flat label is bounded.

Lemma 6.13. *Let $L \subset \mathbb{N}$.*

(a) If the label $\mathcal{M}$ is flat over L and $\mathbf{r}$ is an $\mathbb{N}$-vector with supp $\mathbf{r} \subset L$ then $\mathcal{M} - \mathbf{r}$ is flat over L.

(b) If $\{\mathcal{M}^i\}$ is a collection of labels each flat over L then $\bigcap\{\mathcal{M}^i\}$ is a label which is flat over L.

(c) If $\{\mathcal{M}^i\}$ is a bounded, increasing sequence of labels flat over L then $\bigcup\{\mathcal{M}^i\}$ is a label flat over L.

(d) If $\{\mathcal{M}^i\}$ is a bounded sequence of labels flat over L then $LIMINF$ $\{\mathcal{M}^i\}$ is a label flat over L.

(e) The set of labels which are flat is a closed in the subset of bounded labels.

(f) If $\mathcal{M}$ is a flat label then the elements of $\Theta(\mathcal{M})$ are all flat labels.

Proof: (a) If $\mathbf{m} \in \mathcal{M} - \mathbf{r}$ with $supp\ \mathbf{m} \subset L$ and $F = supp(\mathbf{m} + \mathbf{r})$ then $F \subset L$ and so $\rho(\mathcal{M})|F \in \mathcal{M}$. Hence, $\rho(\mathcal{M})|F - \mathbf{r} \in \mathcal{M} - \mathbf{r}$ and $\rho(\mathcal{M})|F - \mathbf{r} = (\rho(\mathcal{M}) - \mathbf{r})|F$. Clearly, $\rho(\mathcal{M}) - \mathbf{r} \geq \rho(\mathcal{M} - \mathbf{r})$.

(b) If $\mathcal{M}$ is the intersection then $\rho(\mathcal{M}) = min\{\rho(\mathcal{M}^i)\}$. If $\mathbf{m} \in \mathcal{M}$ with $supp\ \mathbf{m} \subset L$ then $\rho(\mathcal{M})|(supp\ \mathbf{m}) \in \mathcal{M}^i$ for all i and so is in $\mathcal{M}$.

(c) If $\mathcal{M}$ is the union then $\rho(\mathcal{M}) = max\{\rho(\mathcal{M}^i)\}$ and this is a non-decreasing sequence of functions. On any finite set F, eventually $\rho(\mathcal{M}) = \rho(\mathcal{M}^i)$. If $\mathbf{m} \in \mathcal{M}$ then eventually $\mathbf{m} \in \mathcal{M}^i$ and so eventually $\rho(\mathcal{M}^i)|(supp\ \mathbf{m}) \in \mathcal{M}$ and eventually these equal $\rho(\mathcal{M})|(supp\ \mathbf{m})$.

(d) Obvious from (b) and (c).

(e) A convergent sequence of bounded labels is a bounded sequence. By (d) the limit of a bounded sequence of flat labels is flat.

(f) If $\mathcal{M}$ is flat then it is bounded and bounds the set $\Theta(\mathcal{M})$. The result then follows from (a) with $L = \mathbb{N}$ and (e).

□

Recall that $Supp\ \mathcal{M}$ f-contains L when $\mathcal{P}_f L \subset Supp\ \mathcal{M}$.

Lemma 6.14. *Let $L \subset \mathbb{N}$. A label $\mathcal{M}$ is flat over L and $Supp\ \mathcal{M}$ f-contains L exactly when for any finite subset F of L, $\rho(\mathcal{M})|F \in \mathcal{M}$. In that case, $\{ \chi(\ell) : \ell \in L \}$ is an independent set in $\mathcal{M}$.*

Proof: The first sentence is clear from the definitions. If F is a finite subset of L and $A \subset F$, then $\rho(\mathcal{M})|F = \rho(\mathcal{M})|A + \rho(\mathcal{M})|(F \setminus A) \in \mathcal{M}$. So $\rho(\mathcal{M})|A \in \mathcal{M} - \rho(\mathcal{M})|(F \setminus A)$ and so $\{ \chi(\ell) : \ell \in A \} \subset \mathcal{M} - \rho(\mathcal{M})|(F \setminus A)$. But if $\ell \in F \setminus A$ then $\chi(\ell) \notin \mathcal{M} - \rho(\mathcal{M})|(F \setminus A)$. This means that $\{ \chi(\ell) : \ell \in L \}$ is an independent set.

□

Proposition 6.15. *Let $\mathcal{M}$ be a bounded label with $L = supp\ \rho(\mathcal{M})$.*

(a) If $\mathcal{M}$ is flat and f-contains L, then $\mathcal{M}$ is a strongly recurrent label.

(b) If $\mathcal{M}$ is a strongly recurrent label, then there exists an infinite set $L_1 \subset L$ such that $\mathcal{M}$ is flat over L_1 and $Supp\ \mathcal{M}$ f-contains L_1.

Proof: (a) In this case, $\mathcal{M}$ is a sublattice of $FIN(\mathbb{N})$ and so it is a strongly recurrent label by Proposition 4.23 (i).

(b) Assume that inductively that we have defined $F_k = \{\ell_1, \dots, \ell_k\}$ of distinct points of $supp\ \rho(\mathcal{M})$ such that $\rho(\mathcal{M})|F_k \in \mathcal{M}$. Because $\mathcal{M}$ is strongly recurrent we can add, in $\mathcal{M}$ any element with support outside of the finite set $F(\rho(\mathcal{M})|F_k)$. So for sufficiently large ℓ_{k+1} we have that $\rho(\mathcal{M})|F_{k+1} = \rho(\mathcal{M})|F_k + \mathbf{r}(\mathcal{M})_{\ell_{k+1}}\chi(\ell_{k+1}) \in \mathcal{M}$ with $F_{k+1} = F_k \cup \{\ell_{k+1}\}$. Let $L_1 = \bigcup_k \{F_k\}$.

□

Corollary 6.16. *Assume that $\mathcal{M}$ is a bounded label not of finite type. If $\mathcal{M}$ is flat or strongly recurrent then it is non-tame.*

Proof: If $\mathcal{M}$ is strongly recurrent then by Proposition 6.15 there exists an infinite subset L_1 of *supp* $\rho(\mathcal{M})$ such that $\mathcal{M}$ is flat over L_1 and *Supp* $\mathcal{M}$ f-contains L_1. By Proposition 4.7(b) any label not of finite type f-contains some infinite set L_1 and if $\mathcal{M}$ is flat then it is flat over L_1. By Lemma 6.14 $\{ \chi(\ell) \ : \ \ell \in L_1 \}$ is an independent set in $\mathcal{M}$.
□

If we define for a bounded label $\mathcal{M}$

$$\mathcal{F}(\mathcal{M}, L) \ = \ \{ \ F \ : F \quad \text{is a finite subset of} \ \ L \ \ \text{and} \ \ \rho(\mathcal{M})|F \in \mathcal{M} \ \}, \tag{6.6}$$

then $\mathcal{M}$ is flat over L and *Supp* $\mathcal{M}$ f-contains L exactly when $\mathcal{F}(\mathcal{M}, L) = \mathcal{P}_f L$.

We conjecture that for every bounded $\mathcal{M}$ not of finite type there exists $\mathcal{N} \in \Theta(\mathcal{M})$ which is non-tame and so that $(X(\mathcal{M}), S)$ is not tame. Beyond the above corollary the best we can do is the following.

Proposition 6.17. *Let $\mathcal{M}$ be a label not of finite type. If there exists $N \in \mathbb{N}$ such that $\rho(\mathcal{M}) \leq N$ then there exists $\mathcal{N} \in \Theta(\mathcal{M})$ which is non-tame.*

Proof: If $K \in \mathbb{N}$ and $\rho(\mathcal{M}) \leq K$ then $\rho(\mathcal{N}) \leq K$ for all $\mathcal{N} \in [[\mathcal{M}]]$. By Proposition 4.23 (d) there is a positive recurrent label in $\Theta(\mathcal{M})$ and so we can assume that $\mathcal{M}$ itself is recurrent. By Proposition 4.23 (h) we can choose an infinite set $L \subset$ *supp* $\rho(\mathcal{M})$ such that $\{ \ \chi(F) \ : \ F \in \mathcal{P}_f(L) \ \}$ is a strongly recurrent set for $\mathcal{M}$. Consider $\mathcal{F}(\mathcal{M}, L)$.

Case (i): If there exists $\{F^i\}$ a strictly increasing sequence of elements of $\mathcal{F}(\mathcal{M}, L)$ with $L_1 = \bigcup \{F^i\}$ then $\mathcal{F}(\mathcal{M}, L_1) = \mathcal{P}_f L_1$ and so $\mathcal{M}$ itself is non-tame by Lemma 6.14.

Case (ii): If F is a maximal element of $\mathcal{F}(\mathcal{M}, L)$ then $\rho(\mathcal{M} - \rho(\mathcal{M})|F)_\ell = 0$ for $\ell \in F$ and for $\ell \in L$ with

$$\rho(\mathcal{M})_\ell \ > \ 0 \qquad \Longrightarrow \qquad \rho(\mathcal{M})_\ell \ > \rho(\mathcal{M} - \rho(\mathcal{M})|F)_\ell \tag{6.7}$$

because for $\ell \in L \setminus F$, $\ \rho(\mathcal{M})_\ell \not\in \mathcal{M} - \rho(\mathcal{M})|F$ by maximality of F. Let $\mathcal{M}_1 = \mathcal{M} - \rho(\mathcal{M})|F$ and $L_1 = L \setminus F(\rho(\mathcal{M})|F)$. We see that $\mathcal{M}_1$ is a recurrent element of $\Theta(\mathcal{M})$ with $\{ \ \chi(F) \ : \ F \in \mathcal{P}_f(L_1) \ \}$ a strongly recurrent set for $\mathcal{M}_1$. Furthermore, $\rho(\mathcal{M}_1) \leq K - 1$ by (6.7). In particular, this cannot happen if $K = 1$.

If Case (ii) occurs then we repeat the procedure with $\mathcal{M}$ and L replaced by $\mathcal{M}_1$ and L_1. Eventually, we must terminate in a Case i situation and so at some $\mathcal{M}_k \in \Theta(\mathcal{M})$ which is non-tame.
□

Remark 6.18. Notice that Case (i) did not require that $\rho(\mathcal{M})$ is bounded by a constant. Furthermore, once we have the set L associated with the strongly recurrent subset of $\mathcal{M}$ we can replace it by any infinite subset. In particular, if there is any infinite subset of L on which $\rho(\mathcal{M})$ is bounded by a constant then the above argument will apply. Thus, the obstruction to proving the conjecture in general arises when $Lim\, \rho(\mathcal{M})_\ell = \infty$ as $\ell \to \infty$ in L and every element of $\mathcal{F}(\mathcal{M}, L)$ is contained in a maximal element of $\mathcal{F}(\mathcal{M}, L)$ and these conditions continue to hold as we replace $\mathcal{M}$ and L by $\mathcal{M} - \rho(\mathcal{M})|F, L_1$ for F any maximal element of $\mathcal{F}(\mathcal{M}, L)$. Finally, we notice that if $\mathcal{F}(\mathcal{M}, L)$ contains sets of arbitrarily large cardinality then $\mathcal{M}$ is at least non-null by Lemma 6.14.

Corollary 6.19. *Let $\mathcal{M}$ be a bounded label not of finite type.*

1. *There is a label $\mathcal{N} \in [[\mathcal{M}]]$ (i.e. $\mathcal{N} \subset \mathcal{M}$) which is not of finite type and with $\rho(\mathcal{N})$ bounded by a constant. In particular, $\mathcal{N}$ is not tame.*
2. *There is a label $\mathcal{N} \supset \mathcal{M}$ which is flat, hence not tame.*

Proof: (1). As $\mathcal{M}$ is not of finite type there is a strictly increasing sequence $\{\mathbf{m}_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$. Let $\mathcal{N} = \langle\{supp\ \mathbf{m}_i : i = 1, 2, \dots\}\rangle$. Then clearly $\mathcal{N}$ is not of finite type and $\rho(\mathcal{N}) \leq 1$. The non-tameness follows from 6.17.

