MINIMAL HYPERSPACE ACTIONS OF HOMEOMORPHISM GROUPS OF H-HOMOGENEOUS SPACES

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Abstract. Let X be a h-homogeneous zero-dimensional compact Hausdorff space, i.e. X is a Stone dual of a homogeneous Boolean algebra. Using the dual Ramsey theorem and a detailed combinatorial analysis of what we call *stable collections* of subsets of a finite set, we obtain a complete list of the minimal sub-systems of the compact dynamical system $(Exp(Exp(X)), Homeo(X))$, where $Exp(X)$ stands for the hyperspace comprising the closed subsets of X equipped with the Vietoris topology. The importance of this dynamical system stems from Uspenskij's characterization of the universal ambit of $G = Homeo(X)$. The results apply to $X = C$ the Cantor set, the generalized Cantor sets $X = \{0,1\}^\kappa$ for non-countable cardinals κ, and to several other spaces. A particular interesting case is $X = \omega^* = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers. This space, called *the corona* or the *remainder* of ω, has been extensively studied in the fields of set theory and topology.

1. Introduction

1.1. **Representative families.** Let G be a (Hausdorff) topological group and X a Hausdorff compact space. We consider compact dynamical systems or G-spaces which we denote by (X, G). The general theory of such systems ensures the existence and uniqueness of a universal G-ambit denoted $(\mathcal{S}(G), e_0, G)$. Here an **ambit** is a G-space (X, x_0, G), with a distinguished point x_0 whose orbit is dense in X. The universality means that for every ambit (X, x_0, G) there is a (necessarily unique) homomorphism of pointed dynamical systems $\pi : (\mathcal{S}(G), e_0, G) \to (X, x_0, G)$. By Zorn's lemma every dynamical system (X, G) admits, at least one, minimal subsystem $Y \subset X$; i.e. Y is closed and invariant and the only invariant

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subsets of Y are Y and $\emptyset$. It then follows that any minimal subset M of $\mathcal{S}(G)$ is a **universal minimal G-system**, in an obvious sense, and moreover up to isomorphism this universal system $\mathcal{M}(G)$ is unique.

The **enveloping semigroup** of a dynamical system (X, G), denoted $E(X)$, is by definition the closure in the compact space X^X of the collection of maps $\{\breve{g} : g \in G\}$, where $\breve{g}$ is the element of the group $Homeo(G)$ which corresponds to g. We refer the reader to books on the abstract theory of topological dynamics for more details. In particular [dV93] is a suitable source from our point of view.

In [Usp09] Uspenskij introduced the following definition

Definition 1.1. A family $\{X_\alpha : \alpha \in A\}$ of compact G-spaces is **representative** if the family of natural maps $\mathcal{S}(G) \to E(X_\alpha)$, where $\mathcal{S}(G)$ is the universal ambit of G and $E(X_\alpha)$ is the enveloping semigroup of X_α, separate points of $\mathcal{S}(G)$ (and hence yields an embedding of $\mathcal{S}(G)$ into $\Pi_{\alpha\in A} E(X_\alpha)$).

In the same article Uspenskij proved:

Theorem 1.2. *If $\{X_\alpha : \alpha \in A\}$ is a representative family of compact G-spaces, the universal minimal compact G-space $\mathcal{M}(G)$ is isomorphic (as a G-space) to a G-subspace of a product $\Pi_\beta Y_\beta$, where each Y_β is a minimal compact G-space isomorphic to a G-subspace of some X_α.*

Denote by $Exp(X)$, the *hyperspace of* X, defined to be the collection all non-empty closed sets of X, equipped with the Vietoris topology. $Exp(X)$ is known to be compact Hausdorff. Let $G = Homeo(X)$ equipped with the compact-open topology. Notice $Exp(X)$ is a G-space. The following is Theorem 4.1 of Uspenskij [Usp09]:

Theorem 1.3. *Let X be a compact space and H a subgroup of $Homeo(X)$. The sequence $\{Exp(Exp(X))^n\}_{n=1}^{\infty}$ of compact H-spaces is representative.* [1]

Remark 1.4. We refer to the action of $Homeo(X)$ on $Exp(Exp(X))^n$, $n = 1, 2, \ldots$ as the **hyperspace actions**.

[1]By this we mean $Exp((Exp(X))^n)$

1.2. **H-homogeneous spaces and homogeneous Boolean algebras.** The following definitions are well know (see e.g. [HNV04] Section H-4):

(1) A zero-dimensional compact Hausdorff topological space X is called **h-homogeneous** if every non-empty clopen subset of X is homeomorphic to the entire space X.
(2) A Boolean algebra B is called **homogeneous** if for any nonzero element a of B the relative algebra $B|a = \{x \in B : x \leq a\}$ is isomorphic to B.

Using Stone's Duality Theorem (see [BS81] IV§4) a zero-dimensional compact Hausdorff h-homogeneous space X is the Stone dual of a homogeneous Boolean algebra, i.e. any such space is realized as the space of ultrafilters B^* over a homogeneous Boolean algebra B equipped with the topology for which $N_a = \{U \in B^* : a \in U\}$, $a \in B$ is a base. Here are some examples of h-homogeneous spaces (see [ŠR89]):

(1) The countable atomless Boolean algebra is homogeneous. It corresponds by Stone duality to the Cantor set $C = \{0,1\}^{\mathbb{N}}$.
(2) Every infinite free Boolean algebra is homogeneous. These Boolean algebras correspond by Stone duality to the generalized Cantor spaces, $\{0,1\}^{\kappa}$, for infinite cardinals κ.

More examples are discussed in section 1.1 of [GG11]. In the next subsection we discuss an especially interesting example:

1.3. **The corona.** For X a Tychonoff space (completely regular Huasdorff space), the *Stone-Čech compactification* βX of X is a compact Hausdorff space, unique up to homeomorphism, such that X densely embeds in βX, $X \hookrightarrow \beta X$ and such that the following universal property holds: Any continuous function $\phi : X \to K$, where K is compact Hausdorff, can be uniquely extended to a continuous function $\tilde{\phi} : \beta X \to K$. When X is discrete and in particular in the case of the integers, $\beta\mathbb{Z}$ has a concrete description as the collection of ultrafilters on $\mathbb{Z}$. The collection $\mathcal{U} = \{U_A : A \subset \mathbb{Z}\}$, where for each $A \subset \mathbb{Z}$ the set U_A is the set of ultrafilters in $\beta\mathbb{Z}$ containing A ($U_A = \{p \in \beta\mathbb{Z} : A \in p\}$), forms a basis for the compact Hausdorff topology on $\beta\mathbb{Z}$. The collection of *fixed* ultrafilters; i.e. ultrafilters of the form $p_n = \{A \subset \mathbb{Z} : n \in A\}$ for $n \in \mathbb{Z}$, forms an open, discrete, dense subset of $\beta\mathbb{Z}$, and one identifies this collection with $\mathbb{Z}$. For more details on the Stone-Čech compactification see [GJ60], [Eng78] and [HS98].

Given a locally compact Hausdorff space Y it can be shown that Y embeds inside βY as an open dense set. One defines the *corona of* Y (or *remainder of* Y*)* to be the compact space $\chi(Y) = \beta Y \setminus Y$. Let ω be the first infinite cardinal which we will identify with $\mathbb{Z}$. The corona of the integers $\chi(\mathbb{Z}) \triangleq \omega^*$, which we will simply call the *corona,* has been extensively studied in the fields of set theory and logic. An excellent survey article is [vM84]. The notion of *P-points* received special attention, see [Kun80] and [BV80]. Specific corona spaces such as $\chi(\mathbb{Z})$, $\chi(\mathbb{Q})$ and $\chi(\mathbb{R})$ appeared in the now classical monograph on commutative C^*-algebras [GJ60]. The name seems to originate in [GP84]. The non-commutative analogue of the corona spaces, namely the *corona algebras* play an important role in the solution of various lifting problems in the theory of C^*-algebras (see [OP89]).

For an infinite subset $A \subset \mathbb{Z}$ let $\hat{A} = \omega^* \cap \mathrm{Cls}_{\beta\mathbb{Z}}(\mathrm{A})$. One sees easily that $\hat{A} = \hat{B}$ iff $A \triangle B$ is finite. The collection $\mathcal{U} = \{\hat{A} : A \subset \mathbb{Z},\ A \text{ infinite}\}$ is a basis consisting of clopen sets for the topology of ω^*. For infinite $A \subset \mathbb{Z}$ one has $\hat{A} \simeq \chi(A)$ and moreover $A \simeq \mathbb{Z}$ implies $\hat{A} \simeq \omega^*$ using the universal property of the Stone-Čech compactification. This shows that ω^* is h-homogeneous. Let $P(\omega)$ be the Boolean algebra of all subsets of ω and let $fin \subset P(\omega)$ be the ideal comprising the finite subsets of ω. Define the equivalence relations $A \sim_{fin} B$, $A, B \in P(\omega)$, if and only if $A \triangle B$ is in fin. The quotient Boolean algebra $P(\omega)/fin$ is homogeneous. This Boolean algebra corresponds by Stone duality to ω^*.

1.4. **The space of maximal chains.** Let K be a compact Huasdorff space. A subset $c \subset Exp(K)$ is a *chain* in $Exp(K)$ if for any $E, F \in c$ either $E \subset F$ or $F \subset E$. A chain is *maximal* if it is maximal with respect to the inclusion relation. One verifies easily that a maximal chain in $Exp(K)$ is a closed subset of $Exp(K)$, and that $\Phi = \Phi(K)$, the space of all maximal chains in $Exp(K)$, is a closed subset of $Exp(Exp(K))$, i.e. $\Phi(K) \subset Exp(Exp(K))$ is a compact space. Note that a G-action on K naturally induces a G-action on $Exp(K)$ and Φ. It is easy to see that every $c \in \Phi$ has a first element F which is necessarily of the form $F = \{x\}$. Moreover, calling $x \triangleq r(c)$ the *root* of the chain c, it is clear that the map $\pi : \Phi \to K$, sending a chain to its root, is a homomorphism of dynamical systems.

1.5. **The main theorem.** In view of Uspenskij's theorems mentioned in Subsection 1.1 one is naturally interested in classifying the G-minimal subspaces of $\{Exp(Exp(X))^n\}_{n=1}^{\infty}$ where $G = Homeo(X)$. The aim of this work is to accomplish this task for the case of a Hausdorff zero-dimensional compact h-homogneous space X and $n = 1$. This turns out to be highly nontrivial and therefore hard to generalize for $n \geq 2$. Fortunately the universal minimal space can be calculated in this case using different methods. We refer the reader to our paper [GG11], where we show that for these spaces $M(G) = \Phi(X)$. Our main result in the present work is the following:

Theorem. *Let X be a Hausdorff zero-dimensional compact h-homogeneous space. The following list is an exhaustive list of the $Homeo(X)$ - minimal spaces of $Exp(Exp(X))$:*

(1) $\{\{X\}\}$.
(2) Φ.
(3) $\{\{\{x_1, x_2, \dots, x_j\}\}_{(x_1,x_2,\dots,x_j)\in X^j}\}$ $(j \in \mathbb{N})$.
(4) $\{\{\{x_1, x_2, \dots, x_j\}, X\}_{(x_1,x_2,\dots,x_j)\in X^j}\}$ $(j \in \mathbb{N})$.
(5) $\{\{\{x_1, x_2, \dots, x_j\}, F\}_{(x_1,x_2,\dots,x_j)\in X^j, F\in\xi}\}_{\xi\in\Phi}$ $(j \in \mathbb{N})$.
(6) $\{\{\{x_1, x_2, \dots, x_j\}_{(x_2,\dots,x_j)\in X^{j-1}}\}\}_{x_1\in X}$ $(j \in \mathbb{N})$.
(7) $\{\{F \cup \{x_1, x_2, \dots, x_q\}\}_{(x_1,x_2,\dots,x_q)\in X^q, F\in\xi}\}_{\xi\in\Phi}(q \geq 1.)$.
(8) $\{\{F \cup \{x_1, x_2, \dots, x_q\}, \{r(\xi), y_2, \dots, x_l\}\}_{(x_1,x_2,\dots,x_q)\in X^q,(y_2,\dots,y_{l-1})\in X^{l-1}, F\in\xi}\}_{\xi\in\Phi}$
$(l > q \geq 1)$.
(9) $\{\{F \cup \{x_1, x_2, \dots, x_q\}, \{z_1, z_2, \dots, z_j\}\}_{(x_1,x_2,\dots,x_q)\in X^q,(z_1,z_2,\dots,z_j)\in X^j, F\in\xi}\}_{\xi\in\Phi}(q, j \geq 1)$.
(10) $\{\{F \cup \{x_1, x_2, \dots, x_q\}, \{r(\xi), y_2, \dots, y_l\},$
$\{z_1, z_2, \dots, z_j\}\}_{(x_1,x_2,\dots,x_q)\in X^q,(y_2,\dots,y_{l-1})\in X^{l-1},(z_1,z_2,\dots,z_j)\in X^j, F\in\xi}\}_{\xi\in\Phi}$
$(l > q \geq 1 \leq j < l)$.
(11) $\{\{X, \{x_1, x_2, \dots, x_j\}_{(x_2,\dots,x_j)\in X^{j-1}}\}\}_{x_1\in X}$ $(j \in \mathbb{N})$.
(12) $\{\{\{\tau(\xi), x_2, \dots, x_j\}, F\}_{(x_2,\dots,x_j)\in X^{j-1}, F\in\xi}\}_{\xi\in\Phi}(j \in \mathbb{N})$.
(13) $\{\{\{y_1, y_2, \dots, y_{j'}\}, \{x_1, x_2, \dots, x_j\}_{(x_2,\dots,x_j)\in X^{j-1},(y_1,y_2,\dots,y_{j'})\in X^{j'}}\}\}_{x_1\in X}$.
(14) $\{\{X, \{y_1, y_2, \dots, y_{j'}\}, \{x_1, x_2, \dots, x_j\}_{(x_2,\dots,x_j)\in X^{j-1},(y_1,y_2,\dots,y_{j'})\in X^{j'}}\}\}_{x_1\in X}$.
(15) $\{\{\{y_1, y_2, \dots, y_{j'}\}, \{\tau(\xi), x_2, \dots, x_j\}, F\}_{(x_2,\dots,x_j)\in X^{j-1}, F\in\xi,(y_1,y_2,\dots,y_{j'})\in X^{j'}}\}_{\xi\in\Phi}$.
(16) $\{\{F|\, F \in Exp(X)\}\}$.
(17) $\{\{F|\, x \in F \in Exp(X)\}\}_{x\in X}$.
(18) $\{\{\{x_1, x_2, \dots, x_j\}, F\}_{(x_1,x_2,\dots,x_j)\in X^j, x\in F\in Exp(X)}\}_{x\in X}$ $(j \in \mathbb{N})$.

