TOPOLOGICAL WEAK MIXING AND QUASI-BOHR SYSTEMS

E. GLASNER

Abstract. A minimal dynamical system (X, T) is called quasi-Bohr if it is a nontrivial equicontinuous extension of a proximal system. We show that if (X, T) is a minimal dynamical system which is not weakly mixing then some minimal proximal extension of (X, T) admits a nontrivial quasi-Bohr factor. (In terms of Ellis groups the corresponding statement is: $AG' = G$ implies weak mixing.) The converse does not hold. In fact there are nontrivial quasi-Bohr systems which are weakly mixing of all orders. Our main tool in the proof is a theorem, of independent interest, which enhances the general structure theorem for minimal systems.

Introduction

A well known and very useful result of the classical theory of topological dynamics is a theorem of Furstenberg which asserts that when the acting group is abelian the property of (2-fold) weak mixing already implies weak mixing of all orders. Another central result — this time for minimal dynamical systems but again with abelian acting group — asserts that the system is not weakly mixing iff it admits a nontrivial equicontinuous factor. Unfortunately these classical and basic results are no longer true for general acting groups. A nonabelian counter example to the first theorem is provided by R. Peleg [17] (see also Weiss [22]), where a weakly mixing system (X, T) is presented such that $(X \times X \times X, T)$ is not topologically transitive. (In fact the system (X, T) in this example is minimal and proximal.) For the second theorem a nonabelian counter example was given by D. McMahon in [15]. He produced there a minimal system which is not weakly mixing, yet does not admit a nontrivial equicontinuous factor.

In my book [9] I introduced the notions of generalized Bohr compactification and generalized strong Bohr compactification of a general group T. The idea was to generalize the notion of an equicontinuous factor and allow instead factors which are *equicontinuous extensions* of proximal (or strongly proximal) systems. Then one considers the associated compact automorphism group as a generalized compactification. In this way topological groups that have a small or even a trivial Bohr compactification may happen to admit a large generalized Bohr compactification. For example it was shown in [9] that the generalized strong Bohr compactification of a connected semisimple Lie group $\mathbb{G}$ with Iwasawa decomposition $\mathbb{G} = \mathbb{K}\mathbb{A}\mathbb{N}$ is isomorphic to $b(\mathbb{A}) \times M$, where $b(\mathbb{A})$ is the Bohr compactification of the abelian group $\mathbb{A}$ and M is the centralizer of $\mathbb{A}$ in $\mathbb{K}$.

Let us say that a minimal dynamical system (X, T) is *quasi-Bohr* if it is a nontrivial equicontinuous extension of a proximal system. In the present paper I show (Theorem 3.1) that for a general group T a minimal system (X, T) which — up to a proximal

Date: October 11, 2003.
1991 *Mathematics Subject Classification.* 54H20.

extension — does not admit a nontrivial quasi-Bohr factor is indeed weakly mixing. Unfortunately the converse does not hold. There are examples of quasi-Bohr systems which are weakly mixing of all orders.

In Section six I prove a relative version (Theorem 6.6) of Theorem 3.1. It is a bit more technical and for that reason is treated in a separate section.

When the acting group T is abelian the algebraic theory of minimal systems shows that for a minimal system (X, T) with Ellis group A the conditions (i) (X, T) is weakly mixing, (ii) (X, T) does not admit a nontrivial equicontinuous factor, and (iii) $AG' = G$ (where G is the automorphism group of the universal minimal T-system) are equivalent. The present work originated from a question raised by R. Ellis and J. Auslander about the relation between these conditions for non-abelian group actions. In Section four I compare the various notions into which weak mixing splits when the commutativity assumption is dropped.

My main (new) tool in the proof of Theorem 3.1 is a theorem of independent interest (Theorem 2.7) which enhances the structure theorem for minimal systems (even in the classical abelian case) in showing that the weakly mixing RIC extension $\pi_\infty : X_\infty \to Y_\infty$ at the top of the PI tower associated with a minimal system (X, T) is in fact weakly mixing of all orders (Corollary 2.9). An analogue of Theorem 2.7 was proved by McMahon in [16]. His proof uses the measure theoretical tool of RIM extensions rather than our RIC extension method. RIM (*relative invariant measure*) extensions were introduced in Glasner [8], and results similar to those of [16] can already be deduced from the work by Furstenberg and Glasner [7].

In the final section I examine the new examples of generalized Bohr compactifications of topological groups that arise as corollaries of recent works of Pestov and Glasner and Weiss.

I am indebted to Joe Auslander for a careful reading of the paper and for many suggestions that improved both the content and the presentation of this work. I also thank Ethan Akin for several helpful e-conversations.

1. A brief survey of abstract topological dynamics

In this section I review the necessary definitions and results from abstract topological dynamics. For details and proofs refer to [9]. See also [2], [21] and [1]. A *topological dynamical system* or briefly a system is a pair (X, T), where X is a compact Hausdorff space and T an abstract group which acts on X as a group of homeomorphisms. A *sub-system* of (X, T) is a closed invariant subset $Y \subset X$ with the restricted action. For a point $x \in X$, we let $\mathcal{O}_T(x) = \{tx : t \in T\}$, and $\bar{\mathcal{O}}_T(x) = \mathrm{cls}\,\{tx : t \in T\}$. These subsets of X are called the *orbit* and *orbit closure* of x respectively. We say that (X, T) is *point transitive* if there exists a point $x \in X$ with a dense orbit. In that case x is called a *transitive point*. If every point is transitive we say that (X, T) is a *minimal system*.

The dynamical system (X, T) is *topologically transitive* if for any two nonempty open subsets U and V of X there exists some $t \in T$ with $tU \cap V \neq \emptyset$. Clearly a point transitive system is topologically transitive and when X is metrizable the converse holds as well: in a metrizable topologically transitive system the set of transitive points is a dense G_δ subset of X.

The system (X,T) is *weakly mixing* if the product system $(X \times X, T)$ (where $t(x,x') = (tx, tx'),\ x, x' \in X,\ t \in T$) is topologically transitive. The system (X,T) is *weakly mixing of all orders* if for every $n \geq 1$ the product system (X^n, T) is topologically transitive (here and in the sequel, A^n, for any set A, denotes the cartesian product $A \times A \times \cdots \times A$ (n times)).

If (Y,T) is another system then a continuous onto map $\pi : X \to Y$ satisfying $t \circ \pi = \pi \circ t$ for every $t \in T$ is called a *homomorphism* of dynamical systems. In this case we say that (Y,T) is a *factor* of (X,T) and also that (X,T) is an *extension* of (Y,T). With the system (X,T) we associate the induced action (the *hyper system* associated with (X,T)) on the compact space 2^X of closed subsets of X equipped with the Vietoris topology. A subsystem Y of $(2^X, T)$ is a *quasifactor* if $\bigcup\{A : A \in Y\} = X$.

The system (X,T) can always be considered as a quasifactor of (X,T) by identifying x with $\{x\}$. Recall that if $(X,T) \xrightarrow{\pi} (Y,T)$ is a homomorphism then in general $\pi^{-1} : Y \to 2^X$ is an upper-semi-continuous map and that $\pi : X \to Y$ is open iff $\pi^{-1} : Y \to 2^X$ is continuous, iff $\{\pi^{-1}(y) : y \in Y\}$ is a quasifactor of (X,T) . When there is no room for confusion we write X for the system (X,T).

We assume for simplicity that our acting group T is a discrete group. βT will denote the Stone-Čech compactification of T. The universal properties of βT make it

- a compact semigroup with right continuous multiplication (for a fixed $p \in \beta T$ the map $q \mapsto qp,\ q \in \beta T$ is continuous), and left continuous multiplication by elements of T, considered as elements of βT (for a fixed $t \in T$ the map $q \mapsto tq,\ q \in \beta T$ is continuous).
- a dynamical system $(\beta T, T)$ under left multiplication by elements of T.

The system $(\beta T, T)$ is universal point transitive T-system; i.e. for every point transitive system (X,T) and a point $x \in X$ with dense orbit, there exists a homomorphism of systems $(\beta T, T) \to (X,T)$ which sends e, the identity element of T, onto x. For $p \in \beta T$ we let px denote the image of p under this homomorphism. This defines an "action" of the semigroup βT on every dynamical system. When dealing with the hyper system $(2^X, T)$ we write $p \circ A$ for the image of the closed subset $A \subset X$ under $p \in \beta T$, to distinguish it from the (usually non-closed) subset $pA = \{px : x \in A\}$. We always have $pA \subset p \circ A$.