(2). Let $\mathcal{N} = \langle\{\rho(\mathcal{M})|(supp\ \mathbf{m}) : \mathbf{m} \in \mathcal{M}\}\rangle$. Clearly $\mathcal{N} \supset \mathcal{M}$ and is flat, hence not tame by 6.16.
□

Questions 6.20.

1. Is it true that for every label $\mathcal{M}$ not of finite type there exists $\mathcal{N} \in \Theta(\mathcal{M})$ which is non-tame (hence also so that $(X(\mathcal{M}), S)$ is not tame) ?
2. Is there a label $\mathcal{M}$ not of finite type such that $X(\mathcal{M})$ is tame or even null ?
3. Is there a recurrent such label ?

A positive answer to the second question (in the null case) will yield an example of a null dynamical system with a recurrent transitive point which is not minimal. The question whether such a system exists is a long standing open question.

In a private conversation Tomasz Downarowicz asked us whether it is the case that every WAP system is null. Our next example shows that there are (a) non-null simple labels, hence topologically transitive WAP subshifts which are non-null; (b) non-tame labels of finite type, hence subshifts arising from finite type labels which are not tame.

Example 6.21. There are simple, finitary labels which are non-null. Accordingly, by Corollary 6.11, the corresponding subshifts are topologically transitive WAP subshifts which are non-null. Also, there are labels of finite type which are non-tame. Again the corresponding subshifts are topologically transitive and non-tame. Note that by Remark 6.10 these latter subshifts are not WAP.

(a) Partition $\mathbb{N}$ into disjoints sets $\{A_n : n \in \mathbb{N}\} \cup \{B_n : n \in \mathbb{N}\}$ with $\#A_n = n$ and $\#B_n = 2^n$ for all n. Define a bijection $A \mapsto \ell_A$ from the power set of A_n onto B_n. Now define $\mathcal{M}_n$ by $\mathcal{M}_n = \langle\{\ \chi(\ell_A) + \chi(i)\ :\ i \in A, A \subset A_n\ \}\rangle$. $\mathcal{M}_n$ is a finite label and since $\mathcal{M}_n - \chi(\ell_A) = \{\chi(i) : i \in A\} \cup \{\emptyset\}$ it follows that $\{\chi(i) : i \in A_n\}$ is an independent set for $\mathcal{M}_n$. Since $\{\mathcal{M}_n\}$ is a pairwise disjoint sequence of finite labels, $\mathcal{M} = \bigcup_n \{\mathcal{M}_n\}$ is a simple, finitary label which is clearly non-null.

Instead we can define $\mathcal{N}_n$ by $\mathcal{N}_n = \langle\ \chi(A_n)\rangle$. Again $\mathcal{N} = \bigcup_n \{\mathcal{N}_n\}$ is a simple, finitary label which is non-null by Lemma 6.14.

(b) Partition $\mathbb{N}$ into two disjoint infinite sets L, B and define a bijection $A \mapsto \ell_A$ from the set of finite subsets of L onto B. Define $\mathcal{M} = \{\ \chi(\ell_A) + \chi(i)\ :\ i \in A, A$ a finite subset of $L\ \}$. Because it is size bounded, the label $\mathcal{M}$ is of finite type. Just as in (a), $\{\chi(i) : i \in L\}$ is an independent set for $\mathcal{M}$.

□

Remark 6.22. By Proposition 6.9 the label $\mathcal{M}$ of example (b) has $\Theta(\mathcal{M})$ uncountable. So this and Example 4.29 are labels $\mathcal{M}$ of finite type with $\Theta(\mathcal{M})$ uncountable. It follows that $(X(\mathcal{M}), S)$ are subshifts which are LE but not HAE (see Remark 2.4).

□

6.3. **Gamow transformations.**

For $L \subset \mathbb{N}$ we let $FIN(L) = \{\ \mathbf{m} \in FIN(\mathbb{N})\ :\ supp\ \mathbf{m} \subset L\ \}$ and $\mathcal{LAB}(L) = \{\ \mathcal{M} \in \mathcal{LAB}\ :\ \bigcup\ Supp\ \mathcal{M}\ \subset L\ \}$. Clearly, $\mathcal{M} \in \mathcal{LAB}(L)$ implies $[[\mathcal{M}]] \subset \mathcal{LAB}(L)$. If $\mathcal{M} \not\in \mathcal{LAB}(L)$ then for some $N \in \mathbb{N}$

$\mathcal{M} \cap \mathcal{B}_N \not\in \mathcal{LAB}(L)$ and so $d(\mathcal{M}, \mathcal{M}_1) < 2^{-N}$ implies $\mathcal{M}_1 \not\in \mathcal{LAB}(L)$. Thus, $\mathcal{LAB}(L)$ is a closed subset of $\mathcal{LAB}$. For example, $\mathcal{LAB}(\emptyset) = \{0, \emptyset\}$.

$FIN(L)$ is a submonoid of $FIN(\mathbb{N})$ and it acts on $\mathcal{LAB}(L)$. Furthermore, if $\mathbf{r} \not\in FIN(L)$ then $P_{\mathbf{r}}(\mathcal{M}) = \emptyset$ for all $\mathcal{M} \in \mathcal{LAB}(L)$. Hence, we can restrict attention to this action and for Φ a closed, invariant subset of $\mathcal{LAB}(L)$, the enveloping semigroup $E(\Phi)$ is the closure of $FIN(L)$ in Φ^{Φ}.

Let $\tau : L_1 \to L_2$ be a bijection with $L_1, L_2 \subset \mathbb{N}$. In honor of the book *One, Two, Three,... Infinity* we will refer to the following as the *Gamow transformation* induced by τ. For an $\mathbb{N}$-vector $\mathbf{m}$ with $supp\, \mathbf{m} \subset L_2$ we let $\tau^*\mathbf{m} = \mathbf{m} \circ \tau$ so that $supp\, \tau^*\mathbf{m} = \tau^{-1} supp\, \mathbf{m} \subset L_1$. Thus, $\tau^* : FIN(L_2) \to FIN(L_1)$ is a monoid isomorphism which also preserves the lattice properties.

For $\mathcal{M} \in \mathcal{LAB}(L_2)$ we let $\tau^*\mathcal{M} = \{\ \tau^*\mathbf{m}\ : \mathbf{m} \in \mathcal{M}\}$ and for $\Phi \subset \mathcal{LAB}(L_2)$ we will let $\tau^*\Phi = \{\tau^*\mathcal{N} : \mathcal{N} \in \Phi\}$. Thus, τ^* is a bijection from $\mathcal{LAB}(L_2)$ to $\mathcal{LAB}(L_1)$ with inverse $(\tau^{-1})^*$.

Given $\ell \in \mathbb{N}$, let $\ell' = max\ \tau([1, \ell]\ \cap\ L_1)$. It follows from the definition (4.5) of the metric on $\mathcal{LAB}$ that $d(\mathcal{M}_1, \mathcal{M}_2) \leq 2^{-\ell'}$ implies $d(\tau^*(\mathcal{M}_1), \tau^*(\mathcal{M}_2)) \leq 2^{-\ell}$. Thus, the τ^* is uniformly continuous on $\mathcal{LAB}(L_2)$ and so is a homeomorphism from $\mathcal{LAB}(L_2)$ onto $\mathcal{LAB}(L_1)$.

Clearly, τ^* preserves all label operations, e.g. $\tau^*(\mathcal{M} - \mathbf{r}) = \tau^*\mathcal{M} - \tau^*\mathbf{r}$ for $\mathcal{M} \in \mathcal{LAB}(L_2)$ and $supp\, \mathbf{r} \subset L_2$. Thus, τ^* is an action isomorphism relating the $FIN(L_2)$ action on $\mathcal{LAB}(L_2)$ to the $FIN(L_1)$ action on $\mathcal{LAB}(L_1)$. Hence, it induces an Ellis semigroup isomorphism from $\mathcal{E}(\Phi)$ to $\mathcal{E}(\tau^*(\Phi))$ where Φ is a compact, invariant subset of $\mathcal{LAB}(L_2)$. Also $\Theta(\tau^*\mathcal{M}) = \tau^*\Theta(\mathcal{M})$ for $\mathcal{M} \in \mathcal{LAB}(L_2)$.

Finally, $\tau^*\mathcal{M}$ is of bounded, finite type, finitary, simple, recurrent or strongly recurrent iff $\mathcal{M}$ satisfies the corresponding property. In the finitary case, $\mathcal{F}$ is an external element for $\mathcal{M}$ iff $\tau^*\mathcal{F}$ is an external element for $\tau^*\mathcal{M}$.

On the other hand, the sets $A(\tau^*\mathbf{m})$ and $A(\mathbf{m})$ are only analogous.

Now define $\overleftarrow{\tau} S^k x[\mathcal{M}] = S^k x[\tau^*\mathcal{M}]$ and $\overleftarrow{\tau} S^k x_+[\mathcal{M}] = S^k x_+[\tau^*\mathcal{M}]$. These are well-defined maps defined on the (not closed) subshifts generated by $x[\mathcal{LAB}]$ and by $x_+[\mathcal{LAB}]$. They each commute with the shift map S. The maps are not at all continuous. However, they have some very nice dynamical properties. If $\mathcal{M}$ is of finite type Corollary 5.33 implies that the map $\overleftarrow{\tau}$ restricts to a bijection from $X(\mathcal{M})$ to $X(\tau^*\mathcal{M})$ and from $X_+(\mathcal{M})$ to $X_+(\tau^*\mathcal{M})$ each of which commutes with the shift. Furthermore, Y is a closed, invariant subset of $X(\mathcal{M})$ (or of $X_+(\mathcal{M})$) iff $\overleftarrow{\tau}(Y)$ is a closed invariant subset of $X(\tau^*\mathcal{M})$ (resp. of

$X_+(\tau^*\mathcal{M}))$. This again follows from Corollary 5.33 because $\Phi(\overleftarrow{\tau}(Y)) = \tau^*\Phi(Y)$ and $\Phi_+(\overleftarrow{\tau}(Y)) = \tau^*\Phi_+(Y)$ Recall that $\mathbf{r}$ induces continuous element $p_{\mathbf{r}} \in E(X(\mathcal{M}), S)$ (and $p_{\mathbf{r}} \in E(X_+(\mathcal{M}), S)$) uniquely defined by $p_{\mathbf{r}}x[\mathcal{M}] = x[P_{\mathbf{r}}\mathcal{M}]$ and which satisfies $p_{\mathbf{r}}x[\mathcal{N}] = x[\mathcal{N} - \mathbf{r}] = x[P_{\mathbf{r}}\mathcal{N}]$ for every $\mathcal{N} \in \Theta(\mathcal{M})$ (resp. $p_{\mathbf{r}}x[\mathcal{M}] = x[P_{\mathbf{r}}\mathcal{M}]$ and $p_{\mathbf{r}}x_+[\mathcal{N}] = x_+[\mathcal{N} - \mathbf{r}] = x_+[P_{\mathbf{r}}\mathcal{N}]$ for every $\mathcal{N} \in \Theta(\mathcal{M})$). When $\mathcal{M}$ is finitary, for each external element $\mathcal{F}$ there is a continuous element $q_{\mathcal{F}}$ of $E(X(\mathcal{M}), S)$ (and $E(X_+(\mathcal{M}), S)$) characterized by $q_{\mathcal{F}}x[\mathcal{M}] = x[\mathcal{F}] = x[Q_{\mathcal{F}}\mathcal{M}]$ (resp. $q_{\mathcal{F}}x_+[\mathcal{M}] = x_+[\mathcal{F}] = x_+[Q_{\mathcal{F}}\mathcal{M}]$). We clearly have

$$\begin{aligned} \overleftarrow{\tau} \circ p_{\mathbf{r}} \ &= \ p_{\tau^*\mathbf{r}} \circ \overleftarrow{\tau} \\ \text{and when } \mathcal{M} \text{ is finitary} \quad \overleftarrow{\tau} \circ q_{\mathcal{F}} \ &= \ q_{\tau^*\mathcal{F}} \circ \overleftarrow{\tau}. \end{aligned} \tag{6.8}$$

Thus, in the finitary case, $\overleftarrow{\tau}$ induces an algebraic isomorphism from $E(X(\mathcal{M}), S)$ to $E(X(\tau^*\mathcal{M}), S)$ which relates the actions on $X(\mathcal{M})$ and $X(\tau^*\mathcal{M})$ and similarly for $X_+(\mathcal{M})$.