The proof of the theorem is achieved by a detailed combinatorial analysis of collections of subsets of a finite set, which we expect will be, in itself, of an independent interest to combinatorists. We thank Noga Alon for his advise pertaining to some aspects of this analysis.

It is interesting to compare this theorem to similar theorems in [Gut08]. In that article it is shown that If X is a closed manifold of dimension 2 or higher, or the Hilbert cube, then

M, the space of maximal chains of continua, is a minimal subspace of $Exp(Exp(X))$ under the action of Homeo(X). Further investigating $Exp(M) \subset Exp(Exp(Exp(X)))$ it is shown:

Theorem. *If X is a closed manifold of dimension 3 or higher, or the Hilbert cube, then the action of* Homeo(X) *on $Exp(M)$, the space of non-empty closed subsets of the space of maximal chains of continua, has exactly the following minimal subspaces:*

(1) $\{M\}$,

(2) $\{M_x\}_{x\in X}$, *where* $M_x = \{c \in M(X) : \bigcap\{c_\alpha : c_\alpha \in c\} = \{x\}\}$,

(3) $\{\{c\} : c \in M\}$.

2. Patterns

The following section deals with results in combinatorics of finite sets. In subsequent sections these results will be used in the context of hyperspace actions.

2.1. Patterns and partitions.

Definition 2.1. A non-empty collection $\mathcal{P}$ of non-empty vectors in $(Exp(\vec{m}))^n$, where $\vec{m} = \{1, 2, \dots, m\}$ is called an m_n**-pattern**. Thus an m_n-pattern has the form

$$\mathcal{P} = \{P_s = (P_s^1, P_s^2, \cdots, P_s^n) : s = 1, 2, \dots, t\},$$

where each P_s^i is a nonempty subset of $\vec{m}$. We denote the collection of m_n-patterns by $C_n(m)$. The number of distinct m_n-patterns is $r = 2^{2^{mn}-1} - 1$ (we exclude the empty set). Denote by $C_n = \bigcup_{m\in\mathbb{N}} C_n(m)$, the collection of n**-dimensional** patterns. Denote by $C(m) = \bigcup_{n\in\mathbb{N}} C_n(m)$, the collection of m**-pattern**.

The idea behind this definition is that patterns represent neighborhoods of element of the space $Exp(Exp(X))^n$. It is much easier to think about, or visualize, a 1-pattern than a higher order ones. So we suggest that, in the sequel, the reader will consider, when each new definition is introduced, the one-dimensional case first.

As a motivating example consider the m_1 pattern $\phi_m \in C_1(m)$:

$$\phi_m = \{\{1\}, \{1, 2\}, \dots, \{1, 2, \dots, m\}\}.$$

Note that given a clopen ordered partition $\alpha = (A_1, A_2 \dots, A_m)$ of the compact zero-dimensional space X, the pattern ϕ_m can serve as a typical neighborhood $\mathfrak{U} = \mathfrak{U}(\phi_m)$

in the Vietoris topology on $Exp(Exp(X)))$ of a maximal chain $c \in \Phi(x)$. Explicitly, set

$$\begin{aligned} \mathfrak{U} &= \langle \langle A_1 \rangle, \langle A_1, A_2 \rangle, \dots, \langle A_1, A_2, \dots, A_m \rangle \rangle \\ &= \langle \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m \rangle, \end{aligned}$$

where for a compact space K and open sets $V_1, \dots, V_k$ we let

$$\langle V_1, V_2, \dots, V_k \rangle = \{F \in Exp(K) : \ F \subset \cup_{j=1}^k V_j, \text{ and } F \cap V_j \neq \emptyset \ \ \forall\, 1 \leq j \leq k\}.$$

In other words, a maximal chain $c \in \Phi(X)$ is in $\mathfrak{U}$ if and only if every $F \in c$ is in at least one of the sets $\mathcal{U}_j = \langle A_1, A_2, \dots, A_j \rangle$ and for every $1 \leq j \leq m$ there is at least one $F \in c \cap \langle A_1, A_2, \dots, A_j \rangle$ (see Lemma 4.5 below). Pictorially we can think of c as a chain of closed subsets of X which grows continuously to fill the sets A_1, then $A_1 \cup A_2$, etc. and eventually the entire space $X = A_1 \cup A_2 \cup \dots \cup A_m$.

An ordered partition $\gamma = (C_1, \dots, C_k)$ of $\{1, \dots, s\}$ into k nonempty sets is said to be **naturally ordered** if for every $1 \leq i < j \leq k$, $\min(C_i) < \min(C_j)$. We denote by $\Pi\binom{s}{k}$ the collection of naturally ordered partitions of $\{1, \dots, s\}$ into k nonempty sets.

Definition 2.2. Let $\mathcal{P}$ be an m_n-pattern and $\gamma = (C_1, \dots, C_k) \in \Pi\binom{m}{k}$. The **induced k_n-pattern** $\mathcal{P}_\gamma$ is defined as the collection $\mathcal{P}_\gamma = \{P_\gamma : P \in \mathcal{P}\}$, where

$$P_\gamma = \{(j_1, j_2, \dots, j_n) : C_{j_1} \times C_{j_2} \times \dots \times C_{j_n} \cap P \neq \emptyset\}.$$

Let $\beta = (B_1, \dots, B_s) \in \Pi\binom{k}{s}$ and $\gamma = (C_1, \dots, C_k) \in \Pi\binom{m}{k}$, we define the **amalgamated partition** $\gamma_\beta = (G_1, \dots, G_s) \in \binom{m}{s}$ by:

$$G_j = \bigcup_{i \in B_j} C_i$$

Notice γ_β is naturally ordered and $(\mathcal{P}_\gamma)_\beta = \mathcal{P}_{\gamma_\beta}$.

2.2. **Notation.** Define $^+ : Exp(\vec{m}) \to Exp(\overrightarrow{m+1})$ by the mapping

$$A \mapsto A^+ = \{j + 1 |\, j \in A\}.$$

In addition $\emptyset^+ = \emptyset$. Define $^- : Exp(\vec{m} \setminus \{1\}) \to Exp(\overrightarrow{m-1})$ by the mapping

$$A \mapsto A^- = \{j - 1 |\, j \in A\}.$$

For $j \in \overrightarrow{m+1}$, define $D_j : Exp(\vec{m}) \to Exp(\overrightarrow{m+1})$ by the mapping

$$A \mapsto (A \cap \overrightarrow{j-1}) \cup (A \setminus \overrightarrow{j-1})^+.$$

For $\gamma \in \Pi\binom{m+1}{m}$, $\gamma = (C_1, C_2, \ldots, C_m)$, denote by $p_\gamma : Exp(\vec{m+1}) \to Exp(\vec{m})$ the mapping $A \mapsto A_\gamma$, where, as above, $A_\gamma = \{j : C_j \cap A \neq \emptyset\}$.

Let $i, j \in \vec{m+1}$ with $i < j$. Define:

$$\gamma_{i,j}^{m+1} = \gamma_{i,j} = (\{1\}, \{2\}, \ldots, \{i-1\}, \{i,j\}, \{i+1\}, \ldots, \{j-1\}, \{j+1\}, \ldots, \{m+1\})$$

Notice $\gamma_{i,j}^{m+1} \in \Pi\binom{m+1}{m}$. For $P \in Exp(\vec{m})$, with $m \in P$ we introduce the notation: $P_{m+1} = P \cup \{m+1\}$ and $\hat{P} = P_{m+1} \setminus \{m\}$. Notice $p^{-1}_{\gamma_{m,m+1}}(P) = \{P, \hat{P}, P_{m+1}\}$.

2.3. **The standard Patterns.**

Definition 2.3. Let $\vec{m} = \{1, \ldots, m\}$ and $\vec{0} = \emptyset$. For $1 \leq i_1 < i_2 < \cdots < i_l \leq m$ let $I^m_{i_1,i_2,\ldots,i_l} = \vec{m} \setminus \{i_1, i_2, \ldots, i_l\}$ and define $e(I^m_{i_1,i_2,\ldots,i_l}) = i_1$. The indices $i_1, i_2, \ldots, i_l$ are referred to as the **holes** of $I^m_{i_1,i_2,\ldots,i_l}$. For $0 \leq h \leq m$, $1 \leq l \leq m$, $h + l \leq m + 1$ define $H^m_{h,l} = \{I^m_{d_1,d_2,\ldots,d_h}\}_{l \leq d_1 < d_2 < \cdots < d_h}$ and $EH^m_{h,l} = \{I^m_{d_1,d_2,\ldots,d_h}\}_{l = d_1 < d_2 < \cdots < d_h}$, where we use the convention $H^m_{0,l} = EH^m_{0,l} = \vec{m}$.

The following m_1-patterns are called **standard** :

(1) $\{\vec{m}\}$.
(2) $\phi_m = \{\{1\}, \{1,2\}, \ldots, \vec{m}\}$.
(3) For every $1 \leq j \leq m$ the collection $\mathcal{A}_{j,m}$ of all subsets of $\vec{m}$ of cardinality $\leq j$.
(4) For every $1 \leq j \leq m-2$ the collection $\mathcal{A}_{j,m} \cup \{\vec{m}\}$.
(5) For every $1 \leq j \leq m-2$ the collection $\mathcal{A}_{j,m} \cup \phi_m$.
(6) For every $1 \leq j \leq m$ the collection $\mathcal{A}^1_{j,m}$ of all subsets of $\vec{m}$ of cardinality $\leq j$ containing 1.
(7) The collection $\mathcal{A}^1_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ for $\emptyset \neq \mathcal{N} \subsetneq H^m_{1,2}$ ($\mathcal{N} = \emptyset$ corresponds to case (12) and the case $\mathcal{N} = H^m_{1,2}$ corresponds to case (6))
(8) $\mathcal{A}_{m-2,m} \cup \mathcal{N}$ for $\mathcal{N} \subset H^m_{1,1}$, $\mathcal{N} \neq H^m_{1,2}$ with $|\mathcal{N}| = m-1$ ($\mathcal{N} = H^m_{1,2}$ corresponds to case (14)).
(9) Let $\emptyset \neq \mathcal{N} \subsetneq H^m_{1,1}$. $\mathcal{N} \neq H^m_{1,2}$. The collection $\mathcal{A}_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ ($\mathcal{N} = \emptyset$ corresponds to case (4), $\mathcal{N} = H^m_{1,1}$ corresponds to case (3)and $\mathcal{N} = H^m_{1,2}$ corresponds to case (14)).
(10) For every $2 \leq r < r+1 < s < m$ the collection $\mathcal{D}^m_{r,s} \triangleq \mathcal{A}^1_{m-r-1,m} \cup \bigcup_{h=1}^r H^m_{h,s-h+1} \cup \{\vec{m}\}$ ($s = m$ corresponds to case (13), $s = r+1$ corresponds to $\mathcal{D}^m_{r-1,s}$ and $r = 1$ corresponds to case (7)).
(11) For every $2 \leq r < r+1 < s < m$ and $1 \leq j \leq m-r-1$, $\mathcal{D}^m_{r,s} \cup \mathcal{A}_{j,m}$.

(12) For every $1 \le j \le m-2$ the collection $\mathcal{A}^1_{j,m} \cup \{\vec{m}\}$.

(13) For every $1 \le j \le m-2$ the collection $\mathcal{A}^1_{j,m} \cup \phi_m$.

(14) For every $2 \le j \le m$ and $j' < j$ the collection $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m}$.

(15) For every $2 \le j \le m-2$ and $j' < j$ the collection $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \{\vec{m}\}$ ($j = m-1, m$ correspond to case (14)).

(16) For every $2 \le j \le m-2$ and $j' < j$ the collection $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \phi_m$ ($j = m-1, m$ correspond to case (14)).

(17) For every $1 \le j < m-2$ the collection $\mathcal{A}_{j,m} \cup \mathcal{A}^1_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ where $\emptyset \ne \mathcal{N} \subsetneq H^m_{1,2}$.

Note:

(1) $\mathcal{D}^m_{r_1,s_1} \cup \mathcal{D}^m_{r_2,s_2} = \mathcal{D}^m_{\min\{r_1,r_2\},\min\{s_1,s_2\}}$.

(2) $D^m_{s,s+1} = D^m_{s-1,s+1}$.

2.4. **Stable patterns.** The notion of a stable pattern which we are about to define is of crucial importance for our analysis. We surmise that it may be relevant for other problems in the combinatorics of finite sets.

Definition 2.4. An m_n-pattern $\mathcal{P}$ is said to be ***k*-stable** if for every partition $\alpha \in \Pi\binom{m}{k'}$ for $2 \le k' \le k$ the induced k'_n-pattern $\mathcal{P}_\alpha$ is a constant pattern (i.e. it does not depend on α). Denote $SP_n(m) = \{\mathcal{P} : \mathcal{P}$ is an m-stable m_n-pattern$\}$.

As an example the reader is advised to check that the m_1-pattern ϕ_m is m-stable.