The compact semigroup βT has a rich algebraic structure. For instance for countable T there are 2^c minimal left (necessarily closed) ideals in βT all isomorphic as systems and each serving as a universal minimal system. Each such minimal ideal, say M, has a subset J of 2^c idempotents such that $\{vM : v \in J\}$ is a partition of M into disjoint isomorphic (non-closed) subgroups. The group of dynamical system automorphisms of (M,T), $G = \mathrm{Aut}\,(M,T)$ can be identified with any one of the groups vM as follows: with $\alpha \in vM$ we associate the automorphism $\hat{\alpha} : (M,T) \to (M,T)$ given by right multiplication $\hat{\alpha}(p) = p\alpha,\ p \in M$. The group G plays a central role in the algebraic theory. It carries a natural T_1 compact topology called by Ellis the *τ-topology*. The τ-closure of a subset A of G consists of those $\beta \in G$ for which the set $\mathrm{graph}\,(\beta) = \{(p, p\beta) : p \in M\}$ is a subset of the closure in $M \times M$ of the set $\bigcup\{\mathrm{graph}\,(\alpha) : \alpha \in A\}$.

It is convenient to fix a minimal left ideal M in βT and an idempotent $u \in M$. As explained above we identify G with uM and it follows that for any subset $A \subset G$,

$$\mathrm{cls}_\tau A = u(u \circ A) = (u \circ A) \cap G.$$

Also in this way we can consider the "action" of G on every system (X,T) via the action of βT on X. With every minimal system (X,T) and a point $x_0 \in uX = \{x \in X : ux = x\}$ we associate a τ-closed subgroup

$$\mathcal{G}(X,x_0) = \{\alpha \in G : \alpha x_0 = x_0\},$$

the *Ellis group* of the pointed system (X,x_0). For a homomorphism $\pi : X \to Y$ with $\pi(x_0) = y_0$ we have

$$\mathcal{G}(X,x_0) \subset \mathcal{G}(Y,y_0).$$

For a τ-closed subgroup F of G the *derived group* F' is given by:

$$F' := \bigcap \{\mathrm{cls}_\tau O : O \text{ a } \tau\text{-open neighborhood of } u \text{ in } F\}.$$

F' is a τ-closed normal (in fact characteristic) subgroup of F and it is characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group.

A pair of points $(x,x') \in X \times X$ for a system (X,T) is called *proximal* if there exists a net $t_i \in T$ and a point $z \in X$ such that $\lim t_i x = \lim t_i x' = z$ (iff there exists $p \in \beta T$ with $px = px'$). We denote by P the set of proximal pairs in $X \times X$. We have

$$P = \bigcap \{TV : V \text{ a neighborhood of the diagonal in } X \times X\}.$$

A system (X,T) is called *proximal* when $P = X \times X$ and *distal* when $P = \Delta$, the diagonal in $X \times X$. It is called *strongly proximal* when the following much stronger condition holds: the dynamical system $(M(X),T)$, induced on the compact space $M(X)$ of probability measures on X, is proximal. A minimal system (X,T) is called *point distal* if there exists a point $x \in X$ such that if x, x' is a proximal pair then $x = x'$.

The *regionally proximal relation* on X is defined by

$$Q = \bigcap \{\overline{TV} : V \text{ a neighborhood of } \Delta \text{ in } X \times X\}.$$

It is easy to verify that Q is trivial — i.e. equals Δ — iff the system is equicontinuous. More generally we set for $n \geq 2$

$$P^{(n)} = \bigcap \{TV : V \text{ a neighborhood of the diagonal in } X^n\}$$

and

$$Q^{(n)} = \bigcap \{\overline{TV} : V \text{ a neighborhood of the diagonal in } X^n\}.$$

An extension $(X,T) \xrightarrow{\pi} (Y,T)$ of minimal systems is called a *proximal extension* if the relation $R_\pi = \{(x,x') : \pi(x) = \pi(x')\}$ satisfies $R_\pi \subset P$ and a *distal extension* when $R_\pi \cap P = \Delta$. One can show that every distal extension is open. The extension π is called an *equicontinuous extension* if for every ϵ, a neighborhood of the diagonal $\Delta = \{(x,x) : x \in X\} \subset X \times X$, there exists a neighborhood of the diagonal δ such that $t(\delta \cap R_\pi) \subset \epsilon$ for every $t \in T$. The extension π is a *weakly mixing extension* when R_π as a subsystem of the product system $(X \times X, T)$ is topologically transitive. It is a *weakly mixing extension of all orders* if $R_\pi^{(n)}$ is topologically transitive for every $n \geq 1$; here $R_\pi^{(n)} = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ 1 \leq i, j \leq n\}$.

The *n-th relative proximal* and *regionally proximal relations* are defined as

$$P_\pi^{(n)} = \bigcap \{TV \cap R_\pi^{(n)} : V \text{ a neighborhood of the diagonal in } X^n\}$$

and

$$Q_\pi^{(n)} = \bigcap \{\overline{TV \cap R_\pi^{(n)}} : V \text{ a neighborhood of the diagonal in } X^n\}$$

respectively.

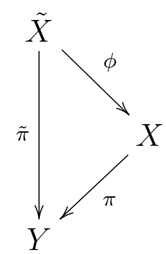
The algebraic language is particularly suitable for dealing with such notions. For example an extension $(X,T) \xrightarrow{\pi} (Y,T)$ of minimal systems is a proximal extension iff the Ellis groups $\mathcal{G}(X, x_0) = A$ and $\mathcal{G}(Y, y_0) = F$ coincide. It is distal iff for every $y \in Y$, and $x \in \pi^{-1}(y)$, $\pi^{-1}(y) = \mathcal{G}(Y, y)x$; iff:

for every $y = py_0 \in Y$, p an element of M, $\pi^{-1}(y) = p\pi^{-1}(y_0) = pFx_0$, where $F = \mathcal{G}(Y, y_0)$.

In particular (X,T) is distal iff $Gx = X$ for some (hence every) $x \in X$. The extension π is an equicontinuous extension iff it is a distal extension and, denoting $\mathcal{G}(X, x_0) = A$ and $\mathcal{G}(Y, y_0) = F$,

$$F' \subset A,$$

in which case the compact group F/F' is the group of the *group extension* $\tilde{\pi}$ associated with the equicontinuous extension π. More precisely, there exists a minimal dynamical system $\tilde{X}$ on which the compact Hausdorff group $K = F/F'$ acts as a group of automorphisms and we have the following commutative diagram



where $\tilde{\pi} : \tilde{X} \to Y \cong \tilde{X}/K$ is a group extension and so is the extension $\phi : \tilde{X} \to X \cong \tilde{X}/L$ with $L = A/F' \subset F/F' = K$. ($\tilde{X} = X$ iff A is a normal subgroup of F.)

A minimal system (X,T) is called *incontractible* if the union of minimal subsets is dense in every product system (X^n, T). This is the case iff $p \circ Gx = X$ for some (hence every) $x \in X$ and $p \in M$. A topological group T is called *strongly amenable* if every minimal T dynamical system is incontractible, or equivalently if every minimal proximal T-system is trivial. This property implies amenability and holds for nilpotent groups. In fact, a group T is amenable iff every minimal strongly proximal T-system is trivial.

We say that $(X,T) \xrightarrow{\pi} (Y,T)$ is a RIC (*relatively incontractible*) *extension* if:

for every $y = py_0 \in Y$, p an element of M, $\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0$, where $F = \mathcal{G}(Y, y_0)$.

It is not hard to see that every RIC extension is open. Every distal extension is RIC and it follows that every distal extension is open.

We have the following theorem from [5] about the interpolation of equicontinuous extensions. For a proof see [9], Theorem X.2.1.

1.1. **Theorem.** *Let $\pi : X \to Y$ be a RIC extension of minimal systems. Fix a point $x_0 \in X$ with $ux_0 = x_0$ and let $y_0 = \pi(x_0)$. Let $A = \mathcal{G}(X, x_0)$ and $F = \mathcal{G}(Y, y_0)$. Then there exists a commutative diagram of pointed systems*

$$\begin{array}{ccc} (X, x_0) & & \\ & \overset{\sigma}{\searrow} & \\ \pi\Big\downarrow & & (Z, z_0) \\ & \underset{\rho}{\swarrow} & \\ (Y, y_0) & & \end{array}$$

such that ρ is an equicontinuous extension with Ellis group $\mathcal{G}(Z, z_0) = AF'$ and the extension ρ is an isomorphism iff $AF' = F$. Moreover if

$$\begin{array}{ccc} (X, x_0) & & \\ & \overset{\sigma'}{\searrow} & \\ \pi\Big\downarrow & & (Z', z'_0) \\ & \underset{\rho'}{\swarrow} & \\ (Y, y_0) & & \end{array}$$

is another such diagram with ρ' an equicontinuous extension then there exists a homomorphism $(Z, z_0) \to (Z', z'_0)$.

