ON HILBERT DYNAMICAL SYSTEMS

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ABSTRACT. Returning to a classical question in Harmonic Analysis we strengthen an old result of Walter Rudin. We show that there exists a weakly almost periodic function on the group of integers $\mathbb{Z}$ which is not in the norm-closure of the algebra $B(\mathbb{Z})$ of Fourier-Stieltjes transforms of measures on the dual group $\hat{\mathbb{Z}} = \mathbb{T}$, and which is **recurrent**. We also show that there is a Polish monothetic group which is reflexively but not Hilbert representable.

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INTRODUCTION

Walter Rudin [22] was the first to show that the algebra $WAP(\mathbb{Z})$, of weakly almost periodic functions on $\mathbb{Z}$, is strictly larger than the algebra $H(\mathbb{Z}) = \overline{B(\mathbb{Z})}$, the norm closure in the Banach space $\ell_\infty(\mathbb{Z})$ of the algebra $B(\mathbb{Z})$ of Fourier-Stieltjes transforms of complex measures on the dual group $\hat{\mathbb{Z}} = \mathbb{T}$. Many other examples of functions in $WAP(\mathbb{Z}) \setminus H(\mathbb{Z})$ followed (see e.g. [5]).

As it turns out, in all of these examples the function in question is non-recurrent. We say that a function $f \in \ell_\infty(\mathbb{Z})$ is *recurrent* if there is a sequence $n_k \nearrow \infty$ with $\lim_{k\to\infty} \sup_{j\in\mathbb{Z}} |f(j)-f(j+n_k)| = 0$. Now in some sense the more interesting functions in $WAP(\mathbb{Z})$ are the recurrent ones, and moreover they are prevalent; e.g. every $\mathbb{Z}$-subalgebra of $WAP(\mathbb{Z})$ which is $\mathbb{Z}$-generated by a recurrent function consists entirely of recurrent functions (see Lemma 6.1 below). Also, a Fourier-Stieltjes transform $\hat{\mu}$

Date: February 10, 2010.
2000 Mathematics Subject Classification. Primary 54H20, 54H15, 43A25, 43A65.
Both author's work is supported by ISF grant #1157/08 and the first author's work is also supported by BSF grant # 2006119.

of a continuous measure μ on $\mathbb{T}$ is recurrent iff μ is Dirichlet [16]. Thus the question whether there are *recurrent* functions in $WAP(\mathbb{Z}) \setminus H(\mathbb{Z})$ is a natural one.

Another related open question is the following. Is there a Polish monothetic group P which can be represented as a group of isometries of a reflexive Banach space but is not representable as a group of unitary operators on a Hilbert space?

In this work we see how these questions are related and show that the answer to both is affirmative (Theorem 4.2 and Corollary 5.2). We also show that if a Polish monothetic group P is Hilbert-representable and $K \subset P$ is a compact subgroup, then the quotient group P/K is also Hilbert-representable (Corollary 5.3).

Our proofs are based on the theory of topological dynamics and rely on a well known construction of Banaszczyk [2]. We also use an idea of Megrelishvili [18] who showed that the topological groups $L_{2n}([0,1])$, for n a positive integer, are reflexively but not Hilbert representable. For the sake of simplicity we state our results in the most basic setup, where the acting group is $\mathbb{Z}$ and the dynamical systems are usually assumed to be point-transitive.

We refer the reader to the following related recent works: Gao and Pestov [10], Megrelishvili [20], Ferri and Galindo [8], and Galindo [9].

1. Some preliminaries from topological dynamics

A *dynamical system* (or sometimes just a *system*) is for us a pair (X,T) where X is a compact Hausdorff space and $T : X \to X$ is a self homeomorphism. With (X,T) we associate an action of the group of integers $\mathbb{Z}$ via the map $n \mapsto T^n$. The *orbit* of a point $x \in X$ is the set $\mathcal{O}_T(x) = \{T^n x : n \in \mathbb{Z}\}$. The *orbit closure* of x is the set $\bar{\mathcal{O}}_T(x) = \overline{\mathcal{O}_T(x)}$. The system (X,T) is *point-transitive* if there is a point $x \in X$ with $\bar{\mathcal{O}}_T(x) = X$. Such a point is called a *transitive point* and the collection of transitive points (when it is not empty) is denoted by X_{tr}. For a point-transitive metric system X_{tr} is a dense G_δ subset of X. We will mostly work in the category of *pointed* dynamical systems (X, x_0, T), where the latter is a point-transitive dynamical system with a distinguished point $x_0 \in X_{tr}$. The restriction of T to a closed invariant subset $Y \subset X$ in a dynamical system (X,T) defines a dynamical system (Y,T). Such a system is called a *subsystem* of (X,T).

A continuous surjective map $\pi : (X,T) \to (Y,S)$ between two dynamical systems (X,T) and (Y,S) which intertwines the $\mathbb{Z}$-actions (i.e. $\pi(Tx) = S\pi(x)$ for every $x \in X$) is called a *homomorphism* of dynamical systems and we sometimes say that Y is a *factor* of X or that X is an *extension* of Y. When dealing with pointed systems a homomorphism $\pi : (X, x_0, T) \to (Y, y_0, S)$ is further assumed to satisfy $\pi(x_0) = y_0$. An extension $\pi : (X,T) \to (Y,S)$ is called *almost 1-1* if there is a dense G_δ subset $X_0 \subset X$ with $\pi^{-1}(\pi(x)) = \{x\}$ for every $x \in X_0$. The extension $\pi : (X, x_0, T) \to (Y, y_0, S)$ is called a *group-extension* if there is a compact subgroup $K \subset \mathrm{Homeo}\,(X)$ such that each $k \in K$ is an automorphism of (X,T) (i.e. $Tk = kT$ for every $k \in K$) and such that the quotient dynamical system $(X/K, T)$ is isomorphic to (Y,S).

A point x in a metric dynamical system (X,T) is called *recurrent* if there is a sequence $\{n_k\} \subset \mathbb{Z}$ with $|n_k| \to \infty$ such that $\lim T^{n_k} x = x$. Note that in a point-transitive system if the set of isolated points is not empty then it coincides with the orbit of a transitive point. On the other hand, when there are no isolated points in a point transitive system then every point of X_{tr} is recurrent. We call a point-transitive system with no isolated points a *recurrent-transitive system*.