Lemma 2.5. *Let $1 \le i < j \le m+1$ and $Q \subset \vec{m}$ with $i \notin Q$ then $p^{-1}_{\gamma_{i,j}}(Q) = \{D_j(Q)\}$.*

Proof. Trivial. □

Lemma 2.6. *Let $m \in \mathbb{N}$ and let $\mathcal{P}$ be an $(m+1)$-pattern. If there exists an m-stable m-pattern $\mathcal{Q}$ so that for every $\gamma \in \Pi\binom{m+1}{m}$, $\mathcal{P}_\gamma = \mathcal{Q}$ then $\mathcal{P}$ is $(m+1)$-stable.*

Proof. For every $\alpha, \alpha' \in \Pi\binom{m+1}{k}$, there exist $\gamma, \gamma' \in \Pi\binom{m+1}{m}$ and $\beta, \beta' \in \Pi\binom{m}{k}$ so that $\mathcal{P}_\alpha = (\mathcal{P}_\gamma)_\beta = \mathcal{Q}_\beta = \mathcal{Q}_{\beta'} = (\mathcal{P}_{\gamma'})_{\beta'}$. □

Lemma 2.7. *Let $m \in \mathbb{N}$ and let $\mathcal{P}$ be an $(m+1)$-pattern. Let $\pi \in \Pi\binom{m+1}{m}$ and $\mathcal{Q}$ an m-pattern so that $\mathcal{P}_\pi = \mathcal{Q}$ then $\mathcal{P} = \bigcup_{Q \in \mathcal{Q}} p^{-1}_\pi(Q) \cap \mathcal{P}$.*

Proof. Trivial. □

Lemma 2.8. *Let $m \in \mathbb{N}$ and let $\mathcal{P}$ be an $(m+1)$-pattern. Let $\gamma \in \Pi\binom{m+1}{m}$ and $\mathcal{Q}$ an m-pattern so that $\mathcal{P}_\gamma = \mathcal{Q}$. Let $Q \in \mathcal{Q}$ and assume $p_\gamma^{-1}(Q) = \{A\}$, then $A \in \mathcal{P}$.*

Proof. Trivial. □

Lemma 2.9. *Let $\mathcal{P}$ be an $(m+1)$-stable $(m+1)_1$-pattern. Let $1 \leq h \leq m$, $1 \leq l \leq m$, $h+l \leq m+1$ and $\pi = \gamma^{m+1}_{m,m+1} \in \Pi\binom{m+1}{m}$. If $\mathcal{P}_\pi \cap EH^m_{h,i} = \emptyset$ for all $i < l$, and $\mathcal{P}_\pi \cap EH^m_{h,l} \neq \emptyset$ then $\mathcal{P} \cap H^{m+1}_{h+1,1} = H^{m+1}_{h+1,l}$ and $\mathcal{P}_\pi \cap H^m_{h,1} = H^m_{h,l}$.*

Proof. Let $Q \in \mathcal{P}_\pi \cap EH^m_{h,l}$ which is non-empty by assumption. Our first goal is to show that $H^{m+1}_{h+1,l} \subset \mathcal{P}$. Let $R = I^{m+1}_{s_1,s_2,\ldots,s_{h+1}} \in H^{m+1}_{h+1,l}$. We will show using induction that for any $l-1 \leq j \leq s_h$ there exists $P = P(j) \in \mathcal{P}_\pi \cap EH^m_{h,l}$ so that $P \cap \vec{j} = R \cap \vec{j}$ and $e(P) \leq e(R)$ (obviously if the statement is true one can choose $P(s_h)$ for all j but this can be concluded only after the induction is carried through). First we verify the base case by choosing $P = Q$ and noticing trivially that $P \cap \overrightarrow{l-1} = R \cap \overrightarrow{l-1} = \overrightarrow{l-1}$ and $l = e(P) \leq e(R)$. Secondly let $l-1 \leq j \leq s_h - 1$ and assume there exist $P \in \mathcal{P}_\pi \cap EH^m_{h,l}$ so that $P \cap \vec{j} = R \cap \vec{j}$ and $e(P) \leq e(R)$. We will prove there exists $P' \in \mathcal{P}_\pi \cap EH^m_{h,l}$ so that $P' \cap \overrightarrow{j+1} = R \cap \overrightarrow{j+1}$ and $e(P') \leq e(R)$. If $P \cap \overrightarrow{j+1} = R \cap \overrightarrow{j+1}$ we are done. Assume $P \cap \overrightarrow{j+1} \neq R \cap \overrightarrow{j+1}$. We distinguish between several cases. We repeatedly use the fact that $D_q(P) \in \mathcal{P}$ for any $e(P) < q \leq m+1$ as seen by Lemma 2.5.

- $j+1 \notin R$ and $j+1 \in P$. As R has at most $h-1$ holes in $\vec{j}$, so does P in $\overrightarrow{j+1}$. Therefore there exists $k > j+1$ so that $k \notin P$. Moreover by assumption $e(P) \leq e(R) \leq j+1$ and as $j+1 \in P$ then $e(P) < j+1$. Define $P' = \pi_{j+2,k+1}(D_{j+1}(P))$ (P' is constructed by adding a hole at $j+1$ and canceling the hole at k). Clearly $P' \in \mathcal{P}_\pi \cap EH^m_{h,l}$ and $e(P') = e(P) \leq e(R)$.
- $j+1 \in R$, $j+1 \notin P$, $e(P) \leq j$ and $\exists k > j+1$, $k \in P$. Define $P' = \pi_{j+1,k}(D_{m+1}(P))$ (P' is constructed by canceling the hole at $j+1$ and adding a hole at m). Clearly $P' \in \mathcal{P}_\pi \cap EH^m_{h,l}$. As P has a hole in $\vec{j}$ so does P' and we have $e(P') = e(P) \leq e(R)$.
- $j+1 \in R$, $e(P) = j+1$ (equivalent to $j+1 \notin P$, $e(P) > j$ and implies $e(R) \geq j+2$) and $j+2 \in P$. Define $P' = \pi_{1,j+1}(D_{j+3}(P))$ (P' is constructed by canceling the hole at $j+1$ and adding a hole at $j+2$) Clearly $P' \in \mathcal{P}_\pi \cap EH^m_{h,l}$. In addition notice $e(P') = j+2 \leq e(R)$.

- $j+1 \in R$, $e(P) = j+1$ (equivalent to $j+1 \notin P$, $e(P) > j$ and implies $e(R) \geq j+2$), $j+2 \notin P$ and $\exists k > j+2$, $k \in P$. $P' = \pi_{j+1,k}(D_{m+1}(P))$ (P' is constructed by canceling the hole at $j+1$ and adding a hole at m). Clearly $P' \in \mathcal{P}_\pi \cap EH^m_{h,l}$. In addition notice $e(P') = j+2 \leq e(R)$.
- $j+1 \in R$, $j+1 \notin P$, $e(P) \leq j$ and $\sim \exists k > j+1$, $k \in P$. Notice that by assumption $j+1 \leq s_h$. However as $j+1 \in R$, we conclude $j+1 < s_h$ which implies $j+2 < s_{h+1} \leq m+1$. Define $P' = \pi_{j+2,m+1}(D_{j+1}(P))$ (P' is constructed by canceling the hole at $j+1$ and adding a hole at $j+2$) Clearly $P' \in \mathcal{P}_\pi \cap EH^m_{h,l}$. In addition notice $e(P') = j+2 \leq e(R)$.
- $j+1 \in R$, $j+1 \notin P$, $e(P) > j$ and $\sim \exists k > j+1$, $k \in P$. As $j \geq l-1$ and $e(P) = l$, we conclude $j = l-1$. As $\sim \exists k > j+1$, $k \in P$ we must have $P = I^m_{l,l+1,\ldots,m}$. This implies $h = m-l+1$. In this case $H^{m+1}_{h+1,l} = \{I^{m+1}_{l,l+1,\ldots,m+1}\}$. As $I^{m+1}_{l,l+1,\ldots,m+1} = D_{m+1}(I^m_{l,l+1,\ldots,m})$, we have $H^{m+1}_{h+1,l} \subset \mathcal{P}$ and we can stop the induction.

At the end of the induction we have either proven $H^{m+1}_{h+1,l} \subset \mathcal{P}$ or shown that for all $R = I^{m+1}_{s_1,s_2,\ldots,s_{h+1}} \in H^{m+1}_{h+1,l}$ there exists $P \in \mathcal{P}_\pi \cap EH^m_{h,l}$ so that $P \cap \vec{s_h} = R \cap \vec{s_h}$ and $e(P) \leq e(R)$. This implies $R = D_{s_{h+1}}(P)$ and therefore we have $R \in \mathcal{P}$. We can thus finally conclude $H^{m+1}_{h+1,l} \subset \mathcal{P}$. This implies $H^m_{h,l} \subset \mathcal{P}_\pi$. Indeed let $P \in H^m_{h,l}$, then $Q = D_{m+1}(P) \in H^{m+1}_{h+1,l}$ and $P = \pi(Q)$. By assumption $\mathcal{P}_\pi \cap EH^m_{h,i} = \emptyset$ for all $i < l$, we can therefore conclude $\mathcal{P}_\pi \cap H^m_{h,1} = H^m_{h,l}$. Assume for a contradiction $A \in \mathcal{P} \cap EH^{m+1}_{h+1,i}$ for some $i < l$. Select $j > i$ so that $j \notin A$ (such j exists as $(h+1) \geq 2$). Notice $p_{\gamma_{i,j}}(A) \in EH^m_{h,j}$ which is a contradiction. We conclude that $\mathcal{P} \cap H^{m+1}_{h+1,1} = H^{m+1}_{h+1,l}$. □

Lemma 2.10. *Let $\mathcal{P}$ be a $(m+1)$-stable $(m+1)_1$-pattern. Assume $j < m$. If $\mathcal{A}_{j,m} \subset \mathcal{P}_\pi$ ($\mathcal{A}^1_{j,m} \subset \mathcal{P}_\pi$) then $\mathcal{A}_{j,m+1} \subset \mathcal{P}$ ($\mathcal{A}^1_{j,m+1} \subset \mathcal{P}$ respectively).*

Proof. Let $Q \in \mathcal{A}_{j,m+1}$. Let $1 \leq i < k \leq m+1$, so that $i, k \notin Q$. Notice $|p_{\gamma_{i,k}}(Q)| = |Q|$ and therefore $p_{\gamma_{i,k}}(Q) \in \mathcal{A}_{j,m}$. As $p^{-1}_{\gamma_{i,k}}(p_{\gamma_{i,k}}(Q)) = \{Q\}$ we have $Q \in \mathcal{P}$. The proof for $\mathcal{A}^1_{j,m+1}$ is similar. □

Lemma 2.11. *Let $\mathcal{P}$ be a $(m+1)$-stable $(m+1)_1$-pattern.* Assume $\mathcal{P}_\pi \cap H^m_{1,1} = H^m_{1,e}$ where $e \geq 3$, then $\mathcal{P} \cap H^{m+1}_{1,1} = H^{m+1}_{1,e+1}$. If in addition $\vec{m} \in \mathcal{P}_\pi$, then $\overrightarrow{m+1} \in \mathcal{P}$.

Proof. Denote $\mathcal{M} = \{I^{m+1}_k : I^{m+1}_k \in \mathcal{P}\}$. Choose $e \leq k \leq m$ and notice that $p^{-1}_{\gamma_{1,2}}(I^m_k) = \{I^{m+1}_{k+1}, I^{m+1}_{1,k+1}, I^{m+1}_{2,k+1}\}$. Clearly $I^{m+1}_{1,k+1} \notin \mathcal{P}$ as $1 \notin p_{\gamma_{2,3}}(I^{m+1}_{1,k+1})$. Also $I^{m+1}_{2,k+1} \notin \mathcal{P}$ as $p_{\gamma_{k,k+1}}(I^{m+1}_{2,k+1})$

$= I_2^m \notin \mathcal{P}_\pi$. Conclude $H_{1,e+1}^{m+1} \subset \mathcal{M}$. Let $k' < e$, then $p_{\gamma_{m,m+1}}(I_{k'}^{m+1}) = I_{k'}^m \notin \mathcal{P}_\pi$. Conclude $\mathcal{M} = H_{1,e+1}^{m+1}$. If $\vec{m} \in \mathcal{P}_\pi$, then as $\vec{m} \notin p_{\gamma_{e-1,e}}(\mathcal{M})$ we conclude $\overrightarrow{m+1} \in \mathcal{P}$. □

Theorem 2.12. *Let $m \geq 2$. The standard patterns are m-stable m_1-patterns and in particular for all $\gamma \in \Pi\binom{m+1}{m}$:*

(1) $\{\vec{m}\}_\gamma = \{\overrightarrow{m-1}\}$.
(2) $(\phi_m)_\gamma = \phi_{m-1}$.
(3) $(\mathcal{A}_{m,m})_\gamma = \mathcal{A}_{m-1,m-1}$.
(4) $(\mathcal{A}_{m,m}^1)_\gamma = \mathcal{A}_{m-1,m-1}^1$.
(5) $(\mathcal{A}_{j,m})_\gamma = \mathcal{A}_{j,m-1}$ for $1 \leq j \leq m-1$.
(6) $(\mathcal{A}_{j,m}^1)_\gamma = \mathcal{A}_{j,m-1}^1$ for $1 \leq j \leq m-1$.
(7) $(\mathcal{D}_{r,s}^m)_\gamma = \mathcal{D}_{r-1,s-1}^{m-1}$ for $2 \leq r < r+1 < s < m$.
(8) $(\mathcal{A}_{m-2,m}^1 \cup \mathcal{N} \cup \{\vec{m}\})_\gamma = \mathcal{A}_{m-1,m-1}^1$ where $\mathcal{N} \subset H_{1,2}^m$.
(9) $(\mathcal{A}_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\})_\gamma = \mathcal{A}_{m-1,m-1}$ for $\mathcal{N} \subset H_{1,1}^m$.
(10) $(\mathcal{A}_{m-2,m} \cup \mathcal{N})_\gamma = \mathcal{A}_{m-1,m-1}$ for $\mathcal{N} \subset H_{1,1}^m$ with $|\mathcal{N}| = m-1$.

Proof.