Let $(X, T) \overset{\pi}{\to} (Y, T)$ be a homomorphism of minimal systems, one constructs a commutative diagram of homomorphisms of minimal systems (the *RIC-shadow diagram*),

$$\begin{array}{ccc} X & \overset{\theta^*}{\longleftarrow} & X^* = X \vee Y^* \\ \pi\Big\downarrow & & \Big\downarrow \pi^* \\ Y & \underset{\theta}{\longleftarrow} & Y^* \end{array}$$

where π^* is RIC and θ, θ^* are proximal (thus we still have $A = \mathcal{G}(X, x_0) = \mathcal{G}(X^*, x_0^*)$ and $F = \mathcal{G}(Y, y_0) = G(Y^*, y_0^*))$. The concrete description of these objects uses quasi-factors and the circle operation:

$$Y^* = \{p \circ F x_0 : p \in M\} \subset 2^X, \qquad X^* = \{(x, y^*) : x \in y^* \in Y^*\} \subset X \times Y^*$$

and

$$\theta(p \circ F x_0) = p y_0, \quad \theta^*(x, y^*) = x, \quad \pi^*(x, y^*) = y^*, \quad (p \in M),$$

where $F = \mathcal{G}(Y, y_0)$. The map θ is an isomorphisms (hence $\pi = \pi^*$) when and only when π is already RIC.

In particular, starting with a trivial map $X \to \{*\}$, where $\{*\}$ is the trivial one point system, we obtain the RIC shadow diagram

(1.1)
$$\begin{CD} X @<\theta<< X^* = X \vee \Pi(X) \\ @VVV @VV\pi V \\ \{*\} @<<< \Pi(X) \end{CD}$$

Thus $\Pi(X) = \{p \circ Gx_0 : p \in M\} \subset 2^X$ and the minimal system X^*, defined as

$$X^* = \{(px_0, p \circ Gx_0) : p \in M\} \subset X \times \Pi(X)$$

coincides with the subsystem

$$\{(x, B) : x \in B \in \Pi(X)\} \subset X \times \Pi(X).$$

The system $(\Pi(X), T)$ is minimal and proximal and the projection map $\pi : X^* \to \Pi(X)$ is RIC, hence also open. The dynamical system (X, T) is incontractible iff $\Pi(X) = \{*\}$ is trivial. Note that if X is metrizable so is 2^X. Thus in this case all the entries of the shadow diagram are metrizable as well.

We say that a minimal system (X, T) is a *strictly PI system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \le \eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \le \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota < \nu}$, and (iv) $W_\eta = X$. We say that (X, T) is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by almost 1-1 extensions we get the notion of *AI-systems*. If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have *I-systems*. In this terminology the structure theorem for distal systems (Furstenberg [6], 1963) can be stated as follows:

1.2**. Theorem.** *A metric minimal system is distal iff it is an I-system.*

And the Veech-Ellis structure theorem for point distal systems (Veech [20], 1970 and Ellis [3], 1973).

1.3**. Theorem.** *A metric minimal dynamical system is point distal iff it is an AI-system.*

Finally we have the structure theorem for minimal systems (Ellis-Glasner-Shapiro [5], 1975, McMahon [15], 1976 and Veech [21], 1977).

1.4**. Theorem** (Structure theorem for minimal systems)**.** *Given a minimal system (X, T), there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*

$$\begin{array}{ccccccccccccc}
X & \xleftarrow{\theta_0^*} & X_0 & & \xleftarrow{\quad\theta_1^*\quad} & & X_1 & \cdots & X_\nu & & \xleftarrow{\quad\theta_{\nu+1}^*\quad} & X_{\nu+1} & \cdots\ X_\eta = X_\infty \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\pi_0}\downarrow & \searrow^{\sigma_1} & & & {\scriptstyle\pi_1}\downarrow & & {\scriptstyle\pi_\nu}\downarrow & \searrow^{\sigma_{\nu+1}} & & {\scriptstyle\pi_{\nu+1}}\downarrow & {\scriptstyle\pi_\infty}\downarrow \\
pt & \xleftarrow[\theta_0]{} & Y_0 & \xleftarrow[\rho_1]{} & Z_1 & \xleftarrow[\theta_1]{} & Y_1 & \cdots & Y_\nu & \xleftarrow[\rho_{\nu+1}]{} & Z_{\nu+1} \xleftarrow[\theta_{\nu+1}]{} & Y_{\nu+1} & \cdots\ Y_\eta = Y_\infty
\end{array}$$

where for each $\nu \leq \eta, \pi_\nu$ is RIC, ρ_ν is isometric, θ_ν, θ^*_ν are proximal and π_∞ is RIC and weakly mixing. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$. Thus X_∞ is a proximal extension of X and a RIC weakly mixing extension of the strictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) iff X is a PI-system.

1.5**. Remark.** Theorem 1.2 was extended by R. Ellis to the non-metrizable case [4].

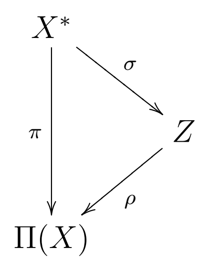
2. Preliminary results

2.1**. Definition.** We will say that a minimal dynamical system (Z, T) is *quasi-Bohr* if it is an equicontinuous extension of a proximal system. We say that it is *nontrivial* if the equicontinuous extension is not 1-1. (Thus a trivial quasi-Bohr system is either proximal or a one point system.)

Our first theorem follows from the Ellis-Glasner-Shapiro general structure theorem for minimal dynamical systems, [5]. In fact it describes the first stage of the canonical PI tower for (X, T) .

2.2**. Theorem.** *Let (X, T) be a minimal system with Ellis group $\mathcal{G}(X) = A$. The following conditions are equivalent.*

1. *There exists a minimal proximal extension $\theta : \tilde{X} \to X$ such that the dynamical system $\tilde{X}$ admits a nontrivial quasi-Bohr factor.*
2. *In the basic RIC shadow diagram* (1.1) *one can interpolate*



   with $\rho : Z \to \Pi(X)$ a nontrivial equicontinuous extension.
3. *$AG' \subsetneq G$.*

We say that two dynamical systems (X, T) and (Y, T) are *weakly disjoint* when the product system $(X \times Y, T)$ is transitive. This is indeed a very weak sense of disjointness as there are systems which are weakly disjoint from themselves. In fact, a dynamical system is weakly mixing iff it is weakly disjoint from itself.

The next result is proved in [9], Theorem II.2.1. (A relative, thus a stronger, version is proved below, Theorem 6.3.)

2.3**. Theorem.** *Let (X, T) be a minimal system. If $Q^{(n)} = X^n$ for every $n \geq 2$ then (X, T) is weakly disjoint from every minimal system and in thus weakly mixing.*

2.4**. Corollary.** *A minimal proximal system is weakly mixing.*

2.5**. Remark.** In [9, Theorem II.2.1] the assumption is that $\overline{P^{(n)}} = X^n(\forall n)$ rather than $Q^{(n)} = X^n$, however the proof given there works also under the latter assumption.

Note that, in general, since $P^{(n)} \subset Q^{(n)}$ the condition $\overline{P^{(n)}} = X^n$ implies $Q^{(n)} = X^n$. When X is metrizable the converse implication holds as well. In fact, choosing a countable basis $\{V_k : k = 1, 2, \dots\}$ for open neighborhoods of the diagonal $\Delta_n \subset X^n$, if $Q^{(n)} = \bigcap_k \overline{TV_k} = X^n$ then $\overline{TV_k} = X^n$ for each k, so that TV_k is an open and dense subset of X^n and by Baire's theorem $P^{(n)} = \bigcap_k TV_k$ is a dense G_δ subset of X^n. Thus for a metrizable system the conditions

$$\overline{P^{(n)}} = X^n(\forall n) \qquad \text{and} \qquad Q^{(n)} = X^n(\forall n)$$

are equivalent.

The theorem we prove next is our main tool. Again it is in the spirit of the general structure theorem. We first need a lemma.

2.6**. Lemma.** *Let $\pi : (X, T) \to (Y, T)$ be a RIC homomorphism of minimal systems. For $n \geq 2$ let*

$$R_\pi^{(n)} = \{(x_1, \dots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ 1 \leq i, j \leq n\}.$$

Let $u \in J$ be a minimal idempotent.

1. *Given a point $y \in Y$ with $uy = y$ we have*

$$u \circ (u\pi^{-1}(y))^n = \pi^{-1}(y)^n.$$

2. *Hence*

$$\mathrm{cls}\, T\big((u\pi^{-1}(y))^n\big) = R_\pi^{(n)}.$$

Proof. The RIC property of π implies that $u \circ u(\pi^{-1}(y)) = \pi^{-1}(y)$, and if $\lim_\nu t_\nu = u$ is a net in T which converges to u in M then this means that in the Vietoris topology on 2^X we have $\lim_\nu t_\nu u(\pi^{-1}(y)) = \pi^{-1}(y)$. Hence also

$$\lim_\nu t_\nu\big(u(\pi^{-1}(y)) \times \cdots \times u(\pi^{-1}(y))\big) = \pi^{-1}(y) \times \cdots \times \pi^{-1}(y) \qquad (n \text{ times})$$

and the first assertion follows.