Let $\ell_\infty(\mathbb{Z})$ be the Banach space (and C^*-algebra) of bounded complex valued functions on $\mathbb{Z}$ with the sup norm: $\|f\|_\infty = \sup_{n\in\mathbb{Z}} |f(n)|$. We write $S : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ for the shift operator, where $Sf(n) = f(n+1)$, $(n \in \mathbb{Z})$. An S-invariant, conjugacy invariant subalgebra of $\ell_\infty(\mathbb{Z})$ containing the constant function $\mathbf{1}$ will be called a $\mathbb{Z}$-*algebra*.

Given a pointed dynamical system (X, x_0, T) we define a map $j_{x_0} : C(X) \to \ell_\infty(\mathbb{Z})$ by

$$j_{x_0}F(n) = F(T^n x_0), \quad F \in C(X),\ n \in \mathbb{Z}.$$

It is easy to see that j_{x_0} is an isometry with $j_{x_0} \circ T = S \circ j_{x_0}$. We denote its image in $\ell_\infty(\mathbb{Z})$ by $\mathcal{A}(X, x_0)$. Clearly $\mathcal{A}(X, x_0)$ is a $\mathbb{Z}$-algebra.

Conversely, given a $\mathbb{Z}$-algebra $\mathcal{A} \subset \ell_\infty(\mathbb{Z})$ we denote its Gelfand space (comprising the non-zero C^*-homomorphisms of $\mathcal{A}(X, x_0)$ into $\mathbb{C}$) by $X = |\mathcal{A}(X, x_0)|$. It is easy to see that the operator S induces a homeomorphism $T : X \to X$ and that the resulting dynamical system (X, T) is a point-transitive system. In fact, the point $x_0 = \mathrm{eva}_0 \in X$, which corresponds to the multiplicative complex valued homomorphism defined on $\mathcal{A}$ by evaluation at 0, is a transitive point.

These operations are inverse to each other and we have

$$(|\mathcal{A}(X, x_0)|, \mathrm{eva}_0, S) \cong (X, x_0, T).$$

A $\mathbb{Z}$-algebra $\mathcal{A}$ is *cyclic* if there is a function $f \in \mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_f$, where the latter is the smallest $\mathbb{Z}$-algebra that contains f.

Given $f \in \ell_\infty(\mathbb{Z})$ we can consider f as an element of the compact metrizable space $\Omega = [-\|f\|_\infty, \|f\|_\infty]^{\mathbb{Z}}$. Again denote by $S : \Omega \to \Omega$ the homeomorphism defined by $Sg(n) = g(n+1)$, $(g \in \ell_\infty,\ n \in \mathbb{Z})$. We let $X_f = \mathrm{cls}\,\{S^n f : n \in \mathbb{Z}\}$, where the closure is taken in Ω with respect to the product topology. It can be easily verified that $\mathcal{A}(X_f, f, S) = \mathcal{A}_f$. Note that as $\mathcal{A}_f$ is always separable, X_f is metrizable.

If $\pi : (X, x_0, T) \to (Y, y_0, S)$ is a homomorphism of pointed transitive systems then the diagram

$$\begin{tikzcd} C(Y) \arrow[r, "j_{y_0}"] \arrow[d, "\pi^*"'] & \mathcal{A}(Y, y_0) \arrow[d, "i"] \\ C(X) \arrow[r, "j_{x_0}"'] & \mathcal{A}(X, x_0) \end{tikzcd}$$

commutes. Here $(\pi^*F)(x) = F(\pi x)$, for $F \in C(Y), x \in X$, and i is the inclusion map. Conversely, if $B \subset C(X)$ is a closed conjugation-invariant, T-invariant subalgebra containing the constant functions, then the restriction map

$$\pi : (|\mathcal{A}(X, x_0)|, \mathrm{eva}_0, T) \to (|j_{x_0}(B)|, \mathrm{eva}_0, T)$$

is a pointed homomorphism of dynamical systems.

With every dynamical system (X, T) we associate its *enveloping semigroup* $E = E(X, T) \subset X^X$. This is the pointwise closure of the set $\{T^n : n \in \mathbb{Z}\}$, as a subset of X^X. $E(X, T)$ is a compact *right topological semigroup*, i.e. for each $p \in E(X, T)$ right multiplication $q \mapsto qp$, $q \in E$ is continuous. The set $\{T^n : n \in \mathbb{Z}\}$ is contained in the center of E. In particular, for each $n \in \mathbb{Z}$ the map $p \mapsto T^n x,\ p \in E$ is a homeomorphism of E, so that via multiplication by T, (E, T) is also a dynamical system.

A function $f \in \ell_\infty(\mathbb{Z})$ is an *almost periodic* (*weakly almost periodic*) function if its orbit $\{S^n f : n \in \mathbb{Z}\}$ is norm precompact (weakly precompact) in the Banach space $\ell_\infty(\mathbb{Z})$. We denote by $AP(\mathbb{Z})$ and $WAP(\mathbb{Z})$ the collections of almost periodic and weakly almost periodic functions in $\ell_\infty(\mathbb{Z})$ respectively. These are $\mathbb{Z}$-algebras with $AP(\mathbb{Z}) \subset WAP(\mathbb{Z})$. A point-transitive dynamical system (X, T) is called *almost periodic* iff $\mathcal{A}(X, x_0, T) \subset AP(\mathbb{Z})$ iff for every F in the Banach space $C(X)$ the orbit $\{F \circ T^n : n \in \mathbb{Z}\}$ is norm pre-compact. The system (X, T) is *weakly almost periodic* iff $\mathcal{A}(X, x_0, T) \subset WAP(\mathbb{Z})$ iff for every $F \in C(X)$ the orbit $\{F \circ T^n : n \in \mathbb{Z}\}$ is weakly pre-compact. It is well known that a point-transitive system (X, T) is almost periodic iff it is equicontinuous and minimal. A theorem of Ellis and Nerurkar [7] based on a theorem of Grothendieck asserts that (X, T) is weakly almost periodic iff its *enveloping semigroup* $E(X, T)$ consists of continuous maps. Another characterization of WAP systems (which again goes back to Grothedieck) is that $E(X, T)$ be a commutative semigroup. Using this observation it is also easy to deduce the following well known *double limit criterion* (stated here for a general topological group G, see e.g. [23]).

