EVERY COUNTABLE GROUP HAS THE WEAK ROHLIN PROPERTY

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Abstract. We present a simple proof of the fact that every countable group Γ is weak Rohlin, that is, there is in the Polish space $\mathbb{A}_\Gamma$ of measure preserving Γ-actions an action $\mathbf{T}$ whose orbit in $\mathbb{A}_\Gamma$ under conjugations is dense. In conjunction with earlier results this in turn yields a new characterization of non-Kazhdan groups as those groups which admit such an action $\mathbf{T}$ which is also ergodic.

1. Introduction

Let $(X, \mathcal{X}, \mu)$ be a standard Lebesgue space, μ a probability measure with no atoms. As usual we denote the group of (classes) of measure preserving transformations on X by $G = \mathrm{Aut}\,(X)$.

Let Γ be a discrete countable infinite group. We denote by $\mathbb{A}_\Gamma$ the collection of Γ measure preserving actions on X. Thus an element $\mathbf{T} \in \mathbb{A}_\Gamma$ is a representation $\mathbf{T} : \Gamma \to G$, $\gamma \mapsto \mathbf{T}_\gamma$.

A countable algebra $\{A_k\}_{k=1}^\infty$ of sets, which separates points of X, gives rise to a metric which induces the coarse topology on G,

$$d(S,T) = \sum_{k=1}^{\infty} 2^{-k} \left(\mu(SA_k \triangle TA_k) + \mu(S^{-1}A_k \triangle T^{-1}A_k) \right).$$

With this metric G is a Polish topological group. Similarly the space $\mathbb{A}_\Gamma$ of Γ-actions is equipped with the metric

$$D(\mathbf{S}, \mathbf{T}) = \sum_{i=1}^{\infty} 2^{-i} d(\mathbf{S}_{\gamma_i}, \mathbf{T}_{\gamma_i}),$$

where $\{\gamma_i : i = 1, 2, \dots\}$ is some enumeration of Γ. Again with this metric $\mathbb{A}_\Gamma$ is a Polish space.

We say that two Γ-actions $\mathbf{S}$ and $\mathbf{T}$ are *isomorphic* if there is $R \in G$ such that $\mathbf{T}_\gamma = R\mathbf{S}_\gamma R^{-1}$ for every $\gamma \in \Gamma$. In other words, if and only if $\mathbf{S}$ and $\mathbf{T}$ belong to the same orbit of the natural action of the Polish group G on the Polish space $\mathbb{A}_\Gamma$ by conjugation. We say that the group Γ has the *weak Rohlin property* if this action of G on $\mathbb{A}_\Gamma$ is topologically transitive; i.e. for any two nonempty open sets U and V in $\mathbb{A}_\Gamma$ there is some $R \in G$ such that $RUR^{-1} \cap V \neq \emptyset$. An equivalent condition is that there is a dense G_δ subset $\mathcal{A}_0$ of $\mathbb{A}_\Gamma$ such that for every $\mathbf{T} \in \mathcal{A}_0$ the G-orbit, $\{R\mathbf{T}R^{-1} : R \in G\}$ is dense in $\mathbb{A}_\Gamma$.

Date: December 4, 2004.

Key words and phrases. weak Rohlin property, Kazhdan group.

2000 Mathematical Subject Classification: 28D15, 22D10.

This notion was introduced by Glasner and King in [2]. The results of [2] apply to groups having the weak Rohlin property and the question as to which groups have that property was left open. In this note we prove the following two theorems.

1.1. **Theorem.** *Every infinite countable group* Γ *has the weak Rohlin property.*

1.2. **Theorem.** *The infinite countable group* Γ *admits an ergodic action* $\mathbf{T} \in \mathbb{A}_\Gamma$ *whose* G*-orbit* $\{R\mathbf{T}R^{-1} : R \in G\}$ *is dense in* $\mathbb{A}_\Gamma$ *if and only if* Γ *does not have the Kazhdan property. Thus for a non-Kazhdan group the set of ergodic actions* $\mathbf{T} \in \mathbb{A}_\Gamma$ *with a dense* G*-orbit is a dense* G_δ*, while for a Kazhdan group* Γ*, the set of ergodic actions forms a meager subset of* $\mathbb{A}_\Gamma$ *and for every ergodic* $\mathbf{T} \in \mathbb{A}_\Gamma$*,* $\operatorname{cls}\{R\mathbf{T}R^{-1} : R \in G\}$ *has an empty interior in* $\mathbb{A}_\Gamma$*.*

We refer to the papers [2] and [3] for further background and motivation.

E. Glasner and J.-P. Thouvenot thank the Max Planck Institute of Mathematics in Bonn, Germany, for its hospitality during the conference on dynamical systems held there in the summer of 2004, where both authors spent a fruitful month and where an exchange of ideas between them started this work.

