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ENVELOPING SEMIGROUPS IN TOPOLOGICAL DYNAMICS

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Abstract. This is a survey of the theory of enveloping semigroups in topological dynamics. We review the, already classical, theory of enveloping semigroups, due mainly to Robert Ellis, and then proceed to describe some new connections which were discovered in the last few years between three seemingly unrelated theories: of enveloping semigroups, of chaotic behavior, and of representation of dynamical systems on Banach spaces.

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Introduction

The enveloping semigroup of a dynamical system was introduced by R. Ellis in [24]. It proved to be a fundamental tool in the abstract theory of topological dynamical systems. However, explicit computations of enveloping semigroups are quite rare. Some examples are to be found in Furstenberg [31] (1963), Glasner [35] (1976) and [38] (1993), Namioka [73] (1984), Milnes [70] (1986) and [71] (1989), Ellis [26] (1993), Berg, Gove & Hadad [9] (1998), Akin [3] (1998), Budak, Işik, Milnes & Pym [14] (2001), and Glasner & Megrelishvili [44] (2004). One reason for the scarcity of concrete examples

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of enveloping semigroups is that these objects are usually non-metrizable (a notable exception is the case of weakly almost periodic metric systems; see Downarowicz [20] (1998) and Glasner [41] (2003), Theorem 1.48).

In the last half a dozen years or so, new fascinating connections were discovered between three seemingly unrelated topics: the theory of enveloping semigroups, the theory of chaotic behavior, and the theory of representation of dynamical systems on Banach spaces.

In an interesting paper [67], Köhler pointed out the relevance of a theorem of Bourgain, Fremlin & Talagrand [11] to the study of enveloping semigroups. She calls a dynamical system, (X, ϕ), where X is a compact Hausdorff space and $\phi : X \to X$ a continuous map, *regular* if for every function $f \in C(X)$ the sequence $\{f \circ \phi^n : n \in \mathbb{N}\}$ does not contain an ℓ_1 sub-sequence (the sequence $\{f_n\}_{n\in\mathbb{N}}$ is an ℓ_1 *sequence* if there are strictly positive constants a and b such that

$$a \sum_{k=1}^{n} |c_k| \le \left\| \sum_{k=1}^{n} c_k f_k \right\| \le b \sum_{k=1}^{n} |c_k|$$

for all $n \in \mathbb{N}$ and $c_1, \dots, c_n \in \mathbb{R}$). Since the word "regular" is already overused in topological dynamics I call such systems *tame*.

It turns out that, for a general topological group G, a metric dynamical system (X, G) has this property if and only if $E(X, G)$, the enveloping semigroup of (X, G), is Rosenthal compact (see [44] and Section 6 below). Following a pioneering paper of Megralishvili [69], Glasner and Megrelishvili in [44] established a Bourgain-Fremlin-Talagrand dynamical dichotomy. In this work we develop a comprehensive study of the interconnections between various (non) chaotic properties of dynamical systems on the one hand and their linear representations on Banach spaces on the other. In [42] I introduced the notion of tame dynamical systems and studied their basic properties. Next came the determinative work of Glasner, Megrelishvili and Uspenskij [47] where, for metrizable systems, we show that metrizability of the enveloping semigroup is equivalent to the HNS (hereditary non-sensitivity) property. The most recent development along this line is to be found in the recent work [46] of Glasner and Megrelishvili where the authors characterize tame dynamical systems as those which can be represented on Banach spaces which do not contain isomorphic copies of l_1 (Rosenthal spaces). Finally in three recent works Huang [59], Kerr and Li [66], and Glasner [43], improve the results of [42] to show that, for Abelian acting group, a metrizable minimal tame system is an almost 1-1 extension of its maximal Kronecker factor. Moreover such a system is uniquely ergodic and measure theoretically isomorphic to its Kronecker factor.

In the following notes I will briefly review the, already classical, theory of enveloping semigroups, due mainly to Robert Ellis, and will then proceed to describe some of the new developments sketched above.

In the first section we recall the necessary background from topological dynamics. In the second we define the enveloping semigroup and review some of the well known results concerning its structure. We describe the close connections which exist between the algebraic and topological properties of the enveloping semigroup and various dynamical properties of the system. In section 3 we discuss the class of WAP (weakly almost periodic) systems. Section 4 presents some concrete computations of enveloping semigroups, and section 5 deals with the special case of nil-systems of class 2. In section 6 we first state the enveloping semigroups version of the Bourgain-Fremlin-Talagrand dichotomy, and then investigate the tame side of this dichotomy. Section 7 deals with operator enveloping semigroups and the corresponding notion of injective systems. Section 8 introduces the basic definitions concerning linear representations of dynamical systems on Banach spaces. We also describe here the aforementioned connections between non-chaotic systems and Banach space representations. Section 9 presents the Glasner-Megrelishvili-Uspenskij characterization of HNS (hereditarily non sensitive) metric systems as those having metrizable enveloping semigroup. Some corollaries of this characterization are discussed in section 10. Section 11 describes the characterization of tame systems as those which can be represented on Rosenthal spaces. Section 12 comprise a short table summarizing the hierarchy of Banach representations. Section 13 is a review of recent results on the structure of minimal tame dynamical systems. In the final section I discuss briefly two related topics: universal point-transitive and minimal systems, and the interplay between topological dynamics and combinatorial number theory.

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1. Topological dynamics background

A *topological dynamical system* or briefly a *system* is a pair (X, G), where X is a compact Hausdorff space and G a topological group which acts on X as a group of homeomorphisms. Thus the action is given by a continuous map $G \times X \to X$, $(g, x) \mapsto gx$, such that, with e the neutral element of G, $ex = x$ and $(hg)x = h(gx)$ for all $g, h \in G$ and $x \in X$. Thus the G-action is given by a continuous homomorphism $i : G \to \mathrm{Homeo}\,(X)$, $i(g) = \breve{g}$ equipped with the uniform topology. Usually, we identify g with $\breve{g}$ and write gx for $\breve{g}x$. In the special case of a *cascade*; i.e. a G-dynamical system with $G = \mathbb{Z}$, the group of integers, we usually write (X, T) instead of $(X, \mathbb{Z})$, where $T = i(1)$.

A *subsystem* of (X, G) is a nonempty closed invariant subset $Y \subset X$ with the restricted action. For a point $x \in X$, we let $\mathcal{O}_G(x) = \{gx : g \in G\}$, and $\overline{\mathcal{O}}_G(x) = \mathrm{cls}\,\{gx : g \in G\}$. These subsets of X are called the *orbit* and *orbit closure* of x respectively. We say that (X, G) is *point transitive* if there exists a point $x \in X$ with a dense orbit. In that case x is called a *transitive point.* If every point is transitive we say that (X, G) is a *minimal system.* Clearly a dynamical system (X, G) is minimal if and only if its only subsystem is (X, G) itself. For a general system (X, G) a point $x \in X$ is a *minimal* (or an *almost periodic*) point if $\overline{\mathcal{O}}_G(x)$ is a minimal subsystem of (X, G) .

The dynamical system (X, G) is *topologically transitive* if for any two nonempty open subsets U and V of X there exists some $g \in G$ with $gU \cap V \neq \emptyset$. Clearly a point transitive system is topologically transitive and when X is metrizable the converse holds as well: in a metrizable topologically transitive system the set of transitive points is a dense G_δ subset of X. The system (X, G) is *weakly mixing* if the product system $(X \times X, G)$ (where $g(x, x') = (gx, gx'),\ x, x' \in X,\ g \in G)$ is topologically transitive.

If (Y, G) is another system then a continuous onto map $\pi : X \to Y$ satisfying $g \circ \pi = \pi \circ g$ for every $g \in G$ is called a *homomorphism* of dynamical systems. In this case we say that (Y, T) is a *factor* of (X, G) and also that (X, G) is an *extension* of (Y, G).

For a collection of systems $\{(X_\alpha, G)\}_{\alpha \in A}$ we define the *product system* (X, G) on the product space $\prod_{\alpha \in A} X_\alpha$ by the diagonal action of G on X; i.e., $(gx)_\alpha = gx_\alpha$.

It is sometimes convenient to work with *pointed dynamical systems*, where one picks a distinguished point $x_0 \in X$. We assume that the system (X, G) is point transitive and the distinguished point is assumed to be transitive: $\overline{\mathcal{O}}_G(x_0) = X$. (Such a pointed system is also called an *ambit*). A homomorphism $\pi : (X, x_0, G) \to (Y, y_0, G)$ sends x_0 onto y_0. The *joint* or *pointed product* of a collection of pointed systems $\{(X_\alpha, x_\alpha, G)\}_{\alpha \in A}$ is the subsystem

$$(X, x_0, G) = \bigvee_{\alpha \in A} (X_\alpha, x_\alpha, G)$$

of the product space $\prod_{\alpha \in A} X_\alpha$, defined as the orbit closure $\overline{\mathcal{O}}_G(x_0) = X$ of the point $x_0 \in \prod_{\alpha \in A} X_\alpha$, with $(x_0)_\alpha = x_\alpha$.

We say that a pointed dynamical system (X, x_0, G) is *point-universal* if it has the property that for every $x \in X$ there is a homomorphism $\pi_x : (X, x_0) \to (\overline{\mathcal{O}}_G(x), x)$.