1.1. **Proposition.** *For a topological group G, a bounded continuous function $f : G \to \mathbb{C}$ is WAP iff whenever g_m, h_n are sequences in G such that the double limits*

$$a = \lim_{n\to\infty} \lim_{m\to\infty} f(g_m h_n) \quad \textit{and} \quad b = \lim_{m\to\infty} \lim_{n\to\infty} f(g_m h_n)$$

exist, then $a = b$.

For WAP $\mathbb{Z}$-systems we have $E(X, T) \cong (X, T)$, see e.g. [4] or [11].
We summarize these results in the following:

1.2. **Theorem.** *Let (X, T) be a point-transitive dynamical system and let $E = E(X, T)$ be its enveloping semigroup. The following conditions are equivalent.*

1. *The system (X, T) is WAP.*
2. *E is a semi-topological semigroup, i.e. both right and left multiplications are continuous.*
3. *E is a commutative semigroup.*
4. *E consists of continuous maps.*

A point x in a metric dynamical system (X, T) is an *equicontinuity point* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, x') < \delta \Rightarrow d(T^n x, T^n x') < \epsilon$ for all $n \in \mathbb{Z}$. A dynamical system with a residual set of equicontinuity points is called an *almost equicontinuous* system. A recurrent-transitive metric almost equicontinuous system (X, T) is uniformly rigid and the the set of equicontinuity points in X coincides with X_{tr}. Moreover, the map $\Lambda_X \twoheadrightarrow X_{tr},\ \lambda \mapsto \lambda x_0$, is a homeomorphism of the Polish group Λ_X onto the dense G_δ subset $X_{tr} \subset X$ [15].

By results of Glasner and Weiss [14] and Akin, Auslander and Berg [1] it follows that if (X, T) is a metric recurrent-transitive WAP dynamical system, then the system (X, T) is *uniformly rigid*, i.e. there is a sequence of positive integers $n_k \to \infty$ such that the sequence $\{T^{n_k} : k \in \mathbb{N}\}$ tends uniformly to the identity map on X (see [12]). In a uniformly rigid system the uniform closure

$$\Lambda_X = \mathrm{cls}\,\{T^n : n \in \mathbb{Z}\} \subset \mathrm{Homeo}\,(X),$$

is a non-discrete Polish monothetic topological group. We then have $X_{tr} = \Lambda_X x_0$.

When (X,T) is recurrent-transitive metric WAP we know much more: The system (X,T) is *hereditarily almost equicontinuous*; i.e., every subsystem $Y \subset X$ is almost equicontinuous (see [14], [1], [15] and [13]).

2. Reflexive and Hilbert representability of groups and dynamical systems

2.1. **Definitions.**
1. We say that a Polish topological group P is *representable* on a reflexive Banach space V if there is a topological isomorphism of P into the group of isometries $\mathrm{Iso}\,(V)$ of V equipped with the strong operator topology. The group P is *reflexibly-representable* if it is representable on some reflexive Banach space.
2. The group P is *Hilbert-representable* if there is a topological isomorphism of P into the unitary group $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ equipped with the strong operator topology (see Megrelishvili [19] and [20]).
3. As in Banaszczyk [2] we say that P is *exotic* if it does not admit any nontrivial continuous unitary representations. We say that P is *strongly exotic* if it does not admit any nontrivial weakly continuous representations in Hilbert spaces.

2.2. **Definitions.**
1. A metric dynamical system (X,T) is called *reflexively-representable* if there is a reflexive Banach space V, a linear isometry $U \in Iso(V)$ and a weakly compact U-invariant subset Z of V such that the dynamical systems (X,T) and (Z,U) are isomorphic.
2. A metric dynamical system (X,T) is called *Hilbert-representable* if there is a Hilbert space $\mathcal{H}$, a unitary operator $U \in \mathcal{U}(\mathcal{H})$ and a weakly compact U-invariant subset Z of $\mathcal{H}$ such that the dynamical systems (X,T) and (Z,U) are isomorphic.
3. Let $B(\mathbb{Z})$ be the sub-algebra of $\ell_\infty(\mathbb{Z})$ which consists of the Fourier-Steiltjes transforms of complex measures on the circle $\mathbb{T}$, i.e. $B(\mathbb{Z}) = \mathcal{F}(M(\mathbb{T}))$ where $\mathcal{F} : M(\mathbb{T}) \to \ell_\infty(\mathbb{Z})$ is the Fourier transform:
$$\mathcal{F}(\mu)(n) = \hat{\mu}(n) = \int_{\mathbb{T}} e^{int}\, d\mu(t).$$
4. Let $H(\mathbb{T}) = \overline{B(\mathbb{Z})}$ be the norm closure of $B(\mathbb{Z})$ in the Banach space $\ell_\infty(\mathbb{Z})$. Clearly $H(\mathbb{Z})$ is a $\mathbb{Z}$-subalgebra of $\ell_\infty(\mathbb{Z})$.
5. A point-transitive system (X, x_0, T) is called *Hilbert* if $\mathcal{A}(X, x_0) \subset H(\mathbb{Z})$. The elements of $H(\mathbb{Z})$ are called *Hilbert functions*.

Suppose (X,T) is reflexively-representable. Thus we assume that there is a reflexive Banach space V, a linear isometry $U \in \mathrm{Iso}\,(V)$ and that $X \subset V$ is a weakly compact U-invariant subset, with $T = U \restriction X$. Let x_0 be a vector in $X \subset V$ and ϕ a vector in V^*. Set $F(x) = \langle x, \phi\rangle$ and $f(n) = F(T^n x_0) = \langle T^n x, \phi\rangle$ (such a function is called a *matrix coefficient* of the representation). Then, it is easy to see how the weak-compactness of the unit ball of V^* implies that the T-orbit of F in $C(X)$ is weakly precompact. Since $\mathcal{A}(X, x_0) \cong C(X)$ it follows that the S-orbit of f in $\ell_\infty(\mathbb{Z})$ is also weakly precompact, i.e. the function f is in $WAP(\mathbb{Z})$.

It turns out that the converse is also true. We have the following basic theorems of Shtern [24] and Megrelishvili [19] concerning reflexive representability. We formulate these results in the context of a general topological group G, where the C^*-algebra $LUC(G)$ of bounded, complex valued, left uniformly continuos functions takes the place of $\ell_\infty(\mathbb{Z})$. Thus e.g. $WAP(G)$ is the G-subalgebra of $LUC(G)$ comprising functions whose G-orbit is weakly pre-compact.