By minimality of (X, T) and the fact that π is open we conclude that $\mathrm{cls}\, T\pi^{-1}(y)^n = R_\pi^{(n)}$ and therefore the second assertion is a direct consequence of the first. $\square$

2.7**. Theorem.** *Let $\pi : (X, T) \to (Y, T)$ be a RIC homomorphism of minimal systems. The following conditions are equivalent*

1. *the relation*

$$R_\pi^{(2)} = \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$$

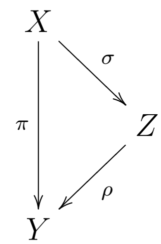
is not topologically transitive.

2. *for some $n \geq 2$ the relation*

$$R_\pi^{(n)} = \{(x_1, \dots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ 1 \leq i, j \leq n\}$$

is not topologically transitive.

3. *there exists a nontrivial equicontinuous intermediate extension* $\rho : Z \to Y$*:*



4. $AF' \subsetneq F$, *where* $F = \mathcal{G}(Y)$ *and* $A = \mathcal{G}(X)$ *are the Ellis groups of* Y *and* X *respectively.*

Proof. The equivalence of the conditions 3 and 4 for a RIC extension follows from Theorem 1.1.

Assuming condition 3 we clearly have that $R^{(2)}_\rho = \{(z_1, z_2) \in Z \times Z : \rho(z_1) = \rho(z_2)\}$ is not topologically transitive and a fortiori $R^{(2)}_\pi = \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$ is not topologically transitive. Thus we have $3 \Rightarrow 1$ and the implication $1 \Rightarrow 2$ is trivial.

It is therefore enough to show that the condition $AF' = F$ implies that for every $n \geq 2$ the relation $R^{(n)}_\pi$ is topologically transitive. The proof will proceed by induction on n. We assume that $AF' = F$ and that we already know that T acts transitively on $R^{(m)}_\pi$ for for $1 \leq m < n$ (where $R^{(1)}_\pi = X$). We then prove that also $R^{(n)}_\pi$ is transitive. Of course, (X, T) being minimal, the topological transitivity for $n = 1$ is clear.

The proof consists of several steps.

Step 1: This is what I called in [9] the "Ellis trick". Refer to [9], Lemma X.6.1 for the proof.

2.8**. Lemma.** *Consider the dynamical system* (M, F) *where the group* F *acts on* M *by right multiplication,* $\alpha : p \mapsto p\,\alpha,\ \alpha \in F, p \in M$.

1. *There exists a minimal idempotent* $w \in J$ *such that the subset* $\overline{wF} \subset M$ *is* F*-minimal.*
2. *If* V *is a nonempty open subset of* $\overline{wF}$ *then in the relative* τ*-topology on* wF

$$\mathrm{int}_\tau \mathrm{cls}_\tau (V \cap wF) \neq \emptyset.$$

For convenience and with no loss of generality I shall assume from now on that $u = w$.

Step 2: Choose an arbitrary point $x_0 \in uX$ and let $y_0 = \pi(x_0)$, so that $ux_0 = x_0$ and $uy_0 = y_0$. Again with no loss in generality we assume that $F = \mathcal{G}(Y, y_0) = \{\alpha \in G : \alpha y_0 = y_0\}$ and that $A = \mathcal{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}$.

Claim: For any nonempty open subset U of $\overline{u\pi^{-1}(y_0)}$ and any point $x' = (x'_2, \ldots, x'_n) \in R^{(n-1)}_\pi$ with $\pi(x'_j) = y_0,\ j = 2, \ldots, n$, we have

$$\mathrm{cls}\, T(U \times \{x'\}) \supset \pi^{-1}(y_0) \times \{ux'\}.$$

Proof of claim. Set $V = \{p \in \overline{F} : px_0 \in U\}$, then V is a nonempty open subset of $\overline{F}$ and by step 1 we conclude that $\tilde{V} = \mathrm{int}_\tau \mathrm{cls}_\tau(V \cap F) \neq \emptyset$. It follows that for some $\alpha \in F$ the set $\alpha^{-1}\tilde{V}$ is a τ-neighborhood of the identity u in F. From the definition of F' as $\bigcap \mathrm{cls}_\tau O$, where the intersection is over all the open τ-neighborhoods of u in F, we conclude that $F' \subset \mathrm{cls}_\tau \alpha^{-1}\tilde{V}$, hence $\alpha F' \subset \mathrm{cls}_\tau \tilde{V}$. Now using our assumption $F = AF' = F'A$ we get

$$\begin{aligned}
\mathrm{cls}\, T(U \times \{x'\}) &\supset u \circ (U \times \{x'\}) \supset u \circ (Vx_0 \times \{x'\}) \\
&\supset u(u \circ V)x_0 \times \{ux'\} \supset u(u \circ (V \cap F))x_0 \times \{ux'\} \\
&= \mathrm{cls}_\tau (V \cap F)x_0 \times \{ux'\} \supset \mathrm{cls}_\tau \tilde{V} x_0 \times \{ux'\} \\
&\supset \alpha F' x_0 \times \{ux'\} = \alpha F' A x_0 \times \{ux'\} \\
&= \alpha F x_0 \times \{ux'\} = F x_0 \times \{ux'\}.
\end{aligned}$$

Now the RIC property of π implies $u \circ Fx_0 = \pi^{-1}(y_0)$, hence

$$\mathrm{cls}\, T(U \times \{x'\}) \supset u \circ (Fx_0 \times \{ux'\}) = \pi^{-1}(y_0) \times \{ux'\}$$

as required. □

Step 3: Let now $W \subset R_\pi^{(n)}$ be a closed invariant set with nonempty interior (relative to $R_\pi^{(n)}$). Since π is an open map so is the projection map $\mathrm{proj}_{(2,\dots,n)} : R_\pi^{(n)} \to R_\pi^{(n-1)}$ and we conclude that $W' = \mathrm{proj}_{(2,\dots,n)}(W)$ is a closed invariant set with nonempty interior (relative to $R_\pi^{(n-1)}$). By our induction hypothesis $W' = R_\pi^{(n-1)}$.

Let U and V be nonempty open subsets of X and $R_\pi^{(n-1)}$ respectively such that $\emptyset \neq (U \times V) \cap R_\pi^{(n)} \subset W$. We will show that there exists a point $x'' \in V$ such that, with $y = \pi(x'')$,

$$W \supset \pi^{-1}(y) \times \{x''\}.$$

By Lemma 2.6 there exists a point $(x_1, x_2, \dots, x_n) \in (u\pi^{-1}(y_0))^n$ and $t \in T$ such that $t(x_1, x_2, \dots, x_n) \in U \times V$, hence $(x_1, x_2, \dots, x_n) \in t^{-1}(U \times V)$. Denote $U_0 = t^{-1}U, V_0 = t^{-1}V$ and $x' = (x_2, \dots, x_n)$, then we have

$$(x_1, x') \in (U_0 \times V_0) \cap R_\pi^{(n)} \subset W \qquad \text{and} \qquad x_1 \in u\pi^{-1}(y_0),\ x' \in (u\pi^{-1}(y_0))^{n-1}.$$

Applying the claim from step 2 to the relatively open set $U_0 \cap \pi^{-1}(y_0)$ and the point $x' \in V_0 \cap (u\pi^{-1}(y_0))^{n-1}$ we conclude that

$$W \supset \mathrm{cls}\, T((U_0 \cap \pi^{-1}(y_0)) \times \{x'\}) \supset \pi^{-1}(y_0) \times \{x'\},$$

hence also

$$W \supset \mathrm{cls}\, T((U \cap \pi^{-1}(ty_0)) \times \{tx'\}) \supset \pi^{-1}(ty_0) \times \{tx'\}.$$

But this is the required property $W \supset \pi^{-1}(y) \times \{x''\}$ (with $x'' = tx'$ and $y = ty_0$).

With no loss in generality we assume that $W = \mathrm{cls}\,\mathrm{int}\, W$. Then the union of sets of the form $(U \times V) \cap R_\pi^{(n)} \subset W$ is dense in W and we finally conclude that $W = R_\pi^{(n)}$. □

2.9**. Corollary.** *In the structure theorem for minimal systems (Theorem 1.4) the final weakly mixing extension $\pi_\infty : X_\infty \to Y_\infty$ is weakly mixing of all orders.*

2.10. **Corollary.** 1. *A minimal incontractible system X is weakly mixing iff it is weakly mixing of all orders iff it does not admit a nontrivial equicontinuous factor iff $AG' = G$ (where $A = \mathcal{G}(X)$ is the Ellis group of X).*

2. *If the acting group T is strongly amenable then every minimal system is incontractible and we conclude that every minimal weakly mixing T-system is weakly mixing of all orders.*

2.11. **Remark.** Regarding part 2 of Corollary 2.10 we remark that results of [16] and [7] imply the stronger statement that even for amenable groups every minimal weakly mixing dynamical system is weakly mixing of all orders. On the other hand part 1 of Corollary 2.10 does not seem to follow from these papers.

