LOCALLY EQUICONTINUOUS DYNAMICAL SYSTEMS

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Abstract. A new class of dynamical systems is defined, the class of "locally equicontinuous systems" (LE). We show that the property LE is inherited by factors as well as subsystems, and is closed under the operations of pointed products and inverse limits. In other words, the locally equicontinuous functions in $l_\infty(\mathbb{Z})$ form a uniformly closed translation invariant subalgebra. We show that WAP $\subset$ LE $\subset$ AE, where WAP is the class of weakly almost periodic systems and AE the class of almost equicontinuous systems. Both of these inclusions are proper. The main result of the paper is to produce a family of examples of LE dynamical systems which are not WAP.

§0. Introduction

A dynamical system is a pair (X, T) where X is a compact Hausdorff space and T a self homeomorphism. Unless stated otherwise we assume that X is metrizable and equipped with a metric $d(\cdot,\cdot)$ bounded by 1. We also assume usually that the system (X, T) is topologically transitive and has a recurrent transitive point. The dynamical system is *equicontinuous* when the homeomorphisms $\{T^n : n \in \mathbb{Z}\}$ act on X as an equicontinuous family of maps. This class of dynamical systems is well understood. The classical theory of equicontinuous dynamical systems characterizes those systems completely. In particular we know that a topologically transitive equicontinuous system is isomorphic to a rotation of a compact monothetic group by a generator. Recently the theory of almost equicontinuous dynamical systems has been treated by several authors (see [AAB1,2],[GW]). A dynamical system (X, T) is called *almost equicontinuous* (AE), if there is a point $x_0 \in X$ which (i) has a dense orbit, (ii) is a recurrent point and (iii) is Lyapunov stable. The latter means that x_0 is an *equicontinuity point* (i.e. for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d(T^n x, T^n x_0) < \epsilon$, $\forall n \in \mathbb{Z}$). It turns out that AE systems which are not equicontinuous are not at all rare. Every AE system is uniformly rigid and every uniformly rigid system has an AE cover (see definitions in the next section). However the class of AE systems in not well behaved in several ways. A subsystem as well as a factor of an AE system may fail to be AE.

There is however a natural subclass of the AE systems which is well behaved. It is the class of weakly almost periodic systems (WAP) (see e.g. [EN]). Every factor as well as every subsystem of a WAP system is WAP. One way to see that the class of WAP systems is closed under these operations, as well as many others such as

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pointed products and inverse limits, is to see that the class of *weakly almost periodic functions* on $\mathbb{Z}$ forms a uniformly closed translation invariant subalgebra of $l_\infty(\mathbb{Z})$.

Since every WAP system is AE, the fact that the WAP property is inherited by subsystems, implies that every WAP system (X,T) has the property:

- For every $x \in X$, the orbit closure $Y = \bar{\mathcal{O}}_T(x)$ is an AE subsystem.

We take this to be the definition of a new class of dynamical systems. A dynamical system (X,T) is called *locally equicontinuous* (LE for short) if each point $x \in X$ is a point of equicontinuity of the subsystem $Y = \bar{\mathcal{O}}_T(x) \subset X$. In other words (X,T) is LE if every transitive sub-system of X is AE. As we will show, the class of LE functions; i.e. those functions $f(n) \in l_\infty(\mathbb{Z})$ that arise as the restriction of continuous functions $F \in C(X)$ to the orbit of a transitive point of a LE system:

$$f(n) = F(T^n x_0),$$

also forms a a uniformly closed translation invariant subalgebra of $l_\infty(\mathbb{Z})$. The main result of the paper is to produce a family of examples of LE dynamical systems which are not WAP.

In the last section we review and augment some of the results of [GW]. Specifically, we show that every uniformly rigid system has an AE cover, and we investigate the question: when the product of two topologically transitive systems is topologically transitive?

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§1. Local equicontinuity

For an AE system the (dense G_δ) subset X_{tr}, of transitive points, coincides with the set of equicontinuity points. Moreover such a system is *uniformly rigid*; i.e. there exists a sequence $\{n_k\}_{k=1}^\infty$ with $|n_k| \to \infty$ such that T^{n_k} tends to the identity uniformly (see [GM] for the theory of rigid systems). For any dynamical system (X,T) the closure of the subgroup $\{T^n : n \in \mathbb{Z}\}$ in the group $H(X)$, equipped with the metric

$$\begin{aligned} D(g,h) &= \sup_{x\in X} d(gx,hx) + \sup_{x\in X} d(g^{-1}x,h^{-1}x) \\ &=: D_+(g,h) + D_-(g,h), \end{aligned}$$

forms a Polish topological group G. When (X,T) is uniformly rigid this Polish group is non-discrete.

Lemma 1.1. *In an almost equicontinuous system (X,T) , for each point $x_0 \in X_{tr}$, the map $g \mapsto gx_0$ from G into X, is a homeomorphism. Conversely, if G is a non-discrete Polish monothetic group and (X,T) a topologically transitive system on which G acts, extending the action of $\mathbb{Z} = \{T^n : n \in \mathbb{Z}\}$, in such a way that for some transitive point $x_0 \in X_{tr}$, the map $g \mapsto gx_0$ from G to X is a homeomorphism, then (X,T) is an AE system.*

Proof. Fix $x_0 \in X_{tr}$ and let $x_1 \in X_{tr}$. Then there exists a sequence $\{m_j\}$ with $T^{m_j}x_0 \to x_1$. Given $\epsilon > 0$ we have by equicontinuity of the point x_1, a $\delta > 0$

with the property: $x \in B_\delta(x_1)$ implies $d(T^n x_1, T^n x) < \epsilon$ for all $n \in \mathbb{Z}$. Let J be a positive integer such that for $l, k > J$, $T^{m_l} x_0$ and $T^{m_k} x_0$ are in $B_\delta(x_1)$. Then for all n $d(T^{m_l+n}x_0, T^{m_k+n}x_0) < 2\epsilon$, hence for all $z \in X$ $d(T^{m_l}z, T^{m_k}z) < 2\epsilon$. Thus T^{m_j} is a Cauchy sequence with respect to the metric $D_+(g,h) = \sup_{x\in X} d(gx, hx)$.