2.3. **Theorem.** 1. *A topological group G can be faithfully represented on a reflexive Banach space V iff the $WAP(G)$ functions separate points and closed sets on G [24], [19].*

2. *Let G be a topological group, then every $f \in WAP(G)$ is a matrix coefficient of a reflexive representation of G [19].*
3. *A metrizable G-system (X, G) is WAP iff it is reflexively-representable [19].*

2.4. **Lemma.** *If a point-transitive metric system (X, x_0, T) is Hilbert-representable then there is a function $F \in C(X)$ such that the corresponding $f \in \mathcal{A}(X, x_0)$, i.e. the function $f \in \ell_\infty(\mathbb{Z})$ defined by $f(n) = F(T^n x_0)$, is positive definite and satisfies $\mathcal{A}(X, x_0) = \mathcal{A}_f$, where the latter is the $\mathbb{Z}$-subalgebra generated by f in $\ell_\infty(\mathbb{Z})$.*

Proof. By assumption we can consider X as a weakly compact subset of a separable Hilbert space $\mathcal{H}$ and $T = U \restriction X$, where $U \in \mathcal{U}(\mathcal{H})$, the unitary group of $\mathcal{H}$. With no loss in generality we also assume that $\|x_0\| = 1$ and that $\mathcal{H} = Z_U(x_0)$, where the latter is the U-cyclic space generated by x_0. Set $F(x) = \langle x, x_0 \rangle$. Then $F \in C(X)$ and

$$f(n) = F(U^n x_0) = \langle U^n x_0, x_0 \rangle$$

is indeed positive definite.

If $x \neq y$ are points in X then, as $\mathcal{H} = Z_U(x_0)$, there is some $n \in \mathbb{Z}$ with

$$F(U^{-n} x) = \langle x, U^{-n} x_0 \rangle \neq \langle y, U^{-n} x_0 \rangle = F(U^{-n} y),$$

so that the sequence of functions $\{F \circ U^n\}_{n \in \mathbb{Z}}$ separates points on X, whence $\mathcal{A}(X, x_0) = \mathcal{A}_f$. □

2.5. **Lemma.** *Suppose $f \in \ell_\infty(\mathbb{Z})$ is positive definite.*

1. *The system (X_f, f, S) is Hilbert-representable, i.e. there exists a Hilbert space $\mathcal{H}$, a unit vector $x_0 \in \mathcal{H}$ and a unitary operator $U \in \mathcal{U}(\mathcal{H})$ such that $\mathcal{H} = Z_U(x_0)$, $X = w\text{-cls}\{U^n x_0 : n \in \mathbb{Z}\}$, and $(X_f, f, S) \cong (X, x_0, U)$.*
2. *Every element g of $X_f = \bar{\mathcal{O}}(f) \subset [-\|f\|, \|f\|]^{\mathbb{Z}}$ has the form $g(j) = \langle U^j x_0, x \rangle$ for some $x \in X$. In particular $g \in B(\mathbb{Z})$.*
3. *For $g \in X_f = \bar{\mathcal{O}}(f)$ we have: $g \in \mathcal{O}(f)$ iff there is $n \in \mathbb{Z}$ with $g(n) = \|g\| = \|f\|$, in which case $g = S^n f$. In particular for $g \in X_f$ we have $g(0) = \|g\| = \|f\| \Rightarrow g = f$.*
4. *$(X_f)_{tr} = \{g \in X_f : \|g\| = \|f\|\}$.*

Proof. 1. This is a consequence of Herglotz' theorem.

2. Let $g \in X_f$. Then there exists a sequence $n_k \to \infty$ with

$$g(j) = \lim_{k \to \infty} f(j + n_k) = \lim_{k \to \infty} S^{n_k} f(j) = \lim_{k \to \infty} \langle U^j x_0, U^{n_k} x_0 \rangle = \langle U^j x_0, x \rangle,$$

where we assume, with no loss in generality, that $x = w\text{-}\lim_{k\to\infty} U^{n_k} x_0$ exists. It follows that $g \in B(\mathbb{Z})$.

3. Suppose $g \in X_f$ satisfies $g(0) = \|g\| = \|f\|$. As we have seen in part 1, there is an $x \in X$ with $g(j) = \langle U^j x_0, x\rangle$ for every $j \in \mathbb{Z}$ and our assumption reads:

$$1 = g(0) = \langle x, x_0\rangle.$$

We conclude that $x = x_0$, hence

$$g(j) = \langle U^j x_0, x_0\rangle = f(j),$$

for all $j \in \mathbb{Z}$, i.e. $g = f$.

4. Let $g \in (X_f)_{tr}$. There is the a sequence n_k with $\lim_{k\to\infty} S^{n_k} g(j) = \lim_{k\to\infty} g(j + n_k) = f(j)$ for every $j \in \mathbb{Z}$. In particular $\lim_{k\to\infty} g(n_k) = f(0) = \|f\|$. Thus $\|f\| \le \|g\| \le \|f\|$, hence $\|f\| = \|g\|$. Conversely, assuming $\|f\| = \|g\|$, we have $\lim_{k\to\infty} g(n_k) = f(0) = \|f\|$. With no loss of generality we can assume that $h = \lim_{k\to\infty} S^{n_k} g$ exists in X_f, so that $h(0) = \|f\|$. By part 3, $h = f$ and we conclude that $g \in (X_f)_{tr}$. ◻

2.6. **Proposition.** *A point-transitive system (X, x_0, T) is Hilbert-representable iff there is a positive definite function $f \in \ell_\infty(\mathbb{Z})$ such that $(X, x_0, T) \cong (X_f, f, S)$, where X_f is the orbit closure of f in $\ell_\infty(\mathbb{Z})$ under the shift S with respect to the pointwise convergence topology. In particular every Hilbert-representable system is Hilbert.*

Proof. Combine Lemmas 2.4 and 2.5. ◻

2.7. **Proposition.** *Every Hilbert system is WAP, i.e. $H(\mathbb{Z}) \subset WAP(\mathbb{Z})$.*

Proof. It is well known (and easy to see) that $B(\mathbb{Z}) \subset WAP(\mathbb{Z})$. Since the algebra $WAP(\mathbb{Z}) \subset \ell_\infty(\mathbb{Z})$ is closed we also have $H(\mathbb{Z}) \subset WAP(\mathbb{Z})$. ◻

3. A structure theorem for Hilbert systems

3.1. **Theorem.** *Every metrizable recurrent-transitive Hilbert system (Y, T) admits an almost 1-1 extension $(\tilde{Y}, \tilde{T})$ which is a compact group-factor of a recurrent-transitive Hilbert-representable system (X, U). Thus there exists a commutative diagram*

$$\begin{tikzcd}
(X,U) \arrow[dr, "\sigma"] \arrow[dd, "\pi"'] & \\
 & (\tilde{Y},\tilde{T}) \arrow[dl, "\rho"] \\
(Y,T) &
\end{tikzcd} \tag{3.1}$$

with σ a group-extension and ρ an almost 1-1 extension.