(1) Trivial.
(2) Trivial.
(3) Trivial.
(4) Trivial.
(5) Fix $\alpha = \gamma_{i,k}$ for some $1 \leq i < k \leq m$. Trivially $(\mathcal{A}_{j,m})_\alpha \subset \mathcal{A}_{j,m-1}$. Let $Q \in \mathcal{A}_{j,m-1}$. If $i \notin Q$, then by Lemma 2.5 $p_\alpha^{-1}(Q) = \{D_k(Q)\}$ and clearly $D_k(Q) \in \mathcal{A}_{j,m}$ as $|D_k(Q)| = |Q|$. If $i \in Q$, then $p_\alpha^{-1}(Q) = \{D_k(Q), D_k(Q) \cup \{k\}, D_k(Q) \cup \{k\} \setminus \{i\}\}$ and again it is enough to note that $D_k(Q) \in \mathcal{A}_{j,m}$.
(6) Similar to the proof of (5).
(7) We start by proving that for all $\gamma \in \Pi\binom{m+1}{m}$, $(\mathcal{D}_{r,s}^m)_\gamma \subset \mathcal{D}_{r-1,s-1}^{m-1}$, where:

$$(\mathcal{D}_{r,s}^m)_\gamma = (\mathcal{A}_{m-r-1,m}^1 \cup \bigcup_{l=1}^{r} H_{l,s-l+1}^m \cup \{\vec{m}\})_\gamma$$

$$\mathcal{D}_{r-1,s-1}^{m-1} = \mathcal{A}_{m-r-1,m-1}^1 \cup \bigcup_{l=1}^{r-1} H_{l,s-l}^{m-1} \cup \{\overrightarrow{m-1}\}$$

This follows from $(\mathcal{A}^1_{m-r-1,m})_\gamma = \mathcal{A}^1_{m-r-1,m-1}$, (as trivially $m-r-1<m$), $\{\vec{m}\}_\gamma = \{\vec{m-1}\}$ and from $(H^m_{l,s-l+1})_\gamma \subset H^{m-1}_{l-1,s-l+1} \cup H^{m-1}_{l,s-l}$ for $l = 1,\ldots,r$. To prove $(\mathcal{D}^m_{r,s})_\gamma \supset \mathcal{D}^{m-1}_{r-1,s-1}$, fix $\gamma = \gamma_{i,j}$ and notice that for $P \in H^{m-1}_{l,s-l}$, one has $p_\gamma(D_j(P)) = P$ and $D_j(P) \in H^m_{l,s-l+1}$.

(8) Trivial.

(9) Trivial.

(10) Notice that as $|\mathcal{N}| = m-1$, for all $\gamma = \gamma_{i,j}$, there exist $P \in \mathcal{N}$ so that $i \notin P$ or $j \notin P$, which implies that $\vec{m} \in (\mathcal{N})_\gamma$. The rest of the proof is trivial.

□

Our next goal is to show that, in fact, the standard patterns are the only m_1-patterns which are m-stable (Theorem 2.14). We begin by analyzing the 3-patterns.

Proposition 2.13. *The 3-stable 3_1-patterns are standard.*

Proof. We enumerate all 3-stable 3_1-patterns. Denote $\alpha = \gamma^3_{1,2}$, $\beta = \gamma^3_{1,3}$ and $\pi = \gamma^3_{2.3}$. Assume $\mathcal{P}$ is a 3-stable 3_1-pattern. Obviously $\mathcal{P}_\pi$ is one of the seven 2-patterns. We analyze the different cases and show $\mathcal{P}$ must be standard:

(1) $\mathcal{P}_\pi = \{\{1\}\} = \mathcal{A}^1_{1,2}$. Notice $p_\pi^{-1}(\{1\}) = \{\{1\}\}$ and conclude by Lemma 2.7 $\mathcal{P} = \mathcal{A}^1_{1,3}$.

(2) $\mathcal{P}_\pi = \{\{2\}\}$. Notice $p_\beta^{-1}(\{2\}) = \{\{2\}\}$. Conclude $\mathcal{P} = \{\{2\}\}$. However $(\{\{2\}\})_\alpha = \{\{1\}\}$. Contradiction.

(3) $\mathcal{P}_\pi = \{\{1,2\}\} = \{\vec{2}\}$. Using case (1) of Theorem 2.14 which holds true for $m \geq 2$, we conclude $\mathcal{P} = \{\vec{3}\}$

(4) $\mathcal{P}_\pi = \{\{1\},\{2\}\} = \mathcal{A}_{1,2}$. Using case (3a.) of Theorem 2.14 which holds true for $m \geq 2$, we conclude $\mathcal{P} = \mathcal{A}_{1,3}$.

(5) $\mathcal{P}_\pi = \{\{1\},\{1,2\}\} = \phi_2$. Notice $p_\pi^{-1}(\{1,2\}) = \{\{1,2\},\{1,3\},\{1,2,3\}\}$. We analyze all $\mathcal{Q} \in p_\pi^{-1}(\phi_2)$ in the following table:

$\mathcal{Q}$	Stable	Identification / Reason for not being stable
$\{\{1\},\{1,2\}\}$	No.	$\mathcal{Q}_\alpha = \{\{1\}\} \neq Q_\pi$
$\{\{1\},\{1,3\}\}$	No.	$\mathcal{Q}_\beta = \{\{1\}\} \neq Q_\pi$
$\{\{1\},\{1,2,3\}\}$	Yes.	$\mathcal{A}^1_{1,3} \cup \{\vec{3}\}$
$\{\{1\},\{1,2\},\{1,3\}\}$	Yes.	$\mathcal{A}^1_{2,3}$
$\{\{1\},\{1,2\}\{1,2,3\}\}$	Yes.	ϕ_3
$\{\{1\},\{1,3\}\{1,2,3\}\}$	Yes.	$\mathcal{A}^1_{1,3} \cup \mathcal{N} \cup \{\vec{3}\}$, where $\mathcal{N} \subset H^3_{1,2}$.
$\{\{1\},\{1,2\},\{1,3\}\{1,2,3\}\}$	Yes.	$\mathcal{A}^1_{3,3}$

(6) $\mathcal{P}_\pi = \{\{2\},\{1,2\}\}$. Notice $p_\beta^{-1}(\{2\}) = \{\{2\}\}$. Conclude $\{2\} \in \mathcal{P}$. However $p_\alpha(\{2\}) = \{1\} \notin \mathcal{P}_\pi$. Contradiction.

(7) $\mathcal{P}_\pi = \{\{1\},\{2\},\{1,2\}\} = \mathcal{A}_{2,2}$. Using case (3b.) of Theorem 2.14 which holds true for $m \geq 2$, we conclude $\mathcal{P} = \mathcal{A}_{1,3} \cup \mathcal{N}'$ for $\mathcal{N}' \subset H^3_{1,1}$ with $|\mathcal{N}'| = 2$ or $\mathcal{P} = \mathcal{A}_{2,3}$ or $\mathcal{P} = \mathcal{A}_{1,3} \cup \mathcal{N} \cup \{\vec{3}\}$, for $\mathcal{N} \subset H^3_{1,1}$.

□

Theorem 2.14. *The standard patterns are the only m_1-patterns which are m-stable.*

Proof. By Theorem 2.12 the standard patterns are m-stable. We prove by induction on m that the standard patterns are the only m_1-patterns which are m-stable. The case $m = 3$ is proven in Theorem 2.13. Assume the theorem is true for $m \geq 3$, we prove it for $m+1$. Let $\pi = \gamma^{m+1}_{m,m+1}$. Let $\mathcal{P}$ be an $(m+1)$-stable $(m+1)_1$-pattern. By the induction assumption $\mathcal{P}_\pi$ must be standard. We analyze the different cases in order to prove $\mathcal{P}$ is standard. Note that if $P \in \mathcal{P}_\pi$ and $m \notin P$, then $p_\pi^{-1}(P) = \{P\}$ and therefore we will be mainly analyzing $P \in \mathcal{P}_\pi$ with $m \in P$.

(1) $\mathcal{P}_\pi = \{\vec{m}\}$. Recall $p_\pi^{-1}(\vec{m}) = \{\overrightarrow{m+1}, \vec{m}, \hat{m}\}$. We claim $\vec{m} \notin \mathcal{P}$. Indeed let $\alpha = \gamma^{m+1}_{1,m}$ and observe that $\vec{m}_\alpha = \overrightarrow{m-1} \notin \mathcal{P}_\pi$ (recall $m \geq 2$). Similarly let $\beta = \gamma^{m+1}_{1,m+1}$ and observe that $\hat{m}_\alpha = \overrightarrow{m-1} \notin \mathcal{P}$ (recall $m \geq 2$). We conclude $\mathcal{P} = \{\overrightarrow{m+1}\}$ using Lemma 2.7.

(2) $\mathcal{P}_\pi = \phi_m$. Recall $p_\pi^{-1}(\vec{m}) = \{\overrightarrow{m+1}, \vec{m}, \hat{m}\}$. We claim $\hat{m} \notin \mathcal{P}$. Indeed let $\alpha = \gamma^{m+1}_{1,2}$ and observe $\hat{m}_\alpha = \{1,2,\ldots,m-2,m\} \notin \mathcal{P}_\pi$(recall $m-2 \geq 1$). We claim $\vec{m} \notin \mathcal{P}$. Indeed in that case $\overrightarrow{m-1} \notin \mathcal{P}_\alpha$. Similarly we claim $\overrightarrow{m+1} \notin \mathcal{P}$ cannot hold as in

that case $\vec{m} \notin \mathcal{P}_\alpha$. This implies $\vec{m}, m \vec{+} 1 \in \mathcal{P}$. Notice $p_\pi^{-1}(\vec{j}) = \{\vec{j}\}$ for $j \leq m-1$. We conclude $\mathcal{P} = \phi_{m+1}$ using Lemma 2.7.

(3) We will divide $\mathcal{P}_\pi = \mathcal{A}_{j,m}$ into two cases:

a. $\mathcal{P}_\pi = \mathcal{A}_{j,m}$ for $j < m$. By Lemma 2.10 $\mathcal{A}_{j,m+1} \subset \mathcal{P}$. We only need to consider $P \in \mathcal{A}_{j,m}$ $|P| = j$ and $m \in P$, for which $p_\pi^{-1}(P) = \{P, \hat{P}, P_{m+1}\}$, and show that $P_{m+1} \notin \mathcal{P}$. Indeed select $k \notin P$ with $k < m$. Let $\alpha = \gamma_{k,k+1}^{m+1}$, then $|(P_{m+1})_\alpha| = j+1$ which implies $(P_{m+1})_\alpha \notin \mathcal{P}_\pi$. Finally we conclude $\mathcal{P} = \mathcal{A}_{j,m+1}$ using Lemma 2.7.

b. $\mathcal{P}_\pi = \mathcal{A}_{m,m}$. By Lemma 2.10 $\mathcal{A}_{m-1,m+1} \subset \mathcal{P}$. We therefore need to determine which elements of $H_{11}^{m+1} \cup \{m \vec{+} 1\}$ belong to $\mathcal{P}$. Recall from article (5) of Theorem 2.12 that $(\mathcal{A}_{m-1,m+1})_\gamma = \mathcal{A}_{m-1,m}$ for all $\gamma \in \Pi\binom{m+1}{m}$, and notice that for any $P \in H_{1,1}^{m+1} \cup \{m \vec{+} 1\}$, $P_\gamma = \vec{m}$. First assume $m \vec{+} 1 \in \mathcal{P}$. Conclude $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{N} \cup \{m \vec{+} 1\}$, for $\mathcal{N} \subset H_{1,1}^{m+1}$, or $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{N}'$ for $\mathcal{N}' \subset H_{1,1}^{m}$. If $|\mathcal{N}'| = m+1$, $\mathcal{P} = \mathcal{A}_{m,m+1}$. If $|\mathcal{N}'| < m$, then there exists $i, j \in m \vec{+} 1$, $i \neq j$ so that for all $P \in \mathcal{N}'$, $i, j \in P$. This implies $\vec{m} \notin (\mathcal{A}_{m-1,m+1} \cup \mathcal{N}')_{\gamma_{i,j}}$. Conclude $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{N}'$ for $\mathcal{N}' \subset H_{1,1}^{m}$ with $|\mathcal{N}'| = m$ or $\mathcal{P} = \mathcal{A}_{m,m+1}$ or $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{N} \cup \{m \vec{+} 1\}$, for $\mathcal{N} \subset H_{1,1}^{m+1}$.

(4) $\mathcal{P}_\pi = \mathcal{A}_{j,m} \cup \{\vec{m}\}$ for $1 \leq j \leq m-2$. $p_\pi^{-1}(\vec{m}) = \{m \vec{+} 1, \vec{m}, \hat{m}\}$. We claim $\vec{m} \notin \mathcal{P}$. Indeed let $\alpha = \gamma_{1,m}$ and observe that $\vec{m}_\alpha = m \vec{-} 1 \notin \mathcal{P}_\pi$. Similarly let $\beta = \gamma_{1,m+1}$ and observe that $\hat{m}_\alpha = m \vec{-} 1 \notin \mathcal{P}$. We now continue as in case (3a). Conclude $\mathcal{P} = \mathcal{A}_{j,m+1} \cup \{m \vec{+} 1\}$.

(5) $\mathcal{P}_\pi = \mathcal{A}_{j,m} \cup \phi_m$ for $1 \leq j \leq m-2$. We start by analyzing $p_\pi^{-1}(P)$ for $P \in \mathcal{A}_{j,m}$. This is done as in case (3) with the sole difference that for the case $P \in \mathcal{A}_{j,m}$, $|P| = j$, $m \in P$ we choose $1 \leq i < k < m$ so that $i, k \notin P$ and notice that for $\alpha = \gamma_{i,k}$, we have $|(P_{m+1})_\alpha| = j+1$ and in addition $i \notin (P_{m+1})_\alpha$ and $m \in (P_{m+1})_\alpha$ which implies $(P_{m+1})_\alpha \notin \mathcal{P}_\pi$. Let now $P \in \phi_m$, so that $j < |P| < m-1$. Notice $p_\pi^{-1}(A) = \{A\}$. For $P = \vec{m}$, we continue as in case (2). Conclude $\mathcal{P} = \mathcal{A}_{j,m+1} \cup \phi_{m+1}$.