2.12. **Remark.** A special case of the implication $2 \Rightarrow 3$ in Theorem 2.7 is the well known result that for a RIC extension of minimal systems $\pi : X \to Y$, if $R_\pi^{(2)}$ is not topologically transitive then there exists a nontrivial equicontinuous intermediate extension $\rho : Z \to Y$ (see [1, Chapter 14, Theorem 27]).

3. The main result

3.1. **Theorem.** *Let (X, T) be a minimal system. Among the conditions*

1. *(X, T) is not weakly mixing,*
2. *for some $n \geq 2$, $Q^{(n)} \neq X^n$,*
3. *there exists a minimal system $\tilde{X}$ which is a proximal extension of X and such that $\tilde{X}$ admits a nontrivial quasi-Bohr factor,*

we have the implications: $1 \Rightarrow 2 \Rightarrow 3$.

In terms of the Ellis group A of X we have: $AG' = G \Rightarrow Q^{(n)} = X^n (\forall n \geq 2) \Rightarrow$ weak mixing.

Proof. $1 \Rightarrow 2$: Suppose (X, T) is not weakly mixing. By Theorem 2.3 there exists an $n \geq 2$ with $Q^{(n)} \neq X^n$.

$2 \Rightarrow 3$: Suppose there exists an $n \geq 2$ with $Q^{(n)} \neq X^n$. This means that for some open neighborhood V of the diagonal $\Delta_n \subset X^n$ the closed invariant set $W = \operatorname{cls} TV$ is a proper subset of X^n with nonempty interior.

Let

$$\begin{array}{ccc} X & \xleftarrow{\ \theta\ } & X^* = X \vee \Pi(X) \\ \downarrow & & \downarrow \pi \\ \{*\} & \longleftarrow & \Pi(X) \end{array}$$

be the RIC shadow diagram corresponding to the trivial map $X \to \{*\}$. Thus $\Pi(X) = \{p \circ Gx_0 : p \in M\} \subset 2^X$ and the minimal system X^*, defined as

$$X^* = \{(px_0, p \circ Gx_0) : p \in M\} \subset X \times \Pi(X)$$

coincides with the subsystem

$$\{(x, B) : x \in B \in \Pi(X)\} \subset X \times \Pi(X).$$

The system $(\Pi(X), T)$ is minimal and proximal and the projection map $\pi : X^* \to \Pi(X)$ is RIC, hence also open.

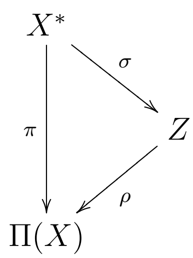
Set

$$\begin{aligned} R_\pi^{(n)} &= \{(x_1^*, \dots, x_n^*) \in (X^*)^n : \pi(x_i^*) = \pi(x_j^*),\ 1 \le i, j \le n\} \\ &\cong \{(x_1, \dots, x_n, B) \in X^n \times \Pi(X) : x_j \in B,\ 1 \le j \le n\}, \end{aligned}$$

and

$$\begin{aligned} W^* &= \{(x_1, \dots, x_n, B) \in W \times \Pi(X) : x_j \in B,\ 1 \le j \le n\} \\ &= R_\pi^{(n)} \cap \{(x_1^*, \dots, x_n^*) \in (X^*)^n : (\theta(x_1^*), \dots, \theta(x_n^*)) \in W\} \\ &= R_\pi^{(n)} \cap (\theta^n)^{-1}(W). \end{aligned}$$

Clearly W^* is a closed invariant subset of $R_\pi^{(n)}$ containing the diagonal $\Delta_n^* \subset (X^*)^n$. Moreover, we have $\Delta_n^* \subset R_\pi^{(n)} \cap (\theta^n)^{-1}(TV) \subset W^*$, so that W^* has nonempty interior in the relative topology of $R_\pi^{(n)}$. And we can not have $W^* = R_\pi^{(n)}$ because this will imply $W = X^n$. Thus Theorem 2.7 applies and we obtain a commutative diagram



where $\rho : Z \to \Pi(X)$ is a nontrivial equicontinuous extension. This completes the proof of the implication 2 ⇒ 3. (The Ellis group picture here is as follows. Let $\mathcal{G}(X) = \mathcal{G}(X^*) = A$ and $\mathcal{G}(Z) = B$. Since $\Pi(X)$ is proximal we have $\mathcal{G}(\Pi(X)) = \mathcal{G}(\{*\}) = G$. Since ρ is an equicontinuous extension we have $G' = \mathcal{G}(\Pi(X))' \subset \mathcal{G}(Z) = B$ and therefore $AG' \subset BG' = B \subsetneq G$.)

Finally, in view of Theorem 2.2 the condition $AG' = G$ is equivalent to the negation of condition 3 and we obtain the last assertion of the theorem as the contrapositive of the first part. □

As we will see in the next section, unlike the case of an abelian group, for a general acting group the converse of Theorem 3.1 is no longer true.

4. The various definitions of weak mixing

4.1. **Theorem.** 1. *There exists a quasi-Bohr system (X, T) (in fact a nontrivial group extension of a proximal system) which is weakly mixing of all orders. (In particular for this system $G' \subset A$ and $AG' = A \neq G$.)*

2. *There exists a quasi-Bohr system (X, T) (again a nontrivial group extension of a proximal system) which is weakly mixing yet with $Q^{(3)} \neq X^3$.*
3. *There exists a quasi-Bohr system (X, S) for which $Q = Q^{(2)} = X \times X$ yet (X, S) is not weakly mixing.*

Proof. 1. Consider the dynamical system (X,T), where $T = \{g \in GL(2,\mathbb{R}) : \det(g) = \pm 1\}$ is acting naturally on the compact space X of rays emanating from the origin in $\mathbb{R}^2$. We have topological transitivity of $X \times X$ because T acts (literally) transitively on $X \times X \setminus (\Delta \cup (\mathrm{id} \times \sigma)\Delta)$, where σ is the map induced by the flip $(x,y) \mapsto (x,-y)$ on $\mathbb{R}^2$.

On the other hand the map $\pi : X \to Y$, which sends a ray to the unique line which contains it, is a group extension (two to one) of the proximal system (Y,T), where T acts on the projective space Y of lines through the origin in $\mathbb{R}^2$. Thus (X,T) is a weakly mixing quasi-Bohr system (with $AG' = A \neq G$).

As was noted by S. Mozes (see [22]) the dynamical system (X,T) presented above has the property that T — whose elements preserve the cross ratio of four points on the circle — acts transitively on X^3 but not on X^4. Even if we give up linearity we still have the problem that the group of all homeomorphisms of the circle is not 4-transitive. However we can overcome this difficulty if we go up one dimension and take X to be the space of rays emanating from the origin in $\mathbb{R}^3$ (i.e. X is homeomorphic to the sphere S^2). We choose a countable dense subgroup T of the Polish group $H_l(X)$ of all (not necessarily linear or orientation preserving) homeomorphisms of X which preserve lines through the origin in $\mathbb{R}^3$. Again (X,T) is a group extension $\pi : X \to Y$, where the proximal factor Y is the projective plane comprising all lines through the origin in $\mathbb{R}^3$. It is not hard to check that (X,T) is weakly mixing of all orders.

2. Let X be the unit circle in the complex plane $\{x \in \mathbb{C} : |x| = 1\}$. Let $m : X \to X$ be the McMahon map with two fixed points, explicitly:

$$m(\exp 2\pi i t) = \exp 2\pi i \left(2\left(t - \frac{[2t]}{2}\right)^2 + \frac{[2t]}{2} \right), \qquad 0 \le t < 1.$$

Let $a : X \to X$ be the antipodal map $a(x) = -x$, let $c : X \to X$ be the conjugation $c(x) = \bar{x}$, and let R be an irrational rotation. We define T to be the group generated by m, c and R. Clearly the map a commutes with each element of T. Thus if $\pi : X \to Y = X/ < a >$ denotes the map from the circle to the projective line, we see that $\pi : (X,T) \to (Y,T)$ is a homomorphism, in fact a $\mathbb{Z}_2$-extension.

Claim 1: (X,T) is minimal and weakly mixing.
Proof: Minimality is clear as already (X,R) is minimal. Weak mixing is most easily seen as follows. Observe first that Y is minimal and proximal hence weakly mixing (Corollary 2.4). Now if $(x,x') \in X \times X$ projects onto a point with dense orbit in $Y \times Y$, then, since on X we can rotate at any angle and reverse the orientation, we see that also (x,x') has a dense orbit in $X \times X$.

