LOCAL ENTROPY THEORY

ELI GLASNER AND XIANGDONG YE

Abstract. In this survey we offer an overview of the so called local entropy theory, developed since the early 1990s. While doing so we emphasize the connections between the topological dynamics and the ergodic theory view points.

Contents

Date: 20.11.2007.

2000 *Mathematics Subject Classification.* Primary: 37B40, 37A35. Secondary: 37B20, 37B05, 37A05, 37A25.

The first author is partially supported by ISF grant 1333.

The second author is partially supported by NSFC (10531010), 973 programm and Ministry of Education (20050358053).

1. Introduction

Dynamical system theory is the study of qualitative properties of group actions on spaces with certain structures. In this survey we are mainly interested in actions by homeomorphims on compact metric spaces with an additional structure of a Borel probability measure invariant under the action. We will mostly consider $\mathbb{Z}$-actions, and only briefly mention more general group actions.

Thus by a *topological dynamical system* (TDS for short) we mean a pair (X, T), where X is a compact metric space and $T : X \longrightarrow X$ is a self homeomorphism. This defines an action $(X, \mathbb{Z})$ of the group of integers $\mathbb{Z}$ on X via the map $n \mapsto T^n$.

By a *measure-theoretical dynamical system* (for short MDS) we mean a quadruple $(X, \mathcal{B}, \mu, T)$, where X is a set, $\mathcal{B}$ is a σ-algebra on X, μ is a probability measure on X and $T : X \longrightarrow X$ is invertible measurable and measure-preserving, that is: $\mu(B) = \mu(T^{-1}(B))$ for each $B \in \mathcal{B}$. For a TDS (X, T), there are always invariant probability measures on X and thus for each such measure μ $(X, \mathcal{B}(X), \mu, T)$, with $\mathcal{B}(X)$ the Borel σ-algebra on X, is a MDS.

Ergodic theory and topological dynamics exhibit a remarkable parallelism. For example the notions of transitivity, weak mixing and strong mixing in TDS have their counterpart in ergodic theory and we speak about topological entropy in topological dynamics and about measure-theoretical entropy in ergodic theory. However, although the notions and results in both theories are similar, the methods one uses to prove them are very different.

In ergodic theory there exists a rich and powerful entropy theory. To mention a few famous results: entropy is a complete invariant for Bernoulli systems [84]; a MDS is a Kolmogorov-system (K-system) if and only if each finite non-trivial partition has positive entropy, if and only if each non-trivial partition with two elements has positive entropy, if and only if it has completely positive entropy (i.e. each of its non-trivial factors has positive entropy [86]), if and only if it is disjoint from every zero entropy system [25]. A K-system is strongly mixing. Each MDS admits a maximal factor with zero entropy, namely the Pinsker factor, as well as a maximal Kronecker factor (i.e. a maximal factor with discrete spectrum). For sequence entropy it is known that an ergodic MDS is null (sequence entropy is zero, for every infinite sequence) if and only if it has discrete spectrum [79]; and that the supremum over all sequence entropies is $\log n$ for some positive integer $n \in \mathbb{N}$ or ∞. Moreover, when this number is $\log n$ then it is a finite to one extension of its Kronecker factor [65]. The classical variational principle states that the topological entropy of a TDS is equal to the supremum of metric entropy taken over all Borel probability invariant measures.

As we have already said, the analogous notion of topological entropy was introduced soon after the measure theoretical one, and was widely studied and applied. Notwithstanding, the level of development of topological entropy theory lagged behind. In recent years however this situation is rapidly changing. A turning point occurred with F. Blanchard pioneering papers [5], [6] in the 1990's.

In [5] Blanchard introduced the notions of completely positive entropy (c.p.e.) and uniformly positive entropy (u.p.e.) as topological analogues of the K-property in ergodic theory. In [6] he defines the notion of entropy pairs and uses it to show that a u.p.e. system is disjoint from all minimal zero entropy systems. The notion of entropy pairs can also be used to show the existence of the maximal zero entropy factor for any TDS, namely the topological Pinsker factor [8]. Later on, it was shown by Glasner and Weiss [39] that a topological system carrying a K-measure is a u.p.e. system. This was generalized in the work of Blanchard, Host, Maass, Martinez and Rudolph [9], where the authors define entropy pairs for an invariant measure and show that for each invariant measure the set of entropy pairs for this measure is contained in the set of entropy pairs. Blanchard, Glasner and Host [11] proved that the converse of this statement holds, namely, there is always an invariant measure such that the set of topological entropy pairs coincides with the set of entropy pairs for this measure. The main tool used in [11] was a new local variational inequality.

A characterization of the set of entropy pairs for an invariant measure as the support of a certain measure on $X \times X$ (which is defined by means of the measure theoretical Pinsker factor of the system) was obtained in Glasner [30]. Finally a proof of the other direction of the local variational principle was obtained by Glasner and Weiss in [43].

In order to facilitate the reading of the survey we divided it into four parts, preceded by a brief survey of classical entropy theory in Section 2.

The early developments of the local theory are described in **Part 1**. In Section 3 we define entropy pairs and u.p.e. systems. In Section 4 the properties of being u.p.e. and K are compared. Finally in Section 5 measure entropy pairs are defined and the local variational principle is described. For these topics see also the relevant chapters in Glasner's book [31] and the survey article [43] by Glasner and Weiss.

Finer properties of the theory ensued and these are described in **Part 2**. In order to gain a better understanding of the topological version of a K-system, Huang and Ye [59] introduced the notion of entropy n-tuples ($n \geq 2$) both in topological and measure-theoretical setting. For a TDS they define the corresponding notions of u.p.e. of order n, for all $n \geq 2$. The property of having u.p.e. of all orders is then called topological K. They show that u.p.e. of order 2 does not, in general, imply topological K.

A local variational relation was proved, and the relation between the set of topological entropy n-tuples and the set of entropy n-tuples for an invariant measure was established. Moreover, a characterization of positive entropy using the dynamical behavior in any neighborhood of an entropy n-tuple is obtained [59]. Recently, the notion of entropy set was introduced in Dou-Ye-Zhang [23] and Blanchard-Huang [7] in order to find where the entropy is concentrated. Particularly it was shown that any TDS with positive entropy has an entropy set with uncountably many points. By studying entropy points Ye and Zhang [103] show that for any TDS there is a compact countable subset such that its Bowen's entropy is equal to the entropy of the original system. Entropy points and entropy sets are defined and discussed in Section 6.

The complexity of a TDS is an important topic in topological dynamics. Building on the idea of entropy pairs, Blanchard, Host and Maass in [13] introduced complexity pairs in TDS and showed that for any TDS the maximal equicontinuous factor is induced by the smallest closed invariant equivalence relation containing the complexity pairs. Moreover, the complexity of a TDS along sequences of $\mathbb{N}$ can be used to describe dynamical properties such as mild mixing, scattering and strong scattering, Huang and Ye [57]. Together with results from Huang-Shao-Ye [54] and Huang-Ye [57], one can show that a minimal topological K-system is strongly mixing. For a related work see Huang-Maass-Ye [50]. Complexity is discussed in Section 7.

The ideas of the local theory of entropy can be used to study sequence entropy. In [48] Huang, Li, Shao and Ye defined sequence entropy pairs and null topological systems. They showed that a null minimal system is an almost 1-1 extension of its maximal equicontinuous factor (topological Kronecker factor), and moreover, that if the topological entropy of an open cover is positive then the sequence entropy along any sequence for the open cover is positive, Huang-Shao-Ye [54]. Recently, Huang and Ye showed [60] that for any TDS the supremum over all sequence entropies is $\log n$ for some $n \in \mathbb{N}$ or ∞. Maass and Shao [81] have shown that if, for a minimal TDS (X, T), this number is $\log n$ with $n \in \mathbb{N}$, then there are proximal extensions (X', T') and (Y', T') of (X, T) and (Y, S) (the maximal equicontinuous factor of (X, T)) such that there is a finite to one equicontinuous extension from (X', T') to (Y', S'), . For related research see [50], [57] and [54]. The works on sequence entropy are described in Sections 8 and 9.

Given a compact metric space X, a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C(X)$ is an ℓ_1-*sequence* if there are positive constants a and b such that

$$a\sum_{k=1}^{n}|c_k| \le \left\|\sum_{k=1}^{n} c_k f_k\right\| \le b\sum_{k=1}^{n}|c_k|$$

for all $n \in \mathbb{N}$ and $c_1, \dots, c_n \in \mathbb{R}$. In a classical work of Rosenthal [89] a notion of independence was introduced which gave rise to a dichotomy for bounded sequences $\{f_n\}_{n\in\mathbb{N}}$ in $C(X)$; they contain either a pointwise convergent sub-sequence, or a ℓ_1 sub-sequence. This dichotomy was later sharpened by Bourgain, Fremlin and Talagrand in [15]. Ideas of independence and ℓ_1 structure were introduced into dynamics in 1995 by Glasner and Weiss in [40]. First by using the local theory of Banach spaces in proving that if a compact topological $\mathbb{Z}$-system (X, T) has zero topological entropy then so does the induced system $(M(X), T_*)$ on the compact space of probability measures on X; and more directly in providing characterizations of positive entropy and of K-systems in terms of interpolation sets (which are the same as independence sets, see Section 11 below), [40] Theorems 3.1 and 3.2.

In the same year Köhler introduced the notion of regularity of a dynamical system, meaning that no sequence of the form $\{f \circ T^{n_i} : i \in \mathbb{N}\}$ is an ℓ_1 sequence [76]. We remind the reader that the *enveloping semigroup* $\mathcal{E}(X, T)$ of a TDS (X, T) is defined as the closure of $\{T, T^2, T^3, \dots\}$ in the compact space X^X. In a recent work of Glasner and Megrelishvili [37], following Köhler [76], a dynamical version of the Bourgain-Fremlin-Talagrand dichotomy was demonstrated. It asserts that the enveloping semigroup $\mathcal{E}(X, T)$ of a compact metric dynamical system is either very large and contains a topological copy of $\beta\mathbb{N}$, or the topology of $\mathcal{E}(X, T)$ is determined by the convergence of sequences. In the latter case we say that the dynamical system is tame [33]. In [37] hereditarily non-sensitive systems (HNS) were shown to be tame and in [38] Glasner, Megrelishvili and Uspenskij show that metric HNS systems are exactly those metric systems whose enveloping semigroup is metrizable. We refer the interested reader to the recent survey of enveloping semigroups in dynamics [35].

Combinatorics, of course, plays a major role in all aspects of entropy theory. It is mainly manifested in various interpolation or independence properties. These subjects are treated in **Part 3**. In Sections 10 and 11 we survey some of the results obtained by Glasner [33], [34], Huang [47], and Kerr-Li [74], concerning tame dynamical systems and various independence notions, including the basic combinatorial lemma of Sauer-Pereles-Shelah, several generalizations of this lemma, and their applications to local entropy.

Following ideas of Kamae and Zamboni [67], Huang and Ye develop in [55] a theory of maximal pattern entropy, which we describe in 12.

The local theory of entropy for a TDS, interesting in its own right, has also many applications. As mentioned above, it can be used to show the existence of a topological Pinsker factor, to prove disjointness theorems, to obtain characterizations of positive entropy, and so on. **Part 4** is dedicated to further applications of the local theory. In Section 13 we discuss Li-Yorke chaos after Blanchard-Glasner-Kolyada-Maass [12], the relation of asymptotic pairs after Blanchard-Host-Ruette [10], and the notions of μ-Pinsker factor and extreme relations according to Kaminski-Siemaszko-Szymanski [70]. A new application to interval maps is proved in Theorem 13.9.

In Section 14 we show how to apply the local theory to the study of sensitivity in dynamical systems. The relevant works here are [49, 94, 104].

The study of local relative entropy in TDS was started in the work of Glasner and Weiss [42]. For related works see Lemanczyk and Siemaszko [80], Park and Siemaszko [83], and the recent papers by Huang, Ye and Zhang [62], [105], [106] and [63]. The relative case is treated in Section 15.

To end the survey we mention in Section 16 some open questions.

We thank Francois Blanchard and Hanfeng Li for a careful reading of the paper and for their helpful suggestions.

2. Preliminaries

A topological dynamical system (X,T) is *transitive* if for every pair of open nonempty subsets U and V, $N(U,V) = \{n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \emptyset\}$ is infinite. (X,T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive. A point $x \in X$ is a *transitive point* if $\{Tx, T^2x, \ldots\}$ is dense in X. It is well known that when (X,T) is transitive the set of transitive points is a dense G_δ set (denoted by $Tran_T$) and when $Tran_T = X$ we say that (X,T) is *minimal.* The orbit of x, $orb(x,T)$, is the set $\{x, Tx, \ldots\}$. The ω-limit set of x, $\omega(x,T)$, is the set $\bigcap_{n\geq 0} \overline{(\{T^i x : i \geq n\})}$. A point $x \in X$ is called a *minimal point* if its orbit closure cls-orb(x,T) is a minimal sub-system.

A MDS $(X, \mathcal{B}, \mu, T)$ it is *ergodic* if every set $B \in \mathcal{B}$ with $T^{-1}B = B$ has measure 0 or 1. It is *weakly mixing* if $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$ is ergodic. It is *strongly mixing* if for every $A, B \in \mathcal{B}$ one has $\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$.

As in many other fields of mathematics, the classification problem is a main goal in the study of dynamical systems. Entropy is a conjugacy invariant both for TDS and MDS. Associated with every TDS (X, T) there is a non-empty collection of invariant probability measures $M_T(X)$ and with each $\mu \in M_T(X)$ one associates the MDS $(X, \mathcal{B}_X, \mu, T)$, where $\mathcal{B}_X$ is the Borel σ-algebra generated by the topology of X. The relationship between these two kinds of entropy has gained a lot of attention.

Measure-theoretical entropy

In order to distinguish two MDS which are spectrally isomorphic Kolmogorov introduced in 1958 an isomorphism invariant called entropy [77]. Let $(X, \mathcal{B}_X, \mu, T)$ be a measure theoretical dynamical system. A *measurable cover* (or just a cover) of X is a finite family of $\mathcal{B}_X$-measurable sets whose union is X. A *partition* of X is a cover of X by pairwise disjoint sets. Let $\mathcal{P}_X$ denote the collection of finite partitions of X.

