CLASSIFYING DYNAMICAL SYSTEMS BY THEIR RECURRENCE PROPERTIES

ELI GLASNER

ABSTRACT. In his seminal paper of 1967 on disjointness in topological dynamics and ergodic theory H. Furstenberg started a systematic study of transitive dynamical systems. In recent years this work served as a basis for a broad classification of dynamical systems by their recurrence properties. In this paper I describe some aspects of this new theory and its connections with combinatorics, harmonic analysis and the theory of topological groups.

CONTENTS

1. INTRODUCTION

At the conference in honor of Hillel Furstenberg, held during two weeks in June 2003, at Jerusalem and Beer-Sheva, I gave a talk sharing the title with the present paper. In fact this paper is an elaboration of that talk and it is mostly a review article.

In his seminal paper of 1967 on disjointness in topological dynamics and ergodic theory [8], Furstenberg started a systematic study of transitive dynamical systems, and the theory was further developed in Furstenberg and Weiss [10] and Furstenberg, [9]. In recent years these works served as a basis for a broad classification of dynamical

Date: October 10, 2003.
1991 *Mathematics Subject Classification.* 54H20.
Key words and phrases. recurrence, transitive, weak mixing, weak disjointness, almost equicontinuity, topologically ergodic, syndetic .

systems by their recurrence properties. In this paper I describe some aspects of the new theory and its connections with combinatorics, harmonic analysis and the theory of topological groups. Works by Glasner & Weiss (1993) [16], Akin, Auslander & Berg (1997) [4], Blanchard, Host & Maass (2000) [6], Weiss (2000) [27], Akin & Glasner (2001) [5] and Huang & Ye (2002) [20], [21], [22] are reviewed.

2. Furstenberg's theorem

A *dynamical system* for us is a pair (X,T) where X is a metrizable compact space and $T : X \to X$ a self homeomorphism. We set, for two non-empty open sets $U,V \subset X$ and a point $x \in X$

$$N(U,V) = \{n \in \mathbb{Z} : T^n U \cap V \neq \emptyset\}, \text{ and}$$
$$N(x,V) = \{n \in \mathbb{Z} : T^n x \in V\}.$$

We say that (X,T) is *topologically transitive* (or just *transitive*) if $N(U,V)$ is nonempty whenever $U,V \subset X$ are two non-empty open sets. Using Baire's category theorem it is easy to see that (X,T) is topologically transitive iff there exists a dense G_δ subset $X_0 \subset X$ such that $\bar{\mathcal{O}}_T(x) = X$ for every $x \in X_0$. Here $\mathcal{O}_T(x) = \{T^n x : n \in \mathbb{Z}\}$ is the orbit of the point x and $\bar{\mathcal{O}}_T(x)$ is the closure in X of $\mathcal{O}_T(x)$. The system (X,T) is *minimal* if $\bar{\mathcal{O}}_T(x) = X$ for every $x \in X$. It is *weakly mixing* if the product system $(X \times X, T \times T)$ is topologically transitive.

2.1. **Theorem** (Furstenberg)**.** *The dynamical system (X,T) is weakly mixing iff the collection*

$$\mathcal{F} = \{N(U,V) : U,V \subset X \quad \text{are non-empty open subsets}\}$$

is a filter base.

Proof. It is easy to see that, for nontrivial (X,T) , both conditions imply that X has no isolated points. Assuming that $\mathcal{F}$ is a filter base we have

$$N(U_1 \times U_2, V_1 \times V_2) = N(U_1,V_1) \cap N(U_2,V_2) \in \mathcal{F}$$

for every $U_1,U_2,V_1,V_2 \subset X$ nonempty open subsets. In particular $N(U_1 \times U_2, V_1 \times V_2)$ is nonempty. This clearly implies that (X,T) is weakly mixing.

Conversely suppose (X,T) is weakly mixing and let $N(U_1,V_1), N(U_2,V_2) \in \mathcal{F}$ be given. Choose $m \in N(U_1,U_2) \cap N(V_1,V_2)$, which is nonempty by weak mixing, and set $A = T^m U_1 \cap U_2,\ B = T^m V_1 \cap V_2$. For any $k \in N(A,B)$

$$\begin{aligned} T^k A \cap B &= T^k(T^m U_1 \cap U_2) \cap (T^m V_1 \cap V_2) \\ &= T^m(T^k U_1 \cap V_1) \cap (T^k U_2 \cap V_2) \neq \emptyset \end{aligned}$$

implies

$$T^k U_1 \cap V_1 \neq \emptyset \quad \text{and} \quad T^k U_2 \cap V_2 \neq \emptyset,$$

i.e.

$$N(A,B) \subset N(U_1,V_1) \cap N(U_2,V_2),$$

so that $\mathcal{F}$ is a filter base. □

We say that a subset $A \subset \mathbb{Z}$ is *thick* if it contains arbitrarily long intervals.

2.2. **Corollary.** *If (X,T) is weakly mixing then every $N(U,V)$ is thick.*

Proof. Given $N(U,V)$ and $k \in \mathbb{N}$, the set $\bigcap_{j=0}^{k} N(U,T^{-j}V)$ is nonempty by Theorem 2.1 and

$$m \in \bigcap_{j=0}^{k} N(U,T^{-j}V) \quad \Rightarrow \quad \forall j,\ 0 \le j \le k,\ T^m U \cap T^{-j}V \neq \emptyset \quad \Rightarrow$$
$$\forall j,\ 0 \le j \le k,\ T^{m+j}U \cap V \neq \emptyset \quad \Rightarrow \quad \{m, m+1, \dots, m+k\} \subset N(U,V).$$

□

With just a little more effort one can show that in fact these classes coincide (see e.g. [3], [13, Theorem 1.11]).

2.3. **Theorem.** *A compact dynamical system (X,T) is weakly mixing iff for every pair of nonempty open subsets $U,V \subset X$ the set $N(U,V) \subset \mathbb{Z}$ is thick.*

3. The standard families

We say that a collection $\mathcal{F}$ of nonempty subsets of $\mathbb{Z}$ is a *family* if it is hereditary upward and *proper* (i.e. $A \subset B$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$, and $\mathcal{F}$ is neither empty nor all of $2^{\mathbb{Z}}$).

With a family $\mathcal{F}$ of nonempty subsets of $\mathbb{Z}$ we associate the *dual family*

$$k\mathcal{F} = \{E : E \cap F \neq \emptyset, \forall\ F \in \mathcal{F}\}.$$

It is easily verified that $k\mathcal{F}$ is indeed a family. Also, for families, $\mathcal{F}_1 \subset \mathcal{F}_2 \ \Rightarrow\ k\mathcal{F}_1 \supset k\mathcal{F}_2$, and $kk\mathcal{F} = \mathcal{F}$.

We say that a family $\mathcal{F}$ is *translation invariant* if for every $F \in \mathcal{F}$ and $j \in \mathbb{Z}$ also $F + j \in \mathcal{F}$. Define the family $\tau\mathcal{F}$ by proclaiming $F \in \tau\mathcal{F}$ iff

$$(F+i_1) \cap (F+i_2) \cap \cdots \cap (F+i_k) \in \mathcal{F}, \quad \forall i_1, i_2, \dots i_k \in \mathbb{Z}.$$

We say that the family $\mathcal{F}$ is *thick* if $\tau\mathcal{F} = \mathcal{F}$. One can easily see that $\tau\mathcal{F}$ is a thick family; i.e. $\tau\tau\mathcal{F} = \tau\mathcal{F}$. And, that $\tau\mathcal{F}$ is the largest thick family contained in $\mathcal{F}$.