Proof. 1. Let (Y, T) be a metrizable recurrent-transitive Hilbert system. Thus, picking a transitive point $y_0 \in Y$ we have $\mathcal{A}(Y, y_0) \subset H(\mathbb{Z})$. Since every function in $B(\mathbb{Z})$ is a linear combination of four positive definite functions, we can find a sequence f_n of positive definite functions (with $f_1 \equiv \mathbf{1}$ and $f_n(0) = \|f_n\|_\infty = 1$ $(n = 1, 2, \dots)$ such that $\mathcal{A}(\{f_n : n \in \mathbb{N}\})$, the closed translation and conjugation invariant algebra generated by the functions f_n, contains $\mathcal{A}(Y, y_0)$. For each n there is a separable Hilbert space $\mathcal{H}_n$, a unit vector $x_n \in \mathcal{H}_n$, and a unitary operator $U_n \in \mathcal{U}(\mathcal{H}_n)$ such that $f_n(k) = \langle U_n^k x_n, x_n \rangle$ $(\forall k \in \mathbb{Z})$. Let $\mathcal{H} = \oplus_{n=1}^\infty \mathcal{H}_n$ (an ℓ_2-sum), $x_0 = \oplus_{n=1}^\infty x_n \in \mathcal{H}$, and $U = \oplus_{n=1}^\infty U_n \in \mathcal{U}(\mathcal{H})$. Set $X = \text{weak-cls}\,\{U^k x_0 : k \in \mathbb{Z}\}$. Then it is easy to verify that the Hilbert-representable system (X, U) admits (Y, T) as a factor, say $\pi : (X, U) \to (Y, T)$.

2. Let $\mathcal{X}$ be the collection of all subsystems (X', U) of (X, U) with $\pi(X') = Y$. By Zorn's lemma there is a minimal element (Z, U) in $\mathcal{X}$. Clearly (Z, U) is point-transitive Hilbert-representable system and, for convenience, we now rename it as (X, U). Also, let us rename $\mathcal{H} = Z_U(x_0)$, the cyclic space generated by $\{U^n x_0 : n \in \mathbb{Z}\}$, where x_0 is now some transitive point in X.

3. Let $X_{tr} \subset X$ be the dense G_δ subset of the transitive points in X. Similarly let $Y_{tr} \subset X$ be the dense G_δ subset of the transitive points in Y. Clearly $\pi(X_{tr}) \subset Y_{tr}$ and we claim that these sets are equal. Suppose $y_1 \in Y_{tr}$. Let $x_1 \in X$ be some point with $\pi(x_1) = y_1$. By assumption y_1 is a transitive point and there is a sequence $T^{n_i} y_1 \to y_0$. We can assume that also $U^{n_i} x_1 \to x_0'$ for some point $x_0' \in X$ with $\pi(x_0') = y_0$. It follows that $\pi(\bar{\mathcal{O}}_U(x_0')) = Y$ and therefore, by minimality, $\bar{\mathcal{O}}_U(x_0') = X$. Thus x_0' is a transitive point and hence so is x_1 hence $y_1 = \pi(x_1) \notin X_{tr}$. We conclude that indeed $\pi(X_{tr}) = Y_{tr}$.

4. Next recall that with every recurrent-transitive metrizable WAP system (W, R) there is an associated non-discrete Polish monothetic group $\Lambda_W = \text{unif-cls}\{R^n : n \in \mathbb{Z}\} \subset \text{Homeo}\,(W)$. Moreover, for every transitive point $w_0 \in W$, the map $g \mapsto g w_0$ is a homeomorphism from Λ_W onto the set W_{tr} of transitive points of W, see [15]. Now using this observations and with notations as in the previous step, we note that the surjection $\pi : X_{tr} \to Y_{tr}$ defines a surjective Polish group homomorphism $\rho : \Lambda_X \to \Lambda_Y$. Moreover, we note that $\ker \rho = K$ is a compact subgroup of Λ_X.

5. Set $\tilde{Y} = X/K$ and let $\sigma : (X, U) \to (\tilde{Y}, \tilde{T})$ denote the corresponding group homomorphism. We now obtained the diagram (3.1). ∎

4. A Polish monothetic group P which is reflexive but not Hilbert-representable

We will use Banaszczyk's construction of some families of Polish monothetic strongly exotic groups [2]. For our purposes it suffices to consider one particular such example which we now proceed to describe. Let $E = \ell_4(\mathbb{N})$ be the Banach space comprising the real valued sequences $u = \{x_j\}_{j=1}^\infty$ with $|u| := (\sum_{j=1}^\infty x_j^4)^{1/4} < \infty$.

Let $\{e_n\}_{n=1}^\infty$ be the standard basis of unit vectors and choose a dense sequence $\{a_n\}_{n=1}^\infty \subset E$ with the property $a_n \in \text{span}\{e_j\}_{j<n}$ for every $n \geq 1$. Let Γ be the subgroup of E generated by the sequence $\{e_n + a_n\}_{n=1}^\infty$. It can be easily checked that $|\gamma| \geq 1$ for every $0 \neq \gamma \in \Gamma$; and that, on the other hand, $E = \Gamma + 2B_1(0)$, where $B_1(0)$ denotes the unit ball in E. Set $P = E/\Gamma$ and equip P with its quotient topology. It follows that P is a Polish topological group. By Theorem 5.1 and Remark 5.2 in [2] P is strongly exotic and monothetic. In Remark 5.2 Banaszczyk leaves the proof of the fact that P is monothetic as an exercise; for completeness we provide a proof of this fact at the end of this section, Proposition 4.3.

4.1. **Theorem.** *The Polish monothetic group $P = \ell_4(\mathbb{N})/\Gamma$ is reflexively-representable.*

Proof. We will follow the ideas of Megrelishvili's proof that the group $L_4([0,1])$ is reflexively-representable ([18], Lemmas 3.3 and 3.4). Note that Megrelishvili's proof works almost verbatim to show that $\ell_4(\mathbb{N})$ is reflexively-representable.