For $\alpha \in \mathcal{P}_X$ define
$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A) \log \mu(A).$$
The *μ-measure theoretical entropy* of α with respect to T is
$$h_\mu(\alpha, T) = \lim_{N\to\infty} \frac{1}{N} H_\mu(\alpha_0^{N-1}) = \inf_{N\in\mathbb{N}} \frac{1}{N} H_\mu(\alpha_0^{N-1}).$$
The *measure theoretical entropy* of T with respect to μ is $h_\mu(T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(\alpha, T)$.

Topological entropy

In 1965 Adler, Konheim and McAndrew [1] defined topological entropy for TDS. A cover is said to be an open cover if it consists of open sets. Let $\mathcal{C}_X$ be the set of finite covers and $\mathcal{C}_X^o$ the set of finite open covers of X. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be *finer than* $\mathcal{V}$ ($\mathcal{U} \succeq \mathcal{V}$) if for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subseteq V$. Let
$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$
It is clear that $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$ and $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{V}$.

Given integers $M \le N$ and $\mathcal{U} \in \mathcal{C}_X$ we denote $\mathcal{U}_M^N = \bigvee_{n=M}^{N} T^{-n}\mathcal{U}$. Given $\mathcal{U} \in \mathcal{C}_X$, define $N(\mathcal{U})$ as the minimum cardinality of a subcovers of $\mathcal{U}$. The *combinatorial entropy of $\mathcal{U}$ with respect to T* is
$$h_c(\mathcal{U}, T) = \lim_{N\to\infty} \frac{1}{N} \log N(\mathcal{U}_0^{N-1}).$$
When $\mathcal{U}$ is an open cover this number is called the *topological entropy of $\mathcal{U}$ with respect to T* and is denoted by $h_{\text{top}}(\mathcal{U}, T)$. If (X, T) is a TDS, the *topological entropy of T* is
$$h_{\text{top}}(T) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{\text{top}}(\mathcal{U}, T).$$

In his 1971 paper [17] Bowen introduced his definition of topological entropy (see also Dinaburg [21]): Let (X, T) be a (not necessarily compact) metric TDS and $K \subset X$ a subset. For $n \in \mathbb{N}$ and $\epsilon > 0$ let $s_n(K, \epsilon)$ be the maximal cardinality among all (n, ϵ)-separating sets taken from K (with respect to the metric d on X). *Bowen's entropy of the set* K is then defined as

$$s(K, \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(K, \epsilon),$$

and

$$h_d(K) = \lim_{\epsilon\to 0} s(K.\epsilon).$$

Finally the d-entropy of (X, T) is defined as

$$h_d(T) = \sup\{h_d(K) : K \text{ is a compact subset of } X\}.$$

When X is compact we have $h_d(T) = h_{\mathrm{top}}(T)$.

The Shannon-McMillan-Breiman and Brin-Katok theorems

Let $(X, \mathcal{B}, \mu, T)$ be a MDS. The Shannon-McMillan-Breiman theorem asserts, given a finite measurable partition α, the convergence almost everywhere of the limit

$$h_\mu(\alpha, x) =: \lim_{N\to\infty} -\frac{1}{N} \log \mu(\alpha_0^{N-1}(x))$$

with $\int h_\mu(\alpha, x)\, d\mu = h_\mu(\alpha)$. When μ is T-ergodic we have $h_\mu(\alpha, x) = h_\mu(\alpha)$ almost everywhere (see e.g. [97]).

Brin and Katok provided a useful topological version of this famous theorem in [16].

Let (X, T) be a TDS and $\mu \in M_T(X)$. Fix a metric d on X and set, for $x \in X$, $\epsilon > 0$ and $n \geq 0$,

$$B_n(x, \epsilon) = \{y \in X : d(T^i x, T^i y) \leq \epsilon, 0 \leq i \leq n-1\},$$

$$h^+_\mu(x, \epsilon) = -\limsup_{n\to\infty} \frac{1}{n} \log \mu(B_n(x, \epsilon)),$$

$$h^-_\mu(x, \epsilon) = -\liminf_{n\to\infty} \frac{1}{n} \log \mu(B_n(x, \epsilon)).$$

Suppose that $h_\mu(T)$ is finite, then for μ-almost every x, (i) $\lim_{\epsilon\to 0} h^+_\mu(x, \epsilon) = \lim_{\epsilon\to 0} h^-_\mu(x, \epsilon) =: h_\mu(x)$, (ii) $h_\mu(x) = h_\mu(Tx)$, and (iii) $\int h_\mu(x)\, d\mu = h_\mu(T)$. If moreover μ is ergodic then $h_\mu(x) = h_\mu(T)$ almost everywhere.

K-systems

Once measure entropy is defined for MDS, two sub-classes immediately stand out: the class of zero entropy systems and the class of systems all of whose non-trivial factors have positive entropy. Members of the latter class are said to have *completely positive entropy* and it is well known that this property is equivalent to the Kolmogorov property. In fact for a MDS $(X, \mathcal{B}, \mu, T)$ the following properties are equivalent:

1. For each non-trivial partition $\mathbb{P}$ consisting of two elements $h_\mu(T, \mathbb{P}) > 0$.
2. For each non-trivial finite partition $\mathbb{P}$, $h_\mu(T, \mathbb{P}) > 0$.
3. Each non-trivial factor has positive entropy.
4. The MDS $(X, \mathcal{B}, \mu, T)$ is K; i.e. there exists a measurable partition ξ satisfying the following conditions:

(i) $T^{-1}\xi \leq \xi$.

(ii) $\bigvee_{n=0}^{\infty} T^n\xi = \epsilon$.
(iii) $\bigwedge_{n=0}^{\infty} T^{-n}\xi = \eta$.
Here ϵ is the point partition and η is the trivial partition $\{X\}$.

The variational principle

For a TDS (X,T) let $M_T(X)$ be the set of all Borel invariant probability measures. In 1969 Goodwyn [46] showed that $h_\mu(T) \le h_{\mathrm{top}}(T)$ for each $\mu \in M_T(X)$. In 1971 Goodman [44] proved $\sup_{\mu\in M_T(X)} h_\mu(T) \ge h_{\mathrm{top}}(T)$, completing the *classic variational principle*. For more details see e.g. [19].

Part 1. Entropy pairs, u.p.e. systems and the local variational principle

3. u.p.e. systems and entropy pairs

In the classical theory of topological dynamics one defines proximal pairs and regionally proximal pairs. These local notions can be used to describe global dynamical behavior such as equicontinuity and distality. At the beginning of the 1990th Blanchard began a search for a topological analogue of K-systems. First he tried to define topological K-systems by means of global notions; then he realized that a local viewpoint may become very useful.

We say that an open cover is *non-trivial* if the closure of each element of the cover is not equal to X. Let (X,T) and (Y,S) be TDS. A continuous surjective map $\pi : X \longrightarrow Y$ is a *factor map* if $\pi \circ T = S \circ \pi$. A TDS is *non-trivial* if it is not a singleton.

Definition 3.1. [5] Let (X,T) be a TDS. We say (X,T) has *uniformly positive entropy* (u.p.e.) if for each non-trivial open cover consisting of two elements $h_{\mathrm{top}}(T,\mathcal{U}) > 0$. We say that (X,T) has completely *positive entropy* (c.p.e.) if each non-trivial factor of (X,T) has positive entropy.

Although in the case of a MDS the two notions are equivalent, in the topological setup we only have the implication u.p.e. $\Rightarrow$ c.p.e. Examples of c.p,e. but not u.p.e. systems were already given by Blanchard in [5]. The long standing problem [5], [31] whether there exists a minimal c.p.e. system which is not u.p.e. was solved recently. In [95] the authors answer the question by constructing a minimal c.p.e. non u.p.e. system.

Moreover, whereas for a MDS — since c.p.e. is equivalent to the system being K — we have the implication c.p.e $\Rightarrow$ strong mixing, this is no longer the case for TDS. In this category we only have:

Theorem 3.2. [5] *Let (X,T) be a TDS.*

1. *If (X,T) has u.p.e. then it is weakly mixing.*
2. *If (X,T) has c.p.e. then there is an invariant measure μ with full support.*

The notion of *disjointness* of two TDS (MDS) was introduced by Furstenberg in [25]. If (X,T) and (Y,S) are two TDS we say that $J \subset X \times Y$ is a *joining* of X and Y if J is a non-empty closed invariant set which is projected onto X and Y. If each joining is equal to $X \times Y$ we then say that (X,T) and (Y,S) are *disjoint* and write $(X,T) \perp (Y,S)$ or $X \perp Y$. It was proved in [25] that if two TDS are disjoint then one of them, say X, is minimal, and in [58] that if moreover (X,T) is non-trivial then the set of recurrent points is dense in Y (for a related paper see [52]). Recall that x is a *recurrent point* if there is an increasing sequence $\{n_i\}$ in $\mathbb{N}$ with $\lim_{i\to\infty} T^{n_i}x = x$. For two MDS $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\nu,S)$, we say that they are *disjoint* if every invariant probability measure on $X\times Y$ projecting to μ and ν is equal to $\mu\times\nu$. It is known that a MDS is a K-system if and only if it is disjoint from all zero entropy systems [25]. Now the question arises: is a u.p.e TDS disjoint from all minimal systems with zero entropy? As a tool for treating this question Blanchard introduced the notion of topological entropy pairs.

Definition 3.3. Let (X,T) be a TDS. A pair $(x_1,x_2)\in X^2$ is an *entropy pair* if $x_1\neq x_2$ and for every disjoint closed neighborhoods U_1,U_2 of x_1 and x_2 respectively, $h_{\rm top}(T,\{U_1^c,U_2^c\})>0$.

The set of entropy pairs is denoted by $E(X,T)$. Entropy pairs enjoy the following properties.

Theorem 3.4. [6] *For a TDS (X,T),*

1. $h_{top}(T)>0$ *if and only if* $E(X,T)\neq\emptyset$.
2. *If* $\pi:(X,T)\longrightarrow(Y,S)$ *is a factor map then* $(\pi(x),\pi(x'))\in E(Y,S)$ *if* $(x,x')\in E(X,T)$ *and* $\pi(x)\neq\pi(x')$.
3. *If* $\pi:(X,T)\longrightarrow(Y,S)$ *is a factor map and* $(y,y')\in E(Y,S)$ *then there is* $(x,x')\in E(X,T)$ *with* $\pi(x)=y$ *and* $\pi(x')=y'$.
4. *If* (W,T) *is a subsystem of* (X,T) *then* $E(W,T)\subset E(X,T)$.

As we can see in the following theorem, the answer to the above question is affirmative; moreover the theory of entropy pairs leads to the existence of a naturally defined maximal factor of zero entropy, a topological analogue of the *Pinsker factor* of a MDS.

Theorem 3.5. *The following statements hold.*

1. [6] *A u.p.e. system is disjoint from every minimal system with zero entropy.*
2. [8] *Let (X,T) be a TDS. The factor map $\pi:(X,T)\to(X/\Pi,T)$, where $\Pi\subset X\times X$ is the smallest closed $T\times T$-invariant equivalence relation containing $E(X,T)$, is the maximal zero entropy factor.*

Definition 3.6. The corresponding TDS $(X/\Pi,T)$ is called the *topological Pinsker factor* of (X,T).

In [8] the authors constructed TDS (X,T) with zero entropy and a u.p.e. (Y,S) such that the maximal zero entropy factor of $(X\times Y,T\times S)$ is not equal to (X,T). We note that the existence an almost equicontinuous scattering system (X,T) established in [3] (see also [55]), allows us to construct similar *transitive* examples. For instance, if (X,T) is such a system and (Y,S) is any minimal u.p.e. system then the maximal Pinsker factor of the transitive product system $(X\times Y,T\times S)$ is not equal to (X,T). To see this fact note that (X,T) has zero topological entropy and that ${\rm supp}\,(X,T) =: {\rm cls}\left(\bigcup_{\mu\in M_T(X)}{\rm supp}\,(\mu)\right)\neq X$ [3]. By [30] $E(X\times Y,T\times S)=\{((x,y),(x,y')) : y\neq y'\in Y, x\in {\rm supp}\,(X,T)\}$, and clearly its closure $R = \{((x, y), (x, y')) : y, y' \in Y, x \in \mathrm{supp}\,(X, T)\}$ is a closed invariant equivalence relation. Thus $X \times Y/R$ is the Pinsker factor of $X \times Y$.

4. u.p.e. versus K

The new theory launched by Blanchard had then two candidates for topological K-systems, the c.p.e and the u.p.e. systems. However, except for the canonical Bernoulli systems and a few variations, there were very few examples. In particular in [6] Blanchard asked whether there are minimal u.p.e. systems.

In [39] Glasner and Weiss provided an affirmative answer:

Theorem 4.1. *Let (X, T) be a TDS. Suppose that there exists a T-invariant probability measure μ on X with* $\mathrm{supp}\,(\mu) = X$*, such that the corresponding MDS $(X, \mathcal{B}_X, \mu, T)$ is a K-system; then (X, T) is u.p.e.*

Combining this result with the representation theorem of Jewett and Krieger, we immediately get a wealth of examples of minimal uniquely ergodic u.p.e. dynamical systems. Another corollary of Theorem 4.1, is obtained when one recalls the construction in [98], of a universal minimal TDS (X, T) with the property that for every measure theoretical ergodic MDS (Ω, m, T), there exists an invariant probability measure μ on X such that (X, μ, T) is measure theoretically isomorphic to (Ω, m, T). By Theorem 4.1, (X, T) is u.p.e. and it therefore serves as a minimal u.p.e. model for *every* ergodic MDS.

On the other hand, as one would expect, the topological property u.p.e. has very little to say about the measure theoretical behavior of invariant measures. Using theorems of Weiss [99] and Furstenberg and Weiss [27], the authors of [39] prove the following:

Theorem 4.2. *Given an arbitrary ergodic MDS (Ω, m, T) of positive entropy, there exists a minimal, uniquely ergodic, uniform positive entropy TDS (X, T) with invariant measure μ, such that the MDS (Ω, m, T) and (X, μ, T) are measure theoretically isomorphic.*

Corollary 4.3. *An ergodic dynamical system has a strictly ergodic, u.p.e. model iff it has positive entropy.*

The main difficulty in proving Theorem 4.1 lies in the way covers differ combinatorially from partitions. In fact, as we shall see, these kind of combinatorial differences will, from now on, dominate the whole subject.