2. Proofs

Proof of Theorem 1.1. Let $\{\mathbf{T}^{(n)} : n \in \mathbb{N}\}$ be a dense sequence in $\mathbb{A}_\Gamma$. Let $\Omega = \prod_{n=1}^\infty X_n = X^{\mathbb{N}}$, where each X_n is a copy of X, and let $\lambda = \prod_{n=1}^\infty \mu = \mu^{\mathbb{N}}$ be the product measure on Ω. Define the diagonal Γ-action $\mathbf{D}$ on Ω by

$$\mathbf{D}_\gamma(x_1, x_2, \dots) = (\mathbf{T}^{(1)}_\gamma x_1, \mathbf{T}^{(2)}_\gamma x_2, \dots).$$

Let $\phi : (X, \mu) \to (\Omega, \lambda)$ be a fixed isomorphism of measure spaces and set

$$\mathbf{T} = \phi^{-1} \circ \mathbf{D} \circ \phi \in \mathbb{A}_\Gamma.$$

We claim that $\{R\mathbf{T}R^{-1} : R \in G\}$ is dense in $\mathbb{A}_\Gamma$.

To see this consider a fixed index $n_0 \in \mathbb{N}$. Given a finite measurable partition $\mathcal{P} = \{P_j\}_{j=1}^N$ of $X_{n_0} = X$, set $Q_j = P_j \times \prod_{n \neq n_0} X_n \subset \Omega$ and $\mathcal{Q} = \{Q_j\}_{j=1}^N$. Then, given a finite set $F \subset \Gamma$ observe that the two finite algebras,

$$\bigvee_{\gamma \in F} \mathbf{T}^{(n_0)}_\gamma \mathcal{P}, \quad \text{and} \quad \bigvee_{\gamma \in F} \mathbf{D}_\gamma \mathcal{Q}$$

are clearly equivariantly isomorphic. We can therefore choose a measurable isomorphism $\psi : (X, \mu) \to (\Omega, \lambda)$ such that

$$\psi(\mathbf{T}^{(n_0)}_\gamma P_j) = \mathbf{D}_\gamma Q_j, \quad \text{for} \ \ 0 \le j \le N, \ \gamma \in F.$$

Set $R = \psi^{-1}\phi$, an element of $G = \operatorname{Aut}(X)$. Then for each $\gamma \in \Gamma$,

$$R \circ \mathbf{T}_\gamma \circ R^{-1}(P_j) = \psi^{-1}(\mathbf{D}_\gamma Q_j) = \psi^{-1}\big(\mathbf{T}^{(n_0)}_\gamma P_j \times \prod_{n \neq n_0} X_n\big),$$

and for $\gamma \in F$

$$\psi^{-1}\big(\mathbf{D}_\gamma Q_j\big) = \mathbf{T}^{(n_0)}_\gamma P_j.$$

Therefore

$$R \circ \mathbf{T}_\gamma \circ R^{-1}(P_j) = \mathbf{T}_\gamma^{(n_0)} P_j, \quad \text{for} \;\; 0 \le j \le N, \; \gamma \in F.$$

Since this procedure can be followed for any finite partition $\{P_1, \dots, P_N\}$ of X, we conclude that $\mathbf{T}^{(n_0)}$ is in the closure of the G-orbit of $\mathbf{T}$. Since $\{\mathbf{T}^{(n)} : n \in \mathbb{N}\}$ is dense in $\mathbb{A}_\Gamma$, it follows that the G-orbit of $\mathbf{T}$ is in fact dense in $\mathbb{A}_\Gamma$. □

Proof of Theorem 1.2. Let $\Omega = \mathbb{T}^\Gamma$, where $\mathbb{T}$ is the circle (or 1-torus). We consider the topological Bernoulli dynamical system (Ω, Γ), where Γ acts by permutations of the coordinates. By [3, Theorem 2] [1] the set $M_\Gamma^{\mathrm{erg}}(\Omega)$ of ergodic measures in the Choquet simplex $M_\Gamma(\Omega)$ of Γ-invariant probability measures on Ω is either closed or dense in $M_\Gamma(\Omega)$, according to whether Γ is a Kazhdan group or not. Since the set of ergodic measures is always a G_δ subset of $M_\Gamma(\Omega)$, we see that in the second case $M_\Gamma^{\mathrm{erg}}(\Omega)$ is a dense G_δ subset of $M_\Gamma(\Omega)$. On the other hand in the first case, when Γ is a Kazhdan group, the closed set $M_\Gamma^{\mathrm{erg}}(\Omega)$ has empty interior and is therefore a meager subset of $M_\Gamma(\Omega)$.