(6) We will divide $\mathcal{P}_\pi = \mathcal{A}_{j,m}^1$ into two cases:

a. $\mathcal{P}_\pi = \mathcal{A}_{j,m}^1$ for $j < m$. Similar to the proof of article (3a). Conclude $\mathcal{P} = \mathcal{A}_{j,m+1}^1$.

b. $\mathcal{P}_\pi = \mathcal{A}_{m,m}^1$. Similar to the proof of article (3b). Notice that $\mathcal{P} = \mathcal{A}_{m-1,m+1}^1 \cup \mathcal{N}'$ for $\emptyset \neq \mathcal{N}' \subsetneq H_{1,2}^{m+1}$ is ruled out because in such a case $|\mathcal{N}'| < m$. Conclude $\mathcal{P} = \mathcal{A}_{m-1,m+1}^1 \cup \mathcal{N} \cup \{m \vec{+} 1\}$, for $\mathcal{N} \subset H_{1,2}^{m+1}$ or $\mathcal{P} = \mathcal{A}_{m,m+1}^1$.

(7) $\mathcal{P}_\pi = \mathcal{A}^1_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ where $\emptyset \neq \mathcal{N} \subsetneq H^m_{1,2}$. Let $e = min_{Q\in\mathcal{N}} e(Q)$. A similar argument to case (9) yields $e \geq 3$, $\mathcal{N} = H^{m+1}_{1,e}$ and $\mathcal{P} = \mathcal{A}^1_{m-2,m+1} \cup H^{m+1}_{2,e} \cup H^{m+1}_{1,e+1} \cup \{\vec{m+1}\} = \mathcal{D}^{m+1}_{2,e+1}$.

(8) $\mathcal{P}_\pi = \mathcal{A}_{m-2,m} \cup \mathcal{N}$ for $\mathcal{N} \subset H^m_{1,1}$, $\mathcal{N} \neq H^m_{1,2}$ with $|\mathcal{N}| = m-1$. This implies $\mathcal{N} = H^m_{1,1} \setminus EH^m_{j,1}$ for some $2 \leq j \leq m$, so this case corresponds to Lemma 2.9 $h = 1, l = 1$ and we conclude $\mathcal{P}_\pi \cap H^m_{1,1} = H^m_{1,1}$ which is a contradiction.

(a) $\mathcal{P}_\pi = \mathcal{A}_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ for $\emptyset \neq \mathcal{N} \subsetneq H^m_{1,1}$, $\mathcal{N} \neq H^m_{1,2}$. Let $e = min_{Q\in\mathcal{N}} e(Q)$. By Lemma 2.9 $\mathcal{N} = H^m_{1,e}$ and $H^{m+1}_{2,e} = \mathcal{P} \cap H^{m+1}_{2,1}$. By assumption $\mathcal{N} \neq H^m_{1,1}$, $H^m_{1,2}$ and therefore $e > 2$. For $e \geq 3$, we use Lemma 2.11 to conclude $H^{m+1}_{1,e+1} = \mathcal{P} \cap H^{m+1}_{1,1}$ and $\vec{m+1} \in \mathcal{P}$. By Lemma 2.10 $\mathcal{A}_{m-2,m+1} \subset \mathcal{P}$. Finally conclude $\mathcal{P} = \mathcal{A}_{m-2,m+1} \cup H^{m+1}_{2,e} \cup H^{m+1}_{1,e+1} \cup \{\vec{m+1}\} = \mathcal{D}^{m+1}_{2,e+1} \cup \mathcal{A}_{m-2,m+1}$.

(9) $\mathcal{P}_\pi = \mathcal{D}^m_{r,s}$, for $2 \leq r < r+1 < s < m$. Recall $\mathcal{D}^m_{r,s} = \mathcal{A}^1_{m-r-1,m} \cup \bigcup_{l'=1}^r H^m_{l',s-l'+1} \cup \{\vec{m}\}$. By Lemma 2.10, $\mathcal{A}^1_{m-r-1,m} \subset \mathcal{P}$. Apply Lemma 2.9 r times w.r.t. pairs $h = l'$ and $l = s - l' + 1$ to conclude $H^{m+1}_{l'+1,s-l'+1} \cap \mathcal{P} = H^{m+1}_{1,s-l'+1}$. Finally conclude $\mathcal{P} = \mathcal{D}^{m+1}_{r+1,s+1}$.

(10) $\mathcal{P}_\pi = \mathcal{D}^m_{r,s} \cup \mathcal{A}_{j,m}$ for $2 \leq r < r+1 < s < m$ and $1 \leq j \leq m-r-1$. $P \in \mathcal{A}_{j,m}$ is analyzed as in case (14). $P \in \mathcal{D}^m_{r,s}$ is analyzed as in case (10). Conclude $\mathcal{P} = \mathcal{D}^{m+1}_{r+1,s+1} \cup \mathcal{A}_{j,m+1}$.

(11) $\mathcal{P}_\pi = \mathcal{A}^1_{j,m} \cup \{\vec{m}\}$ for $1 \leq j \leq m-2$. Similar to case (4). Conclude $\mathcal{P} = \mathcal{A}^1_{j,m+1} \cup \{\vec{m+1}\}$.

(12) $\mathcal{P}_\pi = \mathcal{A}^1_{j,m} \cup \phi_m$ for $1 \leq j \leq m-2$. Similar to case (5). Conclude $\mathcal{P} = \mathcal{A}^1_{j,m+1} \cup \phi_{m+1}$.

(13) $\mathcal{P}_\pi = \mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m}$ for $2 \leq j \leq m$ and $j' < j$. By Lemma 2.10 $\mathcal{A}^1_{j,m+1} \cup \mathcal{A}_{j',m+1} \subset \mathcal{P}$. We treat two cases:

a. $j < m$. Similar to case (14). Conclude $\mathcal{P} = \mathcal{A}^1_{j,m+1} \cup \mathcal{A}_{j',m+1}$.

b. $j = m$. Similar to case (6b). Conclude $\mathcal{P} = \mathcal{A}_{j',m+1} \cup \mathcal{A}^1_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\}$ with $\mathcal{N} \subset H^{m+1}_{1,2}$ or $\mathcal{P} = \mathcal{A}_{j',m+1} \cup \mathcal{A}^1_{m,m+1}$.

(14) $\mathcal{P}_\pi = \mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \{\vec{m}\}$ for $2 \leq j \leq m-2$ and $j' < j$. $P \in \mathcal{A}^1_{j,m} \cup \{\vec{m}\}$ is analyzed as in case (4). For $P \in \mathcal{A}_{j',m}$, we only need to deal with $P \in \mathcal{A}^m_{j',m}$, $1 \notin P$, $|P| = j' < m-2$. This is done as in case (5). Conclude $\mathcal{P} = \mathcal{A}^1_{j,m+1} \cup \mathcal{A}_{j',m+1} \cup \{\vec{m+1}\}$.

(15) $\mathcal{P}_\pi = \mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \phi_m$ for $2 \le j \le m-2$ and $j' < j$. $P \in \mathcal{A}^1_{j,m} \cup \phi_m$ is analyzed as in case (4). $P \in \mathcal{A}_{j',m}$ is analyzed as in case (14). Conclude $\mathcal{P} = \mathcal{A}^1_{j,m+1} \cup \mathcal{A}_{j',m+1} \cup \phi_{m+1}$.

(16) $\mathcal{P}_\pi = \mathcal{A}_{j,m} \cup \mathcal{A}^1_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ where $\emptyset \neq \mathcal{N} \subsetneq H^m_{1,2}$ and $1 \le j < m-2$. $P \in \mathcal{A}_{j,m}$ is analyzed as in case (14). $\mathcal{A}^1_{m-2,m} \cup \mathcal{N} \cup \{\vec{m}\}$ is analyzed as in case (7). Conclude $\mathcal{P} = \mathcal{A}_{j,m+1} \cup \mathcal{A}^1_{m-2,m+1} \cup H^{m+1}_{2,e} \cup H^{m+1}_{1,e+1} \cup \{\vec{m+1}\} = \mathcal{A}_{j,m} \cup \mathcal{D}^{m+1}_{2,e+1}$ for some $e \geq 3$.

□

2.5. **Hereditary patterns.**

Definition 2.15. An m-stable m-pattern $\mathcal{P}$ is said to be **hereditary** if for every $m' > m$ there exist an m'-stable m'-pattern $\mathcal{Q}$ so that for any $\gamma \in \Pi\binom{m'}{m}$ it holds that $\mathcal{P} = \mathcal{Q}_\gamma$. Denote $HSP_n(m) = \{\mathcal{P} : \mathcal{P}$ is a hereditary m-stable m_n-pattern$\}$.

Theorem 2.16. *The following are the only hereditary m-stable m_1-patterns for $m \geq 3$:*

(1) $\{\vec{m}\}$.
(2) ϕ_m.
(3) *For every $1 \le j \le m$ the collection $\mathcal{A}_{j,m}$.*
(4) *For every $1 \le j \le m-2$ the collection $\mathcal{A}_{j,m} \cup \{\vec{m}\}$.*
(5) *For every $1 \le j \le m-2$ the collection $\mathcal{A}_{j,m} \cup \phi_m$.*
(6) *For every $1 \le j \le m$ the collection $\mathcal{A}^1_{j,m}$.*
(7) *For every $1 \le r < r+1 < s < m$* the collection $\mathcal{D}^m_{r,s}$.
(8) *For every $1 \le r < r+1 < s < m$ and $1 \le j \le m-r-1$, $\mathcal{D}^m_{r,s} \cup \mathcal{A}_{j,m}$.*
(9) *For every $1 \le j \le m-2$ the collection $\mathcal{A}^1_{j,m} \cup \{\vec{m}\}$.*
(10) *For every $1 \le j \le m-2$ the collection $\mathcal{A}^1_{j,m} \cup \phi_m$.*
(11) *For every $2 \le j \le m$ and $j' < j$ the collection $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m}$.*
(12) *For every $2 \le j \le m-2$ and $j' < j$ the collection $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \{\vec{m}\}$.*
(13) *For every $2 \le j \le m-2$ and $j' < j$ the collection $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \phi_m$.*

Proof. This follows from the proof of Theorem 2.14. □

Note that this list is the list of standard patterns (Definition 2.3) with the items (7),(8),(9), and (17) removed (notice however that in some cases the allowed indices slightly differ).

Lemma 2.17. *Let* $2 \leq r < r+1 < s < m$*, then*

$$\mathcal{D}^m_{r,s} = \{\{\vec{l} \cup \{j_1, \ldots, j_{m-s}\}\}_{1 \leq l \leq m,\, 1 \leq j_1 \leq \ldots \leq j_{m-s}} \cup \mathcal{A}^1_{m-r-1,m}.$$

Proof. Recall $\mathcal{D}^m_{r,s} \triangleq \mathcal{A}^1_{m-r-1,m} \cup \bigcup_{h=1}^{r} H^m_{h,s-h+1} \cup \{\vec{m}\}$. Notice that for $1 \leq h \leq r$, $H^m_{h,s-h+1} = \{\overrightarrow{s-h} \cup \{j_1, \ldots, j_{m-s}\}\}_{s-h+1 \leq j_1 < j_2 < \ldots < j_{m-s}}$ so clearly the left hand side is contained in the right hand side. Fix $1 \leq l \leq m$ and $1 \leq j_1 \leq \ldots \leq j_{m-s}$. We will show $F \triangleq \vec{l} \cup \{j_1, \ldots, j_{m-s}\} \in \mathcal{D}^m_{r,s}$. If $F \in \mathcal{A}^1_{m-r-1,m}$, we are done, so assume $F \notin \mathcal{A}^1_{m-r-1,m}$. This implies the number of holes of F, which we will denote by h, is less or equal r. We assume w.l.o.g. $h \geq 1$. Let e be the first hole of F. We will show $e \geq s-h+1$ which will imply $F \in H^m_{h,s-h+1}$. Assume for a contradiction that $e < s-h+1$. This implies $l < e \leq s-h$ and $|F| < s-h+m-s = m-h$. However as F has exactly h holes $|F| = m-h$ and we have the desired contradiction. □

Definition 2.18. Let $\mathcal{P} \in HSP_n(m)$. We say that $\mathcal{P}$ is **permutation stable** if $\sigma\mathcal{P} \in HSP_n(m)$ for some $\sigma \in S_m$ implies $\sigma\mathcal{P} = \mathcal{P}$.

Theorem 2.19. *The hereditary* m*-stable* m_1*-patterns for* $m \geq 3$ *are permutation stable.*

Proof. This is proven case by case using the list of Theorem 2.16. The only slightly non-trivial cases are articles (7) and (8) where one uses the representation of Lemma 2.17. □

Definition 2.20. Let $\mathcal{P} \in SP_n(m)$. We say that $\mathcal{P}$ has **unique stable lifts (usl)** if for every $m' > m$ there exists a unique $\mathcal{Q} \in SP_n(m')$ so that for any $\gamma \in \Pi\binom{m'}{m}$ it holds that $\mathcal{P} = \mathcal{Q}_\gamma$.

Remark 2.21. If $\mathcal{P} \in SP_n(m)$ has usl then $\mathcal{P} \in HSP_n(m)$.

Theorem 2.22. *The following* m_1*-patterns* $(m \geq 3)$ *have unique stable lifts:*

(1) $\{\vec{m}\}$.
(2) ϕ_m.
(3) *For every* $1 \leq j \leq m-1$ *the collection* $\mathcal{A}_{j,m}$.
(4) *For every* $1 \leq j \leq m-2$ *the collection* $\mathcal{A}_{j,m} \cup \{\vec{m}\}$.
(5) *For every* $1 \leq j \leq m-2$ *the collection* $\mathcal{A}_{j,m} \cup \phi_m$.
(6) *For every* $1 \leq j \leq m-1$ *the collection* $\mathcal{A}^1_{j,m}$.
(7) *For every* $1 \leq r < r+1 < s < m$ the collection $\mathcal{D}^m_{r,s}$.

(8) *For every* $1 \le r < r+1 < s < m$ *and* $1 \le j \le m-r-1$, $\mathcal{D}^m_{r,s} \cup \mathcal{A}_{j,m}$.

(9) *For every* $1 \le j \le m-2$ *the collection* $\mathcal{A}^1_{j,m} \cup \{\vec{m}\}$.

(10) *For every* $1 \le j \le m-2$ *the collection* $\mathcal{A}^1_{j,m} \cup \phi_m$.

(11) *For every* $2 \le j \le m-1$ *and* $j' < j$ *the collection* $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j'}$.