Finally we have the following characterizations of positive entropy and completely positive entropy (namely the K property) of a dynamical system in [40].

For $a \geq 2$ let $\Omega_a = \{0, 1, \dots, a-1\}^{\mathbb{Z}}$ and $Y \subset \Omega_a$. A subset $I \subset \mathbb{Z}$ is called an *interpolating set for* Y if $Y|I = \Omega_a|I$. More concretely, for each choice $\{b_i : i \in I, b_i \in \{0, 1, \dots, a-1\}\}$ there is some $\omega \in Y$ such that $\omega_i = b_i$ for all $i \in I$. Now suppose that $(X, \mathcal{B}, \mu, T)$ is a MDS and that $\mathcal{P} = \{P_0, P_1, \dots, P_{a-1}\}$ is a finite measurable partition of X. Construct a set $Y_{\mathcal{P}} \subset \Omega_a$ as follows:

$$Y_{\mathcal{P}} = \{\omega \in \Omega_a : \text{for all finite subsets } J \subset \mathbb{Z},\ \mu(\bigcap_{j \in J} T^{-j} P_{\omega_j}) > 0\}.$$

If $\mu_{\mathcal{P}}$ is the image of μ under the mapping $\theta : X \to \Omega_a$ defined by $(\theta x)_n =$ that index b such that $T^n x \in P_b$, i.e. the distribution of the stochastic process defined by $(T, \mu, \mathcal{P})$, then $Y_{\mathcal{P}}$ is simply the closed support of $\mu_{\mathcal{P}}$. Recall that a set $I \subset \mathbb{Z}$ has *positive density* if

$$\lim_{n\to\infty} \frac{|I \cap \{-n, \ldots, n\}|}{2n+1} > 0.$$

Theorem 4.4. [40] *If $\mathcal{P}$ has two elements and $h(T, \mathcal{P}) > 0$ then $Y_{\mathcal{P}}$ has interpolating sets of positive density.*

Theorem 4.5. [40] *The MDS $(X, \mathcal{B}, \mu, T)$ is K if and only if for every non trivial partition $\mathcal{P}$, the set $Y_{\mathcal{P}}$ has interpolating sets of positive density.*

As a corollary we get:

Corollary 4.6. *The MDS $(X, \mathcal{B}, \mu, T)$ is K if and only if for every non trivial measurable partition $\mathcal{P}$ of X, the TDS $Y_{\mathcal{P}}$ is u.p.e.*

5. Measure entropy pairs and local variational principles

Measure entropy pairs

Let (X, T) be a TDS and $\mu \in M_T(X)$. May one define entropy pairs with respect to μ? This was done by Blanchard, Host, Maass, Martinez and Rudolph in [9]. A partition is *non-trivial* if no element of the partition has measure 0 or 1.

Definition 5.1. Let (X, T) be a TDS and $\mu \in M_T(X)$. A pair $(x_1, x_2) \in X^2$ with $x_1 \neq x_2$ is a *μ-entropy pair* if for each non-trivial partition $\mathbb{P}$ with two elements P_1, P_2 such that x_1, x_2 are interior points of P_1, P_2 respectively, the entropy $h_\mu(T, \mathbb{P}) > 0$.

The set of entropy pairs with respect to μ is denoted by $E^\mu(X, T)$. The sets $E(X, T)$ and $E^\mu(X, T)$ are related by:

Theorem 5.2. *Let (X, T) be a TDS. Then*

1. [9] *$E^\mu(X, T) \subset E(X, T)$ for each $\mu \in M_T(X)$.*
2. [11] *There is $\mu \in M_T(X)$ with $E^\mu(X, T) = E(X, T)$.*

We remark that $E^\mu(X, T)$ has properties similar to Theorem 3.4. One can now define the topological μ-Pinsker factor as follows:

Definition 5.3. Let (X, T) be a TDS and $\mu \in M_T(X)$. The factor map $\pi : (X, T) \to (X/\Pi_\mu, T)$, where $\Pi_\mu \subset X \times X$ is the smallest closed $T \times T$-invariant equivalence relation containing $E^\mu(X, T)$, is the maximal zero μ-entropy factor. The corresponding TDS $(X/\Pi_\mu, T)$ is called the *topological μ-Pinsker factor* of (X, T).

Note that Theorem 4.1 is a direct corollary of part 1 of Theorem 5.2. In fact this result from [39] was the motivation for the definition of measure entropy pairs in [9].

Local variational principles

A key point in the proof of the second part of Theorem 5.2 is the following, so called, local variational principles.

Theorem 5.4. *For a given TDS (X,T) and an open cover $\mathcal{U}$*

1. *(Blanchard, Glasner and Host* [11]*)*
$$\sup_{\mu\in M_T(X)} \inf_{\mathbb{P}\succeq\mathcal{U}} h_\mu(T,\mathbb{P}) \geq h_{top}(T,\mathcal{U})$$
2. *(Glasner and Weiss* [43]*)*
$$\sup_{\mu\in M_T(X)} \inf_{\mathbb{P}\succeq\mathcal{U}} h_\mu(T,\mathbb{P}) = h_{top}(T,\mathcal{U})$$
3. *(Huang and Ye* [59]*) If $\mathcal{U}$ is given, then for any $\mu \in M_T(X)$,*
$$h_\mu(T,\mathbb{P}) > 0 \text{ for each } \mathbb{P}\succeq\mathcal{U} \text{ implies } \inf_{\mathbb{P}\succeq\mathcal{U}} h_\mu(T,\mathbb{P}) > 0,$$
where $\mathbb{P}$ is a finite Borel partition. In turn this implies $h_{top}(T,\mathcal{U}) > 0$,

Remark 5.5. In [88] Romangnoli introduced two kinds of measure theoretical entropy for covers, namely

$$h^+_\mu(T,\mathcal{U}) = \inf_{\alpha\succeq\mathcal{U}} h_\mu(T,\alpha) \text{ and } h^-_\mu(T,\mathcal{U}) = \lim_{n\to\infty}\frac{1}{n}\inf_{\alpha\succeq\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U}} H_\mu(\alpha),$$

where α is a finite Borel partition. He showed that for a TDS (X,T) there is $\mu \in M_T(X)$ with $h^-_\mu(T,\mathcal{U}) = h_{\text{top}}(T,\mathcal{U})$. As $h^-_\mu(T,\mathcal{U}) \leq h_{\text{top}}(T,\mathcal{U})$ for each $\mu \in M_T(X)$ we have

$$\max_\mu h^-_\mu(T,\mathcal{U}) = h_{\text{top}}(T,\mathcal{U}). \tag{5.1}$$

The result proved by Glasner and Weiss [43] can be stated as

$$\max_\mu h^+_\mu(T,\mathcal{U}) = h_{\text{top}}(T,\mathcal{U}). \tag{5.2}$$

In [51] the authors showed that if there are a TDS (X,T) and a $\mu \in M_T(X)$ with $h^+_\mu(T,\mathcal{U}) > h^-_\mu(T,\mathcal{U})$ then there is a uniquely ergodic TDS with the same property. Thus, combining (5.1) and (5.2) we conclude that

$$h^+_\mu(T,\mathcal{U}) = h^-_\mu(T,\mathcal{U}).$$

Remark 5.6. In a recent work Shapira [92] shows that for a TDS (X,T) and a measure $\mu \in M_T(X)$, given a finite open cover $\mathcal{U}$ and $\epsilon > 0$,

$$h^+_\mu(\mathcal{U}) = \lim \frac{1}{n}\log \mathcal{N}(\mathcal{U}_0^{n-1},\epsilon).$$

Here $\mathcal{N}(\mathcal{U},\epsilon)$ is the minimal number of elements of $\mathcal{U}$ needed to cover X up to μ-measure $1-\epsilon$. This effective way of computing h^+_μ enables him to provide elegant new proofs to most of the results mentioned above.

We mention here two related recent works. A local variational principle for pressure and applications to equilibrium states are provided by Huang and Yi [61]; and a local variational principle for general countable amenable group actions is to be found in a forthcoming work by Huang, Ye and Zhang [64].

A characterization of μ-entropy pairs

Let $\pi : (X, \mu, T) \longrightarrow (Z, \nu, S)$ be the measure theoretical Pinsker factor of (X, μ, T) and let $\mu = \int_Z \mu_z d\nu(z)$ be the disintegration of μ over (Z, ν). Set

$$\lambda = \int_Z \mu_z \times \mu_z d\nu(z)$$

the independent product of μ with itself over ν. Finally, let $\Lambda_\mu = \mathrm{supp}\,(\lambda)$ be the topological support of λ. In [30] Glasner provides a simple characterization of $E^\mu(X, T)$:

Theorem 5.7. *Let (X, T) be a TDS.*

1. *For a $\mu \in M_T(X)$ of positive entropy $E^\mu(X, T) = \Lambda_\mu \setminus \Delta$.*
2. *When moreover μ is ergodic* $\mathrm{cls}\,(E^\mu(X, T)) = \Lambda_\mu$.
3. *If $h_{top}(T) > 0$, there is $\mu \in M_T(X)$ with $E(X, T) = \Lambda_\mu \setminus \Delta$.*
4. $\mathrm{cls}\,(E(X, T)) = \mathrm{cls}\,(\cup\{\Lambda_\mu : \mu \in M_T^e(X)\})$, *where $M_T^e(X)$ is the collection of ergodic T-invariant ergodic probability measures on X.*

An important corollary of this result is the following:

Theorem 5.8. [30] *The product of two u.p.e. TDS is u.p.e.*

Entropy n-tuples

Once entropy pairs are defined it is natural to inquire what should be a successful definition of entropy n-tuples? Now whereas the Definition 3.3 of topological entropy pairs is easily extended to n-tuples, one finds that the corresponding generalization of the definition of measure entropy pairs (Definition 5.1) is troublesome. To overcome this difficulty one can use Theorem 5.7 and define $E_n^\mu(X, T)$ as the support of

$$\lambda_n = \int_Z \mu_z \times \ldots \times \mu_z d\nu(z).$$

In fact this was the way entropy tuples were defined in a preliminary version of [59]. After awhile though the authors realized that a more direct definition is available.

Definition 5.9. Let (X, T) be a TDS.

1. A cover $\{U_1, \ldots, U_k\}$ of X is *admissible* with respect to $(x_i)_1^n$ if for each $1 \le j \le k$ there is i_j such that $x_{i_j} \notin \overline{U_j}$.
2. An n-tuple $(x_i)_1^n \in X^{(n)}$ is a *topological entropy n-tuple* if at least two of the points $\{x_i\}_{i=1}^n$ are distinct and for any admissible open cover $\mathcal{U}$ with respect to $(x_i)_1^n \in X^{(n)}$ one has $h_{\mathrm{top}}(T, \mathcal{U}) > 0$.
3. $(x_i)_1^n$ is an *entropy n-tuple for μ*, if at least two of the points $\{x_i\}_{i=1}^n$ are distinct and for any admissible partition $\mathbb{P}$ with respect to $(x_i)_1^n$ we have $h_\mu(T, \mathbb{P}) > 0$.

The set of entropy n-tuples is denoted by $E_n(X, T)$, and the set of all entropy n-tuples for μ is denoted by $E_n^\mu(X, T)$. Note that $E_2(X, T) = E(X, T)$ and $E_2^\mu(X, T) = E^\mu(X, T)$. Two TDS are *weakly disjoint* if the product system is transitive. A TDS is *mildly mixing* if the product with any transitive system is transitive (see [57] and [35]). By using Theorem 5.4 one can show [59]:

Theorem 5.10. *Let (X, T) be a TDS. Then*

1. *$E_n^\mu(X, T) \subset E_n(X, T)$ for each μ and each $n \ge 2$.*

2. *There is* μ *with* $E_n^\mu(X,T) = E_n(X,T)$ *for each* $n \geq 2$.
3. *A u.p.e. system is mildly mixing.*

Definition 5.11 ([59])**.** We say that a TDS is *u.p.e. of order* n ($n \geq 2$) if $E_n(X,T) \cup \Delta_n = X^n$, and that it is *topologically* K if it is u.p.e. of order n for every $n \geq 2$.

It now follows that when a TDS (X,T) admits an invariant probability measure μ of full support such that the corresponding MDS (X,μ,T) is K, then it is topologically K. Another case where one can deduce the topological K property is when the TDS is leo. A non-invertible TDS (X,T) is *locally eventually onto* (leo) if for every non-empty open subset there is $n \in \mathbb{N}$ with $T^nU = X$ (see [20] and [29]).

Theorem 5.12. *A leo TDS is topologically K; i.e. it is u.p.e. of all orders.*

Proof. Note that for subsets U and V of X there is $n \in \mathbb{N}$ such that $U \cap T^{-n}V \neq \emptyset$ if and only if $T^nU \cap V \neq \emptyset$. By the leo assumption for every l-tuple of non-empty open subsets $U_1, \ldots, U_l$ of X, there is $n \in \mathbb{N}$ with $T^n(U_i) = X$ for each $i = 1, 2, \ldots, l$. Thus for each $k \in \mathbb{N}$ and each $s \in \{1, 2, \ldots, n\}^k$,

$$T^{-n}U_{s(1)} \cap T^{-2n}U_{s(2)} \cap \ldots \cap T^{-kn}U_{s(k)} \neq \emptyset.$$

This implies that (X,T) is u.p.e. of all orders by Theorem 5.13. □

Note that the property of having u.p.e. of order n is preserved under finite products (see [30] and [59]). Also note that there exists a TDS which is u.p.e. of order 2 but is not u.p.e. of order 3 [59].

To end this section we state a characterization of entropy tuples obtained in [59] which also yields a nice interpretation of positive entropy.

Theorem 5.13. *Let* (X,T) *be a TDS and* $n \geq 2$. *Then* $(x_1, \ldots, x_n) \in E_n(X,T)$ *if and only if for every choice of neighborhoods* U_i *of* x_i *there is a subset* $D = \{d_1, d_2, \ldots\}$ *of* $\mathbb{N}$ *with positive density such that* $\bigcap_{i=1}^{\infty} T^{-d_i}U_{s(i)} \neq \emptyset$ *for each* $s = (s(1), s(2), \ldots) \in \{1, \ldots, n\}^D$.