Next invoke the "equivalence theorem" [2, Theorem 7] (see also [5]) to deduce that the same dichotomy holds for the collection of ergodic elements of $\mathbb{A}_\Gamma$. Namely, this collection is a dense G_δ subset of $\mathbb{A}_\Gamma$ when Γ is not Kazhdan and it is a meager subset of $\mathbb{A}_\Gamma$ when Γ is Kazhdan.

By Theorem 1.1, for a non-Kazhdan group Γ the collection of G-transitive points in $\mathbb{A}_\Gamma$ is also a dense G_δ subset and it follows that there is a dense G_δ subset of actions which are both ergodic and have dense G-orbit in $\mathbb{A}_\Gamma$.

Finally, if Γ is Kazhdan and $\mathbf{T} \in \mathbb{A}_\Gamma$ is ergodic then $\mathrm{cls}\,\{R\mathbf{T}R^{-1} : R \in G\}$ has an empty interior. In fact, if this interior V is non-empty, then by Theorem 1.1 there is an element $\mathbf{S} \in V$ whose G-orbit is dense. This implies that $\mathrm{cls}\,\{R\mathbf{T}R^{-1} : R \in G\} = \mathbb{A}_\Gamma$. Since the set of ergodic actions is G-invariant and G_δ, we conclude that it is a dense G_δ subset of $\mathbb{A}_\Gamma$, a contradiction. □

3. Some applications and extensions

In this section we suggest some possible applications and extensions of the above ideas.

1. Consider the free group $\Gamma = F_2$ on two generators a and b. Let

$$\begin{aligned}\mathfrak{L} = \{\mathbf{T} \in \mathbb{A}_\Gamma : \forall \epsilon > 0, \; \exists\, B \subset X, \; & \text{such that the sets } a^{\pm 1}B, b^{\pm 1}B \\ & \text{are disjoint and } \mu(aB \cup a^{-1}B \cup bB \cup b^{-1}B) > 1 - \epsilon\}.\end{aligned}$$

[1] The results in [3] are proven for the Bernoulli system (Ω, Γ) where $\Omega = \{0,1\}^\Gamma$ rather than $\mathbb{T}^\Gamma$; however it is not hard to check that the same results hold for the Bernoulli system $(\mathbb{T}^\Gamma, \Gamma)$.

It is easy to check that the set

$$\mathcal{L}_\epsilon = \{\mathbf{T} \in \mathbb{A}_\Gamma : \exists\, B \subset X, \text{ such that the sets } a^{\pm 1}B, b^{\pm 1}B \text{ are disjoint and } \mu(aB \cup a^{-1}B \cup bB \cup b^{-1}B) > 1-\epsilon\}.$$

is an open G-invariant subset of $\mathbb{A}_\Gamma$. To see that it is non-empty we proceed as follows.

First recall that a set T *tiles* the group Γ if there is a set $C \subset \Gamma$, whose elements are called the *tiling centers*, such that the collection $\{Tc : c \in C\}$ is a partition of Γ. The following theorem is proved in [4, page 58] (see also [6, Theorem 3.3]).

3.1. **Theorem.** *Let Γ be a countable amenable group acting freely on the measure space $(X, \mathcal{B}, \mu)$ preserving the measure μ. Let $T \subset \Gamma$ be a finite set that tiles Γ. Then, for every $\epsilon > 0$ there is a $B \in \mathcal{B}$ such that*

1. *the sets $\{tB : t \in T\}$ are disjoint, and*
2. *$\mu(\bigcup_{t\in T} tB) > 1 - \epsilon$.*

Now, since $\mathbb{Z}^2$ is a homomorphic image of F_2 it suffices to find an action of $\mathbb{Z}^2$ that has the property defining $\mathcal{L}_\epsilon$. The set $T = \{a^{\pm 1}, b^{\pm 1}\}$ tiles $\mathbb{Z}^2$; explicitly, writing $(\pm 1, 0)$ for $a^{\pm 1}$, $(0, \pm 1)$ for $b^{\pm 1}$, the set

$$C = \{(4k, 4m), (4k, 4m+1), (4k+2, 4m+2), (4k+2, 4m+3) : k, m \in \mathbb{Z}\}$$

provides a set of centers for such a tiling. It follows from the Rohlin lemma for tilings, Theorem 3.1, that any free ergodic $\mathbb{Z}^2$-action satisfies $\mathcal{L}_\epsilon$ and a fortiori $\mathcal{L}_\epsilon$ is non-empty.

By Theorem 1.1 $\mathcal{L}_\epsilon$ is a dense open set for every $\epsilon > 0$. Therefore we conclude that the set $\mathcal{L} = \cap_n \mathcal{L}_{1/n}$ is a dense G_δ subset of $\mathbb{A}_{F_2}$. Since F_2 is non-Kazhdan, by Theorem 1.2, the collection $\mathcal{R}$ of ergodic G-transitive actions is a dense G_δ subset of $\mathbb{A}_\Gamma$ and therefore so is $\mathcal{L} \cap \mathcal{R}$.