A pair of points $(x, x') \in X \times X$ for a system (X, G) is called *proximal* if there exists a net $g_i \in G$ and a point $z \in X$ such that $\lim g_i x = \lim g_i x' = z$. We denote by P the set of proximal pairs in $X \times X$. We have

$$P = \bigcap \{GV : V \text{ a neighborhood of the diagonal in } X \times X\}.$$

A system (X, G) is called *proximal* when $P = X \times X$ and *distal* when $P = \Delta$, the diagonal in $X \times X$.

The *regionally proximal relation* on X is defined by

$$Q = \bigcap \{\overline{GV} : V \text{ a neighborhood of } \Delta \text{ in } X \times X\}.$$

It is easy to verify that Q is trivial — i.e. equals Δ — if and only if the system is *equicontinuous*. A minimal equicontinuous system is called a *Kronecker system*. Every minimal system admits a maximal Kronecker factor. Furthermore when, in addition, G is Abelian the relation Q is a closed G-invariant equivalence relation and the quotient map $X \to X/Q$ realizes the maximal Kronecker factor of (X, G) . The latter is trivial when and only when the minimal system is weakly mixing.

An extension $(X, G) \xrightarrow{\pi} (Y, G)$ of minimal systems is called a *proximal extension* if the relation $R_\pi = \{(x, x') : \pi(x) = \pi(x')\}$ satisfies $R_\pi \subset P$ and a *distal extension* when $R_\pi \cap P = \Delta$. One can show that every distal extension is open. An extension π is an *almost 1-1 extension* if there is a point $y \in Y$ with $\pi^{-1}(y) = \{x\}$ a single point of X. It is easy to see that an almost 1-1 extension is proximal. A metric minimal system (X, G) such that canonical map $\pi : (X, G) \to (Y, G)$, where (Y, G) is the maximal Kronecker factor of (X, G) , is almost 1-1, is called an *almost automorphic* system.

Recall that a function $f \in C(X)$ is *almost periodic* (AP) if its G-orbit $\{{}^g f : g \in G\}$ lies in a norm compact subset of the Banach space $C(X)$. It is *weakly almost periodic* (WAP) if its G-orbit is contained in a weakly compact subset of $C(X)$. Here $C(X)$ is the algebra of continuous real valued functions on X and for $g \in G$, ${}^g f(x) = f(gx)$. The dynamical system (X,G) is called *almost periodic* (AP) if every $f \in C(X)$ is AP. As is well known this is the case if and only if (X,G) is equicontinuous. The system (X,G) is called *weakly almost periodic* (WAP) if every $f \in C(X)$ is WAP.

Suppose now that X is metrizable and let d be a compatible metric on X. We say that (X,G) is *non-sensitive* if for every $\epsilon > 0$ there exists a non-empty open set $O \subset X$ such that for every $g \in G$ the set gO has d-diameter $< \epsilon$. (This property does not depend on the choice of a compatible metric d.) A system (X,G) is *hereditarily non-sensitive* (HNS) if all closed G-subsystems are non-sensitive.

A system (X,G) is *equicontinuous at* $x \in X$ if for every $\epsilon > 0$ there exists a neighborhood O of x such that for every $x' \in O$ and every $g \in G$ we have $d(gx', gx) < \epsilon$. A system is *almost equicontinuous* (AE) if it is equicontinuous at a dense set of points, and *hereditarily almost equicontinuous* (HAE) if every closed subsystem is AE.

Denote by Eq_ϵ the union of all open sets $O \subset X$ such that for every $g \in G$ the set gO has diameter $< \epsilon$. Then Eq_ϵ is open and G-invariant. Let $Eq = \bigcap_{\epsilon>0} Eq_\epsilon$. Note that a system (X,G) is non-sensitive if and only if $Eq_\epsilon \neq \emptyset$ for every $\varepsilon > 0$, and (X,G) is equicontinuous at $x \in X$ if and only if $x \in Eq$. Suppose now that Eq_ϵ is dense for every $\varepsilon > 0$. Then Eq is dense, in virtue of the Baire category theorem, and it follows that (X,G) is AE.

If (X,G) is non-sensitive and $x \in X$ is a *transitive point* — that is, Gx is dense — then for every $\varepsilon > 0$ the open invariant set Eq_ϵ meets Gx and hence contains Gx. Thus $x \in Eq$ and we have shown that *in a metric transitive non-sensitive system* (X,G) *every transitive point is an equicontinuity point and in particular* (X,G) *is AE.* If, in addition, (X,G) is minimal then $Eq = X$. Thus *minimal non-sensitive systems are equicontinuous* (see [7], [48, Theorem 1.3], [4], and [44, Corollary 5.15]).

For the general theory of abstract topological dynamics we refer the reader to the books [54], [25], [84], [35], [13], [6], [86], [1] and [2].

2. The enveloping semigroup of a dynamical system

The *enveloping semigroup* $E = E(X,G) = E(X)$ of a dynamical system (X,G) is defined as the closure in X^X (with its compact, usually non-metrizable, pointwise convergence topology) of the set $\breve{G} = \{\breve{g} : X \to X\}_{g \in G}$ considered as a subset of X^X. With the operation of composition of maps this is a right topological semigroup (i.e. for every $p \in E(X)$ the map $R_p : q \mapsto qp$, $R_p : E(X) \to E(X)$ is continuous). Moreover, the map $i : G \to E(X), g \mapsto \breve{g}$ is a right topological semigroup compactification of G (see the definition in Section 10 below).

Proposition 2.1. *The enveloping semigroup of a dynamical system (X,G) is isomorphic (as a dynamical system) to the pointed product*

$$(E', \omega_0) = \bigvee \{(\overline{\mathcal{O}}_G(x), x) : x \in X\} \subset X^X,$$

where ω_0 is the point in X^X defined by $\omega_0(x) = x$ for every $x \in X$.

Proof. It is easy to see that the map $p \mapsto p\omega_0$, $(E, i(e), G) \to (E', \omega_0, G)$ is an isomorphism of pointed systems. $\square$

Proposition 2.2. *The following conditions on the pointed dynamical system* (X, x_0, G) *are equivalent:*

1. (X, x_0) *is point-universal.*
2. (X, x_0, G) *is isomorphic, as a dynamical system, to its enveloping semigroup* $(E(X), i(e), G)$ *via the map* $p \mapsto px_0$, $E \to X$.

The elements of $E(X, G)$ may behave very badly as maps of X into itself; usually they are not even Borel measurable. However our main interest in the enveloping semigroup lies in its algebraic structure and its dynamical significance. A key lemma in the study of this algebraic structure is the following:

Lemma 2.3 (Ellis-Numakura). *Let* L *be a compact Hausdorff semigroup in which all maps* $p \mapsto pq$ *are continuous. Then* L *contains an* idempotent*; i.e., an element* v *with* $v^2 = v$.

Proof. By Zorn's lemma, there exists a minimal compact subsemigroup $K \subset L$. For any $v \in K$, Kv is a compact subsemigroup of K whence $Kv = K$ and in particular for some $k \in K$, $kv = v$. Now the set $M = \{l \in K : lv = v\}$ is a non-empty closed subsemigroup of K, and again we deduce that $M = K$. In particular $vv = v$. $\square$

In the next series of propositions we state some of the basic properties of the enveloping semigroup $E = E(X, G)$. Most of these are easy consequences of the definitions and Lemma 2.3, but some require deeper arguments, like Ellis' joint continuity theorem [21].

Proposition 2.4.
1. *A subset* M *of* E *is a minimal left ideal of the semigroup* E *if and only if it is a minimal subsystem of* (E, G)*. In particular a minimal left ideal is closed. We will refer to it simply as a* minimal ideal*. Minimal ideals* M *in* E *exist and for each such ideal the set of idempotents in* M*, denoted by* $J = J(M)$*, is non-empty.*
2. *Let* M *be a minimal ideal and* J *its set of idempotents then:*
   (a) *For* $v \in J$ *and* $p \in M$, $pv = p$.
   (b) *For each* $v \in J$, $vM = \{vp : p \in M\} = \{p \in M : vp = p\}$ *is a subgroup of* M *with identity element* v*. For every* $w \in J$ *the map* $p \mapsto wp$ *is a group isomorphism of* vM *onto* wM.
   (c) $\{vM : v \in J\}$ *is a partition of* M*. Thus if* $p \in M$ *then there exists a unique* $v \in J$ *such that* $p \in vM$.
3. *Let* K, L*, and* M *be minimal ideals of* E*. Let* v *be an idempotent in* M*, then there exists a unique idempotent* v' *in* L *such that* $vv' = v'$ *and* $v'v = v$*. (We write* $v \sim v'$ *and say that* v' *is* equivalent *to* v*.) If* $v'' \in K$ *is equivalent to* v'*, then* $v'' \sim v$*. The map* $p \mapsto pv'$ *of* M *to* L *is an isomorphism of* G*-systems.*

Proposition 2.5. *We have the following connections between dynamical properties of the system* (X, G) *and algebraic properties of* $E = E(X, G)$*. Here* M *denotes an arbitrary but fixed minimal ideal in* E*,* J *denotes the collection of idempotents in* M*, and* $\hat{J}$ *is the collection of all minimal idempotents in* E *(i.e. those idempotents which belong to minimal ideals).*