Note that Theorem 5.13 is also valid for non-invertible TDS.

Part 2. Finer structures

6. Entropy sets and entropy points

In this section we investigate the question: where is the entropy concentrated? There are two ways to approach this question. The first is by use of entropy tuples; the second is via Bowen's definition of entropy (see Section 2 above).

Along the first way the notion of entropy set was introduced in [23] by Dou, Ye and Zhang. A subset K with $card(K) \geq 2$ is an *entropy set* if for each $n \geq 2$, every n-tuple of pairwise distinct points from the set K is an entropy n-tuple. Then we can talk about *maximal entropy set* (under inclusion). It is easy to show that a maximal entropy set is closed since the closure of an entropy set is again an entropy set. Moreover, we have:

Theorem 6.1. [23] *Let* (X,T) *be a TDS.*

1. *If $h_{top}(T) > 0$ then there is a maximal entropy set with uncountably many points.*
2. *The topological entropy of T is the supremum of the Bowen entropies over all maximal entropy sets.*

We remark that there is an example for which there is a maximal entropy set with just two points.

Via the second way, namely by using Bowen's definition of entropy, the notion of entropy points was introduced in [103] by Ye and Zhang. For $x \in X$ let

$$h(x) = \lim_{\epsilon \to 0^+} \inf\{s(K, \epsilon) : \text{K is a closed neighborhood of } x\}.$$

It can be shown by using a theorem of Katok [72] that if $x \in \mathrm{supp}\,(\mu)$, for an ergodic measure μ then

$$h(x) \geq h_\mu(T)$$

which, in turn, implies

$$\sup_{x \in X} h(x) = h_{\mathrm{top}}(T).$$

Using this fact we have:

Theorem 6.2. *Let (X, T) be a TDS. There is a countable closed subset whose Bowen entropy is equal to the entropy of T. Moreover, this subset can be chosen so that its limit set has at most one limit point.*

Almost at the same time Blanchard and Huang [7] also defined and studied entropy sets. Their definition of entropy set is the same as in [23] except that they require an entropy set to be closed. Denote by $E_s(X, T)$ the collection of all entropy sets and by $E_s^\mu(X, T)$ the collection of all μ-entropy sets of (X, T). Let $(2^X, d_H)$ be the hyperspace of (X, d), i.e., the collection of all nonempty closed subsets of X equipped with the Hausdorff metric d_H. T induces a homeomorphism $\widehat{T}$ of 2^X, where $\widehat{T}(A) = T(A) = \{Ta : a \in A\}$ for $A \in 2^X$. The sets $E_s(X, T)$ and $E_s^\mu(X, T)$ are $\widehat{T}$-invariant, possibly empty, subsets of 2^X. Define

$$H(X, T) = \overline{E_s(X, T)} \text{ and } H^\mu(X, T) = \overline{E_s^\mu(X, T)}.$$

An *entropy point* is a point $x \in X$ such $\{x\} \in H(X, T)$; it is a *μ-entropy point* if $\{x\} \in H^\mu(X, T)$. The sets of entropy points and of μ-entropy points of (X, T) are denoted by $E_1(X, T)$ and $E_1^\mu(X, T)$ respectively. The authors of [7] prove the following results:

Theorem 6.3. *Let (X, T) be a TDS.*

1. *If $\mu \in \mathcal{M}(X, T)$, then $E_s^\mu(X, T) \subseteq E_s(X, T)$, $H^\mu(X, T) \subseteq H(X, T)$ and $E_1^\mu(X, T) \subseteq E_1(X, T)$.*
2. *There is $\mu \in \mathcal{M}(X, T)$ such that $E_s^\mu(X, T) = E_s(X, T)$, $H^\mu(X, T) = H(X, T)$ and $E_1^\mu(X, T) = E_1(X, T)$.*

Theorem 6.4. *Let (X, T) be a TDS.*

1. *If $\mu \in \mathcal{M}(X, T)$ with $h_\mu(T) > 0$, then $(H^\mu(X, T), \widehat{T})$ has infinite topological entropy and admits an invariant measure with full support.*

2. *If $h_{top}(T) > 0$, then $(H(X,T), \widehat{T})$ has infinite topological entropy and admits an invariant measure with full support.*

7. Complexity pairs and complexity pairs along a sequence

From the definition of the topological entropy of a finite open cover as the limit

$$h_{\text{top}}(\mathcal{U}, T) = \lim_{N\to\infty} \frac{1}{N} \log N(\mathcal{U}_0^{N-1})$$

it follows immediately that this entropy is zero if and only if the growth of the sequence $p_n(\mathcal{U}) = N(\mathcal{U})_0^{n-1}$ is sub-exponential. Can the study of the *complexity function* $n \mapsto p_n(\mathcal{U})$ be helpful in the study of zero-entropy systems? This question was raised by Blanchard, Host and Maass in [13], where the following notion was introduced.

Definition 7.1. [13] A TDS (X,T) is *scattering* if for each non-trivial open cover $\mathcal{U}$,

$$N(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U})) \longrightarrow \infty.$$

It is 2-scattering if for each non-trivial open cover $\mathcal{U}$ consisting of 2-elements, $N(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U})) \longrightarrow \infty$.

The corresponding notion of *complexity pairs* is defined and the collection of complexity pairs is denoted by $Com(X,T)$. The question, left open in [13], whether 2-scattering implies scattering, was later answered in the affirmative in [57].

Theorem 7.2. *2-scattering implies scattering.*

We say that two TDS are weakly disjoint if the product system is transitive. The following results are obtained in [13].

Theorem 7.3.
1. *A TDS is scattering if and only if it is weakly disjoint from all minimal systems.*
2. *A TDS is equicontinuous if and only if for any given $\mathcal{U}$, there is a constant C such that $N(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U})) < C$ for any n.*
3. *Let (X,T) be a TDS. Then the factor defined by the smallest closed invariant equivalence relation containing $Com(X,T)$ is the maximal equicontinuous factor.*

The first part of Theorem 7.3 suggests the following question. Can one characterize weak disjointness from various sub-classes of the class of all transitive systems in terms of complexity properties? These classes were defined and studied in [41], [100], [3] and [55]. We refer the reader to the review article [35] for further information. Now in [57] the authors show that some such characterizations can be achieved by introducing complexity functions along a give sequence.

Definition 7.4. Let $A = \{a_1, a_2, \ldots\}$ be a sequence. A TDS is *A-scattering* if for each non-trivial open cover $\mathcal{U}$, $N(\bigvee_{i=0}^{n-1} T^{-a_i}(\mathcal{U})) \longrightarrow \infty$. Let

$$\mathcal{C}_A(\mathcal{U}) = \lim_{n\to\infty} N(\bigvee_{i=0}^{n-1} T^{-a_i}(\mathcal{U})).$$

As usual, for a TDS and a pair of non-empty open sets U,V let

$$N(U,V) = \{n \in Z_+ : U \cap T^{-n}V\} \neq \emptyset.$$

Theorem 7.5. *Let (X,T) be a TDS. Then the following statements are equivalent:*

1. *(X,T) is mildly mixing, i.e. (X,T) is weakly disjoint from all transitive systems.*
2. *For any standard cover $\mathcal{U}$ of X and any IP-set A, $\mathcal{C}_A(\mathcal{U}) = +\infty$.*
3. *For any non-trivial finite open cover $\mathcal{U}$ of X and IP-set A, $\mathcal{C}_A(\mathcal{U}) = +\infty$.*

A subset S of $\mathbb{Z}_+$ is an *IP-set* if there is a sequence $\{p_i\}$ such that

$$S = \{\sum_{i=1}^{n} p_{i_1} + \ldots + p_{i_n} : i_1 < \ldots < i_n, n \in \mathbb{N}\}$$

A subset is an *IP^** set if it has a non-empty intersection with every IP-set.

Let S be a subset of $\mathbb{Z}_+$. The *upper Banach density of S* is

$$BD^*(S) = \limsup_{|I|\longrightarrow+\infty} \frac{|S\cap I|}{|I|},$$

where I ranges over intervals of $\mathbb{Z}_+$. The *upper density* of S is

$$D^*(S) = \limsup_{n\longrightarrow+\infty} \frac{|S\cap[1,n]|}{n}.$$

We say that a TDS is *strongly scattering* if it is weakly disjoint from every *E-system*, i.e. a transitive system with an invariant measure of full support. The following results are proved in [57].

Theorem 7.6. *Let (X,T) be a TDS. Then the following statements are equivalent:*

1. *(X,T) is strongly scattering.*
2. *For any standard cover $\mathcal{U}$ of X and positive upper density (or positive upper Banach density) subset A, $\mathcal{C}_A(\mathcal{U}) = +\infty$.*
3. *For any non-trivial finite open cover $\mathcal{U}$ of X and positive upper density (or positive upper Banach density) subset A, $\mathcal{C}_A(\mathcal{U}) = +\infty$.*

A subset A of $\mathbb{Z}_+$ is *syndetic* if it has bounded gaps; a subset A of $\mathbb{Z}_+$ is *thick* if it contains arbitrarily long intervals of natural numbers; and a subset A of $\mathbb{Z}_+$ is *piece-wise syndetic* if it is the intersection of a syndetic set with a thick set.

Theorem 7.7. *Let (X,T) be a TDS. Then the following statements are equivalent:*

1. *(X,T) is scattering.*
2. *For any standard cover $\mathcal{U}$ of X and syndetic (or piecewise syndetic) subset A, $\mathcal{C}_A(\mathcal{U}) = +\infty$.*
3. *For any non-trivial finite open cover $\mathcal{U}$ of X and syndetic (or piecewise syndetic) subset A, $\mathcal{C}_A(\mathcal{U}) = +\infty$.*

For a minimal system we have:

Theorem 7.8. *Let (X,T) be a minimal TDS. Then*

1. *(X,T) is weakly mixing if and only if for each pair of non-empty open sets A,B, the lower Banach density of $N(A,B)$ is 1.*
2. *(X,T) is mildly mixing if and only if for each pair of non-empty open sets A,B, $N(A,B)$ is an IP^* set.*

We remark that the corresponding results for MDS were recently obtained by Kuang and Ye [78].

8. Sequence entropy pairs and null systems

It was shown in [5] that u.p.e. implies weak mixing, and that a weakly mixing TDS is scattering [13]. In [13] Blanchard, Host and Maass raised the following question: can we characterize weak mixing in terms of entropy properties?

The answer is yes. To do this we need to introduce the notion of *sequence entropy*. We note that the notion of sequence entropy for a MDS was first introduced by Kushnirenko [79] and that the corresponding definition for TDS was first introduced by Goodman [45].

Let (X,T) be a TDS and $A=\{a_0<a_1<\ldots\}$ be a sequence of integers. Given an open cover $\mathcal{U}$ define

$$h^A_{top}(T,\mathcal{U})=\limsup_{n\longrightarrow\infty}\frac{1}{n}N(\bigvee_{i=0}^{n-1}T^{-a_i}(\mathcal{U}))$$

The *topological entropy along the sequence* A is then defined by

$$h^A_{top}(T)=\sup_{\mathcal{U}\text{ is an open cover}}h^A_{top}(T,\mathcal{U}).$$

Similarly one can define sequence entropy of a MDS. Following the idea of entropy pairs we have in [48]:

Definition 8.1. Let (X,T) be a TDS. A pair $(x_1,x_2)\in X^2$ is a *sequence entropy pair* if $x_1\neq x_2$ and for every pair of disjoint closed neighborhoods U_1,U_2 of x_1 and x_2 respectively, there is a sequence A such that $h^A_{top}(T,\{U_1^c,U_2^c\})>0$. The set of all sequence pairs is denoted by $SE(X,T)$.

Petersen [85] has shown that a TDS is weakly mixing if and only if for every pair of non-empty open sets U,V, $N(U\times U,U\times V)\neq\emptyset$. Motivated by this result we introduce the following:

Definition 8.2. Let (X,T) be a TDS. A pair $(x_1,x_2)\in X^2$ is a *weak mixing pair* if $x_1\neq x_2$ and for every pair of disjoint closed neighborhood U_1,U_2 of x_1 and x_2 respectively,

$$N(U_1^c\times U_1^c,U_1^c\times U_2^c)\neq\emptyset.$$

The set of weak mixing pairs is denoted by $WM(X,T)$.

It is clear that $E(X,T)\subset SE(X,T)\subset WM(X,T)$ and that $SE(X,T)\subset Com(X,T)$. Huang, Li, Shao and Ye [48] proved:

Theorem 8.3. *For a TDS the following statements are equivalent.*

1. (X, T) *is weakly mixing.*
2. $SE(X, T) = X^2 \setminus \Delta_2$*, where* $\Delta_2 = \{(x, x) : x \in X\}$
3. $WM(X, T) = X^2 \setminus \Delta_2$.

Following Kushnirenko [79] and motivated by Theorem 8.3 we define null systems as follows (see also [45]):

Definition 8.4. A TDS is *null* if for any sequence A, $h^A_{\text{top}}(T) = 0$. That is, $SE(X, T) = \emptyset$.

Kushnirenko in [79] defined null MDS and has shown that an ergodic MDS is null if and only if it has discrete spectrum. It turns out that this is not true for TDS even under the minimality assumption, see [45]. However, we have the following:

Theorem 8.5. [48] *If a minimal TDS is null then it is an almost one to one extension of an equicontinuous system.*

In [54] Huang, Shao and Ye studied the question how to characterize mixing properties via sequence entropy, and the question under which entropy conditions a minimal system is strongly mixing. One motivation for asking these questions is the fact that for MDS, a K-system is strongly mixing.

We say that a TDS is *topological K* if it is u.p.e. of order n for each $n \geq 2$. There exists an example which is topological K but not strongly mixing [5]. However, for minimal systems we have:

Theorem 8.6. [54] *A minimal topological K system is strongly mixing.*

The key point in the proof of the above theorem is:

Theorem 8.7. *Assume that* $h_{top}(T, \mathcal{U}) > 0$*. Then then the sequence entropy of* $\mathcal{U}$ *with respect to any infinite sequence* A *is positive.*

Though the same result holds for MDS and the proof is easy, the proof of Theorem 8.7 is rather involved.

9. Sequence entropy and complexity pairs for a measure

In [50] Huang, Maass and Ye considered the question: how to define the notion of sequence entropy and complexity pairs with respect to a measure on a TDS. The definition is similar to that of entropy pairs for a measure.