We first define an invariant metric on P by

$$d(u+\Gamma,0)=\|u+\Gamma\|=\inf_{\gamma\in\Gamma}|u+\gamma|,\qquad u\in E.$$

Note that for $u\in E$ we have

$$|u|\le\frac{1}{2}\Rightarrow\|u+\Gamma\|=|u|. \tag{4.1}$$

Next define the following continuous function on P:

$$F(u+\Gamma)=\min\{\|u+\Gamma\|,\frac{1}{10}\}.$$

We will show that F satisfies Grothendieck's *double limit condition*, i.e. we have to show that given two sequences $\{u_m+\Gamma\}_{m=1}^{\infty},\{v_n+\Gamma\}_{n=1}^{\infty}$ in P, if the limits

$$a=\lim_m\lim_n F(u_m+v_n+\Gamma)\quad\text{and}\quad b=\lim_n\lim_m F(u_m+v_n+\Gamma), \tag{4.2}$$

exist, then necessarily $a=b$.

Of course when $a=b=\frac{1}{10}$ we are done; so we now assume that $a<\frac{1}{10}$. Then for some $\delta>0$, eventually

$$a_m:=\lim_{n\to\infty}F(u_m+v_n+\Gamma)=\lim_{n\to\infty}\|u_m+v_n+\Gamma\|<\frac{1}{10}-2\delta.$$

Thus for some m_0 we have $a_m\le\frac{1}{10}-2\delta$ for every $m\ge m_0$; and for a fixed $m\ge m_0$ there is an $n(m)$ such that for $n\ge n(m)$ we have $\|u_m+v_n+\Gamma\|\le\frac{1}{10}-\delta$.

Now for $n\ge\max\{n(m),n(m_0)\}$ we have

$$\|u_m+v_n+\Gamma\|\le\frac{1}{10}-\delta\quad\text{and}\quad\|u_{m_0}+v_n+\Gamma\|\le\frac{1}{10}-\delta,$$

hence $\|u_m-u_{m_0}+\Gamma\|\le\frac{2}{10}$. Thus by an appropriate choice of representatives we can now assume that $\|u_m-u_{m_0}+\Gamma\|=|u_m-u_{m_0}|\le\frac{2}{10}$ for all $m\ge m_0$. By the same token, for $n\ge n(m_0)$ we can assume that also $\|u_{m_0}+v_n+\Gamma\|=|u_{m_0}+v_n|\le\frac{2}{10}-\delta$. This means that for sufficiently large m and n all the vectors u_m and $-v_n$ lie in a ball of radius $\frac{2}{10}$ around u_{m_0}. By the triangle inequality and (4.1)

$$\|u_m+v_n+\Gamma\|=|u_m+v_n|\le\frac{4}{10}-\delta,$$

for all $m\ge m_0$ and $n\ge n(m_0)$.

Now in E we have:

$$|u_m+v_n|^4=\sum_{i=1}^{\infty}u_m(i)^4+4u_m(i)^3v_n(i)+6u_m(i)^2v_n^2(i)+4u_m(i)v_n(i)^3+v_n(i)^4.$$

As all these vectors lie in a ball we can assume, by passing to subsequences, that

$$\lim_{n\to\infty}v_n=v\quad\text{weakly in }\ell_4$$
$$\lim_{n\to\infty}v_n^2=v^2\quad\text{weakly in }\ell_2$$
$$\lim_{n\to\infty}v_n^3=v^3\quad\text{weakly in }\ell_{\frac{4}{3}},$$

whence

$$a_m = |u_m|^4 + 4\langle u_m^3, v\rangle + 6\langle u_m^2, v^2\rangle + 4\langle u_m, v^3\rangle + V^4,$$

where $V = \lim_{n\to\infty} |v_n|$.

Again we can assume that also

$$\begin{aligned}
\lim_{m\to\infty} u_m = u \quad &\text{weakly in } \ell_4\\
\lim_{m\to\infty} u_n^2 = u^2 \quad &\text{weakly in } \ell_2\\
\lim_{m\to\infty} u_n^3 = u^3 \quad &\text{weakly in } \ell_{\frac{4}{3}}.
\end{aligned}$$

We now get

$$a = \lim_{m\to\infty} a_m = U^4 + 4\langle u^3, v\rangle + 6\langle u^2, v^2\rangle + 4\langle u, v^3\rangle + V^4,$$

where $U = \lim_{n\to\infty} |u_n|$.

Computing the double limit the other way we have similarly

$$b = U^4 + 4\langle u^3, v\rangle + 6\langle u^2, v^2\rangle + 4\langle u, v^3\rangle + V^4,$$

whence $a = b$, as required.

We have thus shown that the function F satisfies the double limit condition. Now observe that, by the same computations, for any fixed $v \in E$ the function $F_v(u) = F(u - v)$ also satisfies the double limit property. By the double limit criterion 1.1 we conclude that the family $F_v, v \in E$, is contained in $WAP(P)$. As clearly the collection $\{F_v\}_{v\in E}$ generates the topology on P it follows that this family separates points and closed sets on P and we conclude from Theorem 2.3.(1), as in [18], that the group P is indeed reflexively-representable. □

4.2. **Theorem.** *There is a Polish monothetic group P which admits a faithful representation on a reflexive Banach space but is not Hilbert-representable (in fact it is strongly exotic).*

Proof. Let $P = \ell_4(\mathbb{N})/\Gamma$ be the strongly exotic Polish monothetic group described by Banaszczyk [2]. By Theorem 4.1 P is reflexively-representable and, being strongly exotic, it is not Hilbert-representable. □

4.3. **Proposition.** *The group P is monothetic.*

Proof. Let $\pi_k : \ell_4(\mathbb{N}) \to \ell_4^k := E_k$ be the projection map onto the first k coordinates. Let $\Gamma_k = \pi_k(\Gamma)$ and $P_k = E_k/\Gamma_k$. Note that for each $k \geq 1$ the group P_k is a k-dimensional torus and by Kronecker's theorem the vectors $z = (z_1, \dots, z_k) \in E_k$ such that $\{nz + \Gamma_k : n \in \mathbb{Z}\}$ is dense in P_k, form a dense subset of E_k. We will construct, by induction, a Cauchy sequence $\{z^{(k)} : k \in \mathbb{N}\}$ in $\ell_4(\mathbb{N})$ whose limit $x = (x_1, x_2, \dots) \in \ell_4(\mathbb{N})$ will have the property that $\{nx + \Gamma : n \in \mathbb{Z}\}$ is a dense subgroup of P.