1. $\overline{\mathcal{O}}_G x = Ex$
2. $\overline{\mathcal{O}}_G x$ *is minimal iff for every minimal ideal* M *in* E, $\overline{\mathcal{O}}_G x = Mx$ *iff in every minimal ideal there is an idempotent* v *such that* $vx = x$. *Thus* $JX = \{vx : v \in J,\ x \in X\}$ *is the set of minimal points of the system* (X, G). *Applying this to the product system we see that* $J(X \times X) = \{(vx, vx') : v \in J,\ (x, x') \in X \times X\}$ *is the set of minimal points in* $X \times X$.
3. *The pair* (x, x') *is proximal iff* $px = px'$ *for some* $p \in E$ *iff there exists a minimal ideal* M *in* E *with* $px = px'$ *for every* $p \in M$.
4. *If* (X, G) *is minimal, then*
$$P[x] = \{x' \in X : (x, x') \in P\} = \{vx : v \in \hat{J}\}.$$
In particular $x \in X$ *is a distal point iff* $vx = x$ *for every* $v \in \hat{J}$.
5. *For* $v \in \hat{J}$ *every pair of distinct points in* vX *is distal (i.e. not proximal).*
6. *The relation* P *is transitive iff* E *contains a unique minimal ideal.*
7. (X, G) *is distal iff* E *is a group.*
8. *A distal system is pointwise minimal (i.e. every point belongs to a minimal set).*
9. (X, G) *is distal iff* $X \times X$ *is pointwise minimal.*
10. *A factor of a distal system is distal.*
11. (X, G) *is equicontinuous iff* E *is a group of homeomorphisms of* X *and the topologies of pointwise and uniform convergence coincide on* E.
12. (X, G) *is equicontinuous iff* E *is a group of continuous maps. (This requires Ellis' joint continuity theorem* [21]*.)*

3. WAP dynamical systems

The following characterization of WAP dynamical systems is due to Ellis and Nerurkar [27] (see also Ellis [23]) and is based on a result of Grothendieck [57] (namely: pointwise compact bounded subsets in $C(X)$ are weakly compact for every compact X).

Theorem 3.1 (Ellis-Nerurkar)**.** *Let* (X, G) *be a dynamical system. The following conditions are equivalent.*

1. (X, G) *is WAP.*
2. *The enveloping semigroup* $E(X, G)$ *consists of continuous maps.*

In their paper [5] Akin, Auslander and Berg obtain the following characterization of AE systems.

Theorem 3.2 (Akin-Auslander-Berg)**.** *Let* (X, G) *be a compact metrizable system. The following conditions are equivalent.*

1. (X, G) *is almost equicontinuous.*
2. *There exists a dense* G_δ *subset* $X_0 \subset X$ *such that every member of the enveloping semigroup* $E(X, G)$ *is continuous on* X_0.

From this result they deduce that every compact metric WAP system is AE, [5]. Since every subsystem of a WAP system is WAP it follows from Theorems 3.1 and 3.2 that every metrizable WAP system is both AE and HAE.

Finally we have the following result of Downarowicz [20] (see also [41, Theorem 1.48]). When the acting group G is Abelian, a point transitive WAP system is always isomorphic to its enveloping semigroup, which in this case is a commutative semitopological semigroup. Thus for such G the class of all metric, point transitive, WAP systems coincides with the class of all metrizable, commutative, semitopological semigroup compactifications of G. In [20] one can find many interesting examples of WAP but not equicontinuous $\mathbb{Z}$-systems.

4. Some concrete examples of enveloping semigroups

Example 4.1. (See e.g. [41]) Let (X, G) be a point transitive system. Then the action of G on X is equicontinuous if and only if $K = E(X, G)$ is a compact topological group whose action on X is jointly continuous and transitive. It then follows that the system (X, G) is isomorphic to the homogeneous system $(K/H, G)$, where H is a closed subgroup of K and G embeds in K as a dense subgroup. When G is Abelian $H = \{e\}$ is trivial, and $E(X, G) = K$. In particular, for $G = \mathbb{Z}$ the collection of Kronecker (= minimal equicontinuous) systems coincides with the collection of compact Hausdorff monothetic topological groups.

Example 4.2. (See [85] and [45] for an enhanced version.) Let G be a semisimple analytic group with finite center and without compact factors. For simplicity suppose further that G is a direct product of simple groups. In his paper [85] Veech shows that the algebra $WAP(G)$, of bounded, right uniformly continuous, weakly almost periodic real valued functions on G, coincides with the algebra $\mathcal{W}^*$ of continuous functions on G which extend continuously to the product of the one-point compactification of the simple components of G ([85, Theorem 1.2]). In particular we have:

Theorem. For a simple Lie group G with finite center (e.g., $SL_n(\mathbb{R})$) $WAP(G) = \mathcal{W}^*$. The corresponding universal WAP compactification is equivalent to the one point compactification $X = G^*$ of G. Thus $E(X, G) = X$.

A similar but a bit more interesting situation occurs in the following example.

Example 4.3. (See [45]) Let $G = S(\mathbb{N})$ be the Polish topological group of all permutations of the set $\mathbb{N}$ of natural numbers (equipped with the topology of pointwise convergence). Consider the one point compactification $X^* = \mathbb{N} \cup \{\infty\}$ and the associated natural G action (G, X^*). For any subset $A \subset \mathbb{N}$ and an injection $\alpha : A \to \mathbb{N}$ let p_α be the map in $(X^*)^{X^*}$ defined by

$$p_\alpha(x) = \begin{cases} \alpha(x) & x \in A \\ \infty & \text{otherwise} \end{cases}$$

We have the following simple claim.

Claim. The enveloping semigroup $E = E(X^*, G)$ of the G-system (X^*, G) consists of the maps $\{p_\alpha : \alpha : A \to \mathbb{Z}\}$ as above. Every element of E is a continuous function so that by the Grothendieck-Ellis-Nerurkar theorem [27], the system (X^*, G) is WAP.

Proof. Let π_ν be a net of elements of $S(\mathbb{N})$ with $p = \lim_\nu \pi_\nu$ in E. Let $A = \{n \in \mathbb{N} : p(n) \neq \infty\}$ and $\alpha(n) = p(n)$ for $n \in A$. Clearly $\alpha : A \to \mathbb{N}$ is an injection and $p = p_\alpha$.

Conversely given $A \subset \mathbb{N}$ and an injection $\alpha : A \to \mathbb{N}$ we construct a sequence π_n of elements of $S(\mathbb{N})$ as follows. Let $A_n = A \cap [1, n]$ and $M_n = \max\{\alpha(i) : i \in A_n\}$. Next define an injection $\beta_n : [1, n] \to \mathbb{N}$ by

$$\beta_n(j) = \begin{cases} \alpha(j) & j \in A \\ j + M_n + n & \text{otherwise.} \end{cases}$$

Extending the injection β_n to a permutation π_n of $\mathbb{N}$, in an arbitrary way, we now observe that $p_\alpha = \lim_{n\to\infty} \pi_n$ in E. The last assertion is easily verified. □

In fact, it is shown in [45] that $E = E(X^*, G)$ is isomorphic to the universal WAP compactification G^{WAP} of G; which, in turn, is also the universal $UC(G)$ compactification G^{UC} of G (where $UC(G) = RUC(G) \cap LUC(G)$ is the algebra of bounded right and left uniformly continuous functions on G).

Example 4.4. (See [47]) The following is an example of a dynamical system $(X, \mathbb{Z})$ which is distal, HNS, and its enveloping semigroup $E(X)$ is a compact topological group isomorphic to the 2-adic integers. However, $(X, \mathbb{Z})$ is not WAP and *a fortiori* not equicontinuous.

Let $\mathbf{S} = \mathbb{R}/\mathbb{Z}$ (reals mod 1) be the circle. Let $X = \mathbf{S} \times (\mathbb{N} \cup \{\infty\})$, where $\mathbb{N} \cup \{\infty\}$ is the one point compactification of the natural numbers. Let $T : X \to X$ be defined by:

$$T(s, n) = (s + 2^{-n}, n), \quad T(s, \infty) = (s, \infty).$$

It is not hard to see that $E(X)$ is isomorphic to the compact topological group $\mathbf{Z}_2$ of 2-adic integers. The fact that X is not WAP can be verified directly by observing that $E(X)$ contains discontinuous maps. Indeed, the map $f_a \in E(X)$ corresponding to the 2-adic integer

$$a = \ldots 10101 = 1 + 4 + 16 + \ldots$$

can be described as follows: $f_a(s, n) = (s + a_n, n)$, where

$$a_{2k} = \frac{2^{2k} - 1}{3 \cdot 2^{2k}} \to \frac{1}{3}, \qquad a_{2k+1} = \frac{2^{2k+2} - 1}{3 \cdot 2^{2k+1}} \to \frac{2}{3}.$$

Geometrically this means that half of the circles are turned by approximately $2\pi/3$, while the other half are turned by approximately the same angle in the opposite direction. The map f_a is discontinuous at the points of the limit circle.