Claim 2: $Q^{(3)}$ is not all of X^3.
Proof: The set

$$W = \{(x,y,z) \in X^3 : x, y \text{ and } z \text{ do not lie in an open semicircle}\}$$

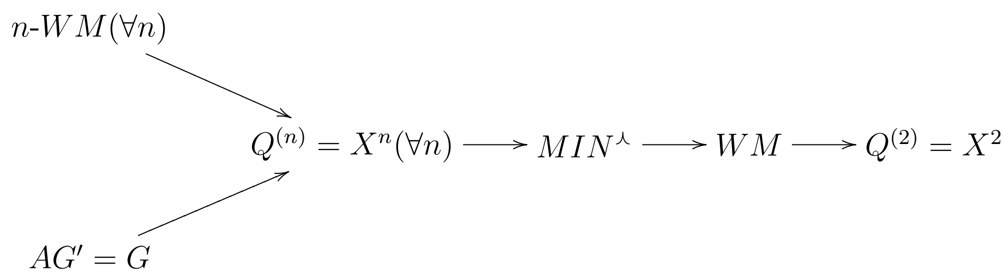
is a closed invariant set with nonempty interior that does not contain the diagonal. Clearly the points of the interior of W do not belong to $Q^{(3)}$.

3. Let (X,S) be as in part 2, where we now restrict the action to the subgroup S of T generated by m and R. For an ordered pair $(x,x') \in X \times X$ let us denote by $\angle(x,x')$

the angle measured counterclockwise from x to x'. We then have $P(X, S) = X \times X \setminus \{(x, -x) : x \in X\}$, hence $Q = \overline{P} = X \times X$. However the set $\{(x, x') : \angle(x, x') \le \pi\}$ is a proper closed invariant subset of $X \times X$ with nonempty interior. Thus (X, S) is not weakly mixing. $\square$

The idea of the example in part 2 of Theorem 4.1 was suggested by J. Auslander. The details were then clarified with the help of E. Akin. I thank them both for letting me include this result in the paper.

I end this section with the following diagram which sums up the known relations between the various "weakly mixing" notions for minimal systems. The class of minimal dynamical systems which are weakly disjoint from every minimal system is denoted by $MIN^{\curlywedge}$. The implication n-$WM \to Q^{(n)} = X^n$, for each $n \ge 2$, follows easily from the definitions. The other implications follow from Theorems 3.1 and 2.3.



The example of a minimal proximal system which is not weakly mixing of order 3 (see the introduction to this paper) shows that the top slanted arrow in this diagram can not be reversed. The example presented in Theorem 4.1.1 shows that the bottom slanted arrow can not be reversed. Theorem 4.1.2 shows that the implication $Q^{(n)} = X^n(\forall n) \to WM$ can not be reversed, and finally Theorem 4.1.3 shows that also the implication $WM \to Q^{(2)} = X^2$ can not be reversed.

5. A property which is an Ellis group invariant

We say that a property of minimal dynamical systems is *an Ellis group invariant* if whenever it holds for a given minimal system it also holds for every other minimal system with a conjugate Ellis group. Equivalently iff it is preserved under proximal factor maps and proximal extensions.

When T is abelian it is well known that for minimal systems weak mixing is an Ellis group invariant. For general acting groups we have:

5.1. **Proposition.** *For minimal systems the property*

$$\overline{P^{(n)}} = X^n(\forall n)$$

is an Ellis group invariant.

In the proof we will use the following definition and lemma (due to J. Auslander).

5.2. **Definition.** Let Y and Z be compact spaces. A continuous map $\rho : Y \to Z$ is called *semi-open* if $\operatorname{int} \rho(U) \ne \emptyset$ whenever $U \subset Y$ is a nonempty open subset of Y.

5.3. **Lemma.** *Let Y be a minimal dynamical system and $\rho : Y \to Z$ a homomorphism. Then the map ρ is semi-open.*

Proof. Let $U \subset Y$ be a nonempty open set. Choose a nonempty open V such that $\overline{V} \subset U$. By minimality there is a finite sequence $\{t_i\}_{i=1}^n$ such that $Y = \cup_{i=1}^n t_i V$. Applying ρ we have $Z = \cup_{i=1}^n t_i \rho(\overline{V})$ and it follows that for some i, $\operatorname{int} t_i \rho(\overline{V}) \neq \emptyset$, whence also $\operatorname{int} \rho(\overline{V}) \neq \emptyset$ and a fortiori $\operatorname{int} \rho(U) \neq \emptyset$. □

Proof of Proposition 5.1. Clearly this property is inherited by factors, hence in particular by proximal factors. Suppose $\pi : (X,T) \to (Y,T)$ is a proximal extension of minimal systems such that $\overline{P(Y)^{(n)}} = Y^n$. Let $U_1 \times U_2 \times \cdots \times U_n$ be a nonempty basic open subset of X^n. Since π is semi-open (Lemma 5.3) we have $V_i = \operatorname{int} \pi(U_i) \neq \emptyset$ for $i = 1, 2, \dots, n$. By assumption there exists a point $(y_1, y_2, \dots, y_n) \in (V_1 \times V_2 \times \cdots \times V_n) \cap P(Y)^{(n)}$. For each i choose some $x_i \in U_i$ with $\pi(x_i) = y_i$. It is easy to see that the fact that π is a proximal extension implies that $(x_1, x_2, \dots, x_n) \in (U_1 \times U_2 \times \cdots \times U_n) \cap P(X)^{(n)}$ and the proof is complete. □

I do not know which of the other properties appearing in the diagram at the end of Section 4 (except of course for the property $AG' = G$) is an Ellis group invariant.

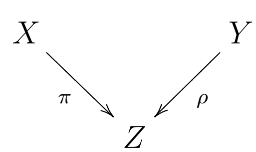
6. The relative case

A relative version of Theorem 2.3 for open extensions holds. It first appeared in J. C. S. P. van der Woude [23]. For completeness I enclose the proof; a straightforward extension of the proof of Theorem II.2.1 in [9].

6.1. **Lemma.** *Let $\rho : Y \to Z$ be a semi-open map, then $\rho(U) \subset \operatorname{cls}(\operatorname{int} \rho(U))$ for every nonempty open $U \subset Y$.*

Proof. Fix $y \in U$, set $z = \rho(y)$ and let V be an arbitrary open neighborhood of z. By continuity of ρ there exists an open neighborhood $A \subset U$ of y such that $\rho(A) \subset V$. Since ρ is semi-open we have $\operatorname{int}(\rho(U)) \supset \operatorname{int}(\rho(A)) \neq \emptyset$. Thus every neighborhood of z meets $\operatorname{int}(\rho(U))$, hence $z \in \operatorname{cls}(\operatorname{int} \rho(U))$. □

6.2. **Lemma.** *Let X and Y be minimal systems,*



a common factor, and suppose that π is open. Then for every nonempty relatively open subset D of the corresponding relation

$$R_{\pi,\rho} = \{(x,y) : \pi(x) = \rho(y)\}$$

there exist open subsets $U \subset X$ and $V \subset Y$ such that

$$\emptyset \neq (U \times V) \cap R_{\pi,\rho} \subset D, \qquad \textit{and} \qquad \pi(U) = \rho(V).$$

Proof. Since we assume that $D \subset R_{\pi,\rho}$ is relatively open and nonempty there are open subsets $U' \subset X$ and $V' \subset Y$ such that $\emptyset \neq (U' \times V') \cap R_{\pi,\rho} \subset D$. Let $\eta : R_{\pi,\rho} \to Z$ be defined by $\eta(x, y) = \pi(x) = \rho(y)$. It is easy to check that

$$\eta((U' \times V') \cap R_{\pi,\rho}) = \pi(U') \cap \rho(V').$$

By Lemma 5.3 ρ is semi-open and by Lemma 6.1 we have $\rho(V') \subset \mathrm{cls}\,(\mathrm{int}\,\rho(V'))$. Since by assumption $\pi(U')$ is open we conclude that $\mathrm{int}\,(\rho(V')) \cap \pi(U') \neq \emptyset$. Thus $O := \mathrm{int}\,(\rho(V') \cap \pi(U')) \neq \emptyset$. We now observe that the sets $U = \pi^{-1}(O) \cap U'$ and $V = \rho^{-1}(O) \cap V'$ satisfy the required properties. □

Here is then a relative version of Theorem 2.3.

6.3. **Theorem.** 1. *Let X and Y be minimal systems,*

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & Z & \end{array}$$

a common factor and suppose that π is open. Suppose that for some $z_0 \in Z$ and every $n \geq 2$ we have $P^{(n)} \cap (\pi^{-1}(z_0))^n$ is dense in $(\pi^{-1}(z_0))^n$, then the system $R_{\pi,\rho}$ is topologically transitive.