(12) *For every* $2 \le j \le m-2$ *and* $j' < j$ *the collection* $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \{\vec{m}\}$.

(13) *For every* $2 \le j \le m-2$ *and* $j' < j$ *the collection* $\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \phi_m$.

Proof. The proof follows easily from the proof of Theorem 2.14. □

Note that this list is the list of Theorem 2.16 with the items (3),(6) and (11) for the case $j = m$ removed. These cases do not have usl as the following lemma shows.

Lemma 2.23. *Let* $m \ge 3$. *The following holds:*

(1) $p_\pi^{-1}(\mathcal{A}_{m,m}) \cap HSP_1(m+1) =$
$\{\mathcal{A}_{m,m+1}, \mathcal{A}_{m+1,m+1}, \mathcal{A}_{m-1,m+1} \cup \mathcal{A}^1_{m,m+1}, \mathcal{A}_{m-1,m+1} \cup \{\vec{m+1}\}\}$
$\cup \{\mathcal{D}^{m+1}_{1,l} \cup \mathcal{A}_{m-1,m+1}\}_{l\in\{2,\dots,m+1\}}$.

(2) $p_\pi^{-1}(\mathcal{A}^1_{m,m}) \cap HSP_1(m+1) =$
$\{\mathcal{A}^1_{m,m+1}, \mathcal{A}^1_{m+1,m+1}, \mathcal{A}^1_{m-1,m+1} \cup \{\vec{m+1}\}\} \cup \{\mathcal{D}^{m+1}_{1,l}\}_{l\in\{3,\dots,m+1\}}$.

(3) *For* $1 \le j \le m-2$, $p_\pi^{-1}(\mathcal{A}^1_{m,m} \cup \mathcal{A}_{j,m}) \cap HSP_1(m+1) =$
$\{\mathcal{A}^1_{m,m+1} \cup \mathcal{A}_{j,m+1}, \mathcal{A}^1_{m-1,m+1} \cup \mathcal{A}_{j,m+1} \cup \{\vec{m+1}\}, \mathcal{A}^1_{m+1,m+1} \cup \mathcal{A}_{j,m+1}\}$
$\cup \{\mathcal{D}^{m+1}_{1,l} \cup \mathcal{A}_{j,m+1}\}_{l\in\{3,\dots,m+1\}}$.

Proof.

(1) Let $\mathcal{P} \in p_\pi^{-1}(\mathcal{A}_{m,m}) \cap HSP_1(m+1)$. According to article (3) in the proof of Theorem 2.14, $\mathcal{P}_\pi = \mathcal{A}_{m,m}$ implies $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{N}'$ for $\mathcal{N}' \subset H^m_{1,1}$ with $|\mathcal{N}'| = m$ or $\mathcal{P} = \mathcal{A}_{m,m+1}$ or $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\}$, for $\mathcal{N} \subset H^{m+1}_{1,1}$. By article (8), if $\mathcal{N} \neq H^m_{2,1}$ in the proof of the Theorem 2.14, $\mathcal{A}_{m-1,m+1} \cup \mathcal{N}' \notin HSP_1(m+1)$. If $\mathcal{N} = H^m_{2,1}$, we have $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \mathcal{A}^1_{m,m+1}$. By article (8a) in the proof of Theorem 2.14, $\mathcal{A}_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\} \in HSP_1(m+1)$ implies for $\mathcal{N} \neq \emptyset$, $\mathcal{N} = H^{m+1}_{1,l}$ for some $l \in \{1, 2, \dots, m+1\}$. $l = 1$ corresponds to $\mathcal{P} = \mathcal{A}_{m+1,m+1}$ and $l \ge 2$ corresponds to $\mathcal{P} = \mathcal{D}^{m+1}_{1,l} \cup \mathcal{A}_{m-1,m+1}$. If $\mathcal{N} = \emptyset$, $\mathcal{P} = \mathcal{A}_{m-1,m+1} \cup \{\vec{m+1}\}$.

(2) Let $\mathcal{P} \in p_\pi^{-1}(\mathcal{A}^1_{m,m}) \cap HSP_1(m+1)$. According to article (6) in the proof of Theorem 2.14, $\mathcal{P}_\pi = \mathcal{A}^1_{m,m}$ implies $\mathcal{P} = \mathcal{A}^1_{m,m+1}$ or $\mathcal{P} = \mathcal{A}^1_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\}$, for $\mathcal{N} \subset H^{m+1}_{1,2}$. By article (7) in the proof of Theorem 2.14, $\mathcal{A}^1_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\} \in$

$HSP_1(m+1)$ for $\mathcal{N} \neq \emptyset$implies $\mathcal{N} = H^{m+1}_{1,l}$ for some $l \in \{2, \dots, m+1\}$. $l = 2$ corresponds to $\mathcal{P} = \mathcal{A}^1_{m+1,m+1}$ and $l \geq 3$ corresponds to $\mathcal{P} = \mathcal{D}^{m+1}_{1,l}$. If $\mathcal{N} = \emptyset$, $\mathcal{P} = \mathcal{A}^1_{m-1,m+1} \cup \{\vec{m+1}\}$.

(3) Let $\mathcal{P} \in p_\pi^{-1}(\mathcal{A}^1_{m,m} \cup \mathcal{A}_{j,m}) \cap HSP_1(m+1)$ for some $1 \leq j \leq m-2$.

According to article (13) in the proof of Theorem 2.14, $\mathcal{P}_\pi = \mathcal{A}^1_{m,m} \cup \mathcal{A}_{j,m}$ implies $\mathcal{P} = \mathcal{A}_{j,m+1} \cup \mathcal{A}^1_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\}$ with $\mathcal{N} \subset H^{m+1}_{1,2}$ or $\mathcal{P} = \mathcal{A}_{j,m+1} \cup \mathcal{A}^1_{m,m+1}$. By article (16) in the proof of Theorem 2.14, $\mathcal{A}_{j,m+1} \cup \mathcal{A}^1_{m-1,m+1} \cup \mathcal{N} \cup \{\vec{m+1}\} \in HSP_1(m+1)$ for $\mathcal{N} \neq \emptyset$ implies $\mathcal{N} = H^{m+1}_{1,l}$ for some $l \in \{2, \dots, m+1\}$. $l = 2$ corresponds to $\mathcal{P} = \mathcal{A}_{j,m+1} \cup \mathcal{A}^1_{m+1,m+1}$ and $l \geq 3$ corresponds to $\mathcal{P} = \mathcal{D}^{m+1}_{1,l} \cup \mathcal{A}_{j,m+1}$. If $\mathcal{N} = \emptyset$, $\mathcal{P} = \mathcal{A}^1_{m-1,m+1} \cup \mathcal{A}_{j,m+1} \cup \{\vec{m+1}\}$.

□

3. An application of the dual Ramsey Theorem to stable patterns

The tool which enables the application of the combinatorial results of the previous section to hyperspace actions is the dual Ramsey Theorem.

3.1. **Ramsey Theorems.** We denote by $\tilde{\Pi}\binom{s}{k}$ the collection of unordered partitions of $\{1, \dots, s\}$ into k nonempty sets. Notice there is a natural bijection $\nu : \tilde{\Pi}\binom{s}{k} \leftrightarrow \Pi\binom{s}{k}$.

Theorem 3.1. *[The dual Ramsey Theorem] Given positive integers k, m, r there exists a positive integer $N = DR(k, m, r)$ with the following property: for any coloring of $\tilde{\Pi}\binom{N}{k}$ by r colors there exists a partition $\alpha = \{A_1, A_2, \dots, A_m\} \in \tilde{\Pi}\binom{N}{m}$ of N into m non-empty sets such that all the partitions of N into k non-empty sets (i.e. elements of $\tilde{\Pi}\binom{N}{k}$) whose atoms are measurable with respect to α (i.e. each equivalence class is a union of elements of α) have the same color.*

Proof. This is Corollary 10 of [GR71]. □

Theorem 3.2. *[The strong dual Ramsey Theorem] Given positive integers $m, r_2, \dots, r_m$ there exists a positive integer $N = SDR(m; r_2, \dots, r_{m-1})$ with the following property: for any colorings of $\tilde{\Pi}\binom{N}{j}$ by r_j colors, $2 \leq j \leq m-1$, there exists a partition $\alpha = (A_1, A_2, \dots, A_m) \in \tilde{\Pi}\binom{N}{m}$ of N into m non-empty sets such that for any $2 \leq j \leq m-1$, any two partitions of N into j non-empty sets whose atoms are measurable with respect to α (i.e. each equivalence class is a union of elements of α) have the same color.*

Proof. [Using the dual Ramsey Theorem] Set $n_2 = DR(2, m; r_2)$, $n_3 = DR(3, n_2; r_3)$ and, by recursion $n_{j+1} = DR(j+1, n_j; r_{j+1})$ for $2 \leq j < m-2$. It is now easy to check that $n = n_{m-1}$ satisfies our claim. (As a demonstration set $n_2 = DR(2, m; r_2)$, $N = n_3 = DR(3, n_2; r_3)$. Start with a partition $\alpha_3 = \{B_1, \ldots, B_{n_2}\}$ of N which is good for all partitions of N into $j = 3$ atoms and the r_3 colors. Next choose a partition $\alpha_2 = \{A_1, \ldots, A_m\}$ of n which is α_3-measurable, and which is good for all partitions of N into $j = 2$ atoms and the r_2 colors. It now follows that $\alpha := \alpha_2 = \{A_1, A_2, \ldots, A_m\}$ has the required property: all 2-partitions of N which are α-measurable are monochromatic and all 3-partitions of N which are α-measurable are monochromatic.) ☐

Corollary 3.3. *For any m, for any number $N \geq SDR(m; r_2, \ldots, r_{m-1})$, with $r_k = 2^{2^{kn}}$, for any N_n-pattern $\mathcal{P}$, there exists a partition $\alpha \in \Pi\binom{N}{m}$ such that the m_n-pattern $\mathcal{P}_\alpha$ is an* m-stable m_n-pattern.

Proof. [Using the strong dual Ramsey Theorem] We define the mapping $f : \Pi_{2 \leq k \leq m-1} \Pi\binom{N}{k} \to \Pi_{i=2}^{m-1} C_n(i)$ by $(\alpha_2, \ldots, \alpha_{m-1}) \mapsto (\mathcal{P}_{\alpha_i})_{i=2}^{m-1}$. By the strong dual Ramsey Theorem applied to $f \circ \nu$ there exists a partition $\alpha = (A_1, A_2, \ldots, A_m) \in \Pi\binom{N}{m}$ of N into m non-empty sets such that for any $(m-2)$-tuple of naturally ordered partitions of N into $k = 2, \ldots, m-1$ non-empty sets (i.e. elements of $\Pi\binom{N}{k}$) respectively whose atoms are measurable with respect to α (i.e. each equivalence class is a union of elements of α) have the same color. Let β_1 and β_2 be two naturally ordered partitions of m into k elements for some $2 \leq k \leq m-1$, then as the amalgamated partitions $\alpha_{\beta_1}, \alpha_{\beta_2}$ are measurable with respect to α and naturally ordered, $(\mathcal{P}_\alpha)_{\beta_1} = \mathcal{P}_{\alpha_{\beta_1}} = \mathcal{P}_{\alpha_{\beta_2}} = (\mathcal{P}_\alpha)_{\beta_2}$ as desired. ☐

4. The minimal subspaces of $Exp(Exp(X))$

4.1. **Clopen partitions.** Let X be a Hausdorff zero-dimensional compact h-homogeneous space. Denote by $\mathcal{D}$ ($\tilde{\mathcal{D}}$) the directed set (semilattice) consisting of all finite ordered (unordered) clopen partitions of X. We denote the members of $\mathcal{D}$ ($\tilde{\mathcal{D}}$) by $\alpha = (A_1, A_2, \ldots, A_m)$ ($\tilde{\alpha} = \{A_1, A_2, \ldots, A_m\}$). The relation is given by refinement: $\alpha \preceq \beta$ ($\tilde{\alpha} \preceq \tilde{\beta}$) iff for any $B \in \beta$ ($B \in \tilde{\beta}$), there is $A \in \alpha$ ($A \in \tilde{\alpha}$) so that $B \subset A$. The join (least upper bound) of α and β, $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$, where the ordering of indices is given by the lexicographical order on the indices of α and β ($\tilde{\alpha} \vee \tilde{\beta} = \{A \cap B : A \in \tilde{\alpha}, B \in \tilde{\beta}\}$). It is convenient to introduce the notations $\mathcal{D}_k = \{\alpha \in \mathcal{D} : |\alpha| = k\}$ and $\tilde{\mathcal{D}_k} = \{\alpha \in \mathcal{D} : |\tilde{\alpha}| = k\}$. There is a natural G-action on $\mathcal{D}$ ($\tilde{\mathcal{D}}$) given by $g(A_1, A_2, \ldots, A_m) = (g(A_1), g(A_2), \ldots, g(A_m))$ ($g\{A_1, A_2, \ldots, A_m\} = \{g(A_1), g(A_2), \ldots, g(A_m)\}$). Let S_k denote the group of permutations of $\{1, \ldots, k\}$. S_k acts naturally on $\mathcal{D}_k$ by $\sigma(B_1, B_2, \ldots, B_k) = (B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(k)})$. This action commutes with the action of G, i.e. $\sigma g\beta = g\sigma\beta$ for any $\sigma \in S_k$ and $g \in G$. Notice one can identify $\tilde{\mathcal{D}}_k = \mathcal{D}_k/S_k$.

For $\alpha = (A_1, A_2, \ldots, A_m) \in \mathcal{D}$ and $\gamma = (C_1, \ldots, C_k) \in \Pi\binom{m}{k}$ define the **amalgamated clopen cover** $\alpha_\gamma = (G_1, G_1, \ldots, G_k)$, where $G_j = \bigcup_{i \in C_j} A_i$. Notice that $(\alpha_\gamma)_\beta = \alpha_{\gamma_\beta}$.