Example 4.5. (See [44]) Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus, and let $\alpha \in \mathbb{R}$ be a fixed irrational number and $R_\alpha : \mathbb{T} \to \mathbb{T}$ is the rotation by α, $R_\alpha \beta = \beta + \alpha \pmod 1$. We define a topological space X and a continuous map $\pi : X \to \mathbb{T}$ as follows. For $\beta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$ the preimage $\pi^{-1}(\beta)$ will be a singleton x_β. On the other hand for each $n \in \mathbb{Z}$, $\pi^{-1}(n\alpha)$ will consist of exactly two points $x^-_{n\alpha}$ and $x^+_{n\alpha}$. For convenience we will use the notation $\beta^\pm$, $(\beta \in \mathbb{T})$ for points of X, where $(n\alpha)^- = x^-_{n\alpha}$, $(n\alpha)^+ = x^+_{n\alpha}$ and $\beta^- = \beta^+ = x_\beta$ for $\beta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$. A neighborhood basis for the topology at a point of the form x_β, $\beta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$, is the collection of sets $\pi^{-1}(\beta - \varepsilon, \beta + \varepsilon)$, $\varepsilon > 0$. For $(n\alpha)^-$ a neighborhood basis will be the collection of sets of the form $\{(n\alpha)^-\} \cup \pi^{-1}(n\alpha - \varepsilon, n\alpha)$, where $\varepsilon > 0$. Finally for $(n\alpha)^+$ a neighborhood basis will be the collection of sets of the form $\{(n\alpha)^+\} \cup \pi^{-1}(n\alpha, n\alpha + \varepsilon)$. It is not hard to check that this defines a compact metrizable zero dimensional perfect topology on X (hence X is homeomorphic to the Cantor set) with respect to which π is continuous.

Next define $T : X \to X$ by the formula $T\beta^\pm = (\beta+\alpha)^\pm$. Again it is not hard to see that $\pi : (X,T) \to (\mathbb{T}, R_\alpha)$ is a homomorphism of dynamical systems and that (X,T) is minimal and not equicontinuous.

We now define for each $\xi \in \mathbb{T}$ two distinct maps $p_\xi^\pm : X \to X$ by the formulas

$$p_\xi^+(\beta^\pm) = (\beta+\xi)^+, \qquad p_\xi^-(\beta^\pm) = (\beta+\xi)^-.$$

None of the following claims is hard to verify.

1. For every $\xi \in \mathbb{T}$ and every *sequence*, $n_i \nearrow \infty$ with $\lim_{i\to\infty} n_i\alpha = \xi$, and $\forall i,\ n_i\alpha < \xi$, we have $\lim_{i\to\infty} T^{n_i} = p_\xi^-$ in $E(T,X)$. An analogous statement holds for p_ξ^+.
2. $E(X,T) = \{T^n : n \in \mathbb{Z}\} \cup \{p_\xi^\pm : \xi \in \mathbb{T}\}$
3. The subspace $\{T^n : n \in \mathbb{Z}\}$ inherits from E the discrete topology.
4. The subspace $E(X,T) \setminus \{T^n : n \in \mathbb{Z}\} = \{p_\xi^\pm : \xi \in \mathbb{T}\}$ is homeomorphic to the "two arrows" space of Alexandroff and Urysohn (see [28, page 212], and also Ellis' example [25, Example 5.29]). It thus follows that E is a separable Rosenthal compact of cardinality $2^{\aleph_0}$.
5. For each $\xi \in \mathbb{T}$ the complement of the set $C(p_\xi^\pm)$ of continuity points of $p_\xi^\pm$ is the countable set $\{\beta^\pm : \beta + \xi = n\alpha, \text{ for some } n \in \mathbb{Z}\}$. In particular each element of E is of Baire class 1.

Example 4.6. (See [37, Lemma 4.1]) Let G be a discrete group. We form the product space $\Omega = \{0,1\}^G$ and let G act on Ω by translations: $(g\omega)(h) = \omega(g^{-1}h),\ \omega \in \Omega,\ g,h \in G$. The corresponding G-dynamical system (Ω, G) is called the *Bernoulli* G-system. The enveloping semigroup of the Bernoulli system (Ω, G) is isomorphic to the Stone-Čech compactification βG (as a G-system but also as a semigroup, when the semigroup structure on βG is as defined e.g. in [25]). To see this recall that the collection $\{\bar{A} : A \subset G\}$ is a basis for the topology of βG consisting of clopen sets. Next identify $\Omega = \{0,1\}^G$ with the collection of subsets of G in the obvious way: $A \longleftrightarrow \mathbf{1}_A$. Now define an "action" of βG on Ω by:

$$p * A = \{g \in G : g^{-1}p \in \overline{A^{-1}}\}.$$

It is easy to check that this action extends the action of G on Ω and defines an isomorphism of βG onto $E(\Omega, G)$.

5. Nil-systems of class 2

For the theory of nil-flows we refer the reader to the book by Auslander, Green and Hahn "Flows on homogeneous spaces" [8], where incidentally a use of Ellis' semigroup theory plays an important role. As we have seen above the enveloping semigroup of a distal system is, in fact, a group. For a special kind of distal systems, namely those that arise from class 2 nil-flows, one can provide an explicit description of the group $E(X,G)$. The first example of such computation was given by Furstenberg in his seminal paper [31].

Example 5.1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-torus and let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $T(z,y) = (z+\alpha, y+z)$, where $\alpha \in \mathbb{R}$ is irrational, and addition is mod 1. Furstenberg shows that $(\mathbb{T}^2, T)$ is a minimal distal but not equicontinuous dynamical system, and exhibits $E(\mathbb{T}^2, T)$ as the collection of all maps $p : \mathbb{T}^2 \to \mathbb{T}^2$ of the form:

$$p(z,y) = (z+\beta, y+\phi(z)),$$

where $\beta \in \mathbb{T}$ and $\phi : \mathbb{T} \to \mathbb{T}$ is a (not necessarily continuous) group endomorphism.

Now let

$$N = \{\left(\begin{smallmatrix} 1 & n & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z},\ z, y \in \mathbb{T}\},$$

so that

$$\begin{pmatrix} 1 & n & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+n' & y+y'+nz' \\ 0 & 1 & z+z' \\ 0 & 0 & 1 \end{pmatrix}.$$

N is a nilpotent group with center $K = \{\left(\begin{smallmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : y \in \mathbb{T}\}$ and $[N, N] \subset K$. Set $a = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{smallmatrix}\right)$, where $\alpha \in \mathbb{T}$ is irrational and let

$$\Gamma = \{\left(\begin{smallmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}.$$

Then Γ is a cocompact discrete subgroup of N and the nil-system $(N/\Gamma, a)$, with $a \cdot g\Gamma = (ag)\Gamma,\ g \in G$, is isomorphic to the minimal system $(\mathbb{T}^2, T),\ T(z, y) = (z + \alpha, y + z)$, described above.

Furstenberg's example and subsequently Namioka's work [72] motivated my work on nil-systems of class 2 [38], where the following theorem is proved. Let X be a compact metric space and $a : X \to X$ a fixed homeomorphism such that the system (X, a) is minimal. Suppose $K \subset \mathrm{Homeo}\,(X)$ is a compact subgroup in the centralizer of a which is topologically isomorphic to a (finite or infinite dimensional) torus. Suppose further that the quotient map $\pi : X \to Z = X/K$ realizes the maximal Kronecker factor of (X, a). Note that under these conditions the system (X, a) is minimal and distal, hence its enveloping semigroup $E = E(X, a)$ is a group.

Theorem 5.2. *The following conditions on the system (X, a) as above are equivalent.*

1. *The enveloping semigroup E is (algebraically) a nilpotent group.*
2. *There exists a nilpotent class 2 subgroup $N \subset \mathrm{Homeo}\,(X)$ and a closed cocompact subgroup $\Gamma \subset N$ such that: (i) $a \in N$, (ii) $K \subset N$ and K is central in N, (iii) $[N, N] \subset K$, and the nil-system $(N/\Gamma, a)$ is isomorphic to (X, a).*
3. *For every $x_0, x_1 \in X$ the subsystem $\Omega = \overline{\mathcal{O}}_{a\times a}(x_0, x_1)$ of the product $X \times X$ is invariant under the action of the group $\Delta_K = \{(k, k) : k \in K\}$ and the quotient map $\pi_1 : \Omega \to \Omega/\Delta_K = Z_1$ realizes the largest Kronecker factor of the system $(\Omega, a \times a)$.*

When these equivalent conditions hold then Γ is isomorphic to a subgroup of the group $\mathrm{Hom}_c(Z, K)$ of continuous homomorphisms of the compact group Z into K. If, in addition, $\hat{K}$, the dual group of K, is finitely generated, then N is locally compact and σ-compact and Γ is a countable discrete subgroup of N.

Remark 5.3. The assumption that K is a torus (rather than any central compact subgroup of N) can be removed for a price: The presentation of (X, a) one obtains is now of the form $(W \setminus N/\Gamma, a)$, where W is a compact Abelian subgroup of N which commutes with a and satisfies $W \cap K = \{e\}$ ([38, Theorem 2.1*]).