3.1. **Examples** (The standard families)**.**

- $\mathcal{B}$ = infinite subsets of $\mathbb{Z}$.
- $k\mathcal{B}$ = co-finite sets; i.e. subsets whose complement is finite.
- $\tau\mathcal{B}$ = thick sets.
- $k\tau\mathcal{B}$ = syndetic subsets. (F is *syndetic* if there exists K, for all $m, F \cap [m, m+K] \neq \emptyset$.)
- $\tau k\tau\mathcal{B}$ = thick-syndetic subsets. (F is *thick-syndetic* if for every M there exists K, $\{m : [m, m+M] \subset F\}$ is K-syndetic.)
- $k\tau k\tau\mathcal{B}$ = piecewise syndetic subsets. (F is *piecewise syndetic* if there exists K such that for every N, there exists $n, [n, n+N] \cap F\}$ is K-syndetic.)

For the family $\mathcal{F} = \mathcal{B}$ we provide the following dictionary:

$\mathcal{B}$	$k\mathcal{B}$	$\tau\mathcal{B}$	$k\tau\mathcal{B}$	$\tau k\tau\mathcal{B}$	$k\tau k\tau\mathcal{B}$
infinite	cofinite	thick	syndetic	thickly syndetic	piecewise syndetic

Table 1. The standard families

4. $\mathcal{F}$ transitivity

For any family $\mathcal{F}$ let $TRS(\mathcal{F})$ be the class of dynamical systems (X,T) such that $N(U,V) \in \mathcal{F}$ for every nonempty open $U,V \subset X$. E.g. in this notation the class of *topologically mixing systems* is $TRS(\text{cofinite})$. We write $\mathbf{RT} = TRS(\text{infinite})$ for the class of *recurrent transitive* dynamical systems. It is not hard to see that when X has no isolated points (X,T) is topologically transitive iff it is recurrent transitive. From this we then deduce that a weakly mixing system is necessarily recurrent transitive. We denote by **WM**, **MIN**, and **E** the classes of weakly mixing, minimal and E-systems, respectively. Recall that (X,T) is an *E-system* if there exists a T-invariant probability measure μ whose support is all of X. By Theorem 2.3 we have $\mathbf{WM} = TRS(\tau\mathcal{B}) = TRS(\text{thick})$. We set $\mathbf{TE} = TRS(k\tau\mathcal{B}) = TRS(\text{syndetic})$, and say that the dynamical systems in this class are *topologically ergodic*.

4.1. **Theorem** (G-W)**.** $\mathbf{MIN}, \mathbf{E} \subset \mathbf{TE}$.

Proof. 1. The claim for **MIN** is immediate by the well known characterization of minimal systems: (X,T) is minimal iff $N(x,U)$ is syndetic for every $x \in X$ and nonempty open $U \subset X$.

2. Given two non-empty open sets U,V in X, choose $k \in \mathbb{Z}$ with $T^kU \cap V \neq \emptyset$. Next set $U_0 = T^{-k}V \cap U$, and observe that $k + N(U_0,U_0) \subset N(U,V)$. Thus it is enough to show that $N(U,U)$ is syndetic for every non-empty open U. We have to show that $N(U,U)$ meets every thick subset $B \subset \mathbb{Z}$. By Poincaré's recurrence theorem, $N(U,U)$ meets every set of the form $A - A = \{n-m : n,m \in A\}$ with A infinite. It is an easy exercise to show that every thick set B contains some $D^+(A) = \{a_n - a_m : n > m\}$ for an infinite sequence $A = \{a_n\}$. Thus $\emptyset \neq N(U,U) \cap \pm D^+(A) \subset N(U,U) \cap \pm B$. Since $N(U,U)$ is symmetric, this completes the proof. □

We remark that most of the claims in this survey about the class **MIN** (including Theorem 4.1) are valid for the larger class of *M-systems*. These are the transitive systems (X,T) with the property that the union of the minimal sets is dense in X (see [16]).

5. Disjointness and weak disjointness

The systematic study of transitive dynamical systems originated in H. Furstenberg's seminal paper [8]. The basic definitions and ideas are there, as well as an outline of many a future development.

Two dynamical systems (X,T) and (Y,T) are *disjoint* if every closed $T\times T$-invariant subset of $X \times Y$ whose projections on X and Y are full, is necessarily the entire space $X \times Y$. It follows easily that when (X,T) and (Y,T) are disjoint, at least one of them must be minimal. If both (X,T) and (Y,T) are minimal then they are disjoint iff the product system is minimal. We say that (X,T) and (Y,T) are *weakly disjoint* when the product system $(X \times Y, T \times T)$ is transitive. This is indeed a very weak sense of disjointness as there are systems which are weakly disjoint from themselves. In fact, by definition a dynamical system is weakly mixing iff it is weakly disjoint from itself.

If $\mathbf{P}$ is a class of transitive dynamical systems (*a property*) we let $\mathbf{P}^{\curlywedge}$ be the class of dynamical systems which are weakly disjoint from every member of $\mathbf{P}$. We clearly have $\mathbf{P} \subset \mathbf{Q} \Rightarrow \mathbf{P}^{\curlywedge} \supset \mathbf{Q}^{\curlywedge}$ and $\mathbf{P}^{\curlywedge\curlywedge\curlywedge} = \mathbf{P}^{\curlywedge}$. As a direct consequence of Furstenberg's Theorem 2.1 we get the following theorem.

5.1. **Theorem.** $TRS(\text{syndetic}) \times TRS(\text{thick}) = \mathbf{TE} \times \mathbf{WM} \subset \mathbf{RT}$, *whence*

$$\mathbf{TE} \subset \mathbf{WM}^{\curlywedge} \quad \textit{and} \quad \mathbf{WM} \subset \mathbf{TE}^{\curlywedge}.$$

The question whether in the last two inclusions we actually have equality naturally presents itself.

QUESTION A: $\mathbf{TE} \subset \mathbf{WM}^{\curlywedge}$, is there an equality?

QUESTION B: $\mathbf{WM} \subset \mathbf{TE}^{\curlywedge}$, are they equal?

6. The complexity function and scattering

Before addressing these questions let us introduce some new definitions (due to Blanchard, Host and Maass [6]).

Let (X, T) be a dynamical system, $\mathcal{U}$ a finite cover. We let $r(\mathcal{U})$ denote the minimal cardinality of a subcover of $\mathcal{U}$ and set $c(n) = c(\mathcal{U}, n) := r(\mathcal{U}_0^n)$ where, as usual $\mathcal{U}_0^n = \mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n}\mathcal{U}$. We call $c(\cdot, \mathcal{U})$ *the complexity function* of the cover $\mathcal{U}$.