Now let $\eta > 0$ be such that $x \in B_\eta(x_1)$ implies $d(T^n x, T^n x_1) < \delta/2$ for all $n \in \mathbb{Z}$. Then for sufficiently large j, $T^{m_j}x_0 \in B_\eta(x_1)$ hence $d(T^{-m_j}(T^{m_j}x_0), T^{-m_j}x_1) = d(x_0, T^{-m_j}x_1) < \delta/2$. Thus we have also $T^{-m_j}x_1 \to x_0$ and as above we conclude that T^{m_j} is also a Cauchy sequence with respect to the metric $D_-(g,h) = \sup_{x\in X} d(g^{-1}x, h^{-1}x)$, whence a Cauchy sequence with respect to D.

Since $H(X)$ is a Polish group with respect to D, we have $\lim T^{m_j} = g$ for some $g \in G$ and clearly $gx_0 = x_1$. Our proof also shows that the map $g \mapsto gx_0$ is a homeomorphism of G onto X_{tr}.

Conversely, assume that $g \mapsto gx_0, G \to \mathcal{O}_T(x_0)$ is a homeomorphism. Given $\epsilon > 0$ there exists a neighborhood V of $e \in G$ such that

$$\sup\{d(gx, x) < \epsilon : x \in X\}, \qquad \forall g \in V. \tag{1}$$

And there exists $\delta > 0$ such that

$$d(x_0, gx_0) < \delta \quad \Rightarrow \quad g \in V. \tag{2}$$

We will show that $x_1 \in B_\delta(x_0)$ implies $\sup\{d(hx_0, hx_1) : h \in G\} \le \epsilon$.

Fix $h \in G$ and chose a sequence $\{n_i\}$ with $T^{n_i}x_0 \to x_1$. Eventually $T^{n_i}x_0 \in B_\delta(x_0)$, hence (by (2)) $T^{n_i} \in V$ and by (1)

$$\sup\{d(x, T^{n_i}x) : x \in X\} < \epsilon. \tag{3}$$

Now

$$\begin{aligned} d(hx_0, hx_1) &\le d(hx_0, hT^{n_i}x_0) + d(hT^{n_i}x_0, hx_1) \\ &= d(hx_0, T^{n_i}hx_0) + d(hT^{n_i}x_0, hx_1). \end{aligned}$$

The first summand is $< \epsilon$ (by (3)) and, by the continuity of h, $d(hT^{n_i}x_0, hx_1) \to 0$. Thus $d(hx_0, hx_1) < \epsilon$. This proves the almost equicontinuity and the proof is complete. □

Theorem 1.2.

1. *WAP ⊂ LE ⊂ AE.*
2. *LE is closed under factors and pointed products. Thus the collection of functions in $l_\infty(\mathbb{Z})$ coming from continuous functions on pointed systems in LE, forms a closed translation invariant algebra, the algebra of locally equicontinuous functions.*

Proof. (1) The inclusion LE ⊂ AE is clear. In [AAB2] it is shown that a system (X,T) is in AE iff each element of the enveloping semigroup $E = E(X)$ is continuous on X_{tr}. Since by [EN] (X,T) is in WAP iff each element of E is continuous on X, the inclusion WAP ⊂ LE follows.

(2) Let $\pi : X \to Y$ be a homomorphism of dynamical systems where (X,T) is LE. Let y_1 be any point of Y and set $Y_1 = \bar{\mathcal{O}}(y_1)$; it suffices to show that the

system Y_1 is AE. By Zorn's lemma there exists a minimal subset X_1 of X which is closed invariant with $\pi(X_1) = Y_1$. If $x_1 \in X_1$ satisfies $\pi(x_1) = y_1$ then clearly $X_1 = \bar{\mathcal{O}}(x_1)$; thus X_1 is transitive and by LE of X, the system X_1 is AE. With no loss of generality we therefore assume that $X = X_1, Y = Y_1$ and we now have the property that $\pi^{-1}(Y_{tr}) = X_{tr}$. Our goal now is to show that there is an equicontinuity point $y_0 \in Y$.

The map $\pi^{-1} : Y \to 2^X$ is an upper-semicontinuous map; therefore there exists a dense G_δ invariant subset Y_0 of Y where π^{-1} is continuous.

Clearly $X_0 = \pi^{-1}(Y_0)$ is a G_δ subset of X. Since $Y_0 \cap Y_{tr} \neq \emptyset$ it follows that X_0 contains transitive points for the system X, so that the set X_0 is a dense G_δ subset of X. Let $x_0 \in X_{tr} \cap X_0$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in B_\delta(x_0)$ and for every $n, d(T^n x_0, T^n x) \leq \epsilon$. Since $y_0 = \pi(x_0)$ is a continuity point for π^{-1}, y_0 is in the interior of the set $\pi(B_\delta(x_0))$. Thus there exists a $\theta > 0$ with $B_\theta(y_0) \subset \operatorname{int} \pi(B_\delta(x_0))$. If $y \in B_\theta(y_0)$ then there exists $x \in B_\delta(x_0)$ with $\pi(x) = y$ whence, for every n, $d(T^n x_0, T^n x) \leq \epsilon$ and finally, if we choose the metric properly, also $d(T^n y_0, T^n y) \leq \epsilon$.

The fact that LE is closed under pointed products follows directly from the definition. Finally the conclusion that the LE functions in $l_\infty(\mathbb{Z})$ form a uniformly closed and invariant subalgebra is a straightforward consequence of the fact that the collection of LE systems is closed under these operations. □

In the next two sections we show that both inclusions in theorem 1.2.(1) are proper.

Theorem 1.3. *Let (X, T) be a LE dynamical system, then*

1. *Every minimal subsystem of (X, T) is equicontinuous, hence isomorphic to a group rotation.*
2. *Every invariant ergodic probability measure on X is supported on a minimal subsystem and is therefore isomorphic to Haar measure on a group rotation. Every invariant probability measure on X is supported by the union of the minimal subsystems of X; in particular if X has a unique minimal subset then (X, T) is uniquely ergodic.*

Proof. (1) This is a direct consequence of the definition of a LE system and theorem 1.3 in [GW].