The easy part of the proof of the theorem consists of yet another concrete computation of an enveloping semigroup:

Example 5.4. Consider the nil-system (X, a) as described in condition 2 of Theorem 5.2. Thus $X = N/\Gamma$ and we let $x_0 = \Gamma$ be the distinguished point of the system (X, a). Let $\phi_0 : N \to K$ be the group homomorphism defined by $\phi_0(g) = [a, g]$. Let $\mathrm{Hom}\,(N, K)$ be the group of all (not necessarily continuous) homomorphisms from N to K. We endow $\mathrm{Hom}\,(N, K)$ with the (compact) topology of pointwise convergence. Now set

$$\Phi = \mathrm{cls}\,\{\phi_0^n : n \in \mathbb{Z}\},$$

and

$$\tilde{E} = \mathrm{cls}\,\{(a^n x_0, \phi_0^n) \in X \times \Phi : n \in \mathbb{Z}\}.$$

Proposition 5.5. *The formulas*

$$(g\Gamma, \phi)(h\Gamma, \psi) = (\phi(h)hg\Gamma, \phi\psi)$$
$$(g\Gamma, \phi)^{-1} = (\phi(g)g^{-1}\Gamma, \phi^{-1}),$$

define a group structure on $\tilde{E}$. The resulting group is nilpotent of class 2. Multiplication on the left by $\tilde{a} = (a\Gamma, \phi_0)$ is continuous and $(\tilde{E}, \tilde{a})$ is isomorphic, as a dynamical system and as a group, to (E, a).

6. A dynamical version of the Bourgain-Fremlin-Talagrand dichotomy and tame dynamical systems

The following theorem of Rosenthal [78], reformulated by Todorcevic [82], was the starting point of the Bourgain-Fremlin-Talagrand theorem.

Let X be a Polish space. Let $C_p(X)$ be the space of real valued continuous functions on X equipped with the pointwise convergence topology.

Theorem 6.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C_p(X)$ which is pointwise bounded (i.e., for each $x \in X$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is bounded). Then, either the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a pointwise convergent subsequence, or it contains a subsequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$.*

A sequence $\{(A_{n,0}, A_{n,1})\}_{n\in\mathbb{N}}$ of disjoint pairs of subsets of X is said to be *independent* if for every finite $F \subset I$ and $\sigma : F \to \{0, 1\}$ we have $\bigcap_{n\in F} A_{n,\sigma(n)} \neq \emptyset$. It is said to be *convergent* if for every $x \in X$, either $x \notin A_{n,0}$ for all but finitely many n, or $x \notin A_{n,1}$ for all but finitely many n.

For example, if $\{f_n\}_{n\in\mathbb{N}}$ is a pointwise convergent sequence in $C_p(X)$ then for every two real numbers $s < t$ the sequence $\{(f_n^{-1}(-\infty, s], f_n^{-1}[t, \infty))\}_{n\in\mathbb{N}}$ is convergent. On the other hand, for $X = \{0, 1\}^{\mathbb{N}}$ the sequence of pairs $\{(A_{n,0}, A_{n,1})\}_{n\in\mathbb{N}}$, with $A_{n,i} = \{x \in X : x(n) = i\}$, is an independent sequence.

The following claim is the combinatorial essence of Rosenthal's theorem: *A sequence of disjoint pairs $\{(A_{n,0}, A_{n,1})\}_{n\in\mathbb{N}}$ always contains either a convergent subsequence or an independent subsequence.*

Ideas of independence and ℓ_1 structure were introduced into dynamics by Glasner and Weiss in [49]. First by using the local theory of Banach spaces in proving that if a compact topological $\mathbb{Z}$-system (X, T) has zero topological entropy then so does the induced system $(\mathfrak{M}(X), T_*)$ on the compact space of probability measures on X; and also in providing a characterization of K-systems in terms of *interpolation sets* which are the same as independence sets in this situation (we refer the reader to [49] and [66] for these important notions; see also Remark 11.3 below).

A topological space K is *Rosenthal compact* [53] if it is homeomorphic to a pointwise compact subset of the space $B_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compactum is the *Helly compact* of all (not only strictly) increasing selfmaps of $[0,1]$ in the pointwise topology. Another is the *two arrows* space of Alexandroff and Urysohn (see Engelking [28]).

A topological space K is a *Fréchet space* if for every $A \subset K$ and every $x \in \overline{A}$ there exists a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$ (see Engelking [28]). Clearly, $\beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers $\mathbb{N}$, cannot be embedded into a Fréchet space (in fact, any convergent sequence in $\beta\mathbb{N}$ is eventually constant).

The following theorem is due to Bourgain, Fremlin and Talagrand [11, Theorem 3F]. As mentioned above it was motivated by results of Rosenthal [78] (see also [19] and [82]). The second assertion (BFT dichotomy) is presented as in the book of Todorčević [82] (see Proposition 1, section 13).

Theorem 6.2. 1. *Every Rosenthal compact space K is Fréchet.*
2. *(BFT dichotomy) Let X be a Polish space and let $\{f_n\}_{n\in\mathbb{N}} \subset C(X)$ be a sequence of real valued functions which is* pointwise bounded. *Let K be the pointwise closure of $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^X$. Then either $K \subset B_1(X)$ (so that K is Rosenthal compact) or K contains a homeomorphic copy of $\beta\mathbb{N}$.*

In [44, Theorem 3.2] the following dynamical Bourgain-Fremlin-Talagrand dichotomy was established.

Theorem 6.3 (A dynamical BFT dichotomy)**.** *Let (X,G) be a metric dynamical system and let $E(X,G)$ be its enveloping semigroup. We have the following dichotomy. Either*

1. *$E(X,G)$ is separable Rosenthal compact, hence with cardinality* $\operatorname{card} E(X) \leq 2^\omega$*; or*
2. *the compact space $E(X,G)$ contains a homeomorphic copy of $\beta\mathbb{N}$, hence*

$$\operatorname{card} E(X,G) = 2^{2^\omega}.$$

In [42] a dynamical system is called *tame* if the first alternative occurs, i.e. $E(X,G)$ is Rosenthal compact. In these terms Theorem 6.3 can be rephrased as saying that *a metric dynamical system (X,G) is either tame or $E(X,G)$ contains a topological copy of $\beta\mathbb{N}$*. When (X,G) is a metrizable system the group G is embedded in the Polish group $\mathrm{Homeo}\,(X)$ of homeomorphisms of X equipped with the topology of uniform convergence. From this fact it is easy to deduce that the enveloping semigroup $E(X,G)$ is separable. If moreover (X,G) is tame then $E = E(X,G)$ is Fréchet and every element $p \in E$ is a limit of a sequence of elements of G, $p = \lim_{n\to\infty} g_n$.

Examples of tame dynamical systems include metric minimal equicontinuous systems, weakly almost periodic (WAP) systems (Akin, Auslander, and Berg [5]), and hereditarily non-sensitive (HNS) systems (Glasner and Megrelishvili [44]).

The cardinality distinction between the two cases entails the first part of the following proposition [42].

Proposition 6.4. 1. *For metric dynamical systems tameness is preserved by taking*
   (a) *subsystems,*
   (b) *countable products, and*
   (c) *factors.*
2. *Every metric dynamical system (X, G) admits a unique maximal tame factor.*

Proof. As pointed out, the first statement follows from cardinality arguments (note that $E(\prod X_i, G) \subset \prod E(X_i, G)$ for every countable family $\{(X_i, G)\}$ of dynamical systems; moreover $E(X, G) = E(X^\kappa, G)$ for any system (X, G) and any cardinal number κ). To prove the second use Zorn's lemma, the first part of the theorem, and the fact that a chain of factors of a metric system is necessarily countable, to find a maximal tame factor. Then use the first part again to deduce that such maximal factor is unique. $\square$

The next result is stated explicitly first in [47, Theorem 6.2].

Theorem 6.5. *A compact metric dynamical system (X, G) is tame if and only if every element of $E(X, G)$ is a Baire class 1 function from X to itself.*

Proof. If Y is a separable metric space and $B_1(X, Y) \subset Y^X$ is the space of Baire 1 functions from X to Y, then every compact subset of $B_1(X, Y)$ is Rosenthal. Indeed, Y embeds in $\mathbb{R}^{\mathbb{N}}$, hence $B_1(X, Y)$ embeds in $B_1(X, \mathbb{R}^{\mathbb{N}}) = B_1(X \times \mathbb{N})$. In particular, if $E(X, G) \subset B_1(X, X)$, then $E(X, G)$ is Rosenthal, which means that (X, G) is tame. Conversely, if $E(X, G)$ is Rosenthal, then by the Bourgain-Fremlin-Talagrand theorem it is Fréchet [11]. In particular, every $p \in E(X, G) = \overline{G}$ (we may assume that $G \subset \mathrm{Homeo}\,(X)$) is the limit of a sequence of elements of G and therefore of Baire class 1 (see e.g. [64]). $\square$

Combining this result with Theorems 8.1 and 8.2 we deduce the following:

Theorem 6.6. *Every metric HAE system is tame.*

From the results in Section 3 we deduce the following:

Theorem 6.7. *Every metric WAP system is tame.*

Reexamining the examples presented in Section 4 we see that:
1. A metrizable minimal and equicontinuous system, as in Example 4.1, is tame.
2. The WAP systems in examples 4.2 and 4.3 are tame.
3. This is also the case with the distal HNS but not WAP $\mathbb{Z}$-system in Example 4.4.
4. The almost automorphic Example 4.5 is again tame, although this one is not HNS.
5. Evidently, the Bernoulli system $\Omega = \{0, 1\}^G$ in Example 4.6 is not tame.
6. As we will see in Section 7 below, a distal minimal system is tame if and only if it is already equicontinuous. Thus the nil-systems presented in Section 5 are tame only when they are equicontinuous.
7. In his paper [26] Ellis, following Furstenberg's classical work [30], investigates the projective action of $GL(n, \mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$. It follows from his results that the corresponding enveloping semigroup is not first countable.