To understand the failure of continuity, observe that $p_{\mathbf{r}}$ is the limit of the sequence S^{t_i} on $X(\mathcal{M})$ where $\mathbf{r}(t^i) = \mathbf{r}$ and $|j_r(t_i)| \to \infty$. The associated sequence S^{s_i} on $X(\tau^*\mathcal{M})$ has $\mathbf{r}^{s_i} = \tau^*\mathbf{r}$ and $|j_r(s^i)| \to \infty$. These are unrelated numerical sequences, except that all the expansions have the same length, $|\mathbf{r}|$.

Let S_∞ denote the group of all permutations on $\mathbb{N}$. On S_∞ we define an ultrametric by

$$d(\tau_1, \tau_2) \ = \ inf \ \{ \ 2^{-\ell} \ : \ \ell \in Z_+ \text{ and } \tau_1|[1,\ell] = \tau_2|[1,\ell] \ \}. \tag{6.9}$$

Clearly, for any $\gamma \in S_\infty$, $d(\gamma \circ \tau_1, \gamma \circ \tau_2) \ = \ d(\tau_1, \tau_2)$. If $\gamma([1,\ell]) \subset [1, \ell_\gamma]$ then $\tau_1|[1,\ell_\gamma] = \tau_2|[1,\ell_\gamma]$ implies $\tau_1 \circ \gamma|[1,\ell] = \tau_2 \circ \gamma|[1,\ell]$ and $\gamma_1|[1,\ell_\gamma] = \gamma|[1,\ell_\gamma]$ then $\gamma_1^{-1}|[1,\ell] = \gamma^{-1}|[1,\ell]$. It follows that S_∞ is a topological group with left invariant ultrametric d. Furthermore, the equivalent metric $\bar{d}$ given by $\bar{d}(\tau_1, \tau_2) = max(d(\tau_1, \tau_2), d(\tau_1^{-1}, \tau_2^{-1}))$ is complete. Finally, the set of permutations S_{fin} consisting of permutations are the identity on the complement of a finite set, is a countable dense subgroup of S_∞. Thus, S_∞ is a Polish group, which is clearly perfect.

Furthermore, if $\mathcal{M}$ and $\mathcal{M}_1$ are labels with $\mathcal{M} \cap \mathcal{N}_{\ell_\gamma} = \mathcal{M}_1 \cap \mathcal{N}_{\ell_\gamma}$ and $\gamma_1|[1,\ell_\gamma] = \gamma|[1,\ell_\gamma]$ then $\gamma^*\mathcal{M} \cap \mathcal{N}_\ell = \gamma_1^*\mathcal{M}_1 \cap \mathcal{N}_\ell$. This implies that the action $S_\infty \times \mathcal{LAB} \to \mathcal{LAB}$ given by $(\tau, \mathcal{M}) \to (\tau^{-1})^*\mathcal{M}$ is a continuous action. The empty label $\emptyset$ is an isolated fixed point for the action. Let $\mathcal{LAB}_+$ denote the perfect set of nonempty labels. We show that this action is topologically transitive on $\mathcal{LAB}_+$ by constructing explicitly a transitive point.

Example 6.23. Let Ξ be the countable set of all pairs $(\mathcal{N}_\xi, \ell_\xi)$ with $\mathcal{N}_\xi$ a finite label such that $\bigcup$ *Supp* $\mathcal{N}_\xi \subset [1, \ell_\xi]$. Partition $\mathbb{N}$ by disjoint intervals indexed by Ξ such that I_ξ has length ℓ_ξ. Let $\tau_\xi : I_\xi \to [1, \ell_\xi]$ be the increasing linear bijection and let $\mathcal{M}_\xi = \tau_\xi^* \mathcal{N}_\xi$ so that $\bigcup$ *Supp* $\mathcal{M}_\xi \subset I_\xi$. Let $\mathcal{M}_{trans} = \bigcup_\xi \mathcal{M}_\xi$. By Theorem 4.43(b) $\mathcal{M}_{trans}$ is finitary and simple and so is of finite type. On the other hand, given any nonempty label $\mathcal{M}$ and any $\ell \in \mathbb{N}$ there exists $\xi \in \Xi$ such that $(\mathcal{N}_\xi, \ell_\xi) = (\mathcal{M} \wedge [1, \ell], \ell)$. It follows that if $\gamma \in S_{fin}$ with $\gamma = \tau_\xi$ on I_ξ then $(\gamma^{-1})^* \mathcal{M}_{trans} \wedge [1, \ell] = \mathcal{M} \wedge [1, \ell]$. Thus, $\mathcal{M}_{trans}$ is a transitive point for the action of S_∞ on $\mathcal{LAB}_+$.

Because $\mathcal{LAB}_+$ is a Cantor set, the set $TRANS$ of transitive points is a dense G_δ subset of $\mathcal{LAB}_+$. By Proposition 4.41 the set $RECUR$ of recurrent labels is a dense G_δ subset of $\mathcal{LAB}$. Hence,
$TRANS \cap RECUR$ is a dense G_δ subset of $\mathcal{LAB}_+$. The transitive point $\mathcal{M}_{trans}$ is of finite type and so is not recurrent. On the other hand, the set of flat labels is a proper, closed S_∞ invariant subset which contains recurrent labels (see Proposition 6.15 (a)) which are thus not transitive with respect to the S_∞ action.

For background regarding our next question we refer the reader to the works [29] and [21].

Question 6.24. Does there exist a label $\mathcal{M}$ such that its S_∞ orbit is residual, or are all the orbits meager? If such a residual orbit exists then it would be unique. It is well known that the adjoint action of S_∞ on itself does have a dense G_δ orbit (see e.g. [21]).

6.4. **Ordinal constructions.**

For a label $\mathcal{M}$ define the label $z_{LAB}(\mathcal{M}) = \mathcal{M} \setminus max\ \mathcal{M} = \{ \mathbf{m} : \mathbf{m} + \mathbf{r} \in \mathcal{M} \text{ for some } \mathbf{r} > \mathbf{0} \}$. If $\mathcal{M}$ is of finite type and nonempty then $z_{LAB}(\mathcal{M})$ is a proper subset of $\mathcal{M}$. If $\mathcal{M}$ is positive, then $\mathbf{0} \in z_{LAB}(\mathcal{M})$ and so $z_{LAB}(\mathcal{M}) \neq \emptyset$. Hence, $z_{LAB}(\mathcal{M}) = \emptyset$ iff $\mathcal{M} = 0$ or $\emptyset$. In general, $z_{LAB}(\mathcal{M}) = \mathcal{M}$ iff $max\ \mathcal{M} = \emptyset$.

Define the descending transfinite sequence of labels by

$$
\begin{aligned}
z_{LAB,0}(\mathcal{M}) &= \mathcal{M}, \\
z_{LAB,\alpha+1}(\mathcal{M}) &= z_{LAB}(z_{LAB,\alpha}(\mathcal{M})), \\
z_{LAB,\beta}(\mathcal{M}) &= \bigcap_{\alpha \langle \beta} \{z_{LAB,\alpha}(X)\} \quad \text{for } \beta \text{ a limit ordinal.}
\end{aligned}
\tag{6.10}
$$

The sequence stabilizes at β when $z_{LAB,\beta}(\mathcal{M}) = z_{LAB,\beta+1}(\mathcal{M})$ in which case $z_{LAB,\alpha}(\mathcal{M}) = z_{LAB,\beta}(\mathcal{M})$ for all $\alpha \geq \beta$. So $\emptyset$ stabilizes at 0 and if $\mathcal{M}$ is nonempty and of finite type then the sequence stabilizes at $\beta + 1$ where β is the first ordinal for which $z_{LAB,\beta} = 0$.

If Φ is a closed, bounded, invariant set of labels define $z_{LAB}(\Phi)$ to be the closure of $\bigcup \{ \ P_{\mathbf{r}}(\Phi) \ : \ \mathbf{r} > \mathbf{0}\}$, a closed, invariant subset of Φ. Thus, $z_{LAB}(\Theta(\mathcal{M})) = \Theta'(\mathcal{M})$.

Define the nonincreasing transfinite sequence of closed, bounded subsets of $\mathcal{LAB}$ by

$$\begin{aligned} z_{LAB,0}(\Phi) \quad &= \quad \Phi, \\ z_{LAB,\alpha+1}(\Phi) \quad &= \quad z_{LAB}(z_{LAB,\alpha}(\Phi)), \\ z_{LAB,\beta}(\Phi) \quad &= \quad \bigcap_{\alpha<\beta} \{z_{LAB,\alpha}(\Phi)\} \quad \text{for } \ \beta \ \text{ a limit ordinal.} \end{aligned} \tag{6.11}$$

Theorem 6.25. *If $\mathcal{M}$ be a label then $z_{LAB,\alpha}(\mathcal{M}) = \bigcup z_{LAB,\alpha}(\Theta(\mathcal{M}))$ for every countable ordinal α. That is, $\mathbf{m} \in z_{LAB,\alpha}(\mathcal{M})$ iff there exists $\mathcal{N} \in z_{LAB,\alpha}(\Theta(\mathcal{M}))$ such that $\mathbf{m} \in \mathcal{N}$.*

Proof: We use transfinite induction. Both procedures stabilize at a countable ordinal and so we need only consider countable ordinals.

Since $\mathcal{M} = \bigcup \Theta(\mathcal{M})$ the result is true for $\alpha = 0$.

If $\mathbf{m} \in z_{LAB,\alpha+1}(\mathcal{M}) = z_{LAB}(z_{LAB,\alpha}(\mathcal{M}))$, then there exists $\mathbf{r} > \mathbf{0}$ such that $\mathbf{m} + \mathbf{r} \in z_{LAB,\alpha}(\mathcal{M})$. By induction hypothesis, there exists $\mathcal{N} \in z_{LAB,\alpha}(\Phi)$ such that $\mathbf{m} + \mathbf{r} \in \mathcal{N}$ and so $\mathbf{m} \in \mathcal{N} - \mathbf{r} \in z_{LAB,\alpha+1}(\Phi)$.

Conversely, if $\mathcal{N} \in z_{LAB,\alpha+1}(\Phi) = z_{LAB}(z_{LAB,\alpha}(\Phi))$ then there exists sequences $\mathcal{N}^i \in z_{LAB,\alpha}(\Phi)$ and $\mathbf{r}^i > 0$ such that $\mathcal{N} = LIM\{\mathcal{N}^i - \mathbf{r}^i\}$. By induction hypothesis $\bigcup\{\mathcal{N}^i\} \subset z_{LAB,\alpha}(\mathcal{M})$. If $\mathbf{m} \in \mathcal{N}$ then eventually $\mathbf{m} \in \mathcal{N}^i - \mathbf{r}^i$ and so $\mathbf{m} \in z_{LAB,\alpha+1}(\mathcal{M})$.