Suppose we have already chosen $w^{(k)} = (z_1^{(k)}, z_2^{(k)}, \dots, z_k^{(k)}) \in E_k$ with the property that for some positive integer N_k

$$\{nw^{(k)} + \Gamma_k : |n| \leq N_k\}, \tag{4.3}$$

is $\frac{1}{k}$-dense in P_k. We set

$$z^{(k)} = (z_1^{(k)}, z_2^{(k)}, \dots, z_k^{(k)}, 0, 0, \dots) \in \ell_4(\mathbb{N}).$$

There exists an $\epsilon_k > 0$ such that the condition (4.3) (with $\pi_k(z)$ instead of $w^{(k)}$) is satisfied for every z in an ϵ_k-ball around $z^{(k)}$. As we have seen there is then a vector $w^{(k+1)} = (z_1^{(k+1)}, z_2^{(k+1)}, \dots, z_{k+1}^{(k+1)}) \in E_{k+1}$ and a positive integer N_{k+1} such that the set $\{nw^{(k+1)} + \Gamma_{k+1} : |n| \le N_{k+1}\}$ is $\frac{1}{k+1}$-dense in P_{k+1}, and such that for $z^{(k+1)} = (z_1^{(k+1)}, z_2^{(k+1)}, \dots, z_k^{(k+1)}, 0, 0, \dots) \in \ell_4(\mathbb{N})$ we have, $|z^{(k)} - z^{(k+1)}| < \min\{2^{-k}, \epsilon_k\}$. This completes the inductive construction and clearly the unique limit point $x = \lim_{k\to\infty} z^{(k)}$ is the required one. ☐

4.4. **Remark.** Consider an arbitrary jointly continuous action of the group P on a compact space X. Clearly the system (X, T), where $T \in P$ is a topological generator, admits a T-invariant probability measure, say μ. It follow that μ is P-invariant as well. Let $g \mapsto U_g$ be the corresponding Koopman representation of P on $L_2(X, \mu)$. As P is exotic, this unitary representation is a trivial representation. This of course means that μ is supported on the set of P-fixed points in X. What we have shown here, is that the group P has the *fixed point on compacta* property, i.e. every jointly continuous action of P on a compact space admits a fixed point. This property is also called *extreme amenability.*

5. A WAP recurrent-transitive system which is not Hilbert

5.1. **Theorem.** *The Banaszczyk group $P = \ell_4/\Gamma$ admits no nontrivial Hilbert system. Thus any nontrivial WAP action of P provides an example of a recurrent-transitive WAP system which is not Hilbert.*

Proof. 1. Let $T \in P$ be a generator of a dense cyclic subgroup in P. Suppose that P admits a nontrivial point-transitive Hilbert system (Y, y_0, P) and consider the $\mathbb{Z}$-system (Y, y_0, T). Since P is exotic it is in particular MAP (= minimally almost periodic, i.e. it admits no nontrivial unitary finite dimensional representations) and it follows that y_0 is a non-periodic, recurrent point of (Y, T).

Let (X, x_0, U) with x_0 a unit vector in a Hilbert space $\mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$ a unitary operator on $\mathcal{H}$, and $X = w\text{-cls}\,\{U^n x_0 : n \in \mathbb{Z}\}$, be the extension provided by Step 5 of Theorem 3.1 (diagram (3.1) above).

Let $G_X = \mathrm{cls}\,\{U^n : n \in \mathbb{Z}\}$, where the closure is taken in the strong operator topology on $\mathcal{U}(\mathcal{H})$. It is not hard to see that there is a canonical topological isomorphism between the groups Λ_X and G_X. There are also canonical topological isomorphisms $\Lambda_X/K \cong \Lambda_Y \cong \Lambda_{\tilde{Y}}$. Note that our assumption that (Y, T) extends to a P-system means that the map $n \mapsto T^n$ extends to a continuous homomorphism $P \to \Lambda_Y$ with a dense image (see Step 4 of the proof of Theorem 3.1).

By Proposition 2.6 we have an isomorphism $(X, U) \cong (X_f, S)$ where f is the positive definite function $f(n) = \langle U^n x_0, x_0\rangle$. Now let $(\Omega, \mathcal{F}, \mu, R)$ be the Gauss dynamical system which corresponds to f; thus there is a $\mathbb{Z}$-sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ of real valued random variables defined on Ω with joint Gauss distribution such that $\xi_n = \xi_0 \circ R^n$ with $\mathbb{E}(\xi_n \xi_m) = f(n - m)$, and such that the σ-algebra generated by the sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ is $\mathcal{F}$. (For more details on Gauss dynamical systems see e.g. [3] and [17].)

Let $\mathcal{H}_0 \subset L_2(\Omega, \mu)$ be the closed subspace spanned by $\{\xi_n\}_{n\in\mathbb{Z}}$ (the first Wiener chaos). Then $\mathcal{H}_0$ is isomorphic, as a Hilbert space, to $\mathcal{H}$ (which is the U-cyclic space spanned by x_0). In particular we can think of Λ_X as a subgroup of $\mathcal{U}_0(\mathcal{H}_0)$ where it commutes with U_R, the Koopman operator on $L_2(\Omega, \mu)$ which corresponds to R.

It follows that Λ_X can be realized a a group of measure preserving transformations (i.e. a subgroup of $\mathrm{Aut}\,(\Omega,\mu)$; see e.g. [17]). In particular we can now view K as a compact subgroup of $\mathrm{Aut}\,(\Omega,\mu)$ and then define the compact group-factor map $\pi_K : (\Omega, \mathcal{F}, \mu, R) \to (\Omega/K, \mathcal{F}/K, \mu_K, R_K)$.