In a later work [3], Akin studies the action of $G = GL(n, \mathbb{R})$ on the sphere $\mathbb{S}^{n-1}$ and shows that here the enveloping semigroup is first countable (but not metrizable). The dynamical systems $D_1 = (G, \mathbb{P}^{n-1})$ and $D_2 = (G, \mathbb{S}^{n-1})$ are tame but not HNS. Note that $E(D_1)$ is Fréchet, being a continuous image of a first countable space, namely $E(D_2)$.

7. Injective dynamical systems

In her paper [67], mentioned in the introduction, Köhler also considers another useful notion, that of the *enveloping operator semigroup*. For a Banach space K and a bounded linear operator $T : K \to K$ this is defined as

$$\mathcal{E}(T) = \mathrm{cls}_{w^*}\{T^{*n} : n \in \mathbb{N}\},$$

where the closure is taken in the space $\mathcal{L}(K^*)$ of bounded linear operators on the dual space K^*, with respect to the weak* operator topology. Köhler shows that when (X, ϕ) is a $\mathbb{Z}$-dynamical system, $K = C(X)$, and $T^* : C(X)^* \to C(X)^*$ is the operator induced by ϕ on the dual space $C(X)^*$, there is a natural surjective homomorphism of dynamical systems

$$\Phi : \mathcal{E}(T) \to E(X, \phi).$$

If we view $\mathfrak{M}(X)$, the compact space of probability measures on X equipped with the weak* topology, as a subset of $C(X)^*$ with $\mathrm{span}(\mathfrak{M}(X)) = C(X)^*$, we see that this map Φ is nothing but the restriction of an element of $\mathcal{E}(T)$ to the subspace of Dirac measures $\{\delta_x : x \in X\}$. Theorem 5.3 of [67] says that for a tame metric dynamical system (X, ϕ), the map Φ is an isomorphism of the enveloping operator semigroup onto the enveloping semigroup. (We will re-prove this theorem below.) In [42] as well as in this section, I call a dynamical system (X, G) for which the corresponding map $\Phi : \mathcal{E}(G) \to E(X, G)$ is an isomorphism, an *injective system*. In [67] there are several other cases where systems are shown to be injective and the author raises the question whether this is always the case. As she points out this question was posed earlier by J. S. Pym (see [77]).

As was mentioned above, the following theorem is due to Köhler [67]; our proof though is different ([42, Theorem 1.5], see also [41, Lemma 1.49]).

Theorem 7.1. *Let (X, G) be a metric tame dynamical system. Let $\mathfrak{M}(X)$ denote the compact convex set of probability measures on X (with the weak* topology). Then each element $p \in E(X, G)$ defines an element $p_* \in E(\mathfrak{M}(X), G)$ and the map $p \mapsto p_*$ is both a dynamical system and a semigroup isomorphism of $E(X, G)$ onto $E(\mathfrak{M}(X), G)$.*

Proof. Since $E(X, G)$ is Fréchet we have for every $p \in E$ a sequence $g_i \to p$ of elements of G converging to p. Now for every $f \in C(X)$ and every probability measure $\nu \in \mathfrak{M}(X)$ we get by the Riesz representation theorem and Lebesgue's dominated convergence theorem

$$g_i\nu(f) = \nu(f \circ g_i) \to \nu(f \circ p) := p_*\nu(f).$$

Since the Baire class 1 function $f \circ p$ is well defined and does not depend upon the choice of the convergent sequence $g_i \to p$, this defines the map $p \mapsto p_*$ uniquely. It is easy to see that this map is an isomorphism of dynamical systems, whence a semigroup isomorphism. Finally as G is dense in both enveloping semigroups, it follows that this isomorphism is onto. $\square$

Definition 7.2. We will say that the dynamical system (X,G) is *injective* if the natural map $E(\mathfrak{M}(X),G) \to E(X,G)$ is an isomorphism.

Since the map $p \mapsto p_*$ described in Theorem 7.1 is the inverse of the map Φ it follows that in these terms the theorem can be restated as follows. *A tame dynamical system is injective.* Our next theorem, which relies on [36], answers the question of J. S. Pym and A. Köhler (see also S. Immervoll [62]).

Theorem 7.3 (Glasner, [42])**.** *A minimal distal metric dynamical system is injective if and only if it is equicontinuous.*

Proof. It is well known that when (X,G) is equicontinuous, $E = E(X,G)$ is a compact topological group and in that case it is easy to see that (X,G) is injective. By a theorem of Ellis (see e.g. [25]), a system (X,G) is distal if and only if $E(X,G)$ is a group. Thus, if (X,G) is distal metric and injective then $E(X,G) = E(\mathfrak{M}(X),G)$ is a group and it follows that the dynamical system $(\mathfrak{M}(X),G)$ is also distal. By Theorem 1.1 of [36], the system (X,G) is equicontinuous. □

Corollary 7.4. *A minimal distal metric system is tame if and only if it is equicontinuous.*

Proof. For a metric minimal equicontinuous system the enveloping semigroup is a compact group of homeomorphisms of X. For the other direction observe that if (X,G) is tame then by Theorem 7.1 it is injective hence, by Theorem 7.3, it is equicontinuous. □

By way of illustration consider, given an irrational number $\alpha \in \mathbb{R}$, the minimal distal dynamical $\mathbb{Z}$-system on the two torus $(\mathbb{T}^2, T)$ given by:

$$T(x,y) = (x+\alpha, y+x) \qquad \pmod 1.$$

Since this system is not equicontinuous Theorem 7.3 and Corollary 7.4 show that it is neither tame nor injective.

Exercise 7.5. Show that, for every discrete countable group G, the Bernoulli G-system $(\{0,1\}^G, G)$, described in Section 4, Example 4.4, is injective.

The fact that tame systems are injective also yields the result that metric tame minimal $\mathbb{Z}$-systems have zero topological entropy [42, Corollary 1.8]. But, see Theorem 13.1.2 below for a much stronger statement.

8. Banach space representations of a dynamical system

With every Banach space V one can associate a dynamical system $S_V = (Y,H)$ as follows: $H = \mathrm{Iso}\,(\mathrm{V})$ is the group of all linear isometries of V onto itself, equipped with pointwise convergence topology (or the compact-open topology, the two topologies coincide on H), and Y is the unit ball of the dual space V^*, equipped with the weak*-topology. The action of H on Y is defined by $g\phi(v) = \phi(g^{-1}(v))$, $g \in H$, $\phi \in Y$, $v \in V$. The continuity of this action can be easily verified. A *representation* of a dynamical system (X,G) on a Banach space V is a homomorphism of (X,G) to $S_V = (Y,H)$, that is, a pair of continuous maps (h,α), $h : G \to \mathrm{Iso}\,(\mathrm{V})$ and $\alpha : X \to Y$, such that h is a group homomorphism and $\alpha(gx) = h(g)\alpha(x)$ for all $g \in G$ and $x \in X$. A representation is *proper* if α is a topological embedding.

An old observation of Teleman [81] is that every dynamical system (X, G) has a proper representation on $C(X)$. Namely

$$\alpha(x) = \delta_x,$$

where δ_x is the point mass at x viewed as an element of $C(X)^*$. Finding representations on geometrically "nicer" Banach spaces (Hilbert, reflexive, etc.) is a more difficult task.

A theorem of Megrelishvili asserts that a metric dynamical system (X, G) is WAP if and only if it admits a proper representation on a reflexive Banach space [69, Corollary 6.10], [44, Theorem 7.6(1)]. A dynamical system is *Radon–Nikodým* (RN) if it admits a proper representation on an Asplund Banach space [69, Definition 3.10], [44, Definition 7.5.2]. (When $G = \{e\}$ one retrieves the class of *Radon–Nikodým compact spaces* in the sense of Namioka [72].) Recall that a Banach space V is *Asplund* if for every separable subspace $E \subset V$ the dual E^* is separable. Reflexive spaces and spaces of the form $c_0(\Gamma)$ are Asplund. Regarding the history and the relevance of Asplund spaces see for example [12, 18, 29]. In [69] Megrelishvili also shows that every metric RN system is LE (locally equicontinuous, see [50]).

The next two theorems are among the main results of [44].

Theorem 8.1 ([44, Theorem 9.14])**.** *For a compact metric G-space X the following conditions are equivalent:*

1. *X is RN.*
2. *X is HNS.*
3. *X is HAE.*
4. *Every nonempty closed G-subspace Y of X has a point of equicontinuity;*
5. *For any compatible metric d on X the metric $d_G(x, y) := \sup_{g \in G} d(gx, gy)$ defines a separable topology on X.*

Theorem 8.2 ([44, Corollary 14.7])**.** *Let (X, G) be a compact metric HNS system. Then $p : X \to X$ is of Baire class 1 for every $p \in E(X, G)$.*

9. When is the enveloping semigroup metrizable?

It was proved in [44] that the equivalent conditions of Theorem 8.1 imply that the enveloping semigroup $E(X)$ must be of cardinality $\leq 2^\omega$. In fact, it was established in [44, Theorem 14.8] that $E(X)$ is Rosenthal compact and the question was posed whether this conclusion can be strengthened to "$E(X)$ is metrizable". We now know that the answer to this question is positive. Moreover, strikingly, it turns out that metrizability of $E(X)$, in fact, is equivalent to the conditions of Theorem 8.1, ([47, Theorem 1.2]).