6.1. **Lemma** (B-H-M). *Let (X, T) be a dynamical system. The following conditions are equivalent.*

1. *(X, T) is equicontinuous.*
2. *For every open cover $\mathcal{U}$, $c(\mathcal{U}, n)$ is bounded.*

Proof. $1 \Rightarrow 2$: Let ϵ be a Lebesgue number for $\mathcal{U}$. By the equicontinuity there exists an $\eta > 0$ satisfying $d(x, x') < \eta \Rightarrow d(T^n x, T^n x') < \epsilon$, $\forall n \in \mathbb{Z}$. Choose $\{x_1, \ldots, x_k\}$ such that $X = \bigcup_{i=1}^k B_\eta(x_i)$. Then

$$\forall i,\ \forall j,\ \exists U_{ij} \in \mathcal{U} \text{ such that } T^j B_\eta(x_i) \subset U_{ij}$$

$$\therefore \qquad B_\eta(x_i) \subset \bigcap_{j=0}^{n} T^{-j} U_{ij}$$

$$\therefore \qquad \forall n,\ c(n) = r(\mathcal{U}_0^n) \le k.$$

$2 \Rightarrow 1$: Assume 2 and suppose to the contrary that $\{T^n : n \in \mathbb{Z}\}$ is not equicontinuous. Then there exist $y_0 \in X$ and $\epsilon > 0$ such that

$$\forall \delta > 0,\ \exists y \in B_\delta(y_0) \text{ and } n \text{ such that } d(T^n y_0, T^n y) \ge \epsilon.$$

Let $\mathcal{U} = \{B_{\epsilon/4}(x_i) : i = 1, \ldots, k\}$ be a cover of X and set $\hat{\mathcal{U}} = \{\overline{B_{\epsilon/4}(x_i)} = A_i : i = 1, \ldots, k\}$. We have $\hat{\mathcal{U}} \prec \mathcal{U}$ and by assumption the complexity of the cover $\mathcal{U}$ is bounded, say $c(\mathcal{U}, n) \le c(\hat{\mathcal{U}}, n) \le M$ for every $n \in \mathbb{Z}$.

Introduce the auxiliary space $\Omega = \{1, 2, \ldots, k\}^{\mathbb{N}}$ and for each $\omega \in \Omega$ set

$$J(\omega) = \{x \in X : T^j x \in A_{\omega(j)},\ \forall j \in \mathbb{N}\} = \bigcap_{j \in \mathbb{N}} T^{-j} A_{\omega(j)}.$$

I claim that there exist M "names" $(\omega^1, \omega^2, \dots, \omega^M) \in \Omega^M$ such that

$$X = \bigcup_{i=1}^{M} J(\omega^i). \tag{6.1}$$

To see this, let for each $n \in \mathbb{N}$

$$J_n(\omega) = \{x \in X : T^j x \in A_{\omega(j)},\ \forall j \in [0,n]\} = \bigcap_{j\in[0,n]} T^{-j} A_{\omega(j)}.$$

Our assumption $c(\hat{\mathfrak{U}}, n) \le M$ implies that for every $n \in \mathbb{N}$ there are M "names" $(\omega^1, \omega^2, \dots, \omega^M) \in \Omega^M$ such that

$$X = \bigcup_{i=1}^{M} J_n(\omega^i). \tag{6.2}$$

Of course this implies that also $X = \bigcup_{i=1}^{M} J_{n-1}(\omega^i)$. Thus, denoting by $H(n)$ the subset of Ω^M satisfying (6.2) we see that (i) $H(n)$ is nonempty, (ii) $H(n)$ is closed (whether a vector $(\omega^1, \omega^2, \dots, \omega^M) \in \Omega^M$ is in $H(n)$ or not, depends only on the first n coordinates of each component ω^i) and (iii) $H(n) \subset H(n-1)$.

By compactness $H = \bigcap_{n\in\mathbb{N}} H(n)$ is nonempty. Fix $(\omega^1, \omega^2, \dots, \omega^M) \in H$ and recall that for $1 \le i \le M$, $J(\omega^i) = \bigcap_{n\in\mathbb{N}} J_n(\omega^i)$. It is now easy to check that (6.2) implies (6.1). (In fact if $x \in X$ then for every n there exists an i_n with $x \in J_n(\omega^{i_n})$. Then, there exists an i for which $x \in J_{n_\ell}(\omega^i)$ for infinitely many ℓ, hence $x \in J(\omega^i)$.) This completes the proof of (6.1).

Note that for each i, every $x, x' \in J(\omega^i)$ — as $J(\omega^i) = \bigcap_{j\in\mathbb{N}} T^{-j} A_{\omega^i(j)}$ — we have for every $j \in \mathbb{N}$,

$$d(T^j x, T^j x') < \epsilon/2. \tag{6.3}$$

By assumption there are sequences $y_n \to y_0$ and $k_n \in \mathbb{N}$ such that

$$d(T^{n_k} y_0, T^{k_n} y_n) \ge \epsilon. \tag{6.4}$$

By (6.1) there exists an $1 \le i \le M$ such that $y_n \in J(\omega^i)$ for infinitely many n. Since $J(\omega^i)$ is closed this implies that also $y_0 \in J(\omega^i)$ and comparing (6.3) with (6.4) we get a contradiction. □

A dynamical system (X, T) is called *scattering* if every finite open cover by nondense sets has unbounded complexity function. We write **SCT** for the class of scattering systems in **RT**.

6.2. **Theorem** (B-H-M)**.** $\mathbf{SCT} = \mathbf{MIN}^{\curlywedge}$.

Note that the inclusion $\mathbf{MIN} \subset \mathbf{TE}$ implies $\mathbf{SCT} = \mathbf{MIN}^{\curlywedge} \supset \mathbf{TE}^{\curlywedge} \supset \mathbf{WM}$.

6.3. **Corollary** (B-H-M)**.** $\mathbf{WM} \subset \mathbf{SCT}$.

QUESTION C: $\mathbf{WM} \subset \mathbf{SCT}$, are they equal?

7. The Weiss-Akin-Glasner theorem

The next theorem (Weiss [27] and Akin & Glasner [5]) will be the key to the solution of some of the above mentioned questions as well as to other problems of a similar nature.

7.1. **Theorem** (W-A-G)**.** *Let $\mathcal{F}$ be a proper translation invariant thick family of subsets of $\mathbb{Z}$. A dynamical system is in $TRS(k\mathcal{F})$ iff it is weakly disjoint from every system in $TRS(\mathcal{F})$:*

$$TRS(k\mathcal{F}) = TRS(\mathcal{F})^{\curlywedge}.$$

In particular, for $\mathcal{F} = \tau\mathcal{B} =$ thick, we get

$$TRS(\text{synd}) = \mathbf{TE} = \mathbf{WM}^{\curlywedge} = TRS(\text{thick})^{\curlywedge},$$

and for $\mathcal{F} = \tau k \tau \mathcal{B} =$ thick-synd, we get

$$TRS(\text{pw-synd}) = (\mathbf{WM} \cap \mathbf{TE})^{\curlywedge} = TRS(\text{thick-synd})^{\curlywedge}.$$

Outline of proof. By definition

$$TRS(k\mathcal{F}) \subset TRS(\mathcal{F})^{\curlywedge}.$$

For the other direction one needs the following lemma whose rather intricate 'combinatorial' proof we omit.

7.2. **Lemma.** *For $\mathcal{F}$ as in the theorem, $A \in \mathcal{F}$, and $0 \in A$ imply that there exists a subshift $(X, \sigma) \in TRS(\mathcal{F})$ (i.e. a subsystem of the Bernoulli system (Ω, σ) where $\Omega = \{0,1\}^{\mathbb{Z}}$ and $\sigma : \Omega \to \Omega$ is the shift) for which*

$$A = N(U_0[1], U_0[1])$$

(here $U_0[1] = \{\omega \in X : \omega(0) = 1\}$).