(2) Let μ be an invariant ergodic probability measure on X and let x be a generic point for μ. Then by LE the subsystem $Y = \bar{\mathcal{O}}(x)$ is an AE system and again theorem 1.3. in [GW] implies that Y is a minimal equicontinuous subsystem and therefore that μ is isomorphic to Haar measure on a group rotation. Finally if μ is any invariant probability measure on X then we obtain the last assertion of the theorem by decomposing μ into its ergodic components. □

§2 Examples

Example 1. We will show that the Katznelson-Weiss example shown to be an AE system in theorem 4.2. of [AAB1] is a WAP system. We do this by computing its enveloping semigroup E and showing that it is commutative. The latter property is easily seen to be equivalent to the continuity of all elements of E, which in turn is a necessary and sufficient condition for a system to be WAP.(see for example [D]). Using the notation of [AAB1] theorem 4.2 we set for $x \in X$, $\mathbf{N}(x) = inf_{n\in\mathbb{Z}} x(n)$.

For $0 \leq s \leq 1$ we let $X_s = \{x \in X : \mathbf{N}(x) \geq s\}$. Clearly X_s is a subsystem of X. Let h_s be the affine map of the unit interval $I = [0,1]$ into itself defined by $h_s(t) = s + t(1-s)$. The function h_s defines a continuous map (also denoted by h_s) from X to $I^{\mathbb{Z}}$ given by $h_s(x)(n) = h_s(x(n))$ and it is easy to see that for every $0 \leq s \leq 1$, the function $\alpha_s =: h_s(\alpha)$ is in X, that $X_s = \bar{\mathcal{O}}(\alpha_s)$ and that $h_s : X \to X_s$ is a homomorphism of dynamical systems (an isomorphism for $s < 1$). Since $h_s h_t = h_{t+s(1-t)}$ we have $h_s h_t = h_t h_s$, and as we shall see soon this commutation relation is the key to our proof. Observe that (say by lemma 4.3 of [AAB1]) if $x \in X$ satisfies $x(0) = \alpha(0)$ then $x = \alpha$ and similarly for $x \in X_s$, $x(0) = \alpha_s(0)$ implies that $x = \alpha_s$.

Now we claim that for $x \in X$, $\mathbf{N}(x) = s$ iff there exists $g \in G$ with $g\alpha_s = x$. Here G is the Polish group which is the closure of $\{T^n : n \in \mathbb{Z}\}$ in the group $H(X)$ with respect to the metric D. To see this observe that $\mathbf{N}(x) = s$ clearly implies the existence of a sequence $\{m_j\}$ with $x(m_j) \to s$, whence (for a subsequence) $T^{m_j}x \to y$ for some $y \in X$ with $y(0) = s$ and by the above remark $y = \alpha_s$. Since α_s is a transitive point of the subsystem X_s it follows that also x is a transitive point of X_s. Now apply lemma 1.1 to get an element $g \in G$ with $g\alpha_s = x$. We conclude that every element $x \in X$ has a unique representation $x = gh_s\alpha$ with $g \in G$ and $s = \mathbf{N}(x)$. We also see that $(X_s)_{tr} = G\alpha_s$.

Now let $\lim T^{m_j}\alpha = x$, for some sequence m_j in $\mathbb{Z}$ and some $x \in X$, and let x' be an arbitrary point in X. Then we have $x = gh_s\alpha$, $x' = g'h_{s'}\alpha$ and

$$\begin{aligned}\lim T^{m_j}x' &= \lim T^{m_j}g'h_{s'}\alpha = g'h_{s'}\lim T^{m_j}\alpha \\ &= g'h_{s'}x = g'h_{s'}gh_s\alpha = gh_s g'h_{s'}\alpha = gh_s x'.\end{aligned}$$

Thus the sequence $T^{m_j}x'$ converges for every $x' \in X$ and therefore defines an element $p \in E$, the enveloping semigroup of X, which by the above calculation coincides with the map gh_s. In this way we identified E algebraically as the direct product $G \times A$, where A is the "affine" (commutative) semigroup $A = \{h_s : 0 \leq s \leq 1\}$. This completes our proof. $\square$

Example 2. (LE $\subsetneqq$ AE) Start with a minimal weakly mixing uniformly rigid system (Y,T) (the existence of such systems is shown in [GM], proposition 6.5). In proposition 1.5 of [GW] we show how, given a uniformly rigid transitive system (Y,T) , one can always construct a transitive AE system (X,T) and a homomorphism $\pi : X \to Y$. Since a minimal AE system is equicontinuous, and a system which is both weakly mixing and equicontinuous is trivial, we conclude that (Y,T) is not AE. Finally since (X,T) has (Y,T) as a factor it follows from theorem 1.(2) that (X,T) is not LE; thus LE $\subsetneqq$ AE. $\square$

§3 The main example: WAP $\subsetneqq$ LE

Our purpose in this section is to construct a LE system (X,T) which is not WAP. Let Ω be the space of continuous maps $x : \mathbb{R} \to 2^I$, where $I = [0,1]$ and 2^I is the compact metric space of closed subsets of I equipped with the Hausdorff metric. The topology we put on Ω is that of uniform convergence on compact sets: $x_n \to x$ if for every $\epsilon > 0$ and every $M > 0$ there exists $N > 0$ such that for all $n > N$, $\sup_{|t|\leq M} d(x_n(t), x(t)) < \epsilon$. This topology makes Ω a compact metrizable space. On Ω there is a natural $\mathbb{R}$-action defined by translations: $(T^t x)(s) = x(s+t)$. We will construct an element $\omega \in \Omega$ and let $X = \text{closure } \{T^n\omega : n \in \mathbb{Z}\}$. Our task then will be to show that (X, T) is LE but not WAP.

Let α_0 be the periodic function in Ω of period 1 whose graph is given in figure 1 bellow.

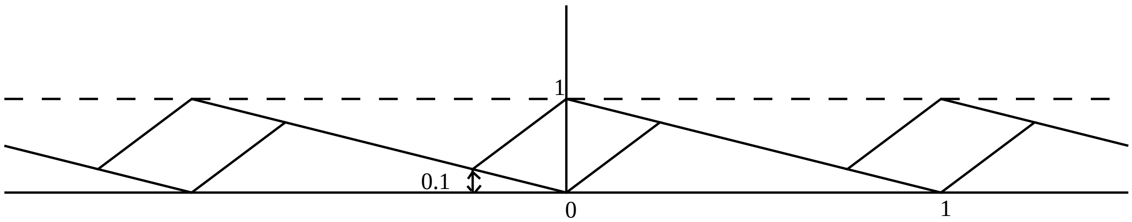

Figure 1

Explicitly, the upper envelope of α_0 is given on $[0, 1]$ by the function:

$$u_0(t) = \begin{cases} 1 - t, & 0 \le t \le 9/10 \\ 9t - 8, & 9/10 \le t \le 1, \end{cases}$$

and the lower envelope by:

$$l_0(t) = \begin{cases} 9t, & 0 \le t \le 1/10 \\ 1 - t, & 1/10 \le t \le 1, \end{cases}$$

(the values $\alpha_0(t)$ are either intervals or points).