Theorem 9.1 (Glasner, Megrelishvili and Uspenskij)**.** *Let X be a compact metric G-space. The following conditions are equivalent:*

1. *The dynamical system (X, G) is hereditarily almost equicontinuous (HAE).*
2. *The dynamical system (X, G) is RN, that is, admits a proper representation on an Asplund Banach space.*
3. *The enveloping semigroup $E(X)$ is metrizable.*

The proof of this theorem relies on results from [44] and, beyond that, mainly on some Banach space tools and Namioka joint-continuity-type results. Here are some signposts for the proof. We begin with the following special case:

Theorem 9.2. *Let V be a Banach space with a separable dual, $H = \mathrm{Iso}\,(\mathrm{V})$, Y the compact unit ball of V^* with the weak* topology. Then the enveloping semigroup $E(Y, H)$ is metrizable.*

Sketch of proof. 1. Let K be the set of all linear operators of norm ≤ 1 on the Banach space V^*. It is easy to see that the closure of H, with respect to the weak* operator topology on the space of bounded linear operators on V^*, is contained in K.

2. Since the weak* operator topology is the one inherited from the product space $(V^*)^{V^*}$ (where each factor V^* is endowed with the weak* topology), and as K is identified with a closed subset of the product $\prod_{f\in V^*} \|f\|Y$, it follows that w^*-cls $H \subset K$ coincides with the enveloping semigroup $E(Y, H) \subset Y^Y$.

3. By assumption V^* is separable and it follows that V is separable as well. In turn this fact implies that Y, the unit ball of V^*, is metrizable.

4. Now choose a norm-dense countable set $F \subset V^*$, and observe that K is homeomorphic to the corresponding subset of the countable product $\prod_{f\in F} \|f\|Y$. The latter is clearly metrizable and therefore so is $K = E(Y, H)$. □

- With a few more technical arguments this takes care of the implication
  $RN \Rightarrow E(X, G)$ is metrizable.
- The implication
  $HAE \Rightarrow RN$
  uses the well known Davis-Figiel-Johnson-Pelczynski Banach space construction [16].
- The remaining implications involve Baire 1 class arguments and a Namioka type theorem.
- Finally it should be pointed out that the equivalence
  $HAE \Leftrightarrow E(X, G)$ is metrizable
  can be proved directly without the use of Banach representations.

For more details see [44] and [47].

10. Some applications of the GMU theorem

Minimal systems with metrizable enveloping semigroup are equicontinuous. Under the additional assumption that (X, G) is minimal Theorem 9.1 now leads to the following definitive result in the spirit of Ellis' joint continuity theorem [21].

Theorem 10.1. [47, Theorem 6.2] *A metric minimal system (X, G) is equicontinuous if and only if its enveloping semigroup $E(X)$ is metrizable.*

Proof. It is well known that the enveloping semigroup of a metric equicontinuous system is a metrizable compact topological group. Conversely, if $E(X)$ is metrizable then, by Theorem 9.1, (X, G) is HAE and being also minimal it is equicontinuous. □

Remark 10.2. Theorem 10.1 answers negatively Problem 3.3 in [42].

Distality and equicontinuity. One version of Ellis' joint continuity theorem says that a compact dynamical system (X, G) whose enveloping semigroup is a group of continuous maps is necessarily equicontinuous (see [21] and [6, page 60]). Using Ellis's characterizations of distality and WAP:

- A dynamical system (X, G) is distal if and only if its enveloping semigroup $E(X)$ is (algebraically) a group, and
- A dynamical system (X, G) is WAP if and only if every element of $E(X)$ is continuous,

we can reformulate the joint continuity theorem as follows:

Theorem 10.3. *A distal WAP system is equicontinuous*

Example 4.4 in Section 4 above shows that the WAP condition can not be much relaxed. Recall that this dynamical system $(X, \mathbb{Z})$ is distal, HAE, with enveloping semigroup $E(X)$ which is a compact topological group isomorphic to the 2-adic integers, but is not WAP hence not equicontinuous.

However, for a point transitive HAE system distality is equivalent to equicontinuity because, as we have seen, a distal point transitive system must be minimal and a minimal HAE system is equicontinuous.

Semigroup compactifications of groups. A semigroup S is *right topological* if it is equipped with such a topology that for every $y \in S$ the map $x \mapsto xy$ from S to itself is continuous. If for every $y \in S$ the self-maps $x \mapsto xy$ and $x \mapsto yx$ of S both are continuous, S is a *semitopological semigroup.* A *right topological semigroup compactification* of a topological group G is a compact right topological semigroup S together with a continuous semigroup morphism $G \to S$ with a dense range such that the induced action $G \times S \to S$ is continuous. A typical example is the enveloping semigroup $E(X)$ of a dynamical system (X, G) together with the natural map $G \to E(X)$. *Semitopological semigroup compactifications* are defined analogously.

We have the following direct corollaries of Theorem 9.1.

Corollary 10.4. *For a metric HAE system (X, G) its enveloping semigroup $E(X)$ is again a metrizable HAE system.*

Proof. This follows from Theorem 9.1 because the enveloping semigroup of the flow $(G, E(X, G))$ is isomorphic to $E(X, G)$. ☐

Corollary 10.5. *The following three classes of semigroups coincide:*

1. *Metrizable enveloping semigroups of G-systems.*
2. *Enveloping semigroups of HAE metrizable G-systems.*
3. *Metrizable right topological semigroup compactifications of G.*

Proof. A dynamical system has the structure of a right topological semigroup compactification of G if and only if it is the enveloping semigroup of some dynamical system (see e.g. [41, Section 1.4] and [44, Section 2]). ☐

For WAP systems we have an analogous statement:

Corollary 10.6. *The following classes of semigroups coincide:*

1. *Enveloping semigroups of WAP metrizable G-systems.*
2. *Metrizable semitopological semigroup compactifications of G.*

Moreover, when the acting group G is commutative, a point transitive WAP system is isomorphic to its enveloping semigroup, which in this case is a commutative semitopological semigroup. Thus for such G the class of all metric, point transitive, WAP systems coincides with that of all metrizable, commutative, semitopological semigroup compactifications of G.

Proof. The enveloping semigroup of a WAP dynamical system is a semitopological semigroup compactification of G (see e.g. [41, Section 1.4] or [44, Section 2]). On the other hand such a compactification yields a point-universal WAP G-system. The second part of the theorem follows from [20] or [41, Theorem 1.48]. $\square$

11. Tame dynamical systems and representations on Rosenthal Banach spaces

In this section I review the main results of [46] which examines representability of dynamical systems on Rosenthal spaces.

Rosenthal's celebrated dichotomy theorem asserts that every bounded sequence in a Banach space either has a weak Cauchy subsequence or a subsequence equivalent to the unit vector basis of l_1 (an l_1-*sequence*). Consequently a Banach space V does not contain an l_1-sequence if and only if every bounded sequence in V has a weak-Cauchy subsequence [78]. In [46] the authors call a Banach space satisfying these equivalent conditions a *Rosenthal space*. There are several other important characterizations of Rosenthal spaces of which we will cite the following two. Rosenthal spaces are exactly those Banach spaces whose dual has the weak Radon-Nikodým property [80, Corollary 7-3-8]. Finally, for a Banach space V with dual V^* and second dual V^{**} one can consider the elements of V^{**} as functions on the weak star compact unit ball $B^* := B_{V^*} \subset V^*$. While the elements of V are clearly continuous on B^* this is not true in general for elements of V^{**}. By a result of Odell and Rosenthal [74], a *separable* Banach space V is Rosenthal iff every element v^{**} from V^{**} is a Baire 1 function on B^*. More generally E. Saab and P. Saab [79] show that V is Rosenthal iff every element of V^{**} has the *point of continuity property* when restricted to B^*; i.e., every restriction of v^{**} to a closed subset of B^* has a point of continuity.

The main result of [46] is that, for metrizable systems, the property of being *tame* is a necessary and sufficient condition for Rosenthal representability.

Theorem 11.1. *Let X be a compact metric G-space. The following conditions are equivalent:*

1. *(G, X) is a tame G-system.*
2. *(G, X) is representable on a separable Rosenthal Banach space.*

An analogous statement is proven for general (not necessarily metrizable) G-systems.

One of the important questions in Banach space theory until the mid 70's was to construct a separable Rosenthal space which is not Asplund. The first counterexamples were constructed independently by James [63] and Lindenstrauss and Stegall [68]. In view of Theorem 11.1 we now see that a fruitful way of producing such distinguishing examples comes from dynamical systems. Just consider a compact metric tame G-system which is not HNS (see e.g. Example 4.5 and remarks (4) and (7) following Theorem 6.7, above) and then apply Theorem 11.1.

12. The hierarchy of Banach representations

In the following table (borrowed from [46]) we encapsulate some features of the trinity: dynamical systems, enveloping semigroups, and Banach representations. Let X be a compact metrizable G-space and $E(X)$ denote the corresponding enveloping semigroup. The symbol f stands for an arbitrary function in $C(X)$ and $fG = \{f \circ g : g \in G\}$ denotes its orbit. Finally, cls(fG) is the pointwise closure of fG in $\mathbb{R}^X$.