Now suppose (Y, S) is transitive but not $k\mathcal{F}$ transitive. Then there exists a nonempty open $U \subset Y$ such that $N(U,U) \notin k\mathcal{F}$, hence $B = N(U,U)^c \in \mathcal{F}$. (In fact $D \notin k\mathcal{F} \Rightarrow \exists\, F \in \mathcal{F}$ such that $D \cap F = \emptyset \Rightarrow F \subset D^c \Rightarrow D^c \in \mathcal{F}$.)

Let $A = B \cup \{0\}$, then $A \in \mathcal{F}$ and applying the above lemma to A we construct a subshift (X, σ) with $A = N(U_0[1], U_0[1])$. Considering the product system $(Y \times X, S \times \sigma)$ we have

$$N(U \times U_0[1], U \times U_0[1]) = N(U,U) \cap A = \{0\}.$$

We conclude that the product system is not transitive and we have thus shown that $(Y,S) \notin TRS(k\mathcal{F}) \Rightarrow (Y,S) \notin TRS(\mathcal{F})^{\curlywedge}$. ∎

This answers question A in the affirmative.

7.3. **Theorem** (Weiss)**.** $\mathbf{WM}^{\curlywedge} = \mathbf{TE}$.

8. Almost equicontinuity, monothetic groups and a fixed point property

Transitive dynamical systems which are *not sensitive to initial conditions* were studied by Glasner and Weiss in [16], where it was shown that under some mild additional condition, such as being an E-system, such systems are isomorphic to a rotation on a compact monothetic group. It was also shown there that any transitive uniformly rigid system admits a transitive non-sensitive extension. The latter fact, when combined with a result of Glasner & Maon [14] which provides examples of nontrivial minimal uniformly rigid and weakly mixing systems, demonstrates the prevalence of non-sensitive systems.

In the paper [4], by Akin, Auslander & Berg, the notion 'non-sensitivity' was given a better name: 'almost equicontinuity'. In this work the authors rediscovered some of the results of [16] and gave the class of almost equicontinuous systems a systematic and comprehensive treatment.

Recall that a point x in a dynamical system (X, T) is an *equicontinuity point* if for every $\epsilon > 0$ there exists a neighborhood U of x such that $\sup_{n\in\mathbb{Z}} d(T^n x, T^n x') \le \epsilon$ for all $x' \in U$. A dynamical system (X, T) is called *almost equicontinuous* (AE for short) if it contains a dense set of equicontinuity points. An AE system is uniformly rigid and the set

$$\Lambda = \Lambda(X, T) = \text{unif-cls}\,\{T^n : n \in \mathbb{Z}\} \subset \text{Homeo}\,(X)$$

is a Polish monothetic group. ((X, T) is *uniformly rigid* iff the Polish group $\Lambda(X, T)$ is not discrete, see [14].) If an AE system (X, T) is also transitive, then the set $EQ(X)$ of equicontinuity points coincides with the dense G_δ set of transitive points. Moreover for a transitive uniformly rigid system (X, T) and any transitive point $x_0 \in X$, the map $S \mapsto Sx_0$ is a homeomorphism of $\Lambda(X, T)$ onto $\Lambda x_0 \subset X$ — with the relative topology it inherits from X — iff (X, T) is AE. Finally, if Λ is any Polish non-discrete monothetic topological group then there exists a transitive AE system (X, T) with $\Lambda = \Lambda(X, T)$.

In [12] the following terminology was introduced. A topological group G has the *fixed point on compacta* property (FPC) if every compact G dynamical system has a fixed point. Recently the theory of Polish groups with the the fixed point on compacta property received a lot of attention and new and exciting connections with other branches of mathematics (like Ramsey theory, Gromov's theory of mm-spaces, and concentration of measure phenomena) were discovered (see V. Pestov's survey paper [25]). In [12] I show that the Polish group G of all measurable functions f from a nonatomic Lebesgue measure space $(\Omega, \mathcal{B}, m)$ into say $[0, 1]$, with pointwise product and the topology of convergence in m-measure, is monothetic and has the FPC property.

I refer the reader to Akin's book 'Recurrence in topological dynamics' [3] where many of the subjects of the present review, including AE systems, are treated in depth. More recently the notion of locally equicontinuous (LE) systems was introduced by Glasner and Weiss in [17]. These are the systems (X, T) with the property that for every $x \in X$ the subsystem $\bar{\mathcal{O}}_T(x)$ is AE. It turns out that every weakly almost periodic (WAP) system is LE and intricate new examples of LE systems which are not WAP were discovered in [17]. The class of LE systems and the related class of hereditarily almost equicontinuous (HAE) systems are studied in details in a work by Glasner and Megrelishvili [15].

9. Scattering but not weakly mixing systems

9.1. **Theorem** (A-G)**.** *Let (X, T) be a transitive AE system. The following conditions are equivalent.*

1. $(X, T) \in \mathbf{MIN}^{\curlywedge} = \mathbf{SCT}$.
2. *The Polish monothetic group $\Lambda(X, T)$ has the fixed point property.*

This theorem provides a negative solution to question C, as follows.

9.2. **Corollary** (A-G)**.** $\mathbf{WM} \subsetneq \mathbf{SCT}$

Proof. Let Λ be any Polish monothetic topological group with the FPC property (such as the one described in Section 8). Let (X, T) be a transitive AE system with $\Lambda = \Lambda(X, T)$. By Theorem 9.1 $(X, T) \in \mathbf{SCT}$. Suppose $(X, T) \in \mathbf{WM}$, so that $(X \times X, T \times T)$ is transitive. Let $(x_0, x_0') \in X \times X$ be a transitive point. If x is an arbitrary point of X, then there exists a sequence $\{n_i\}$ in $\mathbb{Z}$ such that $\lim_{i\to\infty}(T^{n_i}x_0, T^{n_i}x_0') = (x_0, x)$. However, being AE, the topology induced on $\mathbb{Z}$ by the relative topology of $\mathcal{O}_T(x_0)$ is the same as the relative topology induced on $\mathbb{Z}$ when embedded in Λ. Therefore $\lim_{i\to\infty} T^{n_i}x_0 = x_0$ implies $\lim_{i\to\infty} T^{n_i} = e$ in Λ, whence $\lim_{i\to\infty} T^{n_i}x_0' = x = x_0$. Thus $X = \{x_0\}$ is the trivial one point system and Λ is the trivial one element group. This contradiction implies that $(X, T) \notin \mathbf{WM}$ and the proof is complete. □

Recently, Huang and Ye have constructed explicit examples of dynamical systems (in fact subshifts) $(X, T) \in \mathbf{SCT} \setminus \mathbf{WM}$ [20], as well as $(X, T) \in \mathbf{TE}^{\curlywedge} \setminus \mathbf{WM}$ [21]. The latter is a (negative) solution to question B.

9.3. **Theorem** (H-Y)**.** $\mathbf{WM} \subsetneq \mathbf{TE}^{\curlywedge}$.

10. Topological mild mixing

The notion of mild mixing was first introduced in ergodic theory by Furstenberg and Weiss in [11].

10.1. **Definition.** Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be a measure dynamical system.