Finally considering $L_2(\Omega/K, \mu_K)$ we see that the Polish group Λ_X/K is faithfully represented on this Hilbert space. Since we have a continuous homomorphism $P \to \Lambda_Y \cong \Lambda_X/K$ with a dense image, this contradicts the fact that P is exotic and our proof is complete. □

5.2. **Corollary.** *There exist a recurrent-transitive WAP function $f \in WAP(\mathbb{Z}) \setminus H(\mathbb{Z})$.*

Proof. Let P be the Polish monothetic exotic Banaszczyk group. Let $T \in P$ be a generator of a dense cyclic subgroup in P. As we have seen in Theorem 4.1, the functions $\{F_v\}_{v\in E}$ are in $WAP(P)$ and they separate points and closed sets on P. Therefore their restrictions $f_v = F_v \upharpoonright \mathbb{Z}$, to the dense subgroup $\{T^n : n \in \mathbb{Z}\} \subset P$, are in $WAP(\mathbb{Z})$. As elements of $WAP(P)$ the functions F_v are recurrent and hence so are the functions f_v, $v \in E$. Finally by Theorem 5.1 each f_v is an element of $WAP(\mathbb{Z}) \setminus H(\mathbb{Z})$. □

5.3. **Corollary.** *If a Polish monothetic group P is Hilbert-representable and if $K \subset P$ is a compact subgroup, then the quotient group P/K is Hilbert-representable.*

Proof. Follow the last steps of the proof of Theorem 5.1 with P taking the place of Λ_X. □

6. Recurrent functions in $H(\mathbb{Z}) \setminus B(\mathbb{Z})$

6.1. **Lemma.** *For every recurrent $f \in WAP(\mathbb{Z})$ the system (X_f, S) is uniformly rigid and the $\mathbb{Z}$-algebra $\mathcal{A}_f$ consists entirely of recurrent functions.*

Proof. The dynamical system $X_f = \mathrm{cls}\,\{S^n f : n \in \mathbb{Z}\}$ (in the pointwise convergence topology) is WAP and therefore uniformly rigid. It then follows that the non-discrete Polish monothetic group $\Lambda_{X_f} = \mathrm{cls}\,\{S^n : n \in \mathbb{Z}\}$ (in the uniform convergence topology on $\mathrm{Homeo}\,(X_f)$) acts on X_f as a group of automorphisms. In turn this implies that every function $g \in \mathcal{A}(X_f, f) = \mathcal{A}_f$ is a recurrent point: $g = \lim_{k\to\infty} S^{n_k} g$, for every sequence $n_k \nearrow \infty$ with S^{n_k} tending to the identity in Λ_{X_f}. □

6.2. **Lemma.** *If $\mathcal{A}$ is an infinite-dimensional $\mathbb{Z}$-subalgebra of $H(\mathbb{Z}) = \overline{B(\mathbb{Z})}$ then $\mathcal{A} \setminus B(\mathbb{Z})$ is a (norm) dense Borel subset of $\mathcal{A}$.*

Proof. The Fourier transform $\mathcal{F} : M(\mathbb{T}) \twoheadrightarrow B(\mathbb{Z})$ is a 1-1 onto bounded linear map ($\|\mathcal{F}(\mu)\|_\infty \le \|\mu\|$). If $\mathcal{A} \subset \mathcal{F}(M(\mathbb{T})) = B(\mathbb{Z})$ then $\mathcal{F}^{-1}\mathcal{A} \subset M(\mathbb{T})$ is an infinite-dimensional C^*-algebra, which is impossible (see e.g. Theorem 2.1 in [5]). Thus $\mathcal{A} \subsetneq B(\mathbb{Z})$. Moreover, if the algebra $\mathcal{A} \cap B(\mathbb{Z})$ contains a nonempty $\mathcal{A}$-norm-open subset then it must coincide with $\mathcal{A}$ which again is impossible. Thus $\mathcal{A} \setminus B(\mathbb{Z})$ is norm-dense in $\mathcal{A}$.

Finally the algebra $B(\mathbb{Z})$, being a 1-1 continuous image of $M(\mathbb{T})$ (under $\mathcal{F}$), is a Borel subset of $H(\mathbb{Z}) = \overline{B(\mathbb{Z})}$, and hence so are the sets $\mathcal{A} \cap B(\mathbb{Z})$ and $\mathcal{A} \setminus B(\mathbb{Z})$. □

6.3. **Corollary.** *In every infinite dimensional $\mathbb{Z}$-subalgebra $\mathcal{A} \subset H(\mathbb{Z})$ the set $\mathcal{A}_{rec} \setminus B(\mathbb{Z})$, where $\mathcal{A}_{rec}$ is the collection of recurrent functions in $\mathcal{A}$, is a (norm) dense subset of $\mathcal{A}_{rec}$. In particular there are recurrent functions in $H(\mathbb{Z})$ which are not Fourier-Stieltjes transforms (i.e. are not in $B(\mathbb{Z})$).*

Proof. Combine Lemmas 6.1 and 6.2. □

7. A structure theorem for weakly almost periodic systems

7.1. **Theorem.** *Let (X,T) be a metrizable recurrent-transitive WAP dynamical system and $\pi : (X,T) \to (Y,S)$ a factor. Then there is an almost 1-1 extension which is a compact group-factor of a recurrent-transitive subsystem $Z \subset X$. More explicitly there is a commutative diagram*

$$\begin{tikzcd}
(Z,T) \arrow[dd, "\pi"'] \arrow[dr, "\sigma"] & \\
& (\tilde{Y},\tilde{T}) \arrow[dl, "\rho"] \\
(Y,S) &
\end{tikzcd}$$

with $Z \subset X$ a subsystem, σ a group-extension and ρ an almost 1-1 extension.

Proof. Just repeat the steps 2. to 5. of the proof of Theorem 3.1 above. □

8. Some open problems

8.1. **Problem.** In Theorem 3.1 we have shown that every metrizable recurrent-transitive Hilbert system admits an almost 1-1 extension which is a group-factor of a Hilbert-representable system. Can one get rid in this structure theorem of either one of these extensions or maybe of both? Thus, our question is whether a metrizable recurrent-transitive Hilbert system is always: (a) Hilbert-representable, or (b) a group-factor of a Hilbert-representable system, or (c) an almost 1-1 factor of a Hilbert-representable system. Question (a) can be reformulated as follows: Is a factor of a Hilbert-representable system also Hilbert-representable? This latter question is stated as Problem 996 in [21].

8.2. **Problem.** Is there a metrizable recurrent-transitive Hilbert system (X,T) with no nontrivial Hilbert-representable factors? In other words, is there a nontrivial $\mathbb{Z}$-algebra $\mathcal{A} \subset H(\mathbb{Z})$ with $\mathcal{A} \cap B(\mathbb{Z}) = \mathbb{C}$? Such a system will provide a counter-example to option (a) in Problem 8.1.

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