For a sequence of positive integers p_n, let

$$\alpha_n = \alpha_0(\frac{t}{p_n}).$$

We assume $p_0 = 1$ and $p_{n+1} = 10k_n p_n$ for a sequence of integers $k_n \nearrow \infty$ such that

$$\sum_{n=1}^{\infty} \frac{p_n}{p_{n+1}} = \sum_{n=1}^{\infty} \frac{1}{10k_n} < \infty.$$

The upper envelope of the periodic function α_n (of period p_n) is given on the interval $[0, p_n]$ by the function:

$$u_n(t) = \begin{cases} 1 - t/p_n, & 0 \le t \le 9p_n/10 \\ 9t/p_n - 8, & 9p_n/10 \le t \le p_n, \end{cases}$$

and the lower envelope by:

$$l_n(t) = \begin{cases} 9t/p_n, & 0 \le t \le p_n/10 \\ 1 - t/p_n, & p_n/10 \le t \le p_n, \end{cases}$$

Next construct a sequence of affine maps $a_{n+1}(t, \cdot)$. Roughly speaking, the map $a_{n+1}(t, \cdot)$ "squeezes" α_n into α_{n+1}. Then we set $\beta_0 = \alpha_0$ and define inductively

$\beta_{n+1}(t)$ as the image of $\beta_n(t)$ under $a_{n+1}(t,\cdot)$. Finally the element ω will be the limit in Ω of the sequence β_n. Here is the precise construction.

Put $\beta_0 = \alpha_0$, and assume that β_n, periodic of period p_n, is already constructed. We next describe the construction of β_{n+1}. For an integer j with $0 \le jp_n \le p_{n+1}/10$ and $s \in [0,1]$, denote $v = jp_n$ and set

$$\begin{aligned} a_{n+1}(v,s) &= (1 - \frac{10v}{p_{n+1}})s + \frac{9v}{p_{n+1}} \\ &= \lambda_{n+1}(v)s + \mu_{n+1}(v). \end{aligned}$$

For $\frac{9p_{n+1}}{10} \le v = jp_n = p_{n+1} - u \le p_{n+1}$, put

$$\begin{aligned} a_{n+1}(v,s) &= (\frac{10v}{p_{n+1}} - 9)s + (1 - \frac{v}{p_{n+1}}) \\ &= (1 - \frac{10u}{p_{n+1}})s + \frac{u}{p_{n+1}} \\ &= \lambda_{n+1}(v)s + \mu_{n+1}(v), \end{aligned}$$

where $u = p_{n+1} - v$. Define $a_{n+1}(t,s)$ for $0 \le t \le p_{n+1}$ as follows: For $t \in [jp_n - p_n/10, jp_n + p_n/10]$, set $j_n(t) = j$ and $v(t) = j_n(t)p_n$. Now define

$$a_{n+1}(t,s) = a_{n+1}(v(t),s) = \lambda_{n+1}(t)s + \mu_{n+1}(t).$$

These maps define an embedding of the parallelograms of α_n (around the points $0 \le jp_n \le p_{n+1}/10$ and $p_{n+1} - p_{n+1}/10 \le jp_n \le p_{n+1}$), inside the two triangles of α_{n+1} defined on the intervals $[0, p_{n+1}/10]$, and $[p_n - p_{n+1}/10, p_{n+1}]$. Now connect these embedded parallelograms by line segments and for values of t in the rest of $[0, p_{n+1}]$, let $a_{n+1}(s,t)$ be the point on the line segment corresponding to t (this is a constant value independent of s).

Finally define

$$\beta_{n+1}(t) = a_{n+1}(t, \beta_n(t)), \qquad t \in [0, p_{n+1}],$$

and extend it periodically, with period p_{n+1}, over all of $\mathbb{R}$. Note that for every n, j and t

(1) $\beta_n(jp_n) = [0,1]$,
(2) $|\beta_{n+1}(t)| = \lambda_{n+1}(t)|\beta_n(t)|$,
(3) $\lambda_{n+1}(t) = 1 \mp \frac{10j_n(t)p_n}{p_{n+1}}$ where $v(t) = ip_{n+1} \pm j_n(t)p_n$ and ip_{n+1} is the integer multiple of p_{n+1} closest to t.
(4) $\beta_{n+1}(t) \subset \alpha_{n+1}(t)$.

For $t \in \mathbb{R}$ and $1 \le m < n \le \infty$, denote

$$q_k(t) = \frac{10j_k(t)p_k}{p_{k+1}},$$

$$\Lambda_m^n(t) = \prod_{k=m}^{n} \lambda_k(t) = \prod_{k=m}^{n} (1 \mp q_k(t)) = \prod_{k=m}^{n} (1 \mp \frac{10j_k(t)p_k}{p_{k+1}})$$

and $\Lambda(t) = \Lambda_1^\infty(t)$. Thus, $|\omega(t)| = \Lambda(t)$, for all $t \in \mathbb{R}$, with $|\omega(t)| > 0$.

Define the affine map A_m as the composition of the maps $a_k(t,\cdot)$:

$$\begin{aligned}A_m(t,s) =& a_m(t,\cdot)\circ a_{m-1}(t,\cdot)\circ\cdots\circ a_1(t,s)\\ :=&\Lambda_1^m(t)s+M_1^m(t)\\ =&\lambda_m(t)\lambda_{m-1}(t)\cdots\lambda_1(t)s+\lambda_m(t)\lambda_{m-1}(t)\cdots\lambda_2(t)\mu_1(t)+\\ &\lambda_m(t)\lambda_{m-1}(t)\cdots\lambda_3(t)\mu_2(t)+\cdots+\lambda_m(t)\mu_{m-1}(t)+\mu_m(t).\end{aligned}$$

Since $\mu_k(t)\le 10q_k(t)$, it follows that

$$M_1^m(t)<10\sum_{k=1}^m q_k(t).$$

Finally it is easy to see that for $t_1,t_2\in\mathbb{R}$ and $n\in\mathbb{N}$ we have: $j_n(t_2-t_1)=|j_n(t_2)-j_n(t_1)|$ up to ±1.