1. The system $\mathbf{X}$ is *rigid* if there exists a sequence $n_k \nearrow \infty$ such that
$$\lim \mu\left(T^{n_k}A \cap A\right) = \mu(A)$$
for every measurable subset A of X. We say that $\mathbf{X}$ is $\{n_k\}$-*rigid*.
2. An ergodic system is *mildly mixing* if it has no non-trivial rigid factor.

The authors show that the mild mixing property is equivalent to the following multiplier property.

10.2. **Theorem.** *An ergodic system* $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ *is mildly mixing iff for every ergodic (finite or infinite) measure preserving system* $(Y, \mathcal{Y}, \nu, T)$*, the product system*

$$(X \times Y, \mu \times \nu, T \times T),$$

is ergodic.

Since every Kronecker system is rigid and since an ergodic system $\mathbf{X}$ is weakly mixing iff it admits no nontrivial Kronecker factor, it follows that mild mixing implies weak mixing. Clearly strong mixing implies mild mixing. It is not hard to construct rigid weakly mixing systems, so that the class of mildly mixing systems is properly contained in the class of weakly mixing systems. Finally there are mildly but not strongly mixing systems; e.g. Chacón's system is an example (see del Junco, Rahe and Swanson [23] and Aaronson and Weiss [1]).

We say that a subset J of $\mathbb{Z}$ has *uniform density 1* if for every $0 < \lambda < 1$ there exists an N such that for every interval $I \subset \mathbb{Z}$ of length $> N$ we have $|J \cap I| \geq \lambda |I|$. We denote by $\mathcal{D}$ the family of subsets of $\mathbb{Z}$ of uniform density 1.

Let $\mathcal{F}$ be a family of nonempty subsets of $\mathbb{Z}$ which is closed under finite intersections (i.e. $\mathcal{F}$ is a filter). Following [9] we say that a sequence $\{x_n : n \in \mathbb{Z}\}$ in a topological space X *$\mathcal{F}$-converges to a point* $x \in X$ if for every neighborhood V of x the set $\{n : x_n \in V\}$ is in $\mathcal{F}$. We denote this by

$$\mathcal{F}\text{- } \lim x_n = x.$$

We have the following characterization of weak mixing for measure preserving systems.

10.3. **Theorem.** *The dynamical system* $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ *is weakly mixing iff for every* $A, B \in \mathcal{X}$ *we have*

$$\mathcal{D}\text{ - } \lim \mu(T^{-n} A \cap B) = \mu(A)\mu(B).$$

An analogous characterization of measure theoretical mild mixing is obtained by considering the families of *IP* and *IP** sets. An *IP-set* is any subset of $\mathbb{Z}$ containing a subset of the form $IP\{n_i\} = \{n_{i_1} + n_{i_2} + \cdots + n_{i_k} : i_1 < i_2 < \cdots < i_k\}$, for some infinite sequence $\{n_i\}_{i=1}^{\infty}$. We let $\mathcal{I}$ denote the family of *IP*-sets and call the elements of the dual family $k\mathcal{I} = \mathcal{I}^*$, *IP***-sets*. Again it is not hard to see that the family of *IP**-sets is closed under finite intersections. For a proof of the next theorem we refer to [9].

10.4. **Theorem.** *The dynamical system* $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ *is mildly mixing iff for every* $A, B \in \mathcal{X}$ *we have*

$$\mathcal{I}^*\text{ - } \lim \mu(T^{-n} A \cap B) = \mu(A)\mu(B).$$

We now turn to the topological category. It will be convenient here to deal with families of subsets of $\mathbb{Z}_+$ rather than $\mathbb{Z}$. If $\mathcal{F}$ is such a family then

$$TRS(\mathcal{F}) = \{(X, T) : N_+(A, B) \in \mathcal{F} \text{ for every nonempty open } A, B \subset X\}.$$

Here $N_+(A, B) = N(A, B) \cap \mathbb{Z}_+$. Let us call a subset of $\mathbb{Z}_+$ a *SIP-set* (symmetric *IP*-set), if it contains a subset of the form

$$SIP\{n_i\} = \{n_\alpha - n_\beta > 0 : \; n_\alpha, n_\beta \in IP\{n_i\} \cup \{0\}\},$$

for an IP sequence $IP\{n_i\} \subset \mathbb{Z}_+$. Denote by $\mathcal{S}$ the family of SIP sets. It is not hard to show that

$$\mathcal{F}_{\rm thick} \subset \mathcal{S} \subset \mathcal{I},$$

(see [9]). Hence $\mathcal{F}_{\rm syndetic} \supset \mathcal{S}^* \supset \mathcal{I}^*$, hence $TRS(\text{synd}) \supset TRS(\mathcal{S}^*) \supset TRS(\mathcal{I}^*)$, and finally

$$TRS(\text{synd})^{\curlywedge} \subset TRS(\mathcal{S}^*)^{\curlywedge} \subset TRS(\mathcal{I}^*)^{\curlywedge}.$$

10.5. **Definition.** A topological dynamical system (X, T) is called *topologically mildly mixing* if it is in $TRS(\mathcal{S}^*)$ and we denote the collection of topologically mildly mixing systems by $\mathbf{MM} = TRS(\mathcal{S}^*)$.

We will need the following proposition; for a proof refer to [18].

10.6. **Proposition.** *Let (X,T) be a topologically transitive dynamical system; then the following conditions are equivalent:*

1. *$(X,T) \in$ **RT**.*
2. *The recurrent points are dense in X.*

10.7. **Theorem.** *A dynamical system is in **RT** iff it is weakly disjoint from every topologically mildly mixing system:*

$$\mathbf{RT} = \mathbf{MM}^{\curlywedge}.$$

And conversely it is topologically mildly mixing iff it is weakly disjoint from every recurrent transitive system:

$$\mathbf{MM} = \mathbf{RT}^{\curlywedge}.$$

Proof. 1. Since $TRS(\mathcal{S}^*)$ is nonvacuous (for example every topologically mixing system is in $TRS(\mathcal{S}^*)$), it follows that every system in $TRS(\mathcal{S}^*)^{\curlywedge}$ is in **RT**.

Conversely, assume that (X,T) is in **RT** but (X,T) $\notin TRS(\mathcal{S}^*)^{\curlywedge}$, and we will arrive at a contradiction. By assumption there exists (Y,T) $\in TRS(\mathcal{S}^*)$ and a nondense nonempty open invariant subset $W \subset X \times Y$. Then $\pi_X(W) = O$ is a nonempty open invariant subset of X. By assumption O is dense in X. Choose open nonempty sets $U_0 \subset X$ and $V_0 \subset Y$ with $U_0 \times V_0 \subset W$. By Proposition 10.6 there exists a recurrent point x_0 in $U_0 \subset O$. Then there is a sequence $n_i \to \infty$ such that for the IP-sequence $\{n_\alpha\} = IP\{n_i\}_{i=1}^\infty$, $IP\text{-}\lim T^{n_\alpha} x_0 = x_0$ (see [9]). Choose i_0 such that $T^{n_\alpha} x_0 \in U_0$ for $n_\alpha \in J = IP\{n_i\}_{i \ge i_0}$ and set $D = SIP(J)$. Given V a nonempty open subset of Y we have:

$$D \cap N(V_0, V) \neq \emptyset.$$

Thus for some α, β and $v_0 \in V_0$,

$$T^{n_\alpha - n_\beta}(T^{n_\beta} x_0, v_0) = (T^{n_\alpha} x_0, T^{n_\alpha - n_\beta} v_0) \in (U_0 \times V) \cap W.$$

We conclude that

$$\{x_0\} \times Y \subset \operatorname{cls} W.$$

The fact that in an **RT** system the recurrent points are dense together with the observation that $\{x_0\} \times Y \subset \operatorname{cls} W$ for every recurrent point $x_0 \in O$, imply that W is dense in $X \times Y$, a contradiction.