4.2. **Partition Homogeneity.** The following definition was introduced in [GG11]:

Definition 4.1. A zero-dimensional Hausdorff space X is called **partition-homogeneous** if for every two finite ordered clopen partitions of the same cardinality, $\alpha, \beta \in \mathcal{D}_m$, $\alpha = (A_1, A_2, \ldots, A_m)$, $\beta = (B_1, B_2, \ldots, B_m)$ there is $h \in Homeo(X)$ such that $hA_i = B_i$, $i = 1, \ldots, k$.

In [GG11] we proved:

Proposition 4.2. *Let X be a zero-dimensional compact Hausdorff space. If X is h-homogeneous then X is partition-homogeneous.*

4.3. **Signatures and Induced patterns.** For every $n \in \mathbb{N}$ let $E_n = Exp(Exp(X))^n$ equipped with the Vietoris topology.

Definition 4.3. Let $\xi \in E_n$ and $\alpha \in \mathcal{D}$ with m elements $\alpha = (A_1, A_2, \ldots, A_m)$. Define the (α, ξ)**-induced** m_n**-pattern** $\mathcal{P}_{(\alpha,\xi)} = \{P_{(\alpha,F)} : F \in \xi\}$, where $P_{(\alpha,F)} = \{(j_1, j_2, \ldots, j_n) : A_{j_1} \times A_{j_2} \times \cdots \times A_{j_n} \cap F \neq \emptyset\}$. Inversely to an m_n-pattern $\mathcal{P}$ and clopen partition α one associates $\xi_{(\alpha,\mathcal{P})} = \{F_{(\alpha,P)} : P \in \mathcal{P}\} \in E_n$, where

$$F_{(\alpha,P)} = \cup_{(j_1,j_2,\ldots,j_n) \in P} A_{j_1} \times A_{j_2} \times \cdots \times A_{j_n}$$

Denote $\xi_\alpha = \xi_{(\alpha, \mathcal{P}_{(\alpha,\xi)})}$. It is easy to see:.

$$\xi_\alpha = \{F_\alpha : F \in \xi\}$$

where

$$F_\alpha = \cup_{A_{j_1} \times A_{j_2} \times \cdots \times A_{j_n} \cap F \neq \emptyset} A_{j_1} \times A_{j_2} \times \cdots \times A_{j_n}$$

The **signature** of ξ is defined to be the net in E_n, $sig(\xi) = (\xi_\alpha)_{\alpha \in \mathcal{D}}$.

Lemma 4.4. *Let $\xi \in E_n$, $\alpha \in \mathcal{D}_m$, $\sigma \in S_m$ and $g \in G$ then*

(1) $\mathcal{P}_{(\alpha,\xi)} = \mathcal{P}_{(g\alpha,g\xi)}$.
(2) $\sigma^{-1}\mathcal{P}_{(\alpha,\xi)} = \mathcal{P}_{(\sigma\alpha,\xi)}$

Proof. Let $\alpha = (A_1, A_2 \dots, A_m)$. □

(1) Notice that for any $F \in \xi$, $F \cap A_{j_1} \times A_{j_2} \times \cdots \times A_{j_n} \neq \emptyset \Leftrightarrow g(F) \cap gA_{j_1} \times gA_{j_2} \times \cdots \times gA_{j_n} \neq \emptyset$.
(2) Notice that for any $F \in \xi$, $F \cap A_{\sigma(j_1)} \times A_{j_2} \times \cdots \times A_{\sigma(j_n)} \neq \emptyset \Leftrightarrow (j_1, j_2, \dots, j_n) \in \sigma^{-1}P_{(\alpha,F)}$.

4.4. **The topology of E_n.** Recall that for a topological space K, a basis for the Vietoris topology on $Exp(K)$ is given by the collection of subsets of the form

$$\langle V_1, V_2, \dots, V_k\rangle = \{F \in Exp(K) : \ F \subset \cup_{j=1}^k V_j, \text{ and } F \cap V_j \neq \emptyset \ \ \forall\ 1 \leq j \leq k\},$$

where $V_1, \dots, V_k$ are open subsets of K.

Lemma 4.5. *For $\alpha = (A_1, A_2, \dots, A_m)$ let*

$$UE_n(\alpha) = \{\Pi_{i=1}^n \langle A_{j_1^i}, A_{j_2^i}, \dots, A_{j_{k_i}^i}\rangle : \ \forall i, \forall r,\ k_i, j_r^i \in \vec{m}, (\forall s, r \neq r')\ j_r^s \neq j_{r'}^s\}$$

and

$$\mathcal{B}_{E_n}(\alpha) = \{\langle \mathcal{U}_1, \dots, \mathcal{U}_l\rangle : \ \forall i, \mathcal{U}_i \in UE_n(\alpha)\}.$$

Define $\mathcal{B}_{E_n} = \bigcup_{\alpha \in \mathcal{D}} \mathcal{B}_{E_n}(\alpha)$. Then $\mathcal{B}_{E_n}$ is a basis for E_n.

Proof. A basis for the Vietoris topology of E_n is given by $\mathfrak{U} = \langle \mathcal{U}_1, \dots, \mathcal{U}_l\rangle$, where $\mathcal{U}_i$ belong to a fixed basis in $(Exp(X))^n$. □

Remark 4.6. Notice that distinct members of $UE_n(\alpha)$ are disjoint and that for each $F = (F_1, F_2, \dots, F_n) \in (ExpX)^n$, there exists a unique member $\mathcal{U} \in UE_n(\alpha)$, so that $F \in \mathcal{U}$. Denote $\mathcal{U} = UE_n(\alpha)[F]$.

Definition 4.7. For $\alpha = (A_1, \dots, A_m) \in \mathcal{D}$ and $\mathcal{P} \in C_n(m)$ with $\mathcal{P} = \{P_s = (P_s^1, P_s^2, \cdots, P_s^n) : s = 1, 2, \dots, r\}$ and $P_s^i = \{j_{s,1}^i, j_{s,2}^i, \dots, j_{s,k(i,s)}^i\}$ let

$$\begin{aligned} UE_n(\alpha)[\mathcal{P}] =& \langle \Pi_{i=1}^n \langle A_{j_{1,1}^i}, A_{j_{1,2}^i}, \dots, A_{j_{1,k(i,1)}^i}\rangle, \Pi_{i=1}^n \langle A_{j_{2,1}^i}, A_{j_{2,2}^i}, \dots, A_{j_{2,k(i,2)}^i}\rangle, \\ & \dots, \Pi_{i=1}^n \langle A_{j_{r,1}^i}, A_{j_{r,2}^i}, \dots, A_{j_{r,k(i,r)}^i}\rangle\rangle. \end{aligned}$$

Define $U(\xi,\alpha) = UE_n(\alpha)[\mathcal{P}_{(\alpha,\xi)}]$.

Proposition 4.8. *Let $\alpha \in \mathcal{D}$. Let $\xi, \xi' \in E_n$. The following statements are equivalent:*

(1) $U(\xi,\alpha) = U(\xi',\alpha)$.
(2) $\xi \in U(\xi',\alpha)$ or $\xi' \in U(\xi,\alpha)$.
(3) $UE(\alpha)[\mathcal{P}_{(\alpha,\xi)}] = UE(\alpha)[\mathcal{P}_{(\alpha,\xi')}]$.
(4) $\mathcal{P}_{(\alpha,\xi)} = \mathcal{P}_{(\alpha,\xi')}$.
(5) $\xi_\alpha = \xi'_\alpha$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Definition 4.7. Recall $\xi_\alpha = \xi_{(\alpha,\mathcal{P}_{(\alpha,\xi)})}$ which implies (4) $\Rightarrow$ (5). The direction (4) $\Leftarrow$ (5) is trivial. □

Lemma 4.9. *Let $\xi \in E_n$, then $\lim sig(\xi) = \xi$ and $\{\xi\} = \bigcap_{\alpha\in\mathcal{D}} U(\xi,\alpha)$.*

Proof. We will show that for any $\alpha \in \mathcal{D}$, $\xi_\beta \in U(\xi,\alpha)$ for all $\beta \succeq \alpha$. This implies that the limit of the net $(\xi_\alpha)_{\alpha\in\mathcal{D}}$ is ξ. Fix $\beta = (B_0, B_2, \ldots, B_{m-1}) \succeq \alpha$. Acording to Proposition 4.8 it is enough to show $(\xi_\beta)_\alpha = \xi_\alpha$. However as $\beta \succeq \alpha$ this is trivial. In order to prove the second statement of the lemma notice that trivially $\xi \in \bigcap_{\alpha\in\mathcal{D}} U(\xi,\alpha)$. If $\xi' \in \bigcap_{\alpha\in\mathcal{D}} U(\xi,\alpha)$ then $\lim sig(\xi') = \lim sig(\xi)$ which implies by the first statement of the lemma $\xi = \xi'$. □

4.5. **Hereditary stable signatures.** Notice there is a natural action of S_m on $C_n(m)$.

Definition 4.10. Let $\xi \in E_n$. The signature $sig(\xi)$ is called **hereditary stable** if for all $\alpha \in \mathcal{D}$, there exists $\sigma \in S_{|\alpha|}$ so that $\mathcal{P}_{(\sigma(\alpha),\xi)} \in HSP_n(m)$.

Lemma 4.11. *Let $\xi \in E_n$. Let $\beta \in \mathcal{D}$ with $|\beta| = m$. Assume $\mathcal{P}_{(\beta,\xi)} \in HSP_n(m)$. Let $\alpha \in \mathcal{D}$ with $\alpha \preceq \beta$ and $|\alpha| = p$, then there exists $\sigma \in S_m$ so that $\mathcal{P}_{(\sigma(\alpha),\xi)} \in HSP_n(p)$.*

Proof. Denote $\alpha = (A_1, \ldots, A_p)$ and $\beta = (B_1, \ldots, B_m)$. Define $\tilde{C}_i = \{j : B_j \subset A_i\}$, $i = 1, \ldots, p$. Let $\tilde{\gamma} = \{\tilde{C}_i\}_{i=1}^p$. Let $\gamma = (C_i)_{i=1}^p$ be a naturally ordered partition of $\{1, \ldots, m\}$ into p sets, so that there exists $\sigma \in S_m$ so that $C_i = \tilde{C}_{\sigma(i)}$, $i = 1, \ldots p$. Denote $\nu = \beta_\gamma$. It is easy to see that $\sigma(\nu) = \alpha$. Moreover $\mathcal{P}_{(\nu,\xi)} = (\mathcal{P}_{(\beta,\xi)})_\gamma$. The last assertion implies $\mathcal{P}_{(\sigma(\alpha),\xi)} \in HSP_n(p)$. □

Our next theorem is a crucial step that connects the combinatorial condition of being hereditarily stable to the topological dynamical condition of being minimal. Note that whereas our main result (Theorem 4.15 below) handles the case $n = 1$ only, this result applies to all $n \in \mathbb{N}$.

Theorem 4.12. *Let $M \subset E_n$ be minimal, then for all $\xi' \in M$, $sig(\xi')$ is hereditary stable.*

Proof. We start by showing there is $\xi' \in M$ so that $sig(\xi')$ is hereditary stable. Fix $\xi \in M$. Let $\alpha = (A_1, \ldots, A_m) \in \mathcal{D}$. Let $N = SDR(m+1; r_2, \ldots, r_m)$, with $r_k = 2^{2^{kn}}$, $k = 2, \ldots, m$. Let $\beta \in \mathcal{D}$ with $|\beta| = N$ (see subsection 3.1). By Corollary 3.3 there exists a partition $\tilde{\gamma} \in \Pi\binom{N}{m+1}$ such that the $(m+1)$-pattern $(\mathcal{P}_{(\beta,\xi)})_{\tilde{\gamma}} = \mathcal{P}_{(\beta_{\tilde{\gamma}},\xi)}$ is $(m+1)$-stable. Fix $\delta \in \Pi\binom{m+1}{m}$. Denote $\gamma = \tilde{\gamma}_\delta$. Conclude that the m-pattern $(\mathcal{P}_{(\beta,\xi)})_\gamma = \mathcal{P}_{(\beta_\gamma,\xi)}$ is a hereditary m-stable m-pattern. Denote $\beta_\gamma = \{A'_1, \ldots, A'_m\}$. Let $g \in G$ be any element $g = g_\alpha$ of G with $g(A'_j) = A_j,\ j = 1, \ldots, m$. By Lemma 4.4 $\mathcal{P}_{(\beta_\gamma,\xi)} = \mathcal{P}_{(\alpha,g_\alpha\xi)}$. In particular $\mathcal{P}_{(\alpha,g_\alpha\xi)} \in HSP_n(m)$. Let ξ' be a limit point of the net $\{g_\alpha\xi : \alpha \in \mathcal{D}\}$. By definition there is a directed set S and a monotone cofinal mapping $f : S \to \mathcal{D}$, so that $\xi' = \lim_{s\in S} g_{f(s)}\xi$. Fix again some $\alpha \in \mathcal{D}$. By definition there exists $s \in S$ so that for all $\tilde{s} \geq s$, $g_{f(\tilde{s})}\xi \in U(\xi', \alpha)$ (which implies $g_{f(\tilde{s})}\xi \in U(\xi', \sigma(\alpha))$ for any $\sigma \in S_m$) and in particular $\xi'_\alpha = (g_{f(\tilde{s})}\xi)_\alpha$. As f is cofinal there is $r \in S$ so that $f(r) \succeq \alpha \wedge f(s)$. By construction $\mathcal{P}_{(f(r),g_{f(r)}\xi)}$ is a hereditary $|f(r)|$-stable $|f(r)|$-pattern. As $f(r) \geq \alpha$, conclude by Lemma 4.11 $\mathcal{P}_{(\sigma(\alpha),g_{f(r)}\xi)}$ for some $\sigma \in S_m$ is a hereditary m- stable m-pattern. By Proposition 4.8, $\mathcal{P}_{(\sigma(\alpha),g_{f(r)}\xi)} = \mathcal{P}_{(\sigma(\alpha),\xi')}$. We conclude $sig(\xi')$ is hereditary stable. Let now $\xi'' \in M$ and fix $\alpha \in \mathcal{D}$. Using minimality and Proposition 4.8 there is $g \in G$ so that $\mathcal{P}_{(\alpha,\xi'')} = \mathcal{P}_{(\alpha,g\xi')}$. By Lemma 4.4 $\mathcal{P}_{(\alpha,g\xi')} = \mathcal{P}_{(g^{-1}\alpha,\xi')}$. As $sig(\xi')$ is hereditary stable there is $\sigma \in S_m$ so that $\mathcal{P}_{(\sigma g^{-1}\alpha,\xi')} \in HSP_n(m)$. By Lemma 4.4 $\mathcal{P}_{(\sigma g^{-1}\alpha,\xi')} = \sigma^{-1}\mathcal{P}_{(g^{-1}\alpha,\xi')} = \sigma^{-1}\mathcal{P}_{(\alpha,g\xi')} = \sigma^{-1}\mathcal{P}_{(\alpha,\xi'')} = \mathcal{P}_{(\sigma\alpha,\xi'')}$ and we conclude $sig(\xi'')$ is hereditary stable. □

4.6. **The main theorem.**

Definition 4.13. We call a sequence of n-dimensional patterns $\mathbb{P} = \{\mathcal{P}_m\}_{m\in\mathbb{N}}$ so that $\mathcal{P}_m \in HSP_n(m)$ a **pattern-family**. A minimal subspace $M \subset E_n$ is said to be **$\mathbb{P}$-associated** if for every $\xi \in M$, $m \in \mathbb{N}$ and $\alpha \in \mathcal{D}_m$ there exists $\sigma \in S_m$ (depending on ξ) so that $\xi_\alpha = \sigma P_m$. One easily verfies that for a given pattern family $\mathbb{P}$ there is at most one minimal subspace to which it is $\mathbb{P}$-associated. If such a subspace exists it is denoted by $M(\mathbb{P})$.