2. *In particular, for $X = Y$ metric, we conclude that if $Q^{(n)}_\pi = R^{(n)}_\pi$ for every $n \geq 2$ then π is a weakly mixing extension.*

Proof. 1. Let W be a closed invariant subset of $R_{\pi,\rho}$ with a nonempty interior. By Lemma 6.2 there exist open subsets $U \subset X$ and $V \subset Y$ such that

$$\emptyset \neq (U \times V) \cap R_{\pi,\rho} \subset \mathrm{int}\, W, \qquad \text{and} \qquad \pi(U) = \rho(V).$$

Since the property of z_0 is shared by every point of its orbit Tz_0, we now assume with no loss in generality that for some $x_0 \in U$ we have, with $z_0 = \pi(x_0)$, that $P^{(n)} \cap (\pi^{-1}(z_0))^n$ is dense in $(\pi^{-1}(z_0))^n$ for every $n \geq 2$. By minimality of Y there exists a finite sequence $t_1, t_2, \ldots, t_n$ in T such that $\cup_{i=1}^n t_i V = Y$ and by relabelling we have, for some $1 \leq m \leq n$, $\cup_{i=1}^m t_i V \supset \rho^{-1}(z_0)$ and $t_i V \cap \rho^{-1}(z_0) \neq \emptyset$ for every $1 \leq i \leq m$. (By allowing repetitions we can assume that $2 \leq m \leq n$.)

Consider the set $t_1 U \times t_2 U \times \cdots \times t_m U$. We choose, for each $1 \leq j \leq m$, a point $v_j \in V$ with $t_j v_j = y_j \in \rho^{-1}(z_0)$ and then a point $u_j \in U$ with $\pi(u_j) = \rho(v_j)$. Then we have $\pi(t_j u_j) = t_j \pi(u_j) = t_j \rho(v_j) = \rho(t_j v_j) = z_0$, so that

$$(t_1 u_1, t_2 u_2, \ldots, t_m u_m) \in (t_1 U \times t_2 U \times \cdots \times t_m U) \cap (\pi^{-1}(z_0))^m$$

Thus

$$O := (t_1 U \times t_2 U \times \cdots \times t_m U) \cap (\pi^{-1}(z_0))^m$$

is a nonempty relatively open subset of $(\pi^{-1}(z_0))^m$. By assumption there is then a point $(x_1, x_2, \ldots, x_m) \in O \cap P^{(m)}$.

Next let (x, y) be an arbitrary point of $R_{\pi,\rho}$, with say $z = \pi(x) = \rho(y)$, and let D be a relatively open neighborhood of (x, y) in $R_{\pi,\rho}$. A second application of Lemma 6.2 yields open sets $A \subset X$ and $B \subset Y$ such that $\emptyset \neq (A \times B) \cap R_{\pi,\rho} \subset D$ and $\pi(A) = \rho(B)$.

Pick some $a \in A$, then, as X is minimal, there exists a net $s_k \in T$ with

$$\lim s_k(x_1, x_2, \dots, x_m) = (a, a, \dots, a).$$

Eventually, for every $1 \le i \le m$, we have $s_k x_i = s_k t_i u'_i \in A$ with $u'_i \in U$, hence $\pi(s_k x_i) = \pi(s_k t_i u'_i) \in \pi(A) = \rho(B)$. Since $\pi(x_i) = z_0$ for every i, we have $s_k z_0 = \pi(s_k x_i) = \rho(b)$ for some $b \in B$. Thus $\rho(s_k^{-1} b) = z_0$, hence $s_k^{-1} b \in \rho^{-1}(z_0)$ and therefore $s_k^{-1} b \in \cup_{i=1}^m t_i V$. We fix i_0 with $s_k^{-1} b \in t_{i_0} V$.

We now have on the one hand $(s_k t_{i_0} u'_{i_0}, b) \in A \times B$ and on the other

$$\begin{aligned} (s_k t_{i_0} u'_{i_0}, b) &= s_k(t_{i_0} u'_{i_0}, s_k^{-1} b) \\ &\in s_k(t_{i_0} U \times t_{i_0} V) \\ &= s_k t_{i_0}(U \times V) \subset s_k W = W. \end{aligned}$$

Thus $(A \times B) \cap W \neq \emptyset$ and since $D \supset A \times B$ was an arbitrary neighborhood of $(x, y) \in R_{\pi,\rho}$ we conclude that $(x, y) \in W$; i.e. $W = R_{\pi,\rho}$.

2. We note that for metrizable X, as in Remark 2.5, the conditions "$Q_\pi^{(n)} = R_\pi^{(n)}$" and "$P_\pi^{(n)}$ is dense in $R_\pi^{(n)}$" are equivalent. Since the map $\pi : X \to Z$ is open it follows that also $\pi^n : R_\pi^{(n)} \to Z$ is an open map. Next observe that $P_\pi^{(n)}$ is a G_δ subset of $R_\pi^{(n)}$ hence a dense G_δ by our assumption. We can now apply a topological version of Fubini's theorem (see for example [11, Lemma 5.2 and the following remark]) to conclude that for a dense G_δ subset $Z_0 \subset Z$ the intersection $P^{(n)} \cap (\pi^{-1}(z_0))^n$ is dense in $(\pi^{-1}(z_0))^n$ for every $z_0 \in Z_0$. Now apply part 1 to complete the proof. ◻

6.4. **Corollary.** *Let X be a minimal dynamical system and $\pi : X \to Y$ a proximal open homomorphism. Then π is a weakly mixing homomorphism; i.e. the system R_π is topologically transitive.*

6.5. **Definition.** Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal systems. We will say that a minimal dynamical system (Z, T) is a *relative quasi-Bohr factor of X over Y* if there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Z \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\rho} \\ Y & \xleftarrow[\ \theta\ ]{} & Y^* \end{array}$$

where θ is a proximal extension and ρ is an equicontinuous extension. We say that it is *nontrivial* if the equicontinuous extension ρ is not 1-1.

We are now ready to state the relative version of the main theorem (Theorem 3.1).

6.6. **Theorem.** *Assume X is metric and let $\pi : (X, T) \to (Y, T)$ be an open extension of minimal systems (with Ellis groups $A = \mathcal{G}(X)$ and $F = \mathcal{G}(Y)$). If π is not a weakly mixing extension then there exists a minimal system $\tilde{X}$ which is a proximal extension of X and such that $\tilde{X}$ admits a nontrivial relative quasi-Bohr factor over Y. (An equivalent statement is as follows. The condition $AF' = F$ implies that π is a weakly mixing extension.)*

Proof. Repeat the arguments in the proof of Theorem 3.1 with the obvious modifications. Of course we use Theorem 6.3 instead of Theorem 2.3. For the interested reader here are the details.

Suppose π is not a weakly mixing extension. Then by Theorem 6.3 there exists an $n \geq 2$ with $Q_\pi^{(n)} \neq R_\pi^{(n)}$. This means that for some open neighborhood V of the diagonal Δ_n in $R_\pi^{(n)}$ the closed invariant set $W = \operatorname{cls} TV$ is a proper subset of $R_\pi^{(n)}$ with nonempty interior.

Let

$$\begin{array}{ccc} X & \xleftarrow{\theta^*} & X^* = X \vee Y^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ Y & \xleftarrow{\theta} & Y^* \end{array}$$

be the RIC shadow diagram corresponding to the map $\pi : X \to Y$. Thus $Y^* = \{p \circ Fx_0 : p \in M\} \subset 2^X$ and the minimal system X^*, defined as

$$X^* = \{(px_0, p \circ Fx_0) : p \in M\} \subset X \times Y^*$$

coincides with the subsystem

$$\{(x, B) : x \in B \in Y^*\} \subset X \times Y^*.$$

The homomorphism θ and θ^* are proximal and the projection map $\pi^* : X^* \to Y^*$ is RIC, hence also open.

Set

$$\begin{aligned} R_{\pi^*}^{(n)} &= \{(x_1^*, \dots, x_n^*) \in (X^*)^n : \pi^*(x_i^*) = \pi^*(x_j^*),\ 1 \leq i, j \leq n\} \\ &\cong \{(x_1, \dots, x_n, B) \in X^n \times Y^* : x_j \in B,\ 1 \leq j \leq n\}, \end{aligned}$$

and

$$\begin{aligned} W^* &= \{(x_1, \dots, x_n, B) \in W \times Y^* : x_j \in B,\ 1 \leq j \leq n\} \\ &= R_\pi^{(n)} \cap \{(x_1^*, \dots, x_n^*) \in (X^*)^n : (\theta^*(x_1^*), \dots, \theta^*(x_n^*)) \in W\} \\ &= R_{\pi^*}^{(n)} \cap ((\theta^*)^n)^{-1}(W). \end{aligned}$$

Clearly W^* is a closed invariant subset of $R_{\pi^*}^{(n)}$ containing the diagonal $\Delta_n^* \subset (X^*)^n$. Moreover, we have $\Delta_n^* \subset R_{\pi^*}^{(n)} \cap ((\theta^*)^n)^{-1}(TV) \subset W^*$, so that W^* has nonempty interior in the relative topology of $R_{\pi^*}^{(n)}$. And we can not have $W^* = R_{\pi^*}^{(n)}$ because this will imply $W = R_\pi^{(n)}$. Thus Theorem 2.7 applies and we obtain a commutative diagram

$$\begin{array}{ccc} X^* & & \\ & \searrow{\scriptstyle \sigma} & \\ \downarrow{\scriptstyle \pi^*} & & Z \\ & \swarrow{\scriptstyle \rho} & \\ Y^* & & \end{array}$$

where $\rho : Z \to Y^*$ is a nontrivial equicontinuous extension. This completes the proof of the theorem. (The Ellis group picture here is as follows. Let $\mathcal{G}(X) = \mathcal{G}(X^*) = A$ and $\mathcal{G}(Z) = B$. Since Y^* is a proximal extension of Y we have $\mathcal{G}(Y^*) = \mathcal{G}(Y) = F$.