2. From part 1 of the proof we have $\mathbf{RT} = TRS(\mathcal{S}^*)^{\curlywedge}$, hence $\mathbf{RT}^{\curlywedge} = TRS(\mathcal{S}^*)^{\curlywedge\curlywedge} \supset TRS(\mathcal{S}^*)$.

Suppose (X,T) $\in \mathbf{RT}$ but (X,T) $\notin TRS(\mathcal{S}^*)$, we will show that (X,T) $\notin \mathbf{RT}^{\curlywedge}$. There exist $U, V \subset X$, nonempty open subsets and an *IP*-set $I = IP\{n_i\}$ for a monotone increasing sequence $\{n_1 < n_2 < \cdots\}$ with

$$N(U, V) \cap D = \emptyset,$$

where

$$D = \{n_\alpha - n_\beta : n_\alpha, n_\beta \in I,\ n_\alpha > n_\beta\}.$$

If (X,T) is not topologically weakly mixing then $X \times X \notin \mathbf{RT}$ hence (X,T) $\notin \mathbf{RT}^{\curlywedge}$. So we can assume that (X,T) is topologically weakly mixing. Now in $X \times X$

$$N(U \times V, V \times U) = N(U, V) \cap N(V, U) = N(U, V) \cap -N(U, V),$$

is disjoint from $D \cup -D$, and replacing X by $X \times X$ we can assume that $N(U, V) \cap (D \cup -D) = \emptyset$. In fact, if $X \in \mathbf{RT}^{\curlywedge}$ then $X \times Y \in \mathbf{RT}$ for every $Y \in \mathbf{RT}$, therefore $X \times (X \times Y) \in \mathbf{RT}$ and we see that also $X \times X \in \mathbf{RT}^{\curlywedge}$.

By going to a subsequence, we can assume that

$$\lim_{k\to\infty} n_{k+1} - \sum_{i=1}^{k} n_i = \infty.$$

in which case the representation of each $n \in I$ as $n = n_\alpha = n_{i_1} + n_{i_2} + \cdots + n_{i_k};\ \alpha = \{i_1 < i_2 < \cdots < i_k\}$ is unique.

Next let $y_0 \in \{0,1\}^{\mathbb{Z}}$ be the sequence $y_0 = \mathbf{1}_I$. Let Y be the orbit closure of y_0 in $\{0,1\}^{\mathbb{Z}}$ under the shift T, and let $[1] = \{y \in Y : y(0) = 1\}$. Observe that

$$N(y_0, [1]) = I.$$

It is easy to check that

$$IP\text{-}\lim T^{n_\alpha} y_0 = y_0.$$

Thus the system (Y, T) is topologically transitive with y_0 a recurrent point; i.e. (Y,T) $\in \mathbf{RT}$.

We now observe that

$$N([1], [1]) = N(y_0, [1]) - N(y_0, [1]) = I - I = D \cup -D \cup \{0\}.$$

If $X \times Y$ is topologically transitive then in particular

$$N(U \times [1], V \times [1]) = N(U, V) \cap N([1], [1]) =$$
$$N(U, V) \cap (D \cup -D \cup \{0\}) = \text{ infinite set}.$$

But this contradicts our assumption. Thus $X \times Y \notin \mathbf{RT}$ and (X,T) $\notin \mathbf{RT}^{\curlywedge}$. This completes the proof. □

We now have the following:

10.8. **Corollary.** *Every topologically mildly mixing system is weakly mixing and topologically ergodic:*

$$\mathbf{MM} \subset \mathbf{WM} \cap \mathbf{TE}.$$

Proof. We have $TRS(\mathcal{S}^*) \subset \mathbf{RT} = TRS(\mathcal{S}^*)^\curlywedge$, hence for every (X,T) $\in TRS(\mathcal{S}^*)$, $X \times X \in \mathbf{RT}$ i.e. (X,T) is topologically weakly mixing. And, as we have already observed the inclusion $\mathcal{F}_{\text{syndetic}} \supset \mathcal{S}^*$, entails $\mathbf{TE} = TRS(\text{synd}) \supset TRS(\mathcal{S}^*) = \mathbf{MM}$. □

To complete the analogy with the measure theoretical setup we next define a topological analogue of rigidity. This is just one of several possible definitions of topological rigidity and we refer to [14] for a treatment of these notions.

10.9. **Definition.** A dynamical system (X, T) is called *uniformly rigid* if there exists a sequence $n_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \sup_{x\in X} d(T^{n_k}x, x) = 0,$$

i.e. $\lim_{k\to\infty} T^{n_k} = \text{id}$ in the uniform topology on the group of homeomorphism of $H(X)$ of X. We denote by $\mathcal{R}$ the collection of topologically transitive uniformly rigid systems.

In [14] the existence of minimal weakly mixing but nonetheless uniformly rigid dynamical systems is demonstrated. However, we have the following:

10.10. **Lemma.** *A system which is both topologically mildly mixing and uniformly rigid is trivial.*

Proof. Let (X,T) be both topologically mildly mixing and uniformly rigid. Then

$$\Lambda = \text{cls}\,\{T^n : n \in \mathbb{Z}\} \subset H(X),$$

is a Polish monothetic group.

Let T^{n_i} be a sequence converging uniformly to id, the identity element of Λ. For a subsequence we can ensure that $\{n_\alpha\} = IP\{n_i\}$ is an *IP*-sequence such that $IP\text{-}\lim T^{n_\alpha} = \text{id}$ in Λ. If X is nontrivial we can now find an open ball $B = B_\delta(x_0) \subset X$ with $TB \cap B = \emptyset$. Put $U = B_{\delta/2}(x_0)$ and $V = TU$; then by assumption $N(U, V)$ is an SIP^*-set and in particular:

$$\forall \alpha_0\ \exists \alpha, \beta > \alpha_0,\ n_\alpha - n_\beta \in N(U, V).$$

However, since $IP\text{-}\lim T^{n_\alpha} = \text{id}$, we also have eventually, $T^{n_\alpha - n_\beta}U \subset B$; a contradiction. □

10.11. **Corollary.** *A topologically mildly mixing system has no nontrivial uniformly rigid factors.*

We conclude this section with the following result which shows how these topological and measure theoretical notions are related.

10.12. **Theorem.** *Let (X, T) be a topological dynamical system with the property that there exists an invariant probability measure μ with full support such that the associated measure preserving dynamical system $(X, \mathcal{X}, \mu, T)$ is measure theoretically mildly mixing then (X, T) is topologically mildly mixing.*

Proof. Let (Y, S) be any system in **RT**; by Theorem 10.7 it suffices to show that $(X \times Y, T \times S)$ is topologically transitive. Suppose $W \subset X \times Y$ is a closed $T \times S$-invariant set with $\operatorname{int} W \neq \emptyset$. Let $U \subset X, V \subset V$ be two nonempty open subsets with $U \times V \subset W$. By transitivity of (Y, S) there exits a transitive recurrent point $y_0 \in V$. By theorems of Glimm and Effros [19], [7], and Katznelson and Weiss [24] (see also Weiss [26]), there exists a (possibly infinite) invariant ergodic measure ν on Y with $\nu(V) > 0$.