Definition 9.1. Let (X, T) be a TDS and $\mu \in M_T(X)$. An n-tuple $(x_i)_{i=1}^n \in X^{(n)}$, $n \geq 2$, is called

1. a *sequence entropy n-tuple* if at least two points in $(x_i)_{i=1}^n$ are different and for any admissible open cover $\mathcal{U}$ with respect to $(x_i)_{i=1}^n$ there exists an increasing sequence $A \subset \mathbb{N}$ such that $h^A_{\text{top}}(T, \mathcal{U}) > 0$;
2. a *sequence entropy n-tuple for μ*, if at least two points in $(x_i)_{i=1}^n$ are different and for any admissible Borel partition α with respect to $(x_i)_{i=1}^n$ there exists an increasing sequence $A \subset \mathbb{N}$ such that $h^A_\mu(T, \alpha) > 0$.

Denote the set of all entropy tuples (resp. for μ) by $SE_n(X,T)$ (resp. $SE_n^\mu(X,T)$). Since there is no variational principle for sequence entropy (see [45]) one can only show [50]:

Theorem 9.2. *For a TDS and $n \geq 2$, $SE_n^\mu(X,T) \subset SE_n(X,T)$.*

In a similar way one can define *sequence entropy sets* (see Section 6). Unlike the situation with entropy sets it is easy to give an example, say the Morse system, for which there is a finite sequence entropy set, but there is no sequence entropy set with uncountably many points. For the connection between sequence entropy and the cardinalities of sequence entropy sets see section 12. Now we turn to define complexity pairs for a measure.

Definition 9.3. Let (X,T) be a TDS and $\mu \in M_T(X)$.

1. We say that $(x_1,x_2) \in X^2$ is a *weak μ-complexity pair* if $x_1 \neq x_2$ and for every Borel partition $\alpha = \{A_1, A_2\}$ of X with $x_i \in \text{int}(A_i)$, $i \in \{1,2\}$,
$$\lim_{n\to\infty} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha) = \infty.$$
2. We say that $(x_1,x_2) \in X^2$ is a *strong μ-complexity pair* if $x_1 \neq x_2$ and if whenever U_1, U_2, are closed mutually disjoint neighborhoods of the points x_1 and x_2, one has
$$\lim_{n\to\infty} \inf_{\beta \succeq \bigvee_{i=0}^{n-1} T^{-i}\{U_1^c, U_2^c\}} H_\mu(\beta) = \infty.$$

Denote by $Com_\mu^+(X,T)$ the set of all weak μ-complexity pairs and by $Com_\mu^-(X,T)$ the set of all strong μ-complexity pairs. It is clear that $Com_\mu^-(X,T) \subseteq Com_\mu^+(X,T)$.

Theorem 9.4. [50] *Let (X,T) be a TDS and $\mu \in M_T(X)$. Then*
$$SE_2^\mu(X,T) \subseteq Com_\mu^-(X,T) \subseteq Com(X,T).$$

Part 3. Independence properties and combinatorics

10. Tame systems

As explained in the introduction the *enveloping semigroup* $\mathcal{E}(X,T)$ of a TDS (X,T) is the closure of $\{T, T^2, T^3, \ldots\}$ in X^X with the product topology. A dynamical version of the famous Bourgain-Fremlin-Talagrand dichotomy asserts that the enveloping semigroup $\mathcal{E}(X,T)$ is either very large and contains a topological copy of $\beta\mathbb{N}$, or the topology of $\mathcal{E}(X,T)$ is determined by the convergence of sequences [37].

More precisely, a topological space is *Fréchet* if for any $A \subset X$ and any $x \in \overline{A}$, there is a sequence $\{x_n\}$ with $A \ni x_n \longrightarrow x$.

Definition 10.1. A metric TDS (X,T) is tame if its enveloping semigroup is Fréchet.

Tame systems were introduced by Köhler [76] and Glasner in [33]. At a first glance one might wander what has tameness to do with combinatorics and entropy theory. In order to see these connections let us have a brief look at the theorem of Rosenthal which is at the base of the Bourgain-Fremlin-Talagrand dichotomy. For more details we refer the reader to [24], [96].

Definition 10.2. Let X be a set. A sequence (A_0^n, A_1^n) of disjoint pairs of subsets of X is called *independent* if for every finite $F \subset \mathbb{N}$ and every function $\epsilon : F \to \{0,1\}$, $\bigcap_{n\in F} A_{\epsilon(n)}^n \neq \emptyset$. The sequence (A_0^n, A_1^n) is called *convergent* if for every $x \in X$, either $x \notin A_0^n$ for almost all n (i.e. for all but finitely many $n \in \mathbb{N}$) or $x \notin A_1^n$ for almost all n.

Definition 10.3. Let X be a set. A sequence of functions $f_n : X \to \mathbb{R}$ is *independent* if there exists a pair $r_0 < r_1$ of real numbers such that the sequence $(A_0^n, A_1^n) = (\{f_n \le r_0\}, \{f_n \ge r_1\})$ is independent.

The proof of the following lemma is straightforward.

Lemma 10.4. *Let $f_n : X \to \mathbb{R}$ be a sequence of real valued functions on a set X. Then f_n is pointwise convergent iff for any pair $r_0 < r_1$ of rational numbers the sequence of disjoint pairs $(A_0^n, A_1^n) = (\{f_n \le r_0\}, \{f_n \ge r_1\})$ is convergent.*

We can now state (a special case of) a famous dichotomy theorem of Rosenthal.

Theorem 10.5. *Let X be a compact space and $\{f_n\}_{n\in\mathbb{N}}$ a pointwise bounded sequence of functions in $C(X)$. Then either the sequence $\{f_n\}$ contains a pointwise convergent subsequence or there exists a subsequence of $\{f_n\}$ which is independent.*

As it can be shown that a metric TDS (X,T) is not tame when and only when there is a bounded continuous real valued function f on X such that the sequence $\{f \circ T^n : n \in \mathbb{N}\}$ is an independence sequence, the connection of tameness and combinatorial properties becomes apparent.

We have the following results concerning tame systems (for the definition and properties of PI systems see e.g. [4]):

Theorem 10.6. *(Glasner* [33]*) (1) A tame system has zero entropy; (2) A metric minimal tame system is PI; hence a nontrivial metric minimal weakly mixing system is never tame. (3) (Kerr and Li* [73]*) A null system is tame.*

Thus we have the inclusions:

$$\text{zero entropy} \supset \text{tame} \supset \text{null} \supset \text{equicontinuous}$$

In [48] it was shown that a minimal null system is an almost 1-1 extension of an equicontinuous system. The question whether this conclusion is valid for minimal tame systems was answered, independently, by Huang [47], and Kerr and Li [74]. See also Glasner [34] for a different proof.

Theorem 10.7. *A minimal tame system is an almost 1-1 extension $\pi : (X,T) \longrightarrow (Y,S)$ of an equicontinuous system. Moreover (X,T) is uniquely ergodic with unique invariant measure μ and the map π is a measure theoretical isomorphism of (X,μ,T) with the Kronecker system (Y,ν,S), where $\nu = \pi_*(\mu)$ is the Haar measure on Y.*

The way Huang solved the question was via the notion of scrambled pairs. (Warning: this is not the same as the notion of a "scrambled set" from Section 13).

Definition 10.8. A pair $(x_1, x_2) \in X^2$ with $x_1 \neq x_2$ is a *scrambled pair* if there are closed disjoint neighborhood U_i of x_i and an infinite sequence $\{n_1, n_2, \ldots\}$ with

$$\bigcap_{i=1}^{\infty} T^{-n_i} U_{s(i)} \neq \emptyset$$

for any $s = (s(1), s(2), \ldots) \in \{1, 2\}^{\mathbb{N}}$. Denote by $S(X, T)$ the set of all scrambled pairs.

Then he proved:

Theorem 10.9. *If (X, T) is tame then $S(X, T) = \emptyset$.*

As we will see in the next section, the proof provided by Kerr and Li [74] employs a similar notion of independence. Glasner's proof [34], on the other hand, avoids combinatorial machinery and uses only topological methods and the structure theory of minimal systems.

11. Independence pairs

Recall that Theorem 5.13 characterizes an entropy tuple $(x_1, x_2, \ldots, x_k)$ by means of an "interpolation" or "independence" property on a set of neighborhoods $(U_1, U_2, \ldots, U_k)$ along a subset of $\mathbb{N}$ with positive density.

Following ideas and methods developed by Rosenthal [89], Glasner-Weiss [40], and Huang-Ye [57], Kerr and Li, in their recent work [74], introduced various notions of independence. These notions enabled them to give a uniform and elegant treatment of entropy pairs, sequence entropy pairs and scrambled pairs, which were discussed in the previous sections.

In a recent preprint [75] Kerr and Li also introduce various notions of independence for a measure and give a uniform treatment of entropy pairs and sequence entropy pairs for a measure.

Definition 11.1. Let (X, T) be a TDS. For a tuple $\tilde{A} = (A_1, \ldots, A_k)$ of subsets of X, we say a subset $J \subset \mathbb{Z}_+$ is an *independence set* for $\tilde{A}$ if for any nonempty finite subset $I \subset J$, we have

$$\bigcap_{i \in I} T^{-i} A_{s(i)} \neq \emptyset$$

for any $s \in \{1, \ldots, k\}^I$.

Next they define IE-tuples, IT-tuples, and IN-tuples (standing for Entropy, Tame and Null, respectively) as follows.

Definition 11.2. We call a tuple $\tilde{x} = (x_1, \ldots, x_k) \in X^k$

1. an *IE-tuple* if for every product neighborhood $U_1 \times \ldots \times U_k$ of $\tilde{x}$ the tuple $(U_1, \ldots, U_k)$ has an independence set of positive density.
2. an *IT-tuple* if for every product neighborhood $U_1 \times \ldots \times U_k$ of $\tilde{x}$ the tuple $(U_1, \ldots, U_k)$ has an infinite independence sets.

3. an *IN-tuple* if for every product neighborhood $U_1 \times \ldots \times U_k$ of $\tilde{x}$ the tuple $(U_1, \ldots, U_k)$ has arbitrarily long finite independence sets.

Then they prove:

Theorem 11.3. *Let (X, T) be a TDS.*

1. *A tuple is an entropy tuple iff it is a non-diagonal IE-tuple. In particular a system (X, T) has zero entropy if and only if every IE-pair is diagonal (i.e. all its entries are equal).*
2. *A tuple is a sequence entropy tuple iff it is a non-diagonal IN-tuple. In particular a system (X, T) is null if and only if every IN-pair is diagonal.*
3. *(X, T) is tame iff it has no non-diagonal IT-pair.*

The connection between the concepts of independence and the corresponding concept of k-tuples can be discerned, for example, in the following:

Lemma 11.4 ([74], Lemma 3.4)**.** *Let $k \geq 2$. Let $U_1, \ldots, U_k$ be pairwise disjoint subsets of X and set $\mathcal{U} = \{U_1^c, \ldots, U_k^c\}$. Then $(U_1, \ldots, U_k)$ has an independence set of positive density if and only if $h_c(\mathcal{U}, T) > 0$.*

We remark that Part 1 of Theorem 11.3 was proved by Glasner and Weiss [40] for symbolic systems, and by Huang and Ye [59] for general $\mathbb{Z}$-actions. Note that Kerr and Li's result are proven for general (discrete amenable) group actions. Also, one direction of Part 3 of Theorem 11.3 was obtained by Huang [47].

At the heart of the proofs of Glasner-Weiss and Huang-Ye lies a combinatorial fact known as:

The Sauer-Perles-Shelah lemma: If $B \subset \{0,1\}^{\{1,\ldots,n\}}$ satisfies

$$|B| > \sum_{i=0}^{k-1} \binom{n}{i}$$

then there is W with $|W| \geq k$ and $B \supset \{0,1\}^W$.

In Kerr and Li's work the following combinatorial lemma plays a key role. Let $k \geq 2$ and let Z be a nonempty finite set. Let $\mathcal{U}$ be the cover of $\{0,1,\ldots,k\}^Z = \prod_{z \in Z}\{0,1,\ldots,k\}$ consisting of subsets of the form $\prod_{z\in Z}(\{0,1,\ldots,k\} \setminus \{i_z\})$, where $1 \leq i_z \leq k$ for each $z \in Z$. For $S \subset \{0,1\ldots,k\}^Z$ write F_S to denote the minimal number of sets in $\mathcal{U}$ one needs to cover S.

The Kerr-Li lemma: Let $k \geq 2$ and let $b > 0$ be a constant. Then there exists a constant c depending only on k and b such that for every finite set Z and $S \subset \{0,1,\ldots,k\}^Z$ with $F_S \geq k^{b|Z|}$ there is a $W \subset Z$ with $|W| \geq c|Z|$ and

$$S|_W \supset \{1,2,\ldots,k\}^W.$$

In order to demonstrate the use of the Kerr-Li lemma let us next describe how one can deduce from it Lemma 11.4 (which in turn implies Theorem 11.3(1)). We follow [74]. The “only if” part is trivial. For the “if” part, set $b = h_c(\mathcal{U}, T) > 0$ and consider the map $\phi_n : X \longrightarrow \{0,1,\ldots,k\}^{\{1,\ldots,n\}}$ defined by $\phi_n(x)(j) = i$ if $T^j(x) \in U_i$ for some $1 \leq i \leq k$ and 0 otherwise. Then $N(\bigvee_{i=1}^n T^{-i}\mathcal{U}) = F_{\phi_n(X)}$, and so $F_{\phi_n(X)} > e^{\frac{b}{2}n}$ for all large enough n. By the Kerr-Li lemma there exists a constant $c > 0$ depending only on k and b such that $\phi_n(X)|_W \supset \{1,\ldots,k\}^W$ for some $W \subset \{1, \ldots, n\}$ with $|W| \geq cn$ when n is sufficiently large. Thus W is an independence set for the tuple $U = (U_1, \ldots, U_n)$. A simple argument (using symbolic dynamics) shows that U has an independence set of positive density.