Given $0\le t\in\mathbb{R}$, let n_0 be the least n such that $t\le p_n/10$. For every $n>n_0$, the v corresponding to t in the definition of a_n is $v=0$, so that $a_n(t,s)=a_n(0,s)\equiv s,\ s\in[0,1]$, and a similar assertion holds for $t\le 0$. It is now clear that for every $M>0$, the restrictions of β_n to the interval $[-M,M]$ stabilize after a finite number of steps. Therefore the sequence β_n converges uniformly on compact sets, and the limit, $\omega=\lim\beta_n$, is a well defined element of Ω. As asserted above we now let $X=\bar{\mathcal{O}}(\omega)$, where our dynamical system is translation by 1 on Ω, which we denote by $T=T^1$. Given $x\in X$ and $t\in\mathbb{R}$ the set $x(t)\subset I$ is an interval; call such an interval a *rod*. Given $x\in X$ and an interval of length $M>0$, there exists a t in that interval such that the rod $x(t)$ has maximal length.

Lemma 3.1. *Suppose $\omega(r)=[a,b]$, $n\ge1$, and that $\omega(r)$ is a rod of maximal length in an interval of length p_n:*

$$|\omega(r)|=b-a=\max\{|\omega(t)|:t\in[q,q+p_n]\},$$

for some $q\in\mathbb{R}$. Then there exists $j\in\mathbb{Z}$ with $r=jp_n$ and $\omega(t)\subseteq[a,b]=\omega(r)$ for every $t\in[r-0.1p_n,r+0.1p_n]$.

Proof. As we have seen above, if m_0 is the least integer such that $r\le p_{m_0}/10$, then for $|t|\le p_{m_0}/10$ and $l\ge1$,

$$a_{m_0+l}(t,s)=a_{m_0+l}(0,s)\equiv s,$$

hence

$$\omega(t)=\beta_{m_0}(t).$$

There exists a unique $j\in\mathbb{Z}$ with $|jp_n-r|\le p_n/10$. If $jp_n\ne r$ then $\beta_n(r)\subsetneqq[0,1]=\beta_n(jp_n)$. Since for $m\ge n$ the affine contractions $a_m(t,\cdot)$ are the same for all $t\in[jp_n-p_n/10,jp_n+p_n/10]$, this implies also

$$\omega(r)=\beta_{m_0}(r)\subsetneqq\beta_{m_0}(jp_n)=\omega(jp_n),$$

contradicting our assumption. Thus $r=jp_n$ and therefore:

$$\omega(t)=\beta_{m_0}(t)\subseteq\omega(r)=\beta_{m_0}(r)$$

for every $t\in[(j-0.1)p_n,(j+0.1)p_n]$. $\square$

Lemma 3.2. *Let x be an element of X and $\eta > 0$. Suppose $x(0) = [a, b]$ and*

$$|x(0)| = b - a > \sup\{|x(t)| : t \in \mathbb{R}\} - \eta,$$

then

$$x(t) \subseteq [a - 2\eta, b + 2\eta]$$

for every $t \in \mathbb{R}$.

Proof. Suppose that for some t_0 we have $x(t_0) \subsetneqq [a - 2\eta, b + 2\eta]$, we may assume $x(t_0) = [c, d]$ and $d - b - 2\eta = \delta > 0$. Choose n so that $|t_0| < p_n/2$ and choose m with

$$\sup\{d(x(t), \omega(t + m)) : |t| \leq p_n\} < \delta/3.$$

Let $r \in [m - p_n/2, m + p_n/2]$, with $\omega(r) = [e, f]$ satisfy

$$|\omega(r)| = f - e = \max\{|\omega(t)| : t \in [m - p_n/2, m + p_n/2]\}.$$

By the previous Lemma, $\omega(t) \subset [e, f]$ for all $t \in [m - p_n, m + p_n]$, and in particular:

$$\omega(m) \subset \omega(r) = [e, f] \quad \text{and} \quad \omega(t_0 + m) \subset \omega(r) = [e, f].$$

We also have

$$d(x(0), \omega(m)) < \delta/3 \quad \text{and} \quad d(x(t_0), \omega(t_0 + m)) < \delta/3,$$

and it follows that

$$x(0) = [a, b] \overset{\delta/3}{\subset} [e, f] \quad \text{and} \quad x(t_0) = [c, d] \overset{\delta/3}{\subset} [e, f].$$

Thus $d < f + \delta/3$ and since $x(0) = [a, b]$ is, up to η, a maximal rod for x, we deduce that $f < b + \delta/3 + \eta$. We now have

$$d < f + \delta/3 < b + 2\delta/3 + \eta < b + \delta + \eta < d$$

and this contradiction completes the proof. $\square$

Lemma 3.3. *For every $x \in X$ there is a unique interval $[a, b] \subseteq [0, 1]$ such that:*

(1)
$$x(t) \subseteq [a, b], \ \forall t \in \mathbb{R},$$

(2) *there exists a sequence $t_l \in \mathbb{R}$ with*

$$\lim \ x(t_l) = [a, b].$$

We denote

$$\mathbf{N}(x) = [a, b].$$

Proof. Let $d = \sup\{|x(t)| : t \in \mathbb{R}\}$, and choose a sequence $t_l \in R$ satisfying $\lim \ |x(t_l)| = d$. Passing to a subsequence, we can assume that $\lim x(t_l) = [a, b]$ (with $b - a = d$) exists. Our assertions now follow from the previous Lemma. $\square$

Lemma 3.4. *For a sequence $q_k \in \mathbb{R}$, $k = 1, 2, \dots, n$, with $0 \le q_k < 1/10$, denoting $Q = \sum_{k=1}^{n} q_k$ we have for small Q:*

$$1 - Q/2 \ge \exp(-Q) \ge \prod_{k=1}^{n}(1 - q_k) \ge \exp(-2Q) \ge 1 - 2Q.$$

For a compact interval $J \subset \mathbb{R}$ and $x \in \Omega$ we let $\mathbf{N}(x, J)$ denote a rod of maximal length of x restricted to J.