Let μ be the probability invariant measure of full support on X with respect to which $(X, \mathcal{X}, \mu, T)$ is measure theoretically mildly mixing. Then by [11] the measure $\mu \times \nu$ is ergodic. Since $\mu \times \nu(W) \geq \mu \times \nu(U \times V) > 0$ we conclude that $\mu \times \nu(W^c) = 0$ which clearly implies $W = X \times Y$. □

This section is based on [18]. For more on these topics refer to [9], [3], [27], [5], [20], [21] and [22].

11. Monothetic Polish groups admit nontrivial weakly mixing actions

Given a Polish monothetic non-discrete group Λ, we say that a dynamical system (X, T) *extends to* Λ if (X, T) is uniformly rigid and the group Λ acts on X extending the action of $\mathbb{Z} \cong \{T^n : n \in \mathbb{Z}\}$. In other words the map $a^n \mapsto T^n$ (where a is a topological generator of Λ), from $\{a^n\} \subset \Lambda$ into $\Lambda(X, T)$, extends to a continuous surjective homomorphism. The 'dual family theorem', Theorem 7.1, is instrumental in proving the next result.

11.1. **Theorem** (A-G). *Let $\mathcal{F}$ be a proper translation invariant thick family of subsets of $\mathbb{Z}$. Let Λ a Polish monothetic non-discrete group and let (X, x_0, T) be an AE system with transitive point x_0 and with $\Lambda(X, T) = \Lambda$. The following conditions are equivalent.*

1. *The point x_0 is $k\mathcal{F}$-recurrent; i.e. $\forall \epsilon > 0,\ \{n \in \mathbb{Z} : d(x_0, T^n x_0) < \epsilon\} \in k\mathcal{F}$.*
2. *(X, T) is $k\mathcal{F}$-transitive.*
3. *Any transitive system (Y, T) which extends to Λ is $k\mathcal{F}$-transitive.*
4. *The only $\mathcal{F}$-transitive system (Z, T) which extends to Λ is the trivial system.*

Recall our notation $\mathcal{D}$ for the family of subsets of $\mathbb{Z}$ with uniform density 1. One can check that $\mathcal{D}$ is a thick, translation invariant family and that $k\mathcal{D}$ is the family of subsets of $\mathbb{Z}$ with positive upper Banach density.

11.2. **Corollary.** *If (X, T) is a transitive AE but not equicontinuous system (which is the same as not being minimal), then it is neither $k\mathcal{D}$-transitive, nor weakly mixing, nor TE.*

Proof. Let V be an open nonempty subset of X and x_0 a transitive point. It is easy to verify that $N(V, V) = N(x_0, V) - N(x_0, V)$. If $A = N(x_0, V) \in k\mathcal{D}$ then a well known result implies that the difference set $N(V, V) = A - A$ is syndetic so that (X, T) $\in TRS(\text{synd}) = TRS(k\mathcal{F})$, with $\mathcal{F} = \text{thick}$. Applying Theorem 11.1 we conclude that x_0 is $k\mathcal{F}$-recurrent; i.e. syndetically recurrent, hence minimal. Since both weak mixing and topological ergodicity imply $k\mathcal{D}$-transitivity, our claims follow. □

11.3. **Theorem** (A-G). *For every non-discrete, non-compact, Polish monothetic group Λ, there exists a nontrivial $\mathcal{D}$-transitive dynamical system (X,T) to which Λ extends. Such a system is both weakly mixing and TE.*

Proof. Let (X,T) be a transitive AE system with $\Lambda = \Lambda(X,T)$. Since Λ is non-compact (X,T) is not equicontinuous. By Corollary 11.2 (X,T) is neither $k\mathcal{D}$-transitive, nor weakly mixing, nor TE. It follows that the dynamical system (X,T) does not satisfy condition 1 in Theorem 11.1 (with $\mathcal{F} = \mathcal{D}$), and by that theorem neither is the equivalent condition 4 fulfilled. We therefore conclude that there exists a nontrivial transitive system (Z,T) which extends to Λ and is $\mathcal{D}$-transitive, hence both WM and TE. □

Recall that a topological group G is called *minimally almost periodic* (MAP for short) if it admits no nontrivial continuous homomorphism into a compact group. Or equivalently iff it admits no nontrivial minimal equicontinuous action on a compact space. There are many examples of MAP monothetic Polish groups (see e.g. [2]). The following problem was posed in [12].

PROBLEM 1: Is there a Polish monothetic group Λ, which is MAP but does not have the fixed point on compacta property?

In [12] it is shown that a positive answer to problem 1 will provide a negative answer to the following famous problem from combinatorial number theory.

PROBLEM 2: Is it true that for every syndetic subset $S \subset \mathbb{Z}$ the difference set $S-S$ is a Bohr neighborhood? (I.e. is there a finite set of real numbers $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $\epsilon > 0$ such that $\{n \in \mathbb{Z} : \max_j\{\|n\lambda_j\| < \epsilon\}\}$ is contained in $S - S$, where $\|\cdot\|$ denotes the distance to the closest integer.)

In view of Theorem 11.3 it is also natural to ask the following question.

PROBLEM 3: Is there a Polish monothetic group Λ, which does not admit any nontrivial *minimal* weakly mixing dynamical systems but does not have the fixed point on compacta property?

Note that from the general structure theory of minimal systems it follows that if Λ is both MAP and does not admit any nontrivial minimal weakly mixing dynamical systems, then it has the fixed point on compacta property.

12. Various degrees of scattering

In their paper [20] Huang and Ye introduce the following terminology. Applying the $\curlywedge$ operation to the chain of inclusions

$$\mathbf{Equi} \subset \mathbf{MIN} \subset \mathbf{E} \subset \mathbf{TE},$$

(where **Equi** stands for the class of transitive equicontinuous systems) one obtains the corresponding chain

$$\mathbf{Equi}^{\curlywedge} \supset \mathbf{MIN}^{\curlywedge} \supset \mathbf{E}^{\curlywedge} \supset \mathbf{TE}^{\curlywedge}.$$

Motivated by the characterization **SCT** = $\mathbf{MIN}^{\curlywedge}$ they call the class $\mathbf{Equi}^{\curlywedge}$ *weak scattering* and the classes $\mathbf{E}^{\curlywedge}$ and $\mathbf{TE}^{\curlywedge}$ *strong scattering* and *extreme scattering* respectively.

Combining some folklore knowledge with new observations they characterize these classes as follows.

- Weak scattering = $\mathbf{Equi}^{\curlywedge}$ coincides with the class of Bohr-transitive systems, where the latter is the class of all systems for which every $N(U,V)$ meets every Bohr neighborhood.
- Scattering = $\mathbf{MIN}^{\curlywedge}$ coincides with the class of systems in which every $N(U,V)$ is a set of recurrence; i.e. $N(U,V)$ meets every $S-S$ where S is syndetic,
- Strong scattering = $\mathbf{E}^{\curlywedge}$ coincides with the class of systems such that every $N(U,V)$ is a Poincaré set; i.e. $N(U,V)$ meets every $A-A$ where A is a subset of positive upper Banach density.