	Dynamical characterization	Enveloping semigroup	Banach representation
WAP	cls(fG) is a subset of $C(X)$	Every element is continuous	Reflexive
HNS	cls(fG) is metrizable	$E(X)$ is metrizable	Asplund
Tame	cls(fG) is Fréchet	Every element is Baire 1	Rosenthal

Table 1. The hierarchy of Banach representations

13. The structure of a minimal tame system

In [42] I have shown that a *minimal* metrizable tame dynamical system with a commutative acting group is PI and has zero topological entropy. Recently Huang [59], and independently Kerr and Li [66], improved these results to show that under the same conditions a minimal tame system is an almost 1-1 extension of its maximal equicontinuous factor and is uniquely ergodic (see also Huang, Li, Shao & Ye [60], and Huang & Ye [61]). In these works the authors make a heavy use of the structure theory of minimal dynamical systems, as developed by Ellis, Veech, Ellis-Glasner-Shapiro, McMahon and van der Woude (see e.g. the survey [40] and the references thereof). However the main tool in both works (of Huang and Kerr-Li) is the combinatorial notion of independence (see Section 6 above) and the various related notions of independence n-tuples. In fact, Kerr and Li in their work [66], use independence to unify the theory of these various notions and in particular they are able to characterize tame systems as those systems that (in some precise sense) do not admit infinite independence sets ([66, Proposition 6.4.2], see also Remark 13.3 below). In turn they use this characterization to define a notion of relative tameness and develop the whole theory in the relative setup.

In my work [43] — a continuation of [42] — I pursue purely structure theoretical methods and in particular some of the ideas and tools which were developed in my old work [36], to recover the results of Huang and Kerr & Li mentioned above, avoiding the combinatorial treatment. The following theorem is quoted from [43].

Theorem 13.1. *Let G be an Abelian group and (X, G) a metric tame minimal system. Then:*

1. *The system (X, G) is almost automorphic. Thus there exist:*
   (a) *A compact topological group Y with Haar measure η, and a group homomorphism $\kappa : G \to Y$ with dense image.*
   (b) *A homomorphism $\pi : (X, G) \to (Y, G)$, where the G action on Y is via κ.*
   (c) *The sets $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ and $Y_0 = \pi(X_0)$ are dense G_δ subsets of X and Y respectively.*
2. *The system (X, G) is uniquely ergodic with unique invariant measure μ such that $\pi_*(\mu) = \eta$, and $\pi : (X, \mu, G) \to (Y, \eta, G)$ is a measure theoretical isomorphism of the corresponding measure preserving systems.*

The key tool used in the proof is a proposition about diffused measures ([43, Proposition 3.3]), an earlier version of which first appeared in [36]. Another component of the proof is an analogue of an old theorem of Ditor and Eifler [17]. It shows that when a continuous surjection $\pi : X \to Y$, with X and Y compact metric spaces, is semiopen (i.e. it has the property that the image of a nonempty open set has a nonempty interior) then so is the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ on the spaces of probability measures equipped with the weak* topology.

Remark 13.2. The set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is a dense G_δ and G-invariant subset of X and thus has μ measure either zero or one. In [66, Section 11] Kerr and Li construct a minimal Toeplitz system which is tame and not null. Since in this construction the growth of the sequence $\{n_1 < n_2 < \cdots\}$ is arbitrary it follows that the resulting Toeplitz system can be made not regular in the sense that the densities of the periodic parts converge to $d < 1$. For such nonregular systems $\mu(X_0) = 0$. This shows that the unique invariant measure of a minimal tame system need not be supported by the set X_0 where π is 1-1.

Remark 13.3. Huang and Kerr & Li, following the works of Rosenthal [78] and Glasner-Weiss [49], base their works on the notion of independence (see the Section 6 above). For example, following Kerr and Li [66], given a dynamical system (X, G) and a pair $\mathbf{A} = (A_0, A_1)$ of subsets of X, a set $J \subset G$ is called an *independence set* for $\mathbf{A}$ if for every nonempty finite set $J \subset I$ and function $\sigma : I \to \{0, 1\}$ we have $\bigcap_{g \in J} g^{-1} A_{\sigma}(g) \neq \emptyset$. A pair $(x_0, x_1) \in X \times X$ is called an IT-*pair* if for any neighborhood $U_0 \times U_1$ of (x_0, x_1) the pair (U_0, U_1) has an infinite independence set $I \subset G$. The following is one of many similar characterizations given in [66].

Theorem 13.4. [66, Proposition 6.4.2] *A dynamical system (X, G) is untame if and only if there exists a non-diagonal IT-pair in $X \times X$.*

14. Brief remarks on some related topics

On the interplay with combinatorial number theory. In his path-breaking article [32] on Szemerédi's theorem, Furstenberg initiated a new branch of ergodic theory: the interplay between dynamics and combinatorial number theory. The recent spectacular achievement in this field is the Green-Tao theorem on the existence of arbitrarily long arithmetical progressions of primes [55]. Furstenberg's paper was followed by the work of Furstenberg and Weiss [34], where topological dynamics follows ergodic theory with its share of related combinatorial results. About the same time, it was realized by Glazer (see e.g. [15]) and independently by Glasner [37] that ultrafilters — and along with them, Stone-Čech compactifications of groups, enveloping semigroups, minimal left ideals, idempotents, etc. — form a convenient language and provide a powerful tool for working in this theory. Since then great advances were made. This short subsection is hardly the place for a detailed account of these new and exciting developments. I refer the reader to some of the authorities on the subject. Foremost comes Furstenberg's book [33], and then Bergelson's comprehensive review article [10] and its 141 items reference list. See also Akin [2], and Hindman & Strauss [58] for related research areas.

Universal ambits and universal minimal systems. For an arbitrary topological group G, the Gelfand space $\mathcal{S}(G)$ of the algebra $RUC(G)$ of bounded real valued right uniformly continuous functions, is a model for the universal ambit. (When G is discrete this is βG, the Stone-Čech compactification of G). This dynamical system is point universal and thus has a structure of an enveloping semigroup (for instance it is always isomorphic to its own enveloping semigroup). It follows then that any minimal left ideal $M = M(G)$ of $\mathcal{S}(G)$ is a model for the universal minimal G-system. Usually there are many minimal left ideals but they are all isomorphic as semigroups as well as dynamical systems and each of them is *coalescent*; i.e. every endomorphism of (M, G) is an automorphism. An old result of Veech shows that for a locally compact G the G-action on $\mathcal{S}(G)$ and hence also on M, are free. Moreover, when G is locally compact but not compact, $M(G)$ is non-metrizable [65].

In view of these results it is surprising to discover that for many familiar and well investigated (mostly Polish) topological groups, $M(G)$ is the trivial one point system; that is, *every compact G-system has a fixed point*. Such a group is said to have the *fixed point on compacta* property, or to be *extremely amenable*. Typical examples of such groups are $U(\mathcal{H})$ the unitary group of a separable infinite dimensional Hilbert space with the strong operator topology [56], $L_0(\mathbb{I}, S^1)$ the group of measurable maps from the unit interval $\mathbb{I}$ to the circle S^1 with pointwise multiplication and the topology of convergence in Lebesgue measure [39], and the group $\mathrm{Aut}\,(\mathbb{Q}, <)$ of order preserving homeomorphisms of the rational numbers $\mathbb{Q}$ with the topology of pointwise convergence [75] (where $\mathbb{Q}$ is considered as a discrete space). These results are often intimately connected with combinatorial Ramsey theory and also with the phenomenon of concentration of mass. Whereas in the previous subsection we have seen topological dynamics in the service of combinatorial number theory, here the situation is reversed and one sees results from combinatorial Ramsey theory applied in order to prove theorems in topological dynamics.

Perhaps even more unexpected is the fact that for many Polish groups G the universal minimal system (M, G) is metrizable and the G-action is easy to describe and understand. This is the case for example for the groups, $G = \mathrm{Homeo}\,(S^1)$ [75], $G = S_\infty$ the group of all permutations of a countable set [51], and $G = \mathrm{Homeo}\,(C)$ where C is the Cantor set [52].

The computations of $M(G)$ for S_∞ and $\mathrm{Homeo}\,(C)$ were followed by the outstanding work of Kechris, Pestov and Todorcevic who used model theory to give a unified and elegant theory of the $M(G)$ spaces for many closed subgroups of S_∞ [65].

The first result of this kind was Pestov's who, for $G = \mathrm{Homeo}\,(S^1)$, identified $(M(G), G)$ as the circle S^1 with the natural G-action. The possibility that a similar results will hold for, say S^n the unit sphere with $n \geq 2$, the Cantor set, or the Hilbert cube, was proved to be wrong by Uspenskij who showed that the action of a topological group G on its universal minimal system $M(G)$ is never 3-transitive [83].

Again, this is not the place for a detailed exposition of this quickly developing theory. Fortunately, there is now a new book by Pestov which will give the interested reader a panoramic overview of the theory [76].

Note: See http://www.math.umd.edu/∼mmb/md02/photos/ for Joe Auslander and an enveloping semigroup.

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