Lemma 3.5. *Given $\epsilon > 0$ choose $n \in \mathbb{N}$ such that*

$$\sum_{k=n}^{\infty} 10 \frac{p_k}{p_{k+1}} < \epsilon/2,$$

if $m \in \mathbb{Z}$ and $t_1, t_2, r \in \mathbb{R}$ are such that $r, t_1, t_2 \in J = [m - p_n, m + p_n]$, and

(1)
$$\mathbf{N}(\omega, J) = \omega(r) = [a, b]$$

(2)
$$\frac{|\omega(t_i)|}{b - a} > 1 - \epsilon/10, \qquad i = 1, 2,$$

then for a function $\epsilon' = \epsilon'(\epsilon, b - a)$ with $\lim_{\epsilon \to 0} \epsilon' = 0$,

(3)
$$|\beta_n(t_i)| > 1 - \epsilon', \qquad i = 1, 2$$

(4)
$$\sum_{k=1}^{\infty} 10 \frac{j_k(s_0) p_k}{p_{k+1}} < 4\epsilon',$$

where $s_0 = t_2 - t_1$.

Proof. By Lemma 3.1 there exists an integer j such that $r = jp_n$, hence $\beta_n(r) = [0, 1]$, and $\omega(t) \subset [a, b]$, $\forall t \in [r - p_n, r + p_n]$. By Lemma 3.4,

$$\sum_{k=n}^{\infty} 10 \frac{p_k}{p_{k+1}} < \epsilon/2$$

implies

$$1 - \prod_{k=n}^{\infty}\left(1 - 10 \frac{p_k}{p_{k+1}}\right) < \epsilon.$$

It follows that on an interval of radius p_n around r, the numbers $\Lambda_n^{\infty}(t)$ can't vary by more than ϵ. Thus, for $i = 1, 2$

$$\begin{aligned} \frac{\Lambda_n^{\infty}(t_i)|\beta_n(t_i)|}{\Lambda_n^{\infty}(r)|\beta_n(r)|} &= \frac{\Lambda_n^{\infty}(t_i)|\beta_n(t_i)|}{\Lambda_n^{\infty}(r)} \\ &= \frac{\Lambda_n^{\infty}(t_i)|\beta_n(t_i)|}{b - a} = \frac{|\omega(t_i)|}{|\omega(r)|} > 1 - \epsilon/10. \end{aligned}$$

Hence

$$\prod_{k=1}^{n}(1-q_k(t_i)) = |\beta_n(t_i)| > (1-\epsilon/10)\left(\frac{b-a}{(b-a)\pm\epsilon}\right) > 1-\epsilon', \qquad i=1,2.$$

Use Lemma 3.4 again to deduce,

$$\sum_{k=1}^{n} q_k(t_i) < 2\epsilon'.$$

Now for $k \geq n+1$ we have $j_k(t_i) = j_k(r)$, $i=1,2$, hence for $s_0 = t_2 - t_1$

$$\begin{aligned}\sum_{k=1}^{\infty} 10\frac{j_k(s_0)p_k}{p_{k+1}} &\sim \sum_{k=1}^{\infty} 10\frac{|j_k(t_2)-j_k(t_1)|p_k}{p_{k+1}} \\ &= \sum_{k=1}^{n} 10\frac{|j_k(t_2)-j_k(t_1)|p_k}{p_{k+1}} \\ &= \sum_{k=1}^{n} |q_k(t_2)-q_k(t_1)| < 4\epsilon'.\end{aligned}$$

□

Lemma 3.6. *If $s_0 \in \mathbb{R}$ satisfies*

$$\sum_{k=1}^{\infty} 10\frac{j_k(s_0)p_k}{p_{k+1}} < \epsilon,$$

then

$$\sup\{d(\omega(t+s_0),\omega(t)) : t \in \mathbb{R}\} < 3\epsilon.$$

Proof. Fix $t_0 \in \mathbb{R}$ and choose $m, n \in \mathbb{N}$ with

(1) $t_0, t_0 + s_0 \in J = [-p_n, p_n]$
(2) $d(\beta_m(t), \omega(t)) < \epsilon/2, \quad \forall t \in J$.

Now for every k and t, $|\beta_{k+1}(t)| = \lambda_{k+1}(t)|\beta_k(t)|$, hence $|\beta_m(t)| = \prod_{k=1}^{m}\lambda_k(t)$. Thus denoting $t_1 = t_0$ and $t_2 = t_0 + s_0$, we have

$$\begin{aligned}||\beta_m(t_2)| - |\beta_m(t_1)|| &= |\prod_{k=1}^{m}\lambda_k(t_2) - \prod_{k=1}^{m}\lambda_k(t_1)| \\ &\leq \sum_{k=1}^{m} |\lambda_k(t_2)-\lambda_k(t_1)| \\ &= \sum_{k=1}^{m} 10\frac{j_k(s_0)p_k}{p_{k+1}} < \epsilon.\end{aligned}$$

A similar argument shows that

$$|M_m(t_2) - M_m(t_1)| < \epsilon,$$

and since

$$\beta_m(t_i) = A_m(t_i, [0,1]) = \Lambda_m(t_i)[0,1] + M_m(t_i),$$

we get

$$d(\beta_m(t_2), \beta_m(t_1)) < 2\epsilon,$$

hence

$$d(\omega(t_0 + s_0), \omega(t_0)) < 3\epsilon.$$

□

Theorem 3.7. *The dynamical system (X, T) is LE but not WAP. It contains $2^{\aleph_0}$ minimal sets, namely the constant functions $x(t) \equiv a,\ a \in [1/10, 9/10]$.*

Proof. (1) It is enough to show that given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that if

$$d(x(t_0), \mathbf{N}(x)) < \delta, \quad \text{and} \quad d(x(t_0 + s_0), \mathbf{N}(x)) < \delta,$$

then

$$\sup_{t\in\mathbb{R}} d(x(t + s_0), x(t)) < 4\epsilon.$$

Choose $n \in \mathbb{N}$ such that

(1)

$$\sum_{k=n}^{\infty} 10\frac{p_k}{p_{k+1}} < \epsilon/10,$$

(2) for the interval $J = [-p_n, p_n]$, $\mathbf{N}(x, J) \stackrel{\epsilon}{=} \mathbf{N}(x) = [a, b]$.