By constructing appropriate subshifts Huang and Ye show that

$$\mathbf{WM} \subsetneq \mathbf{TE}^{\curlywedge} \subsetneq \mathbf{E}^{\curlywedge}.$$

The question whether the classes of strong scattering, scattering, and weak scattering are equal is open and in fact depends on the solution of problem 2.

Equi	**MIN**	**E**	**TE**
$\mathbf{Equi}^{\curlywedge}$ Weak scattering TRS(Bohr)	$\mathbf{MIN}^{\curlywedge}$ Scattering TRS(Recurrence)	$\mathbf{E}^{\curlywedge}$ Strong scattering TRS(Poincaré)	$\mathbf{TE}^{\curlywedge}$ Extreme scattering

TABLE 2. Degrees of scattering

13. THE STANDARD CLASSES OF TRANSITIVE DYNAMICAL SYSTEMS

I have described in this survey, some of the key ideas and results which were produced recently in the subject of classification of transitive dynamical systems. The diagram in Table 3 supplies further information, albeit in a rather concise and incomplete form. For more details the reader is advised to consult the original papers, some of which are indicated in the references list.

The entries in the diagram appear as names of classes with their $\mathcal{F}$-transitive characterization below (when one is available).

$A^{\curlywedge}$ denotes the class of systems which are weakly disjoint from the class A, $\hookrightarrow$ is just $\subset$, and $\dashrightarrow$ means taking $\curlywedge$ of a class. The k above an arrow indicates that, in addition, the passage is to the dual family. Recall that $\mathcal{B}$ is the family of infinite subsets of $\mathbb{Z}$. The various classes are:

- **WM** = weak mixing = thick-transitive
- **TE** = topologically ergodic = syndetic-transitive
- $\mathbf{TE}^{\curlywedge}$ = synd$^{\curlywedge}$ (is there an $\mathcal{F}$-transitivity description for this class?),
- $\mathbf{WM} \cap \mathbf{TE}$ = (thick-syndetic)-transitive

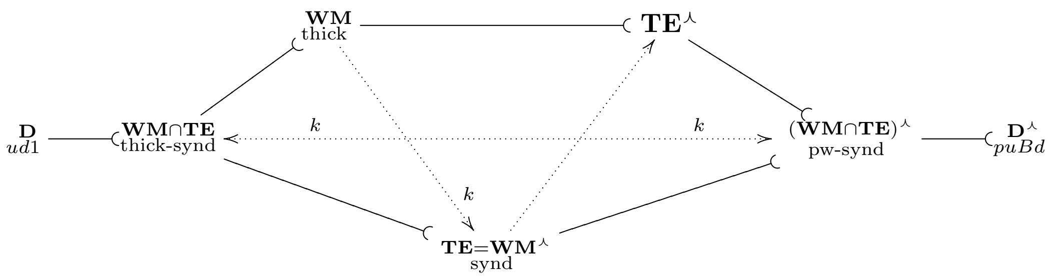
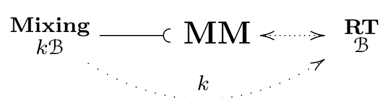


TABLE 3. The standard classes of dynamical systems

- $(\mathbf{WM} \cap \mathbf{TE})^{\curlywedge}$ = (piecewise-syndetic)-transitive
- $\mathbf{D}$ = (uniform density 1)-transitive
- $\mathbf{D}^{\curlywedge}$ = (positive upper Banach density)-transitive

Further information, which is hard to fit into the diagram, is as follows:

1. By the W-A-G theorem
$$\mathbf{D} \overset{k}{\dashrightarrow} \mathbf{D}^{\curlywedge} \, .$$
2. By the W-A-G theorem and the results of Section 10



REFERENCES

[1] J. Aaronson and B. Weiss, *Remarks on tightness of cocycles*, Collloq. Math. **84/85**, Part 2, (2000), 363-376.

[2] M. Ajtai, I. Havas and J. Komlos, *Every group admits a bad topology*, in *Studies in pure mathematics* (P. Erdos, ed.) Birkhäuser, Basel (1983).

[3] E. Akin, *Recurrence in topological dynamics, Furstenberg families and Ellis actions*, Plenum Press, New York and London, 1997.

[4] E. Akin, J. Auslander, and K. Berg, *When is a transitive map chaotic* , Convergence in Ergodic Theory and Probability, Walter de Gruyter & Co. 1996, pp. 25-40.

[5] E. Akin and E. Glasner, *Residual properties and almost equicontinuity*, J. d'Analyse Math. **84**, (2001), 243-286.

[6] F. Blanchard, B. Host and A. Maass *Topological complexity*, Ergod. Th. Dynam. Sys. **20**, (2000), 641-662.

[7] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. **81**, (1976), 38-55.

[8] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. System Theory **1**, (1967), 1-49.

[9] ———, *Recurrence in ergodic theory and combinatorial number theory*, Princeton university press, Princeton, N.J., 1981.

[10] H. Furstenberg and B. Weiss, *Topological dynamics and combinatorial number theory*, J. d'Analyse Math. **34**, (1978), 61-85.

[11] H. Furstenberg and B. Weiss, *The finite multipliers of infinite transformation*, Spriger Verlag Lecture Notes in Math. **688**, (1978), 127-132.

[12] E. Glasner, *On minimal actions of Polish groups*, Topology and Its Applications, **85**, (1998), 119-125.

[13] ———, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, **101**, American Mathematical Society, Providence, RI, 2003.

[14] E. Glasner and D. Maon, *Rigidity in topological dynamics*, Ergod. Th. Dynam. Sys. **9**, (1989), 309-320.

[15] E. Glasner and M. Megrelishvili, *Linear representations of hereditarily non-sensitive dynamical systems*, forthcoming.

[16] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity **6**, (1993), 1067-1075.

[17] ——— *Locally equicontinuous dynamical systems*, Collloq. Math. **84/85 Part 2** dedicated to the memory of Anzelm Iwanik, (2000), 345-361.

[18] ——— *On the interplay between measurable and topological dynamics*, to appear in: Handbook of dynamical systems, **Vol. 1.B**. Editors: B. Hasselblatt and A. Katok.

[19] J. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101**, (1961), 124-138.

[20] W. Huang and X. Ye, *An explicite sacttering, non-weakly mixing example and weak disjointness*, Nonlinearity **15**, (2002), 849-862.

[21] ———, *Dynamical systems disjoint from any minimal system*, to appear in Trans. Amer. Math. Soc.

[22] ———, *Topological complexity, return times and weak disjointness*, to appear in Ergod. Th. Dynam. Sys.

[23] A. del Junco, M. Rahe and L. Swanson *Chacón's automorphism has minimal self-joinings*, J. d'Analyse Math. **37**, (1980), 276-284.

[24] Y. Katznelson and B. weiss, *When all points are recurrent/generic* , Ergodic theory and dynamical systems I, Proceedings, Special year, Maryland 1979-80, Birkhäuser, Boston, 1981.

[25] V. G. Pestov, *Topological groups: where to from here?*, Topology Proceedings, **24**, (1999), 421-502.

[26] B. Weiss, *Measurable dynamics*, Contemporary Mathematics **26**, Conference in Modern Analysis and Probability, (1984), 395-421.

[27] ———, *A survey of generic dynamics*, In *Descriptive set theory and dynamical systems*, LMS Lecture note Series **277**, Cambridge University Press, 2000, 273-291.

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel

E-mail address: glasner@math.tau.ac.il