12. Maximal pattern entropy

In their work [67] Kamae and Zamboni introduce a variation of the complexity function which they call *maximal pattern complexity.* For a sequence $\alpha = (a_1, a_2, \ldots) \in \{1, \ldots, k\}^{\mathbb{Z}_+}$ and any $n \in \mathbb{N}$ this number is defined by

$$p^*_\alpha(n) := \sup_{(t_1 < \ldots < t_n) \in \mathbb{Z}^n_+} \text{card}\, \{a_{i+t_1} \ldots a_{i+t_n} : i \in \mathbb{Z}_+\}.$$

In a recent preprint by Huang and Ye [60] the authors introduce a notion, called *maximal pattern entropy.* By extending the combinatorial lemma of Kerr and Li [74] they show that if (X, T) is a TDS then

$$\sup_A \, h_A(T) = \log k \text{ with } k \in \mathbb{N} \cup \{\infty\}.$$

This should be compared with the fact that for an ergodic MDS $\sup_A h^A_\mu(T) = \log k$ with $k \in \mathbb{N} \cup \{\infty\}$ [65]. The authors of [60] also show that a zero dimensional system is null if and only if the maximal pattern entropy with respect to any open cover is of polynomial order. We now proceed to give some more details.

For a TDS (X, T), $n \in \mathbb{N}$ and a finite open cover $\mathcal{U}$ let

$$p^*_{X,\mathcal{U}}(n) = \max_{(t_1 < t_2 < \ldots < t_n) \in \mathbb{Z}^n_+} N(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}).$$

The *maximal pattern entropy of* T *with respect to* $\mathcal{U}$ is defined by

$$h^*_{top}(T, \mathcal{U}) = \limsup_{n \longrightarrow +\infty} \frac{1}{n} \log p^*_{X,\mathcal{U}}(n).$$

It is easy to see that $\{\log p^*_{X,\mathcal{U}}(n)\}$ is a sub-additive sequence. Hence

$$h^*_{top}(T, \mathcal{U}) = \lim_{n \longrightarrow +\infty} \frac{1}{n} \log p^*_{X,\mathcal{U}}(n) = \inf_{n \geq 1} \frac{1}{n} \log p^*_{X,\mathcal{U}}(n).$$

The *maximal pattern entropy* of (X, T) is

$$h^*_{top}(T) = \sup_{\mathcal{U}} h^*_{top}(T, \mathcal{U}),$$

where supremum is taken over all finite open covers of X.

Analogously, given a MDS $(Y, \mathcal{C}, \nu, T)$ and a finite measurable partition α of Y we define

$$p^*_{Y,\alpha,\nu}(n) = \max_{(t_1 < t_2 < \ldots < t_n) \in \mathbb{Z}^n_+} \sum_{A \in \bigvee_{i=1}^{n} T^{-t_i}\alpha} -\nu(A) \log \nu(A),$$

and the *maximal pattern entropy of* T *with respect to* α is defined by

$$h^*_\nu(T, \alpha) = \limsup_{n \longrightarrow +\infty} \frac{1}{n} p^*_{X,\alpha,\nu}(n).$$

Again one checks that $\{p^*_{X,\alpha,\nu}(n)\}$ is a sub-additive sequence, hence

$$h^*_\nu(T,\alpha) = \lim_{n\longrightarrow+\infty} \frac{1}{n} p^*_{X,\alpha,\nu}(n) = \inf_{n\geq 1} \frac{1}{n} p^*_{X,\alpha,\nu}(n).$$

The *maximal pattern entropy* of (X,T,ν) is

$$h^*_\nu(T) = \sup_\alpha h^*_\nu(T,\alpha),$$

where the supremum is taken over all finite measurable partitions.

The first theorem below tells us that $h^*_{\rm top}(T)$ and $h^*_\mu(T)$ can be interpreted by known notions. We denote by $\mathcal{S}$ the collection of increasing sequences in $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$.

Theorem 12.1. *Let (X,T) be a TDS and $(Y,\mathcal{C},\nu,T)$ be a MDS. Then*

1. *for each open cover $\mathcal{U}$ of X, $h^*_{top}(T,\mathcal{U}) = \sup_A h^A_{top}(T,\mathcal{U})$ and there is $A\in\mathcal{S}$ with $h^*_{top}(T,\mathcal{U}) = h^A_{top}(T,\mathcal{U})$. Moreover, $h^*_{top}(T) = \sup_A h^A_{top}(T)$.*
2. *for each finite partition α of Y, $h^*_\nu(T,\alpha) = \sup_{\mathcal{A}} h^A_\nu(T,\alpha)$ and there is $A\in\mathcal{S}$ with $h^*_\nu(T,\alpha) = h^A_\nu(T,\alpha)$. Moreover, $h^*_\nu(T) = \sup_A h^A_\nu(T)$.*

The next theorem states some properties of maximal pattern entropy.

Theorem 12.2. *Let (X,T) be a TDS and $(Y,\mathcal{C},\nu,T)$ be a MDS.*

1. *$h^*_{top}(T)$ and $h^*_\mu(T)$ are conjugacy and isomorphism invariants.*
2. *For every open cover $\mathcal{U}$, $h^*_{top}(T^k,\mathcal{U}) = h^*_{top}(T,\mathcal{U})$ for each $k\in\mathbb{Z}\setminus\{0\}$ and thus*
$$h^*_{top}(T^k) = h^*_{top}(T)$$
for every $k\in\mathbb{Z}\setminus\{0\}$.
3. *For every finite partition α of Y, $h^*_\nu(T^k,\alpha) = h^*_\nu(T,\alpha)$ for each $k\in\mathbb{Z}\setminus\{0\}$ and thus*
$$h^*_\nu(T^k) = h^*_\nu(T)$$
for every $k\in\mathbb{Z}\setminus\{0\}$.
4. *If $h_{top}(T) > 0$, then $h^*_{top}(T) = \infty$; if $h_\nu(T) > 0$, then $h^*_\nu(T) = \infty$.*

It was shown by Goodman [45] that when X has a finite covering dimension, then $h^A_\mu(T) \leq h^A_{\rm top}(T)$ for any $\mu\in M_T(X)$ and any sequence A of $\mathbb{Z}_+$. In fact, the assumption of finite dimension is not necessary, as the following theorem shows.

Theorem 12.3. *Let (X,T) be a TDS. Then*

$$\sup_{\mu\in M_T(X)} h^A_\mu(T) \leq h^A_{top}(T)$$

for any sequence A of $\mathbb{Z}_+$. Consequently, $\sup_{\mu\in M_T(X)} h^*_\mu(T) \leq h^*_{top}(T)$.

The following lemma obtained in [60] is an extension of the Kerr-Li lemma in [74]. We use the notation introduced at the end of Section 11.

Lemma 12.4. *Let $k,\ell\geq 2$. Then there are C_k, D_k depending on k such that for every finite set Z with $|Z|\geq \max\{\ell C_k, D_k\}$ and $S\subseteq\{0,1,2,\cdots,k\}^Z$ with $F_S\geq N(k,|Z|,\ell)$ there exists a $W\subseteq Z$ with $|W|\geq\ell$ and $S|_W\supseteq\{1,2,\cdots,k\}^W$, where $N(\cdot,\cdot,\cdot)$ is a map from $\mathbb{N}^3$ to $\mathbb{N}$.*

In a similar way to entropy tuples one defines *maximal pattern entropy tuples.* A maximal pattern entropy tuple $(x_1, \dots, x_n)$ is *intrinsic* if $x_i \neq x_j$ if $i \neq j$. By using Lemma 12.4 and other arguments Huang and Ye prove:

Theorem 12.5. *Let (X,T) be a TDS. Then $h^*_{top}(T) = \sup_A h^A_{top}(T)$ is the logarithm of the maximal length of an intrinsic maximal pattern entropy tuple. Consequently it is log n with $n \in \mathbb{N} \cup \{\infty\}$.*

We remark that there is a recent result by Maass and Shao [81] which gives the structure of a minimal TDS with $h^*_{\rm top}(T) < \infty$. Roughly speaking, such a system is a bounded finite to one extension of its maximal equicontinuous factor up to a proximal extension.

Using the Sauer-Perles-Shelah lemma and some other arguments one can prove [60]:

Theorem 12.6. *Let (X,T) be zero dimensional. Then it is null if and only if $p^*_{X,\mathcal{U}}$ is of polynomial order for each open cover $\mathcal{U}$.*

Part 4. Applications and open problems

13. Li-Yorke chaos, asymptotic pairs, extreme relations and interval maps

Li-Yorke chaos

Positive entropy can be interpreted as a manifestation of chaotic behavior and an extensive work has been done along these lines (see for example [41]). In the present section we examine the notion of Li-Yorke chaos and the related concept of asymptotic pairs.

Definition 13.1. Let (X,T) be a TDS. A pair of points $\{x,y\} \subseteq X$ is said to be a *Li-Yorke pair* (with modulus δ) if one has simultaneously

$$\limsup_{n\to\infty} d(T^n x, T^n y) = \delta > 0 \quad \text{and} \quad \liminf_{n\to\infty} d(T^n x, T^n y) = 0.$$

A set $S \subseteq X$ is called *scrambled* if any pair of distinct points $\{x,y\} \subseteq S$ is a Li–Yorke pair. Finally, a system (X,T) is called *chaotic in the sense of Li and Yorke* if there is an uncountable scrambled set.

It turns out that the existence of uncountable scrambled sets actually follows from relative weak mixing rather than positive entropy. The following theorem is proved in Blanchard-Glasner-Kolyada-Maass [12]. A set K of a compact space X is called a *Mycielski set* if it has the form $K = \bigcup_{j=1}^{\infty} C_j$ with C_j a Cantor set for every j [12].

Theorem 13.2. *Let (X,T) be a TDS and assume that for some T-ergodic probability measure μ the corresponding measure preserving system $(X, \mathcal{B}_X, \mu, T)$ is not measure distal. Denote $Z = \mathrm{supp}\,(\mu)$; then there exists a closed invariant set $W \subseteq Z \times Z$ such that the subsystem $(W, T \times T)$ is topologically transitive and for every open set $U \subseteq X$ with $U \cap Z \neq \emptyset$ there exists a Mycielski set $K \subseteq U$ which satisfies $(K \times K) \setminus \Delta_Z \subseteq W_{trans}$. Finally, every such set K is a scrambled set of the system (X,T).*

The proof of Theorem 13.2 relies on the Furstenberg-Zimmer structure theorem for ergodic MDS which represents every such system as a weakly mixing extension of a measure-distal system. Using Glasner's characterization of entropy pairs [30], the authors of [12] obtain the following corollary, the second part of which solved a long standing open problem.

Theorem 13.3. *Let (X, T) be a TDS.*

1. *If (X, T) admits a T-invariant ergodic measure μ with respect to which the measure preserving system $(X, \mathcal{B}_X, \mu, T)$ is not measure distal then (X, T) is Li-Yorke chaotic.*
2. *If (X, T) has positive topological entropy then it is Li-Yorke chaotic.*

For related results see Huang-Ye [56] and Kerr-Li [74].

Asymptotic pairs

Similar questions are treated in the work [10] of Blanchard, Host and Ruette. Before describing their results let us recall the following classical results (see [87] and [82])).

In a MDS $(X, \mathcal{B}, \mu, T)$ a measurable partition ξ is called *extreme* if it satisfies the following conditions:

1. $T^{-1}\xi \le \xi$.
2. $\bigvee_{n=0}^{\infty} T^n \xi = \epsilon$.
3. $\bigwedge_{n=0}^{\infty} T^{-n} \xi = \pi_\mu$.

Here ϵ is the point partition and π_μ is the Pinsker measurable partition. If in addition one has:

(4) $h_\mu(T) = H_\mu(\xi | T^{-1}\xi)$,

then ξ is called *perfect*. It was shown by Rokhlin and Sinai in [87] that every MDS admits a perfect measurable partition.

Definition 13.4. Let (X, T) be a TDS. A pair of points $\{x, y\} \subseteq X$ is *asymptotic* if

$$\lim_{n\to\infty} d(T^n x, T^n y) = 0.$$

We let $\mathbf{A}$ denote the set of asymptotic pairs in $X \times X$.

Given a TDS (X, T) of positive entropy, the authors of [10] fix an invariant measure $\mu \in M_T(X)$. Then using, on the one hand, the measure theoretical Pinsker factor π_μ and the related perfect partition, and on the other hand the relative product measure $\lambda = \mu \underset{\pi_\mu}{\times} \mu$, they prove the following results.

Theorem 13.5. [10] *Let (X, T) be a TDS with positive topological entropy.*

1. *The set $\{x \in X : \exists\, x' \neq x,\ (x, x') \in \mathbf{A}\}$ has measure 1 for every $\nu \in M_T(X)$.*
2. *The set of entropy pairs is contained in the closure of the set of asymptotic pairs:*
$$E(X, T) \subset \overline{\mathbf{A}}.$$
3. *Let $R \subset X \times X$ be the smallest closed T-invariant equivalence relation containing $\mathbf{A}$. Then the quotient TDS $(X/R, T)$ has zero topological entropy.*

Using these methods they also obtain results concerning Li-Yorke pairs similar to those in [12].

Extreme relations

Building on results from [10] and [31] and from their earlier work [69] Kaminski, Seimaszko and Szymanski, in [70], tie up local entropy theory with classical results of Rohlin and Sinai [87].

Definition 13.6. Let (X,T) be a TDS and $\mu \in M_T(X)$. A closed equivalence relation $R \subset X \times X$ is called *extreme* (respectively *μ-extreme*) if it satisfies the following conditions:

1. $(T \times T)R \subset R$.
2. $\bigcap_{n=0}^{\infty}(T \times T)^n R = \Delta$.
3. $\bigvee_{n=0}^{\infty}(T \times T)^{-n} R = \Pi$ (resp. Π_μ).

Here Δ is the diagonal in $X \times X$, Π is the closed T-invariant equivalence relation corresponding to the topological Pinsker factor of (X,T), and Π_μ is the closed T-invariant equivalence relation corresponding to the μ-Pinsker factor of (X,T). (Note that $\bigvee$ in condition 3 means: the $T \times T$-invariant closed equivalence relation generated by the corresponding union).