In the following definition, use Lemma 2.17 to understand articles (7), (8), (9) and (10)

Definition 4.14. The pattern-families in the following list are called **standard**. Each of these is associated with a minimal subspaces of E_1 and we call these minimal subspaces **the standard minimal spaces**.

(1) $M(\{\vec{m}\}_{m\in\mathbb{N}}) = \{\{X\}\}$.
(2) $M(\{\phi_m\}_{m\in\mathbb{N}}) = \Phi$.
(3) $M(\{\mathcal{A}_{j,m}\}_{m\in\mathbb{N}}) = \{\{\{x_1, x_2, \ldots, x_j\}\}_{(x_1,x_2,\ldots,x_j)\in X^j}\}$ $(j \in \mathbb{N})$.
(4) $M(\{\mathcal{A}_{j,m} \cup \{\vec{m}\}\}_{m\in\mathbb{N}}) = \{\{\{x_1, x_2, \ldots, x_j\}, X\}_{(x_1,x_2,\ldots,x_j)\in X^j}\}$ $(j \in \mathbb{N})$.
(5) $M(\{\mathcal{A}_{j,m} \cup \phi_m\}_{m\in\mathbb{N}}) = \{\{\{x_1, x_2, \ldots, x_j\}, F\}_{(x_1,x_2,\ldots,x_j)\in X^j, F\in\xi}\}_{\xi\in\Phi}$ $(j \in \mathbb{N})$.
(6) $M(\{\mathcal{A}^1_{j,m}\}_{m\in\mathbb{N}}) = \{\{\{x_1, x_2, \ldots, x_j\}_{(x_2,\ldots,x_j)\in X^{j-1}}\}\}_{x_1\in X}$ $(j \in \mathbb{N})$.
(7) Let $q \geq 1$. Let $\mathbb{P}^q = \{\mathcal{P}^q_m\}_{m\in\mathbb{N}}$ be given by

$$\mathcal{P}^q_m = \begin{cases} \mathcal{A}^1_{m,m} & 1 \leq m \leq q+2 \\ \mathcal{D}^m_{m-q-3,m-q} & m > q+2 \end{cases}$$

$M(\mathbb{P}^q) = \{\{F \cup \{x_1, x_2, \ldots, x_q\}\}_{(x_1,x_2,\ldots,x_q)\in X^q, F\in\xi}\}_{\xi\in\Phi}$.

(8) Let $l > q \geq 1$. Let $\mathbb{P}^{q,l} = \{\mathcal{P}^{q,l}_m\}_{m\in\mathbb{N}}$ be given by

$$\mathcal{P}^{q,l}_m = \begin{cases} \mathcal{A}^1_{m,m} & 1 \leq m \leq l \\ \mathcal{D}^m_{m-l-1,m-q} & m > l \end{cases}$$

$M(\mathbb{P}^{q,l}) = \{\{F\cup\{x_1, x_2, \ldots, x_q\}, \{r(\xi), y_2, \ldots, y_l\}\}_{(x_1,x_2,\ldots,x_q)\in X^q,(y_2,\ldots,y_l)\in X^{l-1}, F\in\xi}\}_{\xi\in\Phi}$.

(9) Let $q, j \geq 1$. Let $\mathbb{P}^{q,j} = \{\mathcal{P}^{q,j}_m\}_{m\in\mathbb{N}}$ be given by

$$\mathcal{P}^{q,j}_m = \begin{cases} \mathcal{A}_{j,m} \cup \mathcal{A}^1_{m,m} & 1 \leq m \leq q+2 \\ \mathcal{A}_{j,m} \cup \mathcal{D}^m_{m-l-1,m-q} & m > q+2 \end{cases}$$

$M(\mathbb{P}^{q,l}) = \{\{F\cup\{x_1, x_2, \ldots, x_q\}, \{z_1, z_2, \ldots, z_j\}\}_{(x_1,x_2,\ldots,x_q)\in X^q,(z_1,z_2,\ldots,z_j)\in X^j, F\in\xi}\}_{\xi\in\Phi}$.

(10) Let $l > q \geq 1$ and $1 \leq j < l$. Let $\mathbb{P}^{q,l,j} = \{\mathcal{P}^{q,l,j}_m\}_{m\in\mathbb{N}}$ be given by

$$\mathcal{P}^{q,l,j}_m = \begin{cases} \mathcal{A}_{j,m} \cup \mathcal{A}^1_{m,m} & 1 \leq m \leq l \\ \mathcal{A}_{j,m} \cup \mathcal{D}^m_{m-l-1,m-q} & m > l \end{cases}$$

$M(\mathbb{P}^{q,l,j}) = \{\{F \cup \{x_1, x_2, \ldots, x_q\}, \{r(\xi), y_2, \ldots, y_l\},$
$\{z_1, z_2, \ldots, z_j\}\}_{(x_1,x_2,\ldots,x_q)\in X^q,(y_2,\ldots,y_l)\in X^{l-1},(z_1,z_2,\ldots,z_j)\in X^j, F\in\xi}\}_{\xi\in\Phi}$

(11) $M(\{\mathcal{A}^1_{j,m} \cup \{\vec{m}\}\}_{m\in\mathbb{N}}) = \{\{X, \{x_1, x_2, \ldots, x_j\}_{(x_2,\ldots,x_j)\in X^{j-1}}\}\}_{x_1\in X}$, $(j \in \mathbb{N})$.
(12) $M(\{\mathcal{A}^1_{j,m} \cup \phi_m\}_{m\in\mathbb{N}}) = \{\{\{\tau(\xi), x_2, \ldots, x_j\}, F\}_{(x_2,\ldots,x_j)\in X^{j-1}, F\in\xi}\}_{\xi\in\Phi}$ $(j \in \mathbb{N})$.

(13) $M(\{\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m}\}_{m\in\mathbb{N}}) = \{\{\{y_1, y_2, \dots, y_{j'}\},$
$\{x_1, x_2, \dots, x_j\}_{(x_2,\dots,x_j)\in X^{j-1},(y_1,y_2,\dots,y_{j'})\in X^{j'}}\}\}_{x_1\in X}$ $(j, j' \in \mathbb{N})$.

(14) $M(\{\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \{\vec{m}\}\}_{m\in\mathbb{N}}) = \{\{X, \{y_1, y_2, \dots, y_{j'}\},$
$\{x_1, x_2, \dots, x_j\}_{(x_2,\dots,x_j)\in X^{j-1},(y_1,y_2,\dots,y_{j'})\in X^{j'}}\}\}_{x_1\in X}$.

(15) $M(\{\mathcal{A}^1_{j,m} \cup \mathcal{A}_{j',m} \cup \phi_m\}_{m\in\mathbb{N}}) = \{\{\{y_1, y_2, \dots, y_{j'}\},$
$\{\tau(\xi), x_2, \dots, x_j\}, F\}_{(x_2,\dots,x_j)\in X^{j-1}, F\in\xi,(y_1,y_2,\dots,y_{j'})\in X^{j'}}\}_{\xi\in\Phi}$.

(16) $M(\{\mathcal{A}_{m,m}\}_{m\in\mathbb{N}}) = \{\{F |\, F \in Exp(X)\}\}$.

(17) $M(\{\mathcal{A}^1_{m,m}\}_{m\in\mathbb{N}}) = W_X = \{\{F |\, x \in F \in Exp(X)\}\}_{x\in X}$.

(18) $M(\{\mathcal{A}^1_{m,m} \cup \mathcal{A}_{j,m}\}_{m\in\mathbb{N}}) = \{\{\{x_1, x_2, \dots, x_j\}, F\}_{(x_1,x_2,\dots,x_j)\in X^j, x\in F\in Exp(X)}\}_{x\in X}$ $(j \in \mathbb{N})$.

Theorem 4.15. *The standard minimal spaces are the only minimal subspaces of E_1.*

Proof. Let $M \subset E_1$ be a minimal subspace and $\xi \in M$. By Theorem 4.12 $sig(\xi)$ is hereditary stable. For a subspace $N \subset E_1$ and $\alpha \in \mathcal{D}$ denote $U(N, \alpha) = \bigcup_{\xi\in N} U(\xi, \alpha)$. Fix $\alpha \in \mathcal{D}_m$ with $m \geq 3$. By Lemma 4.9, $N = \bigcap_{\beta\in S} U(N, \beta)$ for any cofinal set $S \subset \mathcal{D}$, therefore it is enough to to show for any $\beta \succeq \alpha$ that $\xi_\beta \in U(N, \beta)$ for N a standard space. By the definition of hereditary stability there exists $\sigma_1 \in S_m$ so that $\mathcal{P}_{(\xi,\sigma_1(\alpha))} \triangleq \mathcal{P}$ is hereditary stable. Assume first $\mathcal{P}$ has usl (see Definition 2.20). Let $\mathbb{P} = \{\mathcal{P}_k\}_{k\in\mathbb{N}}$ be the unique pattern-family so that $\mathcal{P}_m = \mathcal{P}$ (it corresponds to one of the first 15 cases in Definition 4.14). Let $\beta \succeq \alpha$ with $\beta \in \mathcal{D}_{m'}$. Again there exists $\sigma_2 \in S_{m'}$ so that $\mathcal{P}_{(\xi,\sigma_2(\beta))}$ is hereditary stable. Let $\gamma \in \Pi\binom{m'}{m}$ and $\sigma_3 \in S_m$ so that $(\sigma_2(\beta))_\gamma = \sigma_3\sigma_1(\alpha)$. Conclude by Lemma 4.4 that $(\mathcal{P}_{(\xi,\sigma_2(\beta))})_\gamma = \mathcal{P}_{(\xi,\sigma_3\sigma_1(\alpha))} = \sigma_3^{-1}(\mathcal{P}_m)$. As $\mathcal{P}_m$ is permutation stable by Theorem 2.19, conclude $(\mathcal{P}_{(\xi,\sigma_2(\beta))})_\gamma = \mathcal{P}_m$. As $\mathcal{P}_m$ has usl one has that $\mathcal{P}_{(\xi,\sigma_2(\beta))} = \mathcal{P}_{m'}$, i.e. $\mathcal{P}_{(\xi,\beta)} = \sigma_2^{-1}(\mathcal{P}_{m'})$, which implies $\xi_\beta \in U(M(\mathbb{P}), \beta)$.

We now treat the case when $\mathcal{P}$ does not have usl. Considering Theorem 2.16 and the proof of Theorem 2.12 it is clear that $\mathcal{P} = \mathcal{A}^1_{m,m}$ or $\mathcal{P} = \mathcal{A}_{m,m}$ or $\mathcal{P} = \mathcal{A}^1_{m,m} \cup \mathcal{A}_{j,m}$ for $1 \leq j \leq m-2$. As in the case that $\mathcal{P}$ has usl, we have $(\mathcal{P}_{(\xi,\sigma_2(\beta))})_\gamma = \mathcal{P}_m$ for all $\gamma \in \Pi\binom{m'}{m}$. Let us consider the case $m' = m + 1$. Assume w.lo.g. $\mathcal{P} = \mathcal{A}_{m,m}$ and denote $\mathbb{P} = \{\mathcal{A}_{m,m}\}_{m\in\mathbb{N}}$. By Lemma 2.23, $p_\pi^{-1}(\mathcal{A}_{m,m}) \cap HSP_1(m+1) = \{\mathcal{A}_{m,m+1}, \mathcal{A}_{m+1,m+1}, \mathcal{A}_{m+1,m+1} \cup \{\overrightarrow{m+1}\}, \mathcal{A}^1_{m,m+1}, \mathcal{A}_{m-1,m+1}\} \cup \{\mathcal{D}^{m+1}_{1,l} \cup \mathcal{A}_{m-1,m+1}\}_{l\in\{2,\dots,m+1\}}$. Note that except for $\mathcal{A}_{m+1,m+1}$ all members in the list have usl. This implies $\mathcal{P}_{(\xi,\sigma_2(\beta))}$ has usl or $\mathcal{P}_{(\xi,\sigma_2(\beta))} = \mathcal{A}_{m+1,m+1}$. This means that either we have reduced to the usl case or $\xi_\beta \in U(M(\mathbb{P}), \beta)$. Using induction we see that only three more types of minimal subspaces are possible, namely the three last cases of the list in the statement of the theorem. □

Theorem 4.16. *The only minimal spaces of E_1 up to isomorphism are $\{*\}$, X and Φ.*

Proof. The result follows easily from Theorem 4.15. □

5. An Open Question

In view of Theorem 4.12 the following problem is interesting:

Problem 5.1. Classify all hereditary m-stable m_n patterns for $n \geq 2$ and $m \in \mathbb{N}$.

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