Choose $m \in \mathbb{Z}$ with

$$\max\{d(x(t), \omega(t + m)) : t \in J\} < \delta,$$

where δ is small enough so that the assumptions of Lemma 3.5 are satisfied with $t_1 = t_0 + m$ and $t_2 = t_0 + s_0 + m$. We conclude that

(1)

$$|\beta_n(t_i)| > 1 - \epsilon', \qquad i = 1, 2$$

(2)

$$\sum_{k=1}^{\infty} 10\frac{j_k(s_0)p_k}{p_{k+1}} < 4\epsilon'.$$

Now Lemma 3.6 yields

$$\sup\{d(\omega(t + s_0), \omega(t)) : t \in \mathbb{R}\} < 4\epsilon'.$$

This concludes the proof that x is an equicontinuity point.

(2) The claim that each constant function a for $a \in [1/10, 9/10]$ is an element of X is easy to see. Thus for each such a the singleton $\{a\}$ is a minimal subset of X. Since a WAP system has a unique minimal set, it follows that (X,T) is not WAP.

□

Remark. If (Y,T) is any LE system with transitive point y_0, then the system $Z = \bar{\mathcal{O}}(\omega, y_0) \subset X \times Y$ is a LE system. In particular if we take for (Y,T) a Kronecker (i.e. a minimal equicontinuous) system, then for each fixed point $a \in X$ the subsystem $\{a\} \times Y \subset Z$ is a minimal subsystem of Z.

Thus the minimal sets in an LE system that is not WAP can be any Kronecker system, and not only points as in the construction above. Moreover if we let $(Y,T) = (\mathbb{T}, R_\alpha)$, with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $R_\alpha y = y + \alpha \pmod 1$ for $\alpha \in \mathbb{R}$ an irrational number such that $p_n\alpha \to 0$ fast enough, then the LE systems X and Z have the same Polish group $G(X) = G(Z)$.

§4 Appendix

In this appendix we would like to clarify and augment some points from our paper [GW]. We first restate Proposition 1.5 of [GW] as Theorem 4.1 below. and provide it with a modified version of the original "constructive" proof using more precise notation:

Theorem 4.1. *Any infinite topologically transitive uniformly rigid system (X,T) has an extension (Y,S) that is AE.*

Proof. We assume that T^{n_i} tends uniformly to the identity map and that x_0 has a dense orbit. Define for $x, x' \in X, \rho(x,x') = \sup_{n\in\mathbb{N}} d(T^n x, T^n x')$ and notice that by uniform rigidity, the sequence $\rho(T^{n_i}x, x)$ tends to 0 with i. Let now $\Omega = (X \times \mathbb{R})^{\mathbb{N}}$. For $\bar\omega \in \Omega$ we denote by $\bar\omega = (\xi, \omega)$ the decomposition into $\xi \in X^{\mathbb{N}}$ and $\omega \in \mathbb{R}^{\mathbb{N}}$. For $\bar\omega, \bar\omega' \in \Omega$ let

$$\hat{d}(\bar\omega, \bar\omega') = \sum_{k=0}^{\infty} 2^{-k}\{d(\xi(k), \xi'(k)) + |\omega(k) - \omega'(k)|\}.$$

Let $\bar\omega_0$ be the point of Ω whose n-th coordinate is $(T^n x_0, \rho(T^n x_0, x_0))$ and let Y be the orbit closure of $\bar\omega_0$ under the shift map S of Ω.

The points $\bar\omega \in Y$ have the form $\bar\omega(k) = (T^k x, \omega(k))$ for some $x \in X$, and

$$(S\bar\omega)(n) = (T^{n+1}x, \omega(n+1)).$$

It turns out, as is always the case for a transitive system, that in checking the non-sensitivity we will be dealing with only one point $\bar\omega_0$. Given $\epsilon > 0$, let U be the neighborhood of $\bar\omega_0$ defined by

$$U = \{\bar\omega \in Y : \omega(0) < \epsilon/2\}.$$

Since $\bar\omega_0$ has a dense orbit, in order to verify that for all $\bar\omega \in U$ and all n

$$\hat{d}(S^n\bar\omega_0, S^n\bar\omega) \le \epsilon,$$

it suffices to do so for points $\bar\omega$ of the form $S^j\bar\omega_0$. Suppose then that $S^{j_0}\bar\omega_0 \in U$. Since $\rho(T^{j_0}x_0, x_0) < \epsilon/2$ we have $d(T^{i+j_0}x_0, T^i x_0) < \epsilon/2$ for all $i \ge 0$, hence also $\rho(T^{i+j_0}x_0, T^i x_0) < \epsilon/2$ for all $i \ge 0$. By the triangle inequality we find that $|\rho(T^i x_0, x_0) - \rho(T^{i+j_0}x_0, x_0)| \le \epsilon/2$ for all $i \ge 0$. For any n we therefore have

$$\begin{aligned}
&\hat{d}(S^n\bar\omega_0, S^n(S^{j_0}\bar\omega_0)) = \hat{d}(S^n\bar\omega_0, S^{n+j_0}\bar\omega_0)\\
&= \sum_{k=0}^{\infty} 2^{-k}\{d(T^{k+n}x_0, T^{k+n+j_0}x_0) + |\rho(T^{k+n}x_0, x_0) - \rho(T^{k+n+j_0}x_0, x_0)|\}\\
&\le \sum_{k=0}^{\infty} 2^{-k}\{\epsilon/2 + \epsilon/2\} = 2\epsilon.
\end{aligned}$$

We observe that the only reason for requiring (X, T) to be rigid, is to make sure that the point $\bar{\omega}_0$, is not an isolated point. $\square$

A more abstract proof of Theorem 4.1 can be given using in an explicit way the Polish group that is associated with an AE system. First we prove:

Theorem 4.2. *Every infinite Polish monothetic group G, admits an almost equicontinuous action, where each dense orbit is homeomorphic to G.*

Proof. Let G be a Polish monothetic group generated by T. Let $UC_b(G)$ be the Banach algebra of uniformly continuous bounded real-valued functions on G with the sup norm. We choose a countable collection of elements of $UC_b(G)$ which separate between points and closed sets in G (see e.g. [HR, page 68]), and let $\mathcal{A}$ be the smallest closed, T (hence G) invariant algebra which contains this collection and the constant functions. If we denote by Z the Gelfand space corresponding to $\mathcal{A}$ we see that G acts on Z and that (Z, G) is a topologically transitive system. Since the natural embedding of G into Z is a homeomorphism we conclude, by Lemma 1.1, that the restricted system (Z, T) is almost equicontinuous with $G(Z, T) = G$. $\square$

The construction of the system (Z, T) above provides us with the following:

Alternative proof for Theorem 4.1. Let $G(X, T) = G$ be the non-discrete Polish group corresponding to the uniformly rigid system (X, T) . As in the proof of Theorem 4.2, construct an algebra $\mathcal{A} \subset UC_b(G)$ and let Z be the Gelfand space corresponding to $\mathcal{A}$. We now set $Y = \text{closure } \{g(x_0, z_0) : g \in G\} \subset X \times Z$, where $x_0 \in X$ and $z_0 \in Z$ are transitive points. Since the natural embedding of G into Z is a homeomorphism, a fortiori this is true also for the embedding of G into Y and we conclude that the system (Y, T), which is by construction an extension of (X, T) , is almost equicontinuous (Lemma 1.1). $\square$

The second subject from [GW] that we would like to treat here is the question: when two topologically transitive dynamical systems have the property that their product is also topologically transitive?