The main result of [70] is the following:

Theorem 13.7. *Let (X,T) be a TDS and $\mu \in M_T(X)$ an ergodic measure. Then here exits a μ-extreme closed equivalence relation $R_\mu \subset X \times X$. If (X,T) is uniquely ergodic it is also extreme.*

In [69] the authors consider TDS (X,T) for which there is a closed equivalence relation R satisfying the following properties:

1. $(T \times T)R \subset R$.
2. $\bigcap_{n=0}^{\infty}(T \times T)^n R = \Delta$.
3. $\mathrm{cls} \bigcup_{n=0}^{\infty}(T \times T)^{-n} R = X \times X$.

They call such TDS a Kolmogorov flow (or a K-flow); however, as they show in this same paper, such a TDS can be nontrivial and yet have zero entropy. This fact makes the name K-flow seems incongruous. We will therefore refer to this property as KSS. It is shown in [70] that a uniquely ergodic u.p.e. system is KSS. We make the following observations.

Theorem 13.8. *The following statements hold.*

1. *If (X,T) is KSS then the asymptotic relation $\mathbf{A}$ is dense in $X \times X$. Thus a minimal KSS system is weakly mixing.*
2. *There is a c.p.e. KSS system which is not u.p.e.*

Proof. 1. Assume that (X,T) is KSS. Then for each pair of non-empty open subsets U and V of X, there is $n \in \mathbb{N}$ such that $U \times V \cap (T \times T)^{-n} R \neq \emptyset$. So there are $x \in U$ and $y \in V$ with $(T^n x, T^n y) \in R$. This implies that $(x,y) \in \mathbf{A}$.

2. Let $\Sigma = \{0,1\}^{\mathbb{Z}}$ be the full shift. Take two copies of Σ, namely $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, and identify $\mathbf{0} \times \{0\}$ with $\mathbf{0} \times \{1\}$, where $\mathbf{0} = (\ldots,0,0,0,\ldots)$. The new space is denoted by X. Define a relation R on X as follows: $(x,y) \in R$ if and only if $x = y$ or $x = (\ldots,x_1,x_0,0,\ldots) \times \{i\}$ and $y = (\ldots,y_1,y_0,0,\ldots) \times \{i\}$ for some

$i \in \{0,1\}$. It is easy to verify that X is KSS (evidenced by R) and c.p.e. It is clear that X is not u.p.e. $\square$

An application to interval maps

We end this section with the following observation (we thank W. Huang for suggesting this question). In [28] the authors have shown that if $T : [0,1] \longrightarrow [0,1]$ is a continuous map then T is null if and only if T is not chaotic in the sense of Li and Yorke; and in [18] the author has shown that $h^*_{top}(T) \in \{0, \log 2, \infty\}$. In fact we can prove the following:

Theorem 13.9. *Let $X = [0,1]$ and $T : X \longrightarrow X$ be continuous. Then (X,T) is tame if and only if (X,T) is null.*

To prove this theorem we need (Proposition 25 from [14]):

Lemma 13.10. *Let $X = [0,1]$ and suppose $T : X \longrightarrow X$ is continuous. If $h_{top}(T) = 0$ and T is not null then there exist points $y < z$ and an increasing sequence $\{k_i\}$ in $\mathbb{N}$ such that for any $\alpha = (a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}}$ there is $w = w_\alpha \in X$ for which $T^{k_i}w < y$ if $a_i = 0$ and $T^{k_i}w > z$ if $a_i = 1$.*

Proof of Theorem 13.9. It remains to show that if T is not null then T is not tame. Thus we may assume that T has zero entropy, is tame and is not null. Then we have $y < z$ and $\{p_i\}$ as in Lemma 13.10. Since the enveloping semigroup $\mathcal{E}(X,T)$ is sequentially compact we can assume, with no loss in generality, that $T^{k_i} \longrightarrow p \in \mathcal{E}(X,T)$. Choose a sequence $\alpha = (a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}}$ with $a_{k_i} = 0$ when i is odd and $a_{k_i} = 1$ when i is even. Let $w = w_\alpha$ be the point provided in the lemma. Then $T^{k_i}w < y$ if i is odd and $T^{k_i}w > z$ when i is even. Since $\lim_{i\to\infty} T^{k_i}w = pw$ exists, this is a contradiction. $\square$

14. Sensitive sets

In this section we shall review some further applications of local entropy theory.

Sensitivity is an important notion in describing the complexity of a TDS. It is known that a transitive system is either almost equicontunous or sensitive, [41] and [2]. If it is almost equicontinuous then it is *uniformly rigid*. Recently, using ideas and results from local entropy theory, Shao, Ye and Zhang [94], Ye and Zhang [104] and Huang, Lu and Ye [49] developed a theory of sensitive sets, which measures the "degree" of sensitivity both in the topological and the measure-theoretical setting. See [102] for the motivation of the research.

Definition 14.1. Let (X,T) be a TDS and K be a subset of X with $\text{card}\,(K) \geq 2$. We say that (X,T) is *sensitive relative to* K if for every $(x_1, \dots, x_n) \in K^n \setminus \Delta_n$ and neighborhoods U_i of x_i and a neighborhood U of $x \in X$, there are $k \in \mathbb{N}$ and $x'_i \in U$ with $T^k x'_i \in U_i$ for each $1 \leq i \leq n$. Such a K is called a *sensitive set* or *a S-set.*

It is not hard to show that a transitive TDS is sensitive if and only if there is a sensitive set in X. Using results from [23], [60] and a simple combinatorial lemma one can prove the following:

Theorem 14.2. [104] *For a transitive TDS, every entropy set is a sensitive set. Consequently, a transitive TDS having no uncountable sensitive set has zero topological entropy.*

A notion of sensitive set can also be defined for a measure $\mu \in M_T(X)$.

Definition 14.3. Let (X, T) be a TDS, $\mu \in M_T(X)$ and $n \geq 2$. Then

1. $(x_i)_1^n \in X^n$ is a *sensitive n-tuple* for μ, if $(x_i)_1^n$ is not on the diagonal $\Delta_n(X)$, and for any open neighborhood U_i of x_i and any Borel subset A of positive measure there is $k \geq 0$ such that $A \cap T^{-k}U_i \neq \emptyset$ for $i = 1, 2, \cdots, n$.
2. A subset K of X with card$(K) \geq 2$ is a sensitive set for μ (μ-S-set) if each n-tuple from K is a sensitive n-tuple for μ.

It is easy to see that if (X, T) is an E-system, then each sensitive set for μ is an sensitive set. Recall that a TDS is an *E-system* if it is transitive and there exists a $\mu \in M_T(X)$ with full support; and it is an *M-system* if it is transitive and the set of minimal points is dense (see [41] or [31]).

Theorem 14.4. [49] *For an E-system, each sequence entropy set is a sensitive set for μ, and for a non-minimal M-system there is an infinite sequence entropy set. Thus, a non-minimal M-system has infinite sequence entropy. In particular, a Devaeny chaotic system has infinite sequence entropy.*

Recall that a TDS (X, T) is chaotic in the sense of Devaney if it is infinite, transitive, and the set of periodic points is dense in X. It is clear that such a system is an M-system.

Let $\Delta_n(\epsilon) = \{(x_1, \ldots, x_n) \in X^n : d(x_i, x_j) < \epsilon \text{ for all } 1 \leq i \leq j \leq n\}$ and let

$$Q_n(X, T) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=1}^{\infty} T^{-j}\Delta_n(1/k)}.$$

A subset $K \subset X$ is a *Q-set* if card$(K) \geq 2$ and each n-tuple from K is in $Q_n(X, T)$.

Theorem 14.5. [94, 104, 49] *Let (X, T) be a minimal TDS and $\pi : (X, T) \longrightarrow (Y, S)$ be the factor map onto the maximal equicontinuous factor.*

1. *For every $y \in Y$, $\pi^{-1}(y)$ is a Q-set.*
2. *For every $y \in Y$, $\pi^{-1}(y)$ is a maximal sensitive set.*

15. The relative case

Soon after the introduction of the notions of c.p.e. and u.p.e. for TDS, Glasner and Weiss [42] initiated a study of the corresponding notions in the relative case. The following definitions were introduced.

Let $\pi : (X, T) \longrightarrow (Y, S)$ be a factor map, and let $R_\pi = \{(x, x') : \pi x = \pi x'\}$. Put $E_\pi = E(X, T) \cap R_\pi$, an let F_π be the smallest closed invariant equivalence relation containing E_π. The extension π is:

1. a null-entropy extension, if $E_\pi = \emptyset$,
2. a u.p.e. extension, if $E_\pi \cup \Delta = R_\pi$

3. an entropy-generated extension, if $F_\pi = R_\pi$
4. a c.p.e. extension, if whenever $(X,T) \xrightarrow{\sigma} (Z,T) \xrightarrow{\rho} (Y,T)$, where $\rho \circ \sigma = \pi$ and ρ is not an isomorphism, then $h(Z,T) > h(Y,T)$.

Theorem 15.1. [42] *Every extension $\pi : (X,T) \longrightarrow (Y,T)$ can be decomposed uniquely as follows:*

$$(X,T) \xrightarrow{\sigma} (Z,T) \xrightarrow{\rho} (Y,T)$$

where $\rho \circ \sigma = \pi$ and ρ is a null-entropy extension and σ is an entropy-generated extension.

In this decomposition we call $\sigma : (X,T) \longrightarrow (Z,T)$ the *relative Pinsker factor* of X over Y. The disadvantage of these definitions is that they do not involve the relative entropy. For example, if we consider the 2 to 1 factor map $\pi : (X,T) \to (X,T)$ where $X = \{0,1\}^{\mathbb{Z}}$, T is the shift and $\pi(x)_n = x_n + x_{n+1}$, then π is an isometric extension but nonetheless, according to the above definition, it is also a u.p.e. extension.

Noticing this, Lemanczyk and Siemaszko [80] introduced another definition of relative Pinsker factor by using relative measure-theoretical entropy. Later, Park and Siemaszko [83] reconsidered this relative factor by introducing relative m-entropy pairs but the question how to define this factor topologically remained open. It was answered in Huang, Ye and Zhang [62, 63] by introducing relative topological entropy pairs and proving local relative variational principles which we now describe briefly.

We begin by defining the *topological conditional entropy of an open cover $\mathcal{U}$ of X with respect to* (Y,S). Let

$$N(\mathcal{U}|Y) = \sup_{y \in Y} N(\mathcal{U}|y) \text{ and } H(\mathcal{U}|Y) = \log N(\mathcal{U}|Y).$$

It is easy to see that $a_n = H(\mathcal{U}_0^{n-1}|Y)$ is a non-negative sub-additive sequence. The *conditional entropy of $\mathcal{U}$ with respect to* (Y,S) is then defined by

$$h_{\text{top}}(T,\mathcal{U}|Y) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}|Y) = \inf_{n\geq 1} \frac{1}{n} H(\mathcal{U}_0^{n-1}|Y).$$

The *topological conditional entropy of* (X,T) *with respect to* (Y,S) is defined by

$$h_{\text{top}}(T,X|Y) = \sup_{\mathcal{U}\in \mathcal{C}_X^o} h_{\text{top}}(T,\mathcal{U}|Y),$$

where $\mathcal{C}_X^o$ is the set of all open covers.

Given a partition α, $\mu \in M_T(X)$ and a T-invariant σ-algebra $\mathcal{A} \subset \mathcal{B}(X)$ define

$$H_\mu(\alpha|\mathcal{A}) = \sum_{A\in\alpha} \int_X -\mathbb{E}(1_A|\mathcal{A}) \log \mathbb{E}(1_A|\mathcal{A}) d\mu$$

where $\mathbb{E}(1_A|\mathcal{A})$ is the conditional expectation of 1_A with respect to $\mathcal{A}$. We define the *conditional entropy of α with respect to $\mathcal{A}$* by

$$h_\mu(T,\alpha|\mathcal{A}) = \lim_{n\to+\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\mathcal{A}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\mathcal{A}),$$

and the *measurable conditional entropy of (X,T,μ) with respect to $\mathcal{A}$* by

$$h_\mu(T,X|\mathcal{A}) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha|\mathcal{A}).$$

Particularly, if $\pi : (X,T) \to (Y,S)$ is a factor map between TDS and $\alpha \in \mathcal{P}_X$, we define the *conditional entropy of α with respect to* (Y,S) by

$$h_\mu(T,\alpha|Y) = h_\mu(T,\alpha|\pi^{-1}(\mathcal{B}(Y))) = \lim_{n\to+\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\pi^{-1}(\mathcal{B}(Y))),$$

and the *measurable conditional entropy of* (X,T,μ) *with respect to* (Y,S) by

$$h_\mu(T,X|Y) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha|Y).$$

The usual process can now be applied to define relative entropy pairs both in the topological and in the measure-theoretical setting. The relative Pinsker factor is now defined as the factor generated by the smallest closed invariant equivalence relation containing the relative entropy pairs. The following theorem tells us that the Pinsker factor defined in this way is the same as the one defined in [80] and [83].

Theorem 15.2. *Let $\pi : (X,T) \longrightarrow (Y,S)$ be a factor map. Then there is $\mu \in M_T(X)$ such that the set of relative entropy pairs for μ coincides with the set of relative entropy pairs.*

For other results concerning the relative case see the two recent papers [105, 106] by Zhang.

16. Some open problems

In the previous sections we described the development of the local entropy theory. However, there still are many remaining open questions. Here is a sample:

First recall a question from [31]: Does there exist a minimal c.p.e. system which is not u.p.e.? This long standing problem was solved recently. In [95] the authors answered the question by constructing a minimal c.p.e. non u.p.e. system. We remind the readers that a TDS is called *diagonal* if for every $x \in X$, the pair $(x, T(x))$ is an entropy pair. The example constructed in [95] is a diagonal system (hence is clearly c.p.e.).

Question: Does there exist a minimal c.p.e. system which is not diagonal?

It was shown in [54] that a minimal topological K-system is strongly mixing.

Question: Is a minimal u.p.e. system strongly mixing?

We remark that it is known that u.p.e. (= u.p.e. of order 2) does not imply topological K [59].

It is known that there exists a transitive non-minimal tame TDS [33].

Question: Does there exist a transitive non-minimal null TDS?

It is well known that for a minimal TDS the regional proximal relation is an equivalence relation (see e.g. Auslander [4]). It was shown in Glasner-Maon [36] that the regional proximal relation along an IP-set need not be an equivalence relation. Still, a better understanding of this relation and the closed invariant equivalence relation generated by it may contribute to a solution of the following:

Question: Is it true that a minimal TDS is mildly mixing if and only if the only uniformly rigid factor it admits is the trivial one point factor?