Of course this kind of "generic Rohlin lemma" can be obtained in many similar situations. In fact the next application provides a large family of such instances.

2. Let Γ be a residually finite countable group. This means that for every $e \neq \gamma \in \Gamma$ there exists a normal subgroup $N \lhd \Gamma$ such that $[\Gamma : N] < \infty$ and $\gamma \notin N$. Let $\mathcal{N}$ be the collection of normal co-finite subgroups of Γ and for each $N \in \mathcal{N}$ let $\theta_N : \Gamma \to \Gamma/N = X_N$ denote the canonical homomorphism. Note that when Γ is finitely generated the collection $\mathcal{N}$ is countable. This is not necessarily true when Γ is not finitely generated. For example it is not hard to see that for $\Gamma = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ the collection $\mathcal{N}$ is uncountable.

We now assume that $\mathcal{N}$ is countable. The collection $\mathcal{N}$ is partially ordered by inclusion and the inverse limit

$$X = \hat{\Gamma} = \lim_{\leftarrow\, N\in\mathcal{N}} \Gamma/N$$

over the directed set $\mathcal{N}$ is a compact topological group called the *pro-finite compactification* of Γ. Let λ denote the Haar measure on $X = \hat{\Gamma}$ and let $(X, \mathcal{B}, \lambda, \Gamma)$ denote the measure preserving action of Γ on $\hat{\Gamma}$.

For a fixed $N \in \mathcal{N}$ let $T_N = \{\gamma_h : h \in \Gamma/N\}$ be a section for θ_N; that is $\theta_N(\gamma_h) = h$ for every $h \in \Gamma/N$ and $|T_N| = |\Gamma/N|$. Then clearly T_N tiles Γ with N as a set of centers. Moreover the set $B = \mathrm{cls}\, N \subset X$ satisfies $\gamma_h B \cap \gamma_{h'} B = \emptyset$ for $h \neq h'$ and $\lambda(\bigcup_{h\in\Gamma/N} \gamma_h B) = 1$.

Thus the set

$$\mathcal{L}_{N,\epsilon} = \{\mathbf{T} \in \mathbb{A}_\Gamma : \exists\, B \subset X, \text{ such that the sets } \gamma_h B, \\ h \in \Gamma/N \text{ are disjoint and } \lambda(\bigcup_{h\in\Gamma/N} \gamma_h B) > 1 - \epsilon\}.$$

is an open non-empty G-invariant subset of $\mathbb{A}_\Gamma$.

By Theorem 1.1 $\mathcal{L}_{N,\epsilon}$ is a dense open set for every $N \in \mathcal{N}, \epsilon > 0$. Therefore the set

$$\begin{aligned}\mathcal{L} &= \bigcap_{N\in\mathcal{N}} \bigcap_{\epsilon\searrow 0} \mathcal{L}_{N,\epsilon} \\ &= \{\mathbf{T} \in \mathbb{A}_\Gamma : \forall \epsilon > 0,\ \forall N \in \mathcal{N},\ \exists\, B \subset X, \text{ such that the sets} \\ &\qquad \gamma_h B,\ h \in \Gamma/N \ \text{ are disjoint and } \lambda(\bigcup_{h\in\Gamma/N} \gamma_h B) > 1 - \epsilon\}\end{aligned}$$

is a dense G_δ subset of $\mathbb{A}_\Gamma$.

If moreover Γ is non-Kazhdan then by Theorem 1.2 the collection $\mathcal{R}$ of ergodic G-transitive actions is a dense G_δ subset of $\mathbb{A}_\Gamma$ and therefore so is $\mathcal{L} \cap \mathcal{R}$.

3. It seems that Theorem 1.1 can be easily extended to the case when Γ is a locally compact topological group. An extension of Theorem 1.2 to this category of groups is probably possible as well, where in the proof the Bernoulli system (Ω, Γ) is replaced by the analogous system (Σ, Γ) on the space of closed subsets of Γ (see [3]). The more delicate task here is to extend the equivalence theorem of [2] to locally compact groups.

4. In a recent paper O. Ageev shows ([1, Proposition 3]) that for a countable group Γ with the weak Rohlin property and any fixed element $\gamma \in \Gamma$, the maximal spectral multiplicity $m(\mathbf{T}_\gamma) = \sup\{n : n \in M(\mathbf{T}_\gamma)\}$ (where $M(U)$ is the multiplicity function of the unitary operator U) is a constant on a comeager set of actions $\mathbf{T} \in \mathbb{A}_\Gamma$. Theorem 1.1 implies that the weak Rohlin assumption in this statement is redundant.

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