In [F1] H. Furstenberg has shown that for a weakly mixing system (X, T) and two nonempty open subsets U, V of X, the set

$$N(U, V) = \{n \in \mathbb{Z} : T^n U \cap V \neq \emptyset\},$$

is a *thick* subset of $\mathbb{Z}$, i.e. it contains arbitrarily long intervals. Since for any minimal system (Y, T) and nonempty open subsets A, B of Y the set $N(A, B)$ is a *syndetic* subset of $\mathbb{Z}$, i.e. a set with bounded gaps, it follows that for all such (X, T), (Y, T) , U, V, A, B, the set

$$N(U \times A, V \times B) = N(U, V) \cap N(A, B),$$

is nonempty. In other words the product system is topologically transitive.

We say that a dynamical system (Y, T) is *syndetically transitive* if for any two nonempty open subsets U, V of X, the set

$$N(A, B) = \{n \in \mathbb{Z} : T^n A \cap B \neq \emptyset\},$$

is a syndetic subset of $\mathbb{Z}$. The argument above immediately implies:

Theorem 4.3. *Let (X,T) be a weakly mixing system and (Y,T) a syndetically transitive one, then the product system $(X \times Y, T \times T)$ is topologically transitive.*

In Proposition 2.2.(2) of [GW] we claimed, without proof, that the product of a weakly mixing system and an *E-system*, i.e. one which carries a T-invariant probability measure which is positive on every nonempty open set, is topologically transitive.

Theorem 4.4. *Let (X,T) be a weakly mixing system and (Y,T) an E-system, then the product system $(X \times Y, T \times T)$ is topologically transitive.*

Proof. Again we describe two proofs. The first consists of showing that every E-system is syndetically transitive. Once we have this, Theorem 4.3 completes the proof. As was explained in [GW], in an E-system with invariant measure μ with full support, the generic points for ergodic measures are dense (take the set of generic points for the ergodic components of μ in its ergodic decomposition). Now given two nonempty open sets U, V in X, we choose $k \in \mathbb{Z}$ with $V_0 = T^k U \cap V \neq \emptyset$. Next set $U_0 = T^{-k} V_0 \cap U$, and observe that $k + N(U_0, U_0) \subset N(U,V)$. Thus it is enough to show that $N(U,U)$ is syndetic for every nonempty open U. Let x_0 be a generic point for an ergodic measure ν with $\nu(U) > 0$. Then the set

$$A = \{n \in \mathbb{Z} : T^n x_0 \in U\},$$

has positive upper density and it follows that the set $A - A$ is syndetic (see for example [F2]. p.75). Since clearly $A - A \subset N(U,U)$, this completes the proof.

For a second proof let $A, B \subset X, U, V \subset Y$ be nonempty open sets. We have to show that for some $l \in \mathbb{Z}$, $T^l A \cap B \neq \emptyset$ and also $T^l U \cap V \neq \emptyset$. Let $W = \cup_{n \in \mathbb{Z}} T^n U$, then W is a nonempty T-invariant open subset of Y. By assumption there exists a T-invariant probability measure μ on Y which assigns positive measure to every nonempty open set, and in particular $\mu(W) = a > 0$. Since Y is transitive the set $O = W \cap V$ is a nonempty open subset and we have $\mu(O) = b > 0$. We now choose a positive integer N such that

$$\mu\Big(\bigcup_{|n| \leq N} T^n U\Big) > a - b/2.$$

Now the system (X,T) is topologically weakly mixing, hence by [F1] the set $N(A,B) = \{k \in \mathbb{Z} : T^k A \cap B \neq \emptyset\}$ contains arbitrarily long intervals. We can therefore find some $j \in \mathbb{Z}$ with

$$T^{j+k} A \cap B \neq \emptyset, \qquad \forall |k| \leq N.$$

By T-invariance of μ we have

$$\mu\Big(T^j\Big(\bigcup_{|n| \leq N} T^n U\Big)\Big) = \mu\Big(\bigcup_{|n| \leq N} T^n U\Big) > a - b/2.$$

This implies $T^j(\bigcup_{|n| \leq N} T^n U) \cap V \neq \emptyset$, and there exists n_0 with $|n_0| \leq N$ and $T^{j+n_0} U \cap V \neq \emptyset$ as well as $T^{j+n_0} A \cap B \neq \emptyset$. This completes the proof. $\square$

Example: Taking (Y, S) to be the one point compactification of the translation on $\mathbb{Z}$ it is easy to see that the assumption of topological transitivity of Y is not enough for this result to hold. A more interesting example is obtained as follows. Take (X, T) to be a weakly mixing rigid minimal system (see [GM]), and (Y, S) the AE system constructed from it in Theorem 4.1. Although the system (Y, S) is transitive and pointwise recurrent, the product system $(X \times Y, T \times S)$ is not transitive. To see this, suppose on the contrary that there exists a point (x_0, y_0) whose orbit is dense in $X \times Y$. Let x be an arbitrary point of X and choose a sequence n_k with

$$\lim T^{n_k}(x_0, y_0) = (x, y_0).$$

Then, since Y is an AE system, we deduce from $\lim T^{n_k} y_0 = y_0$ that $\lim T^{n_k} = \mathrm{id}$ in the corresponding Polish group G (Lemma 1.1). Since the T action on X extends to a G action, we conclude that also $\lim T^{n_k} x_0 = x_0 = x$. Thus $X = \{x_0\}$, a contradiction. $\square$

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