Now let β be a limit ordinal. If $\mathbf{m} \in \mathcal{N} \in z_{LAB,\beta}(\Phi)$ then by definition $\mathcal{N} \in z_{LAB,\alpha}(\Phi)$ for all $\alpha < \beta$. So by induction hypothesis, $\mathbf{m} \in z_{LAB,\alpha}(\mathcal{M})$ for all $\alpha < \beta$ and hence $\mathbf{m} \in z_{LAB,\beta}(\mathcal{M})$.

Conversely, if $\mathbf{m} \in z_{LAB,\beta}(\mathcal{M})$ and so in $z_{LAB,\alpha}(\mathcal{M})$ for all $\alpha < \beta$. Let $\{\alpha^i\}$ be an increasing sequence of ordinals converging to β. By induction hypothesis there exists $\mathcal{N}^i \in z_{LAB,\alpha^i}(\Phi)$ such that $\mathbf{m} \in \mathcal{N}^i$. Since $\{\alpha^i\}$ is increasing $\mathcal{N}^i \in z_{LAB,\alpha^j}(\mathcal{M})$ for all $j < i$. Let $\{\mathcal{N}^{i'}\}$ be a convergent subsequence with limit $\mathcal{N}$. Since $\mathbf{m} \in \mathcal{N}^i$ for all i, $\mathbf{m} \in \mathcal{N}$. For every $\alpha < \beta$ there exists $\alpha^j > \alpha$. For $i \geq j$ the sequence $\mathcal{N}^i \in z_{LAB,\alpha^j}(\Phi) \subset z_{LAB,\alpha}(\Phi)$ and so the limit of the subsequence $\mathcal{N}$ is in the closed set $z_{LAB,\alpha}(\Phi)$. Since this is true for all $\alpha < \beta$, $\mathcal{N} \in z_{LAB,\beta}(\Phi)$.

$\square$

It follows that the sequences stabilize at the same countable ordinal. If $\mathcal{M} = \emptyset$ then $\Theta(\mathcal{M}) = \{\emptyset\}$ and the sequence stabilizes at 0. If $\mathcal{M}$ is nonempty and of finite type then the sequence stabilizes at $\beta+1$ where β is the ordinal with $z_{LAB,\beta}(\mathcal{M}) = 0$ and $z_{LAB,\beta}(\Theta(\mathcal{M})) = \{0, \emptyset\}$. In this nonempty finite type case, we call β the *height* of $\Theta(\mathcal{M})$.

Theorem 6.26. *Assume that $\mathcal{M}$ is a label of finite type. For every countable ordinal α, every closed, invariant $Y \subset X(\mathcal{M})$ and every closed, invariant $Y_+ \subset X_+(\mathcal{M})$,*

$$\begin{aligned} \Phi(z_{LIM,\alpha}(Y)) \quad &= \quad z_{LAB,\alpha}(\Phi(Y)), \\ \Phi_+(z_{LIM,\alpha}(Y_+)) \quad &= \quad z_{LAB,\alpha}(\Phi_+(Y_+)). \end{aligned} \tag{6.12}$$

Proof: The equation is clear for $\alpha = 0$.

Since $\Phi(Y)$ is the preimage of Y with respect to the continuous map $x[\cdot]$ it follows that $\Phi(z_{LIM}(Y))$ is a closed invariant set containing $P_{\mathbf{r}}\mathcal{N}$ whenever $x[\mathcal{N}] \in Y$ and $\mathbf{r} > \mathbf{0}$ since $x[P_{\mathbf{r}}\mathcal{N}] = p_{\mathbf{r}}x[\mathcal{N}]$. That is, $z_{LAB}(\Phi(Y)) \subset \Phi(z_{LIM}(Y))$.

On the other hand, Corollary 5.33 (b) implies that $z_{LAB}(\Phi(Y)) = \Phi(\tilde{Y})$ for a closed, invariant subspace $\tilde{Y}$ of $X(\mathcal{M})$. If $\mathbf{r} > 0$ and $y \in Y$ then $y = S^k(x[\mathcal{N}])$ with $k \in \mathbb{Z}$ and $\mathcal{N} \in \Phi(Y)$. So $p_{\mathbf{r}}y = S^k(x[P_{\mathbf{r}}\mathcal{N}])$. Since $P_{\mathbf{r}}\mathcal{N} \in z_{LAB}(\Phi(Y))$, it follows that $p_{\mathbf{r}}y \in \tilde{Y}$. Hence, $z_{LIM,\alpha}(Y) \subset \tilde{Y}$ and so $\Phi(z_{LIM,\alpha}(Y)) \subset \Phi(\tilde{Y}) = z_{LAB}(\Phi(Y))$.

This proves equation (6.12) with $\alpha = 1$. Assuming the result for an ordinal α it follows for $\alpha + 1$. For a limit ordinal, β we use the fact that Φ commutes with intersection and so by the induction hypothesis

(6.13)

$$\begin{aligned} \Phi(z_{LIM,\beta}(Y)) \ &= \ \Phi(\bigcap_{\alpha<\beta} z_{LIM,\alpha}(Y)) \ = \ \bigcap_{\alpha<\beta} \Phi(z_{LIM,\alpha}(Y)) \\ &= \ \bigcap_{\alpha<\beta} z_{LAB,\alpha}(\Phi(Y)) \ = \ z_{LAB,\beta}(\Phi(Y)). \end{aligned}$$

This completes the induction.

□

We will say that $\Phi \subset \mathcal{LAB}$ is Θ *invariant* when it is nonempty and $\mathcal{M} \in \Phi$ implies $\Theta(\mathcal{M}) \subset \Phi$. If Φ is closed then it is invariant iff it is Θ invariant. Θ invariance always implies invariance but is usually a stronger condition since $\{P_{\mathbf{r}}(\mathcal{M})\}$ is usually a proper subset of $\Theta(\mathcal{M})$.

Let $\mathcal{M}$ be a label of finite type. For a Θ invariant $\Phi \subset \Theta(\mathcal{M})$, we define $z^*_{\mathcal{M}}(\Phi) = \{ \ \mathcal{N} \in \Theta(\mathcal{M}) \ : \ \Theta'(\mathcal{N}) \subset \Phi\}$. Equivalently, $\mathcal{N} \in z^*_{\mathcal{M}}(\Phi)$ iff $Q(\mathcal{N}) \in \Phi$ for all $Q \in \mathcal{A}(\Theta(\mathcal{M})) = \mathcal{E}(\Theta(\mathcal{M})) \setminus \{id_{\Theta(\mathcal{M})}\}$. For example, $z^*_{\mathcal{M}}(\{\emptyset\}) = \{\emptyset, 0\} = [[0]]$.

Starting with a Θ invariant $\Phi \subset \Theta(\mathcal{M})$, define the nondecreasing transfinite sequence of Θ invariant subsets of $\Theta(\mathcal{M})$ by

$$\begin{aligned} z^*_{\mathcal{M},0}(\Phi) &= \Phi, \\ z^*_{\mathcal{M},\alpha+1}(\Phi) &= z^*_{\mathcal{M}}(z^*_{\mathcal{M},\alpha}(\Phi)), \\ z^*_{\mathcal{M},\beta}(\Phi) &= \bigcup_{\alpha<\beta} \{z^*_{\mathcal{M},\alpha}(\Phi)\} \quad \text{for } \beta \text{ a limit ordinal.} \end{aligned} \tag{6.14}$$

Recall that for a dynamical system (X,T), $Y \subset X$ is called orbit-closed when $x \in Y$ implies $\overline{O_T(x)} \subset Y$. For $\mathcal{M}$ of finite type it is easy to adjust the proof of Corollary 5.33 to show that $Y \subset X(\mathcal{M})$ is orbit-closed iff $\Phi(Y)$ is Θ closed and $Y_+ \subset X_+(\mathcal{M})$ is orbit-closed iff $\Phi_+(Y_+)$ is Θ closed.

Theorem 6.27. *Assume that $\mathcal{M}$ is a label of finite type, that Y is an orbit closed subset of $X(\mathcal{M})$ and that Y_+ is an orbit closed subset of $X_+(\mathcal{M})$. For every countable ordinal α,*

$$\begin{aligned} \Phi(R^*_{S,\alpha}(Y)) &= z^*_{\mathcal{M},\alpha}(\Phi(Y)), \\ \Phi_+(R^*_{S,\alpha}(Y_+)) &= z^*_{\mathcal{M},\alpha}(\Phi_+(Y_+)), \end{aligned} \tag{6.15}$$

Proof: This is obvious for $\alpha = 0$.

A point $x \in R^*_S(Y)$ iff $qx \in Y$ for all $q \in A(X(\mathcal{M}), S)$. Since Y is S-invariant, Corollary 5.51 implies that this is true iff $J_{\mathcal{M}}(Q)x \in Y$ for all $Q \in \mathcal{A}(\Theta(\mathcal{M}))$. Hence, $x[\mathcal{N}] \in R^*_S(Y)$ iff $x[Q(\mathcal{N})] \in Y$ for all $Q \in \mathcal{A}(\Theta(\mathcal{M}))$, i.e. iff $Q(\mathcal{N}) \in \Phi(Y)$ for all $Q \in \mathcal{A}(\Theta(\mathcal{M}))$ and so iff $\mathcal{N} \in z^*_{\mathcal{M}}(\Phi(Y))$. This proves equation (6.15) for $\alpha = 1$ and so inductively for any $\alpha + 1$.

Since Φ is the preimage operator with respect to the map $x[\cdot]$, it commutes with union. So the equation for a limit ordinal β follows because it is assumed, inductively, to hold for all $\alpha < \beta$. The equations are analogous to those of (6.13) with intersection replaced by union.

$\Box$

The constructions of (6.10), (6.11) and (6.14) are label constructions and so they commute with Gamow transformations. We can use Gamow transformations to assure that a countable number of labels all occur with supports on disjoint sets. Let $\tau_0 : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection and let $L_i = \tau_0^{-1}(\mathbb{N} \times \{i\})$ for $i \in \mathbb{N}$. Define $\tau_i : L_i \to \mathbb{N}$ to be the bijection $\tau_i = \pi_1 \circ \tau_0$ where $\pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the first coordinate projection. Given a sequence $\{\mathcal{M}^i\}$ of labels, the label $\{\tau_i^* \mathcal{M}^i\}$ is Gamow equivalent to $\mathcal{M}^i$ and $\bigcup$ *Supp* $\mathcal{M}^i \subset L_i$.

For nonempty labels $\mathcal{M}_1, \mathcal{M}_2$ we have $(\bigcup \mathit{Supp}\, \mathcal{M}_1) \cap (\bigcup \mathit{Supp}\, \mathcal{M}_2) = \emptyset$ iff $\mathcal{M}_1 \cap \mathcal{M}_2 = 0 = \{\mathbf{0}\}$. In that case we will say that $\mathcal{M}_1$ and $\mathcal{M}_2$

are disjoint. Recall that for labels $\mathcal{M}_1, \mathcal{M}_2$, $\mathcal{M}_1 \oplus \mathcal{M}_2 = \{\mathbf{m}_1 + \mathbf{m}_2 : \mathbf{m}_1 \in \mathcal{M}_1, \mathbf{m}_2 \in \mathcal{M}_2\}$.