Since ρ is an equicontinuous extension we have $F' = \mathcal{G}(Y^*)' \subset \mathcal{G}(Z) = B$ and therefore $AF' \subset BF' = B \subsetneq F$.)

$\square$

6.7. **Remark.** The condition that π be an open extension in Theorem 6.6 is clearly necessary. There are easy examples of extensions $\pi : (X, T) \to (Y, T)$ of minimal systems such that π is an almost 1-1 (hence a proximal) extension yet π is not a weakly mixing extension. To mention one specific class of examples we observe that every Toeplitz minimal $\mathbb{Z}$-system X is an almost 1-1 extension of its maximal equicontinuous factor Y and that for some such systems the extension is weakly mixing while for others it is not.

7. Generalized Bohr compactifications

Recall the following definition\theorem from [9], Chapters VIII and IX.

7.1. **Definition.** Let T be a topological group.

1. A minimal dynamical system (X, T) is called a *compactification system for T* if it is a group extension of a proximal system. Equivalently, iff the group $\mathrm{Aut}\,(X, T)$ of automorphisms of (X, T) is compact (in the topology of uniform convergence) and for every pair of points $x, y \in X$ there exists an automorphism $\phi \in \mathrm{Aut}\,(X, T)$ such that $\phi(x)$ is proximal to y.
2. There exists a universal compactification system for T
$$X \to X/K \cong \Pi(T),$$
and the compact Hausdorff topological group $K = \mathrm{Aut}\,(X, T)$ is called the *generalized Bohr compactification of T*.
3. The Ellis group of (X, T) is the derived group $\mathcal{G}(X) = G'$, where $G = \mathrm{Aut}\,(M)$ is the group of automorphisms of the universal minimal system (M, T). Therefore $K \cong G/G'$.

Using the terminology of Section 3, we see that every compactification system is quasi-Bohr, and the canonical minimal group extension associated with every quasi-Bohr system is a compactification system.

In this short section I would like to point out some new results concerning generalized Bohr compactifications of some Polish topological groups which follow from recent works of Pestov, [18], and Glasner and Weiss, [13] and [14]. I remind the reader that a topological group T has the *fixed point on compacta property* if it has a fixed point whenever it acts on a compact space ([12]). This of course is equivalent to the fact that the universal minimal system $(M(T), T)$ is the trivial one point system. Recently a large supply of new examples of such groups, including a monothetic Polish group, was discovered (see e.g. [12] and [19]).

7.2. **Theorem.** 1. *There exist topological groups with trivial generalized Bohr compactification; in fact this is the case for every group with the fixed point on compacta property.*

2. *Let $L = \mathrm{Homeo}(S^1)$ be the Polish group of homeomorphisms of the circle S^1 with the topology of uniform convergence. The natural action of L on S^1 is the universal minimal system as well as the universal minimal proximal system. In particular $L = \mathrm{Homeo}(S^1)$ has a trivial generalized Bohr compactification (see* [18]*).*

3. *Let $H = \mathrm{Homeo}(E)$ be the Polish group of homeomorphisms of the Cantor set E equipped with the topology of uniform convergence. The universal minimal system $(M(H), H)$ is nontrivial and the topological space $M(H)$ is homeomorphic to a Cantor set. Explicitly it is Uspenskij's space of maximal chains on E. Moreover the system $(M(H), H)$ is proximal so that $(M(H), H) \cong (\Pi(H), H)$ is also the universal minimal proximal H-system. Finally, the generalized Bohr compactification of $H = \mathrm{Homeo}(E)$ is trivial (see* [14]*).*

4. *Let $S = S_\infty(\mathbb{Z})$ be the Polish topological group of all permutations of the integers equipped with the topology of pointwise convergence. The universal minimal S system is again a Cantor set. Explicitly it is isomorphic to the natural action of S on the compact space of linear orders on $\mathbb{Z}$. Moreover $M(S)$ is also the universal compactification system for S with $M(S) \to M(S)/\mathbb{Z}_2 \cong \Pi(S)$ so that $\mathbb{Z}_2$, the group with two elements, is the generalized Bohr compactification of $S_\infty(\mathbb{Z})$ (see* [13]*).*

The search for examples of Polish groups with more interesting generalized Bohr compactifications is an intriguing project. An outstanding question in this plan actually regards semisimple Lie groups. In [10] I have shown, using M. Ratner's machinery, that for $SL(2,\mathbb{R})$ there are minimal proximal actions which are not strongly proximal. This result however depends very much on the existence of non-arithmetic lattices. Thus the question whether for a higher rank semisimple Lie group every minimal proximal action is actually strongly proximal is still open. And, if indeed this is the case, then the identification of the generalized strong Bohr compactification of a connected semisimple Lie group $\mathbb{G}$ as $b(\mathbb{A}) \times M$ mentioned in the introduction will hold true for the generalized Bohr compactification as well.

References

[1] J. Auslander, *Minimal Flows and their Extensions*, Mathematics Studies 153, Notas de Matemática, 1988.

[2] R. Ellis *Lectures on Topological Dynamics*, W. A. Benjamin, Inc. , New York, 1969.

[3] ______, *The Veech structure theorem*, Trans. Amer. Math. Soc. **186**, (1973), 203-218.

[4] ______, *Furstenberg structure theorem*, Pacific J. Math. **76**, (1978), 345-349.

[5] R. Ellis, E. Glasner and L. Shapiro, *Proximal-Isometric flows*, Advances in Math., **17**, (1975), 213–260

[6] H. Furstenberg, *The structure of distal flows*, Amer. J. Math., **85**, (1963), 477-515.

[7] H. Furstenberg and E. Glasner, *On the existence of isometric extensions*, Amer. J. of Math. **100**, (1978), 1185-1212.

[8] E. Glasner, *Relatively invariant measures*, Pacific J. Math. Math. **58**, (1975), 393-410.

[9] ______, *Proximal flows*, Lecture Notes in Math. **517**, Springer-Verlag, 1976.

[10] ______, *Proximal flows of Lie groups*, Israel J. Math. **45**, (1983), 97-99.

[11] ______, *A topological version of a theorem of Veech and almost simple flows*, Ergod. Th, & Dynam. Sys. **10**, (1990), 463-482.

[12] ———, *On minimal actions of Polish groups*, Topology and its applications **85**, (1998), 119-125.

[13] E. Glasner and B. Weiss, *Minimal actions of the group* $\mathbb{S}(\mathbb{Z})$ *of permutations of the integers*, Geom. Funct. Anal. **12**, (2002), 964-988.

[14] ———, *The universal minimal system for the group of homeomorphisms of the Cantor set*, Fund. Math. **176**, (2003), 277-289.

[15] D. C. McMahon, *Weak mixing and a note on a structure theorem for minimal transformation groups*, Illinois J. of Math., **20**, (1976), 186-197.

[16] ———, *Relativized weak mixing of uncountable order*, Can. J. Math., **32**, (1980), 559-566.

[17] R. Peleg, *Weak disjointness of transformation groups*, Proc. Amer. Math. Soc. **33**, (1972), 165-170.

[18] V. Pestov, *Some universal constructions in abstract topological dynamics*, Contemporary Mathematics **215**, (1998), 83-99.

[19] ———, *Topological groups: where to from here?*, Proceedings of the 14th Summer Conference on General Topology and its Applications, Topology Proc. **24** (1999), 421-502.

[20] W. A. Veech, *Point-distal flows*, Amer. J. Math., **92**, (1970), 205-242.

[21] ———, *Topological dynamics*, Bull. Amer. Math. Soc. **83**, (1977), 775-830.

[22] B. Weiss, *A survey of generic dynamics*, Descriptive set theory and dynamical systems, LMS Lecture Notes Series **277**, Cambridge University Press, Cambridge, 2000, 273-291.

[23] J.C.S.P. van der Woude, *Topological dynamics*, CWI Tracts **22**, Center for Mathematics and Computer Science, Amsterdam, 1986.

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel
E-mail address: glasner@math.tau.ac.il