Proposition 6.28. *If $\{\mathcal{M}_a\}$ is a finite or infinite pairwise disjoint collection of nonempty labels of finite type, then $\mathcal{M} = \bigcup\{\mathcal{M}_a\}$ is a label of finite type which is finitary (or simple) if the $\mathcal{M}_a$'s are all finitary (resp. are all simple). Moreover $\mathcal{M}$ has the following properties:*

(1) *If $\mathbf{r} > \mathbf{0}$ then $\mathcal{M} - \mathbf{r} = \emptyset$ unless $\mathbf{r} \in \mathcal{M}_a$ for some a. If $\mathbf{r} \in \mathcal{M}_a$, then $\mathcal{M} - \mathbf{r} = \mathcal{M}_a - \mathbf{r}$. Furthermore,*

$$\begin{aligned} \Theta'(\mathcal{M}) \ &= \ \bigcup \{\Theta'(\mathcal{M}_a)\}, \\ max\ \mathcal{M} \ &= \ \bigcup \{max\ \mathcal{M}_a\}. \end{aligned} \tag{6.16}$$

(2) *For $\mathbf{r} > \mathbf{0} \in \mathcal{M}_a$, the map sending $P_{\mathbf{r}}$ on $\Theta(\mathcal{M}_a)$ to $P_{\mathbf{r}}$ on $\Theta(\mathcal{M})$ extends to a continuous injective homomorphism $j_a : \mathcal{A}(\Theta(\mathcal{M}_a)) \to \mathcal{A}(\Theta(\mathcal{M}))$. If $a \neq b$ then:*

$$j_a(\mathcal{A}(\Theta(\mathcal{M}_a))) \ \cap \ j_b(\mathcal{A}(\Theta(\mathcal{M}_b))) \subset \{q : q(\mathcal{A}(\Theta(\mathcal{M}))) \subset \{0, \emptyset\}\}.$$

Furthermore,

$$\mathcal{A}(\Theta(\mathcal{M})) \setminus \bigcup\{j_a(\mathcal{A}(\Theta(\mathcal{M}_a)))\} \tag{6.17}$$

contains at most one point Q^* in which case $Q^*\mathcal{M} = 0$ and $Q^* = \emptyset$ on $\Theta'(\mathcal{M})$.

(3) *For every $\alpha \geq 1$*

$$z_{LAB,\alpha}(\Theta(\mathcal{M})) \ = \ \bigcup \{z_{LAB,\alpha}(\Theta(\mathcal{M}_a))\}. \tag{6.18}$$

(4) *If for all a, Φ_a is a Θ invariant subspace of $\Theta'(M_a)$ then $\bigcup \{\Phi_a\}$ is a Θ invariant subset of $\Theta'(\mathcal{M})$. If it is a proper subset then*

$$z^*_{\mathcal{M}}(\bigcup \{\Phi_a\}) \ = \ \bigcup \{z^*_{\mathcal{M}_a}(\Phi_a)\}. \tag{6.19}$$

Proof: By Theorem 4.43 $\mathcal{M}$ is of finite type and is finitary or simple if all of the $\mathcal{M}_a$'s are.

If $\{\mathbf{r}^i > \mathbf{0}\}$ is a sequence in $\mathcal{M}$ and $\mathbf{m} > \mathbf{0}$ is an $\mathbb{N}$-vector in $\mathcal{M}$ then $\mathbf{m} \in \mathcal{M}_a$ for some a. If $\{\mathcal{M} - \mathbf{r}^i\}$ converges with $\mathbf{m}$ in the limit then eventually $\mathbf{r}^i \in \mathcal{M}_a$ in which case $\mathcal{M} - \mathbf{r}^i = \mathcal{M}_a - \mathbf{r}^i$ and $LIM\{\mathcal{M} - \mathbf{r}^i\} = LIM\{\mathcal{M}_a - \mathbf{r}^i\} \in \Theta'(\mathcal{M}_a)$. Hence, $\Theta'(\mathcal{M}) \ \subset \ \bigcup \{\Theta'(\mathcal{M}_a)\}$. The reverse inclusion is obvious. That *max* $\mathcal{M}$ is the union of the *max* $\mathcal{M}_a$'s is obvious.

Now suppose that $\{P_{\mathbf{r}^i}\}$ is a net with $\mathbf{r}^i > \mathbf{0}$ in $\mathcal{M}$ converging to $Q \in \mathcal{A}(\Theta(\mathcal{M}))$. Assume first that for some $\mathcal{N} \in \mathcal{A}(\Theta(\mathcal{M}))$ that $\mathbf{m} > \mathbf{0} \in Q\mathcal{N}$. Let M_b be the unique member to the sequence which contains $\mathbf{r}$.

Eventually $\mathbf{r}^i \in \mathcal{M}_b$ so that eventually $\mathcal{M}_1 - \mathbf{r}^i = (\mathcal{M}_1 \cap \mathcal{M}_b) - \mathbf{r}^i$ for all $\mathcal{M}_1 \in [[M]]$. In particular, $Q(\mathcal{M}) = LIM\ \{\mathcal{M} - \mathbf{r}^i\} = LIM\ \{\mathcal{M}_b - \mathbf{r}^i\}$. Notice that $\mathcal{M}_1 = \bigcup\{\mathcal{M}_1 \cup \mathcal{M}_a\}$ and by (6.16) $\mathcal{M}_1 \in \Theta'(\mathcal{M})$ iff $\mathcal{M}_1 = \mathcal{M}_1 \cap \mathcal{M}_a \in \Theta(M_a)$ for some a. Clearly $Q(\mathcal{M}_1) = \emptyset$ if $a \neq b$ and $Q(\mathcal{M}_1) = LIM\{\mathcal{M}_1 - \mathbf{r}_b\}$ for $\mathcal{M}_1 \in \Theta(M_a)$. Hence, $Q = j_b(\tilde{Q})$ where $\tilde{Q}$ is the pointwise limit of $\{P_{\mathbf{r}^i}\}$ in $\mathcal{A}(\Theta(\mathcal{M}_b)$. On the other hand if $\{P_{\mathbf{r}^i}\}$ in $\mathcal{A}(\Theta(\mathcal{M}_b))$ converges to $\tilde{Q}$ with $\mathbf{r}^i > \mathbf{0}$ in $\mathcal{M}_b$ then $P_{\mathbf{r}^i}$ in $\mathcal{A}(\Theta(\mathcal{M}))$ converges pointwise to $Q(\mathcal{N}) = LIM\{\mathcal{N} \cap \mathcal{M}_b - \mathbf{r}^i\}$ for all $\mathcal{N} \in \Theta(\mathcal{M})$. This defines the injection $j_b : \mathcal{A}(\Theta(\mathcal{M}_b)) \to \mathcal{A}(\Theta(\mathcal{M}))$. Because $\{u\} \cup \{P_\mathbf{r} : \mathbf{r} > \mathbf{0}\}$ is dense in $\mathcal{A}(\Theta(\mathcal{M}_b))$ it follows that j_b is continuous and is a homomorphism. It is obviously injective.

Now assume that $Q(\mathcal{M}) = 0$. If for some cofinal set $\{\mathbf{r}^{i'} \in \mathcal{M}_b\}$ then $Q = j_b(\tilde{Q})$ with $\tilde{Q} = LIM\{P_{\mathbf{r}_{i'}}\}$ in $A(\Theta(\mathcal{M}_b))$. Otherwise, eventually $\mathbf{r}^{i'} \notin M_b$ for every b and so $Q(\mathcal{N}) = \emptyset$ for all $\mathcal{N} \in \Theta'(\mathcal{M})$. This is Q^* which exists iff the sequence $\{\mathcal{M}_a\}$ is infinite. Notice that if some $\mathcal{M}_b$ is infinite then $Q^* = j_b(\tilde{Q})$ where $Q = LIM\{P_{\mathbf{s}^i}\}$ with $\{\mathbf{s}^i\}$ a sequence of distinct members of $max\ \mathcal{M}_b$. However, if the sequence $\{\mathcal{M}_a\}$ is infinite and each $\mathcal{M}_a$ is finite then Q^* is not in $\bigcup\ \{j_a(\mathcal{A}(\Theta(\mathcal{M}_a)))\}$.

In general, let $\{\mathcal{M}^i\}$ be a convergent sequence of nonempty labels with $\mathcal{M}^i \subset \mathcal{M}_{a(i)}$. If $a(i')$ has some common value a for i' in an infinite subset, then $\{M^{i'}\}$ is a sequence in $[[\mathcal{M}_a]]$ and the limit is in $[[\mathcal{M}_a]]$. Otherwise, the limit is 0. It follows that if Φ_a is a closed subset of $\Theta'(\mathcal{M}_a)$ for each a, and $0, \emptyset \in \bigcup\{\Phi_a\}$ then $\bigcup\{\Phi_a\}$ is a closed subset of $\Theta'(\mathcal{M})$.

If Φ is a closed subset of $\mathcal{A}(\Theta(\mathcal{M}))$ then $z_{LAB}(\Phi)$ is the closure of $\mathcal{A}(\Theta(\mathcal{M}))\Phi$. Hence, (6.18) follows by transfinite induction starting with $z_{LAB}(\Theta(\mathcal{M})) = \Theta'(\mathcal{M})$.

Finally, the Θ invariance of $\bigcup\ \{\Phi_a\}$ and (6.19) follow from (6.16) and (6.17).

□

Recall that $(\mathcal{M}_1, \mathcal{M}_2) \mapsto \mathcal{M}_1 \oplus \mathcal{M}_2$ is a continuous map from $\mathcal{LAB} \times \mathcal{LAB} \to \mathcal{LAB}$. So if $\Phi_1, \Phi_2 \subset \mathcal{LAB}$ are compact then $\Phi_1 \oplus \Phi_2$ is compact and is therefore closed.

Proposition 6.29. *If $\mathcal{N}$ and $\mathcal{M}$ are positive disjoint labels with $\mathcal{N}$ finite and $\mathcal{M}$ of finite type then $\mathcal{N} \oplus \mathcal{M}$ is a label of finite type which is finitary (or simple) if $\mathcal{M}$ is finitary (resp. simple). Moreover $\mathcal{N} \oplus \mathcal{M}$ has the following properties:*

(1) *If $\mathbf{r} \in \mathcal{N} \oplus \mathcal{M}$ then $\mathbf{r} = \mathbf{n} + \mathbf{m}$ with $\mathbf{n} \in \mathcal{N}$ and $\mathbf{m} \in \mathcal{M}$ uniquely determined by $\mathbf{r}$. In that case, $\mathcal{N} \oplus \mathcal{M} - \mathbf{r} = (\mathcal{N} - \mathbf{n}) \oplus (\mathcal{M} - \mathbf{m})$.*

Furthermore,

$$\begin{aligned} \Theta'(\mathcal{N} \oplus \mathcal{M}) &= \Theta'(\mathcal{N}) \oplus \Theta(\mathcal{M}) \ \cup \ \Theta(\mathcal{N}) \oplus \Theta'(\mathcal{M}), \\ max\ \mathcal{N} \oplus \mathcal{M} &= (max\ \mathcal{N}) \oplus (max\ \mathcal{M}). \end{aligned} \tag{6.20}$$

$\Theta(\mathcal{N})$ is the finite set $\{\mathcal{N} - \mathbf{r}\}$.

(2) *The map $\oplus : \Theta(\mathcal{N}) \times \Theta(\mathcal{M}) \to \Theta(\mathcal{N} \oplus \mathcal{M})$ is a homeomorphism inducing an isomorphism of compact semigroups $j_{\oplus} : \mathcal{E}(\Theta(\mathcal{N})) \times \mathcal{E}(\Theta(\mathcal{M})) \to \mathcal{E}(\Theta(\mathcal{N} \oplus \mathcal{M}))$.*

(3) *If $\Psi \subset [[\mathcal{N}]]$ and $\Phi \subset [[\mathcal{M}]]$ are closed and invariant then*

$$z_{LAB}(\Psi \oplus \Phi) \ = \ z_{LAB}(\Psi) \oplus \Phi \cup \Psi \oplus z_{LAB}(\Phi). \tag{6.21}$$

And for every limit ordinal α

$$z_{LAB,\alpha}(\Theta(\mathcal{N} \oplus \mathcal{M})) \ = \ \{\mathcal{N}\} \oplus z_{LAB,\alpha}(\Theta(\mathcal{M})). \tag{6.22}$$

For $\alpha = 0$ or a limit ordinal and $k \in \mathbb{N}$

$$z_{LAB,\alpha+k}(\Theta(\mathcal{N} \oplus \mathcal{M})) \ = \ \bigcup_{r=0}^{k} \ z_{LAB,\alpha+r}(\Theta(\mathcal{N})) \oplus z_{LAB,\alpha+k-r}(\Theta(\mathcal{M})). \tag{6.23}$$

(4) *For all $k \in \mathbb{N}$*

$$z^*_{\mathcal{N}\oplus\mathcal{M},k}([[0]]) \ = \ \bigcup_{r=0}^{k} \ z^*_{\mathcal{N},r}([[0]]) \oplus z^*_{\mathcal{M},k-r}([[0]]). \tag{6.24}$$

For every limit ordinal α and $k \in \mathbb{Z}_+$

$$z^*_{\mathcal{N}\oplus\mathcal{M},\alpha+k}([[0]]) \ = \ \{\mathcal{N}\} \oplus z^*_{\mathcal{M},\alpha}([[0]]) \cup (\bigcup_{r=1}^{k} \ z^*_{\mathcal{N},k-r}([[0]]) \oplus z^*_{\mathcal{M},\alpha+r}([[0]])). \tag{6.25}$$

Proof: Again Theorem 4.43 says that $\mathcal{M}$ is of finite type and is finitary if $\mathcal{M}$ is.

The closed set $\Theta'(\mathcal{N}) \oplus \Theta(\mathcal{M}) \cup \Theta(\mathcal{N}) \oplus \Theta'(\mathcal{M})$ contains $\mathcal{N} \oplus \mathcal{M} - \mathbf{r}$ for all $\mathbf{r} > 0$ and so contains $\Theta'(\mathcal{N} \oplus \mathcal{M})$. On the other hand, if either $\mathbf{n} > 0$ or $\mathbf{m} > 0$ then $\mathbf{r} > 0$ and so $\Theta'(\mathcal{N} \oplus \mathcal{M})$ contains all $(\mathcal{N} - \mathbf{n}) \oplus (\mathcal{M} - \mathbf{m})$ with either $\mathbf{n} > 0$ or $\mathbf{m} > 0$. Since $\oplus$ is continuous, it follows that $\Theta'(\mathcal{N} \oplus \mathcal{M})$ contains $\Theta'(\mathcal{N}) \oplus \Theta(\mathcal{M}) \ \cup \ \Theta(\mathcal{N}) \oplus \Theta'(\mathcal{M})$.

That $\Theta(\mathcal{N}) = \{\mathcal{N} - \mathbf{r}\}$ follows from Corollary 4.31.

Because of the disjoint support assumption, the continuous map $\oplus : [[\mathcal{N}]] \times [[\mathcal{M}]] \to [[\mathcal{N} \oplus \mathcal{M}]]$ is injective and so, by (6.19), it restricts to a homeomorphism from $\Theta(\mathcal{N}) \times \Theta(\mathcal{M})$ onto $\Theta(\mathcal{N} \oplus \mathcal{M})$. Notice that labels such as $\mathcal{N} \cup \mathcal{M}$ do not lie in $\Theta(\mathcal{N} \oplus \mathcal{M})$. Furthermore, if $\{\mathcal{N}^i\}$ and $\{\mathcal{M}^i\}$ are nets in $[[N]]$ and $[[M]]$ then $\{\mathcal{N}^i \oplus \mathcal{M}^i\}$ converges iff both $\{\mathcal{N}^i\}$ and $\{\mathcal{M}^i\}$ do, in which case the limit is $LIM\ \{\mathcal{N}^i\} \oplus LIM\ \{\mathcal{M}^i\}$.

In particular, $P_{\mathbf{r}^i}$ converges in $\mathcal{E}(\Theta(\mathcal{N} \oplus \mathcal{M}))$ iff $(P_{\mathbf{n}^i}, P_{\mathbf{m}^i})$ converges in $\mathcal{E}(\Theta(\mathcal{N})) \times \mathcal{E}(\Theta(\mathcal{M}))$. This implies that $j_{\oplus}$ is continuous and surjective. Hence, it is a homeomorphism. It is a homomorphism by continuity since it is clearly a homomorphism on the dense submonoid $\{(P_{\mathbf{n}}, \mathcal{P}_{\mathbf{m}})\}$.

It therefore follows that for $\Phi_1 \subset \Theta(\mathcal{N})$ and $\Phi_2 \subset \Theta(\mathcal{M})$ closed invariant subspaces, $z_{LAB}(\Phi_1 \oplus \Phi_2) = z_{LAB}(\Psi) \oplus \Phi \cup \Psi \oplus z_{LAB}(\Phi)$. Then (6.23) follows by induction on k since the operator z_{LAB} commutes with union. For (6.22) we use induction on the limit ordinals together with 0 starting with $\alpha = 0$. Assume the result for all $\beta < \alpha$. From (6.23) we have

$$\begin{aligned} \{\mathcal{N}\} \oplus z_{LAB,\beta+k}(\Theta(\mathcal{M})) \ &\subset \ z_{LAB,\beta+k}(\Theta(\mathcal{N} \oplus \mathcal{M})) \\ &\subset \ \{\mathcal{N}\} \oplus z_{LAB,\beta+k-r_0}(\Theta(\mathcal{M})) \end{aligned} \tag{6.26}$$

where $z_{LAB,r_0}(\Theta(\mathcal{N})) = [[0]]$. Intersecting we obtain (6.22) for $\alpha = \beta + \omega$. Otherwise, α is an increasing limit of limit ordinals and the result follow from the induction hypothesis by intersecting.

For (6.24) and (6.25) we proceed by transfinite induction. First, starting from 0 or a limit ordinal α we use induction on $k \geq 1$. Observe that if $\mathbf{n} > \mathbf{0}$ and $\mathbf{m} \in z^*_{\mathcal{M},\alpha+r+1}([[0]]) \setminus z^*_{\mathcal{M},\alpha+r}([[0]])$ then $\mathbf{n} + \mathbf{r} \notin z^*_{\mathcal{N}\oplus\mathcal{M},\alpha+r+1}([[0]])$. This yields for β a limit ordinal less than α and $k \in \mathbb{N}$ that

$$\begin{aligned} \{\mathcal{N}\} \oplus z^*_{\mathcal{M},\beta+k-r_0}([[0]]) \ &\subset \ z^*_{\mathcal{N}\oplus\mathcal{M},\beta+k}([[0]]) \\ &\subset \ \{\mathcal{N}\} \oplus z^*_{\mathcal{M},\beta+k}([[0]]) \end{aligned} \tag{6.27}$$

where $z^*_{\mathcal{N},r_0}([[0]]) = \Theta(\mathcal{N})$. So by taking the unions we obtain the result for a limit ordinal α and $k = 0$.

□

Definition 6.30. For $\mathcal{M}$ a nonzero finitary or simple label, the *height* of $\Theta(\mathcal{M})$ is $\alpha + 1$ where α is the ordinal with $z_{LAB,\alpha}(\Theta(\mathcal{M})) = [[0]]$. The *height** is $\alpha + 1$ where α is the ordinal where $z^*_{\mathcal{M},\alpha}(\{\emptyset\}) = \Theta'(\mathcal{M})$. Notice that $z^*_{\mathcal{M},\alpha}(\{\emptyset\}) = z^*_{\mathcal{M},1+\alpha}([[0]])$ and so if $\alpha \geq \omega$ then $z^*_{\mathcal{M},\alpha}(\{\emptyset\}) = z^*_{\mathcal{M},\alpha}([[0]])$.

Theorem 6.31. *For any countable ordinal α there exists a label $\mathcal{M}$ which is both simple and finitary with height = height* $= \alpha+1$. Hence, $(X(\mathcal{M}), S)$ and $(X_+(\mathcal{M}), S)$ are topologically transitive WAP subshifts with height = height* $= \alpha + 1$.*

Proof: First, let $\mathcal{N}_n = \{k\chi(\ell_1) : 0 \leq k \leq n\}$. It is easy to see that $\Theta(\mathcal{N}_n) = \{\mathcal{N}_k : k \leq n\} \cup \{\emptyset\}$ has height and height* equal to $n + 1$. These are finite labels and so are both simple and finitary.

Now suppose that α is a countable limit ordinal, the limit of an increasing sequence β^i. By inductive hypothesis, we can choose for each i a finitary and simple label $\mathcal{M}^i$ so that $\Theta(M_i)$ has height $\beta^i + 1$, and height* $\beta^i + 1$. By using a Gamow transformation we can assume that $\{\mathcal{M}^i\}$ is a sequence with disjoint supports. By Proposition 6.28 $\mathcal{M} = \bigcup\{\mathcal{M}^i\}$ is finitary and simple and by (6.18) $\zeta_{LAB,\alpha}(\Theta(\mathcal{M})) = \{[[0]]\}$ and so $\Theta(\mathcal{M})$ has height $\alpha + 1$. By (6.19) it follows that $\zeta^*_{\mathcal{M},\alpha}(\{\emptyset\}) = \zeta^*_{\mathcal{M},\alpha}([[0]]) = \Theta'(\mathcal{M})$ and so $\Theta(\mathcal{M})$ has height* equal to $\alpha + 1$.

Now for a countable limit ordinal α assume that $\mathcal{M}$ is a finitary and simple label with height and height* equal to $\alpha+1$. By using a Gamow transformation, we can assume that ℓ_1 is not in the support of $\mathcal{M}$. For $n \geq 1$, $\mathcal{N}_n$ is a positive, finite label disjoint from $\mathcal{M}$. By Proposition 6.28 $\mathcal{N}_n \oplus \mathcal{M}$ is finitary and simple and by (6.22) $z_{LAB,\alpha}(\Theta(\mathcal{N}_n \oplus \mathcal{M})) = \Theta(\mathcal{N}_n \oplus 0)$. Hence, $\Theta(\mathcal{N}_n \oplus \mathcal{M})$ has height $\alpha + n + 1$. From (6.25) it follows that $\Theta(\mathcal{N}_n \oplus \mathcal{M})$ has height* $\alpha + n + 1$.

The results for $(X(\mathcal{M}), S)$ follow from Theorem 6.26 and Theorem 6.27.

□

7. Scrambled sets

Following Li and Yorke [31] a subset $S \subset X$ is called *scrambled* for a dynamical system (X, T) when every pair of distinct points of S is proximal but not asymptotic.

Recall that $A(X,T)$ is the ideal of the enveloping semigroup $E(X,T)$ consisting of the limit points of $\{T^n\}$ as $|n| \to \infty$. Let $A_+(X,T)$ be the set of limit points of $\{T^n\}$ as $n \to \infty$, that is, we move only in the positive direction. Thus, $\omega T(x) = A_+(X,T)x$ for every $x \in X$.

Definition 7.1. For a compact, metric dynamical system (X,T) let (x, y) be a pair in $X \times X$.

(a) We call the pair (x, y) *proximal* when it satisfies the following equivalent conditions:

- (i) $\liminf_{n>0} \ d(T^n(x), T^n(y)) \ = \ 0$.
- (ii) There exists a sequence $n_i \to \infty$ such that $\lim \ d(T^{n_i}(x), T^{n_i}(y)) = 0$.
- (iii) There exists $p \in A_+(X,T)$ such that $px \ = \ py$.
- (iv) There exists u a minimal idempotent in $A_+(X,T)$ such that $ux \ = \ uy$.

We denote by $PROX(X,T)$ (or just $PROX$ when the system is clear) the set of all proximal pairs.

(b) We call the pair (x,y) *asymptotic* when it satisfies the following equivalent conditions:

(i) $\limsup_{n>0} \ d(T^n(x),T^n(y)) \ = \ 0$.
(ii) $\lim_{n>0} \ d(T^n(x),T^n(y)) \ = \ 0$.
(iii) For all $p \in A_+(X,T) \ \ px \ = \ py$.

We denote by $ASYMP(X,T)$ (or just $ASYMP$ when the system is clear) the set of all asymptotic pairs.

(c) We call the pair (x,y) a *Li-Yorke pair* when it is proximal but not asymptotic.

(d) The system (X,T) is called *proximal* when all pairs are proximal, i.e. $PROX = X \times X$. It is called *completely scrambled* when all non diagonal pairs are Li-Yorke. That is, the system is proximal, but $ASYMP = \Delta_X$.

Observe that the set $\{p \in A_+(X,T) : px = py\}$ is a closed left ideal if it is nonempty and so it then contains minimal idempotents. This shows that $(iii) \Leftrightarrow (iv)$ in (a). The remaining equivalences are obvious.

Remark 7.2. This notion of proximality actually refers to the action of the semigroup $\{T^n : n \in \mathbb{Z}_+\}$. The usual definition of proximality would be: x and y are proximal if there exists a sequence $n_i \in \mathbb{Z}$ with $|n_i| \ \to \infty$ such that $\ lim \ d(T^{n_i}(x),T^{n_i}(y)) \ = \ 0$.

Lemma 7.3. *For $x \in X$ and $n \in \mathbb{N}$ the pair $(x,T^n(x))$ is proximal iff $T^n(y) = y$ for some $y \in \omega T(x)$. The pair $(x,T^n(x))$ is asymptotic iff $T^n(y) = y$ for every $y \in \omega T(x)$.*

Proof: $pT^n(x) = T^n(px)$ and so $px = pT^n(x)$ iff $px = T^n(px)$. The pair is proximal (or asymptotic) iff $px = T^n(px)$ for some $p \in A_+(X,T)$ (resp. for all $p \in A_+(X,T)$).
$\square$

We recall the following, see, e.g. [7] Proposition 2.2.

Proposition 7.4. *A compact, metric dynamical system (X,T) is proximal iff there exists a fixed point $e \in X$ which is the unique minimal subset of X, i.e. (X,T) is a minCT system. Consequently, (X,T^{-1}) is proximal if (X,T) is.*

Proof: If u is a minimal idempotent then ux is a minimal point for every $x \in X$. So if e is the unique minimal point of X, then $ux = e$

for every $x \in X$ and every minimal idempotent u. Hence, every pair is proximal.

Assume now that (X, T) is proximal. For any $x \in X$, the pair $(x, T(x))$ is proximal and so there exists $p \in A_+(X, T)$ such that $px = pT(x) = T(px)$ and so $e = px$ is a fixed point. A pair of distinct fixed points is not proximal and so e is the unique fixed point. Hence, e is in the orbit closure of every point and so $\{e\}$ is the only minimal set.

Since the minimal subsets for T and T^{-1} are the same, it follows that (X, T^{-1}) is proximal.

□

Thus, we obtain the following obvious corollary. Compare Proposition 1.16

Corollary 7.5. *A compact, metric dynamical system (X, T) is completely scrambled iff it is a minCT system and $A_+(X, T)$ distinguishes points of X.*

□

Completely scrambled systems were introduced by Huang and Ye [27] who provided a rich supply of examples, but all appear to be of height the first countable ordinal.

In contrast with proximal systems there exist completely scrambled systems (X, T) whose inverse (X, T^{-1}) is not completely scrambled.

Example 7.6. Begin with (Y, F) a completely scrambled system with fixed point e. Let (X, T) be the quotient space of the product system $(Y \times \{0, 1\}, F \times id_{\{0,1\}})$ obtained by identifying $(e, 0)$ with $(e, 1)$ to obtain the fixed point denoted e in X. Let X_0 and X_1 be the images of $Y \times \{0\}$ and $Y \times \{1\}$ in X. Since (X, T) has a unique fixed point e we can construct a sequence $\{x^n : n \in \mathbb{Z}_+\}$ so that

- $x^0 = e$.
- $\{d(x^n, T(x^n))\} \to 0$ as $n \to \infty$.
- For every $N \in \mathbb{N}$ the set $\{x^i : i \geq N\}$ is dense in X.

Now for $n \in \mathbb{Z}$ let $z^n \in X \times [0, 1]$ be defined by

$$z^n = \begin{cases} (x^n, 1/(n+1)) & \text{for } n \geq 0, \\ (e, 1/(-n+1)) & \text{for } n < 0. \end{cases}$$

Let

$$\hat{X} = X \times \{0\} \cup \{ z^n : n \in \mathbb{Z} \}.$$

Let $\hat{T}(x, 0) = (T(x), 0)$ and $\hat{T}(z^n) = z^{n+1}$.

The system $(\hat{X}, \hat{T})$ is topologically transitive with fixed point $(e, 0)$ the unique minimal set. Hence, the system is proximal. Since every orbit in X is confined to either X_0 or X_1 it follows that no point z^n is asymptotic to a point in $X \times \{0\}$. By Lemma 7.3 no two distinct points on the z^n orbit are asymptotic. Hence, $(\hat{X}, \hat{T})$ is completely scrambled. However, the inverse $(\hat{X}, \hat{T}^{-1})$ is not since $\{z^n\} \to (e, 0)$ as $n \to -\infty$.

□

By a result of Schwartzman (see Gottschalk and Hedlund [25, Theorem 10.36]) an expansive system admits non-diagonal asymptotic pairs. We can sharpen this result a bit.

If e is a fixed point of an expansive system (X, T) then $\{e\}$ is an *isolated invariant set*. That is, there exists a neighborhood U of e which contains no nonempty, invariant set other than $\{e\}$.

Proposition 7.7. *If (X, T) is a nontrivial minCT system with fixed point e an isolated invariant set then there exists $x \in X$ with $x \neq e$ and with $\lim_{n\to\infty} T^n(x) = e$, i.e. the pair (x, e) is asymptotic.*

Proof: By Theorem 3.6 (b.3) of [1], if A is a closed, isolated invariant set such that $\omega T(x) \subset A$ implies $x \in A$ then A is a repellor for (X, T). In particular, if $\{e\}$ is an isolated invariant set and the pair (x, e) is asymptotic only when $x = e$ then $\{e\}$ is a repellor. A minCT system is chain transitive and so contains no proper attractors or repellors.

□

It follows that no nontrivial subshift can be completely scrambled. However, we note that the subshifts which arise from labels of finite type are pretty close.

Proposition 7.8. *If $\mathcal{M}_1, \mathcal{M}_2$ are two different labels then the following are equivalent.*

- (i) *For some $n_1, n_2 \in \mathbb{Z}$ the pair $(S^{n_1}x[\mathcal{M}_1], S^{n_2}x[\mathcal{M}_2])$ is asymptotic for S or S^{-1}.*
- (i+) *For some $n_1, n_2 \in \mathbb{Z}$ the pair $(S^{n_1}x_+[\mathcal{M}_1], S^{n_2}x_+[\mathcal{M}_2])$ is asymptotic for S or S^{-1}.*
- (ii) *For all $n_1, n_2 \in \mathbb{Z}$ the pairs $(S^{n_1}x[\mathcal{M}_1], S^{n_2}x[\mathcal{M}_2])$ are asymptotic for both S and S^{-1}.*
- (ii+) *For all $n_1, n_2 \in \mathbb{Z}$ the pairs $(S^{n_1}x_+[\mathcal{M}_1], S^{n_2}x_+[\mathcal{M}_2])$ are asymptotic for both S and S^{-1}.*
- (iii) $\{\mathcal{M}_1, \mathcal{M}_2\} = \{\emptyset, 0\}$.

Proof: Since $R_S(x[0]) = e = x[\emptyset]$, it is clear that (iii) implies (ii), (ii+). That (ii) implies (i) and (ii+) implies (i+) are obvious.

Now assume that $\{\mathcal{M}_1, \mathcal{M}_2\} \neq \{\emptyset, 0\}$. By renumbering we can assume that there exists $\mathbf{r} > \mathbf{0}$ such that $\mathbf{r} \in \mathcal{M}_1 \setminus \mathcal{M}_2$. Let $\{t^i\}$ be a sequence of positive expanding times with length vector $\mathbf{r}$ and such that $\{j_r(t^i) \to \infty$. Then by Theorem 5.28 $Lim\, S^{t^i}(x[\mathcal{M}_1]) = x[\mathcal{M}_1 - \mathbf{r}]$ and $Lim\, S^{t^i}(x_+[\mathcal{M}_1]) = x_+[\mathcal{M}_1 - \mathbf{r}]$. Neither limit point is the fixed point e since $\mathbf{r} \in \mathcal{M}_1$. Hence, for any $n_1 \in \mathbb{Z}$, $Lim\, S^{t^i}(S^{n_1}(x[\mathcal{M}_1])) = S^{n_1}(x[\mathcal{M}_1 - \mathbf{r}]) \neq e$. On the other hand, since, $\mathbf{r} \notin \mathcal{M}_2$,

$$Lim\, S^{t^i}(S^{n_2}(x[\mathcal{M}_2])) = S^{n_2}(x[\mathcal{M}_2 - \mathbf{r}]) = e$$

and

$$Lim\, S^{t^i}(S^{n_2}(x_+[\mathcal{M}_2])) = S^{n_2}(x_+[\mathcal{M}_2 - \mathbf{r}]) = e.$$

Thus, the pairs $(S^{n_1}x[\mathcal{M}_1], S^{n_2}x[\mathcal{M}_2])$ $(S^{n_1}x_+[\mathcal{M}_1], S^{n_2}x_+[\mathcal{M}_2])$ are not asymptotic for S or for S^{-1}. This prove the contrapositive of $(i), (i{+}) \Rightarrow (iii)$.

□

Corollary 7.9. *For any positive label $\mathcal{M}$ the set $\{ S^n(x[\mathcal{N}]) : \mathcal{N} \neq 0 \in \Theta(\mathcal{M}), n \in \mathbb{Z} \}$ is a scrambled subset for $(X(\mathcal{M}), S)$ and for $(X(\mathcal{M}), S^{-1})$. The set $\{ S^n(x_+[\mathcal{N}]) : \mathcal{N} \neq 0 \in \Theta(\mathcal{M}), n \in \mathbb{Z} \}$ is a scrambled subset for $(X_+(\mathcal{M}), S)$ and for $(X_+(\mathcal{M}), S^{-1})$. If $\mathcal{M}$ is a label of finite type then these sets are the complement of the orbit of $x[0]$ in $X(\mathcal{M})$ and $X_+(\mathcal{M})$, respectively.*

Proof: That the set is scrambled is clear from Proposition 7.8. In the finite type case, Corollary 5.33 (a) implies that we are excluding only the orbit of $x[0]$ from the set.

□

Definition 7.10. An *inverse sequence* in $\mathcal{LAB}$ is a sequence $\{\mathcal{M}^i, \mathbf{r}^i : i \in \mathbb{Z}_+\}$ with $\mathbf{r}^i > \mathbf{0}$ in $\mathcal{M}^i$ and such that $\mathcal{M}^i = \mathcal{M}^{i+1} - \mathbf{r}^{i+1}$ for $i > 0$. For the associated inverse sequences $p_{\mathbf{r}^{i+1}} : (X(\mathcal{M}^{i+1}), S) \to (X(\mathcal{M}^i), S)$ and $p_{\mathbf{r}^{i+1}} : (X_+(\mathcal{M}^{i+1}), S) \to (X_+(\mathcal{M}^i), S)$ we let $(X(\{\mathcal{M}^i, \mathbf{r}^i\}), S)$ and $(X_+(\{\mathcal{M}^i, \mathbf{r}^i\}), S)$ denote the respective inverse limits.

Theorem 7.11. *Let $\{\mathcal{M}^i, \mathbf{r}^i\}$ be an inverse sequence in $\mathcal{LAB}$. The inverse limit system $(X(\{\mathcal{M}^i, \mathbf{r}^i\}), S)$ and $(X_+(\{\mathcal{M}^i, \mathbf{r}^i\}), S)$ are topologically transitive, compact metrizable systems. If each $\mathcal{M}^i$ is of finite type then the limit systems their inverses are completely scrambled. If each $\mathcal{M}^i$ is either finitary or simple then the limit systems are WAP.*

Proof: Topologically transitive and WAP systems are closed under inverse limits. In this case, each map $p_{\mathbf{r}^{i+1}}$ is surjective as required because it maps the transitive point $x[\mathcal{M}^{i+1}]$ of $X(\mathcal{M}^{i+1})$ onto the transitive point $x[\mathcal{M}^i]$ of $X(\mathcal{M}^i)$ since $\mathcal{M}^i = \mathcal{M}^{i+1} - \mathbf{r}^{i+1}$. In particular, the sequence $x^* = \{x[\mathcal{M}^i]\}$ is a transitive point for the inverse limit.

The point e associated with the sequence $\{x[\emptyset]\}$ is a fixed point in $X(\{\mathcal{M}^i, \mathbf{r}^i\})$. A minimal subset of the limit space projects to a minimal subset of $X(\mathcal{M}^i)$ for each i. If $\mathcal{M}^i$ is of finite type then this minimal subset is $\{e\} \subset X(\mathcal{M}^i)$. Thus, if all are of finite type, the fixed point is the only minimal subset of the limit and so $(X(\{\mathcal{M}^i, \mathbf{r}^i\}), S)$ is proximal by Proposition 7.4.

Notice that if $x \in X(\mathcal{M}^i)$ is not equal to the fixed point e then $x[0] \notin p^{-1}_{\mathbf{r}^{i+1}}(x)$. If x, y are distinct points of $X(\{\mathcal{M}^i, \mathbf{r}^i\})$ then for sufficiently large i they project to distinct points of $X(\mathcal{M}^i)$ with neither projecting to $x[0]$ in $X(\mathcal{M}^i)$. In the finite type case it then follows from Corollary 7.9 that for sufficiently large i, x and y project to a non-asymptotic pair. Consequently, the pair (x, y) is not asymptotic in $X(\{\mathcal{M}^i, \mathbf{r}^i\})$.
□

Remark 7.12. Since a transitive point for $X(\{\mathcal{M}^i, \mathbf{r}^i\})$ projects to a transitive point on each $X(\mathcal{M}^i)$ it follows that the transitive points for $X(\{\mathcal{M}^i, \mathbf{r}^i\})$ are all on the orbit of x^* described above and so x^* is isolated when the labels $\mathcal{M}^i$ are of finite type. Similarly, for $X_+(\{\mathcal{M}^i, \mathbf{r}^i\})$.

For the construction of our examples, we need the following. Recall that (6.20) implies that if $\mathcal{M}_1$ is a finite label and $\mathcal{M}_2$ is a label with supports disjoint from those of $\mathcal{M}_1$, then

$$\Theta'(\mathcal{M}_1 \oplus \mathcal{M}_2) \ = \ \Theta'(\mathcal{M}_1) \oplus \Theta(\mathcal{M}_2) \cup \Theta(\mathcal{M}_1) \oplus \Theta'(\mathcal{M}_2).$$

Lemma 7.13. *Let* $\mathbf{r}$ *be a positive finite vector with support disjoint from those in Supp* $\mathcal{M}$ *for some nonempty label* $\mathcal{M}$*. Then* $P_{\mathbf{r}}(\Theta'(\langle \mathbf{r} \rangle) \oplus \Theta(\mathcal{M})) = \{\emptyset\}$ *and on* $\{\langle \mathbf{r} \rangle\} \oplus \Theta(\mathcal{M})$ $P_{\mathbf{r}}$ *is a bijection onto* $\Theta(\mathcal{M})$*.*

Proof: Since $\mathbf{r}$ is not an element of any label in $\Theta'(\langle \mathbf{r} \rangle) \oplus \Theta(\mathcal{M})$ it follows that all of these labels are mapped to $\emptyset$.

By (6.20) every label of $\Theta(\langle \mathbf{r} \rangle \oplus \mathcal{M})$ is of the form $\mathcal{N}_1 \oplus \mathcal{N}_2$ with $\mathcal{N}_1 \in \Theta(\langle \mathbf{r} \rangle)$ and $\mathcal{N}_2 \in \Theta(\mathcal{M})$. If $\mathcal{N}_1 \neq \langle \mathbf{r} \rangle$ then $P_{\mathbf{r}}$ maps $\mathcal{N}_1 \oplus \mathcal{N}_2$ to $\emptyset$. If $\mathcal{N}_1 = \langle \mathbf{r} \rangle$ then $P_{\mathbf{r}}$ maps $\mathcal{N}_1 \oplus \mathcal{N}_2$ to $\mathcal{N}_2$. Hence, for any $\mathcal{N}_2 \in \Theta(\mathcal{M})$ the unique label of the form $\langle \mathbf{r} \rangle \oplus \mathcal{N}$ which is mapped to $\mathcal{N}_2$ has $\mathcal{N} = \mathcal{N}_2$.
□

Example 7.14. Let $\{\mathbf{r}^i\}$ a sequence of positive $\mathbb{N}$-vectors all with disjoint supports and let $\mathcal{M}$ be a finitary label with the sets in *Supp* $\mathcal{M}$ disjoint from the supports of the sequence.

Let $\mathcal{N}^0 = \{0\}$ and $\mathcal{N}^{i+1} = \langle \mathbf{r}^{i+1} \rangle \oplus \mathcal{N}^i$ define an increasing sequence of finite labels. Define $\{\mathcal{M}^i = \mathcal{N}^i \oplus \mathcal{M}, \mathbf{r}^i\}$, an inverse sequence of finitary labels. For each i Lemma 7.13 implies that the preimage of $\emptyset$ by $P_{\mathbf{r}^{i+1}} : \Theta(\mathcal{M}^{i+1}) \to \Theta(\mathcal{M}^i)$ is countable and the preimage of every other point is a singleton. It follows that the limit system $(X(\{\mathcal{M}^i, \mathbf{r}^i\}), S)$ and its inverse $(X(\{\mathcal{M}^i, \mathbf{r}^i\}), S^{-1})$ are completely scrambled, topologically transitive, countable WAPs.

Notice that $X(\mathcal{M})$ and $X_+(\mathcal{M})$ are factor of $X(\{\mathcal{M}^i, \mathbf{r}^i\})$ and $X_+(\{\mathcal{M}^i, \mathbf{r}^i\})$ respectively. Hence, if we choose $\mathcal{M}$ with height greater than some countable ordinal α then $X(\{\mathcal{M}^i, \mathbf{r}^i\})$ and $X_+(\{\mathcal{M}^i, \mathbf{r}^i\})$ have height greater than α.

$\square$

Following Huang and Ye we can take countable products of copies of these examples to get completely scrambled WAP systems on the Cantor set with arbitrarily large heights. However, these examples will not be topologically transitive.

References

1. Ethan Akin, *The general topology of dynamical systems*, Graduate Studies in Mathematics, **1**, American Mathematical Society, Providence, RI, 1993.
2. Ethan Akin, *Recurrence in topological dynamical systems: Furstenberg families and Ellis actions*, Plenum Press, New York, 1997.
3. Ethan Akin, *Lectures on Cantor and Mycielski sets for dynamical systems*, Contemporary Mathematics, **356**, (2004), 21–79.
4. Ethan Akin, Joseph Auslander and Kenneth Berg, *When is a transitive map chaotic*, Convergence in Ergodic Theory and Probability, Walter de Gruyter & Co. 1996, 25–40.
5. Ethan Akin, Joseph Auslander and Kenneth Berg, *Almost equicontinuity and the enveloping semigroup*, Topological dynamics and applications, Contemporary Mathematics **215**, a volume in honor of R. Ellis, 1998, 75–81.
6. Ethan Akin, Joseph Auslander and Eli Glasner, *The topological dynamics of Ellis actions*, Mem. Amer. Math. Soc. **195** (2008), no. 913.
7. Ethan Akin and Sergii Kolyada, *Li-Yorke sensitivity*, Nonlinearity, **16**, (2003), 1421–1433.
8. Joseph Auslander and James A. Yorke, *Interval maps, factors of maps and chaos*, Tohoku Math. Journal **32**, (1980), 177–188.
9. Tomasz Downarowicz, *Weakly almost periodic flows and hidden eigenvalues,* Contemp. Math., **215** (1998), 101–120.
10. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67**, (1949), 217–240.

11. Robert Ellis and Mahesh Nerurkar, *Weakly almost periodic flows*, Trans. Amer. Math. Soc. **313**, (1989), 103–119.
12. H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton U. Press, Princeton, 1981.
13. Eli Glasner, *Ergodic theory via joinings*, Math. Surveys and Monographs, AMS, **101**, 2003.
14. Eli Glasner, *On tame dynamical systems*, Colloq. Math. **105** (2006), 283–295.
15. Eli Glasner and David Maon, *Rigidity in topological dynamics*, Ergod. Th. Dynam. Sys. **9**, (1989), 309–320.
16. Eli Glasner and Michael Megrelishvili, *Linear representations of hereditarily non-sensitive dynamical systems*, Colloq. Math., **104** (2006), no. 2, 223–283.
17. Eli Glasner and Michael Megrelishvili, *Eventual nonsensitivity and tame dynamical systems*, arXiv:1405.2588.
18. Eli Glasner and Benjamin Weiss, *Sensitive dependence on initial conditions*, Nonlinearity **6** (1993), 1067–1075.
19. Eli Glasner and Benjamin Weiss, *Quasi-factors of zero-entropy systems*, J. Amer. Math. Soc. **8**, (1995), no. 3, 665–686.
20. E. Glasner and B. Weiss, *Locally equicontinuous dynamical systems*, Colloquium Mathematicum, part 2, **84/85** (2000), 345–361.
21. Eli Glasner and , Benjamin Weiss, *Topological groups with Rokhlin properties*, Colloq. Math., **110**, (2008), no. 1, 51–80.
22. Eli Glasner and Benjamin Weiss, *On the interplay between measurable and topological dynamics*, Handbook of dynamical systems. Vol. **1B**, pp. 597–648, Elsevier B. V., Amsterdam, 2006.
23. Eli Glasner and Xiangdong Ye, *Local entropy theory*, Ergodic Theory Dynam. Systems, **29**, (2009), no. 2, 321–356.
24. T. N. T. Goodman, *Topological sequence entropy*, Proc. London Math. Soc., **29** (1974), 331–350.
25. W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, AMS Colloquium Publications, Vol. 36, 1955.
26. Neil Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combinatorial Theory Ser. **A 17**, (1974), 1–11.
27. Wen Huang and Xiangdong Ye, *Homeomorphisms with the whole compacta being scrambled sets*, Ergodic Theory Dynam. Systems, **21**, (2001), no. 1, 77–91.
28. Yitzhak Katznelson and Benjamin Weiss, *When all points are recurrent/generic*, Ergodic theory and dynamical systems, I (College Park, Md., 1979–80), 195–210, Progr. Math., **10**, Birkhäuser, Boston, 1981.
29. Alexander Kechris and Christian Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc., (3), **94**, (2007), no. 2, 302–350.
30. David Kerr and Hanfeng Li, *Independence in topological and C^*-dynamics*, Math. Ann. **338** (2007), 869–926.
31. T. Li and J. Yorke, *Period 3 implies chaos*, Amer. Math. Monthly, **82** (1975), 985–992.
32. Wolfgang Ruppert, *On weakly almost periodic sets,* Semigroup Forum, 32 (1985) 267–281.

33. S. A. Shapovalov, *A new solution of one Birkhoff problem*, J. Dynam. Control Systems, **6** (2000), no. 3, 331–339.

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