For related papers see [36], [57] and [35].

It is known that for a TDS (X, T) the condition "every pair $(x, x') \in X \times X$ is *positively recurrent*" implies zero entropy. The next question is particularly tantalizing.

Question: Suppose one only knows that every pair $(x, x') \in X \times X$ is recurrent; does this weaker condition imply zero entropy? (See [101])

It is known that for a null TDS defined on a zero dimensional space, $p^*_{X,\mathcal{U}}$ is of polynomial order for each open cover $\mathcal{U}$.

Question: Assume (X, T) is null. Is it true that $p^*_{X,\mathcal{U}}$ is of polynomial order for each open cover $\mathcal{U}$?

This may be related to our final question. It is known that a TDS with zero entropy has a zero dimensional extension with zero entropy.

Question: Does every null system admit a zero dimensional extension which is null?

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc., **114** (1965), 309-319.

[2] E. Akin, J. Auslander and K. Berg, *When is a transitive map chaotic*? Convergence in Ergodic Theory and Probability (Columbus, OH, 1993) (Ohio University Math. Res. Inst. Pub., 5). de Gruyter, Berlin, 1996, pp. 25-40.

[3] E. Akin and E. Glasner, *Residual properties and almost equicontinuity*, J. d'Anal. Math., **84** (2001), 243-286.

[4] J. Auslander, *Minimal flows and their extensions*, (North-Holland Mathematics Studies, 153). North-Holland, Amsterdam, 1988.

[5] F. Blanchard, *Fully positive topological entropy and topological mixing*, Symbolic Dynamics and its Applications, Amer. Math. Soc., Contem. Math., **135** (1992), 95-105.

[6] F. Blanchard, *A disjointness theorem involving topological entropy*, Bull. de Math. Soc. France, **121** (1993), 565-578.

[7] F. Blanchard and W. Huang, *Entropy set, extremely chaotic and entropy capacity*, Dis. and Cont. Dynam. Sys., to appear.

[8] F. Blanchard and Y. Lacroix, *Zero-entropy factors of topological flows*, Proc. Amer. Math. Soc., **119** (1993), 985-992.

[9] F. Blanchard, B. Host, A. Maass, S. Martinez and D. Rudolph, *Entropy pairs for a measure*, Ergod. Th. and Dynam. Sys., **15** (1995), 621-632.

[10] F. Blanchard, B. Host and S. Ruette, *Asymptotic pairs in positive-entropy systems*, Ergod. Th. and Dynam. Sys., **22** (2002), No. 3, 671-686.

[11] F. Blanchard, E. Glasner and B. Host, *A variation on the variational principle and applications to entropy pairs*, Ergod. Th. and Dynam. Sys., **17** (1997), 29-43.

[12] F. Blanchard, E. Glasner, S. Kolyada and A. Maass, *On Li-Yorke pairs*, J. für die Reine und Angewandte Mathematik **547**, (2002), 51-68.

[13] F. Blanchard, B. Host and A. Maass, *Topological complexity*, Ergod. Th. and Dynam. Sys., **20** (2000), 641-662.

[14] L. Block and W. Coppel, *Dymanics in one dimension*, Lec. Notes in Math., 1513, Springer-Verlag, 1992.

[15] J. Bourgain, D.H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. **100** (1978), 845-886.

[16] M. Brin and A. Katok, *On local entropy*, Geometric dynamics (Rio de Janeiro, 1981), 30-38, Lecture Notes in Math., **1007**, Springer, Berlin, 1983.

[17] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153**, (1971), 401-414.

[18] J. S. Cánovas, *Topological sequence entropy of interval maps*, Nonlinearity, **17** (2004), 49-56.

[19] M Denker, C. Grillenberger and C. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. **527**, Springer-Verlag, New York.

[20] M. Denker and M. Urbanski, *Ergodic theory of equilibrium states for rational maps*, Nonlinearity, **4** (1991), 103-134.

[21] E. I. Dinaburg, *The relation between topological and metric entropy*, Dokl. Akad. Nauk SSSR **190** (1970), = Soviet Math. Dokl. **11**, (1970), 13-16.

[22] D. Dou, W. Huang and X. Ye, *Null flows and null functions on R*, J. of Dynam. and Diff. Equa., **18** (2006), 197-221.

[23] D. Dou, X. Ye and G. Zhang, *Entropy sequences and maximal entropy sets*, Nonlinearity, **19** (2006), 53-74.

[24] D. van Dulst, *Characterization of Banach spaces not containing l^1*, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[25] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory, **1** (1967), 1-49.

[26] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981.

[27] H. Furstenberg and B. Weiss, *On almost 1-1 extensions*, Israel J. of Math. **65** (1989), 311-322

[28] N. Franzová and J. Smital, *Positive sequence topological entropy charaterizes chaotic maps*, Proc. Amer. Math. Soc., **112** (1991), 1083-1086.

[29] W. Geller and M. Misiurewicez, *Rotation and entropy*, Trans. Amer. Math. Soc. **251** (1999), 2927-2984.

[30] E. Glasner, *A simple characterization of the set of μ-entropy pairs and applications*, Israel J. of Math., **192** (1997), 13-27.

[31] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs **101**, Amer. Math. Soc., 2003.

[32] E. Glasner, *Classifying dynamical systems by their recurrence properties*, Topol. Methods Nonlinear Anal. **24** (2004), 21-40.

[33] E. Glasner, *On tame dynamical systems*, Colloq. Math., **105** (2006), 283-295.

[34] E. Glasner, *The structure of tame minimal dynamical systems*, Ergod. Th. and Dynam. Sys. **27**, (2007), 1819-1837

[35] E. Glasner, *Enveloping semigroups in topological dynamics*, Topology Appl. **154**, (2007), 2344–2363.

[36] E. Glasner and D. Maon, *Rigidity in topological dynamics*, Ergod. Th. Dynam. Sys. **9**, (1989), 309-320.

[37] E. Glasner and M. Megrelishvili, *Hereditarily non-sensitive dynamical systems and linear representations*, Colloq. Math., **104** (2006), 223-283.

[38] E. Glasner, M. Megrelishvili and V. V. Uspenskij, *On metrizable enveloping semigroups*, to appear in the Israel J. of Math. ArXiv:math.DS/0606373.

[39] E. Glasner and B. Weiss, *Strictly ergodic uniform positive entropy models*, Bull. de la Soc. Math. de France, **122** (1994), 399-412.

[40] E. Glasner and B. Weiss, *Quasi-factors of zero-entropy systems*, J. Amer. Math. Soc., **8** (1995), 665-686.

[41] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity, **6** (1993), 1067-1075.

[42] E. Glasner and B. Weiss, *Topological entropy of extensions*, Ergodic Theory and its connections with Harmonic Analysis, Cambridge University Press, 1995, 299-307.

[43] E. Glasner and B. Weiss, *On the interplay between measurable and topological dynamics*, Handbook of Dynamical systems, Vol. 1B, Hasselblatt and Katok, eds., Elsevier, Amsterdam, 2006, 597-648.
[44] T.N.T. Goodman, *Relating topological entropy with measure theoretical entropy*, Bull. London Math. Soc., **3** (1971), 176-180
[45] T.N.T. Goodman, *Topological sequence entropy*, Proc. London Math. Soc. (3) **29**, 1974, 331-350.
[46] L.W. Goodwyn, *Topological entropy bounds measure theoretical entropy*, Proc. Amer. Math. Soc., **23** (1969), 679-688.
[47] W. Huang, *Tame systems and scrambled pairs under an Abelian group action*, Ergod. Th. and Dynam. Sys., **26** (2006), 1549-1567.
[48] W. Huang, S. Li, S. Shao and X. Ye, *Null systems and sequence entropy pairs*, Ergod. Th. and Dynam. Sys., **23** (2003), 1505-1523.
[49] W. Huang, P. Lu and X. Ye, *Measure-theoretical sensitivity and equicontinuity*, preprint, 2007.
[50] W. Huang, A. Maass and X. Ye, *Sequence entropy pairs and complexity pairs for a measure*, Ann. Inst. Fourier (Grenoble), **54** (2004), 1005-1028.
[51] W. Huang, A. Maass, P. Romagnoli and X. Ye, *Entropy pairs and a local Abramov formula for a measure theoretical entropy of open covers*, Erg. Th. and Dynam. Sys., **24** (2004), 1127-1153.
[52] W. Huang, K. Park and X. Ye, *Dynamical systems disjoint from all minimal systems with zero entropy*, Bull. de Math. Soc. France, to appear.
[53] W. Huang, S. Shao and X. Ye, *Mixing and proximal cells along sequences*, Nonlinearity, **17** (2004), 1245-1260.
[54] W. Huang, S. Shao and X. Ye, *Mixing via sequence entropy*, Contemporary Mathematics, **385**, 2005, 101-122.
[55] W. Huang and X. Ye, *An explicit scattering, non-weakly mixing example and weak disjointness*, Nonlinearity, **15** (2002), 1-14.
[56] W. Huang and X. Ye, *Devaney's chaos or 2-scattering implies Li-Yorke's chaos*, Topology Appl., **117**(2002), 259-272.
[57] W. Huang and X. Ye, *Topological complexity, return times and weak disjointness*, Erg. Th. Dynam. Sys., **24** (2004), 825-846.
[58] W. Huang and X. Ye, *Dynamical systems disjoint from all minimal systems*, Trans. Amer. Math. Soc., **357** (2005), 669-694.
[59] W. Huang and X. Ye, *A local variational relation and applications*, Israel J. Math., **151** (2006), 237-280.
[60] W. Huang and X. Ye, *Combinatorial lemmas and applications*, preprint, 2006.
[61] W. Huang and Y. Yi, *A local variational principle of presure and its applications to equilibrium states*, Israel J. of Math., to appear.
[62] W. Huang, X. Ye and G.H. Zhang, *Relative entropy tuples, relative u.p.e. and c.p.e. extensions*, Israel. J. of Math., **158**(2007), 249-283..
[63] W. Huang, X. Ye and G.H. Zhang, *A local variational principle for conditional entropy*, Erg. Th. and Dynam. Sys., **26** (2006), 219-245.
[64] W. Huang, X. Ye and G.H. Zhang, *Local entropy theory for countable discrete amenable group actions*, preprint, 2007.
[65] P. Hulse, *Sequence entropy and subsequence generators*, J. London Math. Soc., **26** (1982), 441-450.
[66] T. Kamae, *Maximal pattern complexity as a topological invariant*, Preprint, 2002.
[67] T. Kamae and L. Zamboni, *Sequence entropy and the maximal pattern complexity of infinite words*, Ergod. Th. and Dynam. Sys., **22** (2002), 1191-1199.
[68] T. Kamae and L. Zamboni, *Maximal pattern complexity for discrete systems*, Ergod. Th. and Dynam. Sys., **22** (2002), 1201-1214.

[69] B. Kaminski, A. Siemaszko and J. Szymanski, *The determinism and the Kolmogorov property in topological dynamics*, Bull. Polish Acad. Sci. Math., **51** (2003), 401-417.

[70] B. Kaminski, A. Siemaszko and J. Szymanski, *Extreme relations for topological flows*, Bull. Polish Acad. Sci. Math., **53** (2005), 17-24.

[71] M. G. Karpovsky and V. D. Milman, *Coordinate density of sets of vectors*, Discrete Math., **24** (1978), 177-184.

[72] A. B. Katok, *Lyapunov exponents, entropy and periodic points for diffeomorphisms*, IHES Pub. **51**(1980), 137-173.

[73] D. Kerr and H. Li, *Dynamical entropy in Banach spaces*, Invent. Math., **162** (2005), 649-686.

[74] D. Kerr and H. Li, *Independence in topological and C^*-dynamics*, Math. Ann. **338** (2007), 869-926.

[75] D. Kerr and H. Li, *Combinatorial independence in measurable dynamics*, math.DS (math.FA). arXiv:0705.3424.

[76] A. Köhler, *Enveloping semigroups for flows*, Proceedings of the Royal Irish Academy, **95A**, (1995), 179-191.

[77] A. N. Kolmogorov, *A new metric invariant of transient dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Akad. Sci. SSSR., **119** (1958), 861-864(Russian).

[78] R. Kuang and X. Ye, *The return times set and mixing for mpt*, Dis. and Cont. Dynam. Sys., to appear.

[79] A. G. Kushnirenko, *On metric invariants of entropy type*, Russian Math. Surveys **22**, 1967, No.5, 53-61.

[80] M. Lemanczyk and A. Siemaszko, *A note on the existence of a largest topological factor with zero entropy*, Proc. Amer. Math. Soc., **129** (2001), 475-482.

[81] A. Maass and S. Shao, *Sequence entropy in minimal systems*, to appear in J. London of Math. Soc..

[82] W. Parry, *Entropy and generators in ergodic theory*, Benjamin, New York, 1969.

[83] K. K. Park and A. Siemaszko, *Relative topological Pinsker factors and entropy pairs*, Monatsh. Math., **134** (2001), 67-79.

[84] D. S. Ornstein, *Bernoulli shifts with the same entropy are isomorphic*, Adv. in Math, **4** (1970), 337-352.

[85] K. Petersen, *Disjointness and weak mixing of minimal sets*, Proc. Amer. Math. Soc., **24** (1970), 278-280.

[86] V. A. Rohklin, *Lectures on ergodic theory*, Russian Math . Surveys, **22** (1967), 1-52.

[87] V. A. Rohklin and Ya. G. Sinai, *Construction and properties of measurable partitions*, Dokl. Akad. Nauk SSSR **141** (1961), 1038-1041 (in Russian).

[88] P. Romagnoli, *A local variational principle for the topological entropy*, Ergodic Th. and Dynam. Sys., **23** (2003), 1601-1610.

[89] H. P. Rosenthal, *A characterization of Banach spaces containing ℓ_1*, Proc. Nat. Acad. Sci. (USA) **71** (1974), 2411-2413.

[90] A. Saleski, *Sequence entropy and mixing*, J. of Math. Anal. and Appli., **60** (1977), 58-66.

[91] N. Sauer, *On the density of families of sets*, J. Combinatorial Theory Ser. A, **23** (1972), 145-147.

[92] U. Shapira, *Measure theoretical entropy for covers*, Isr. J. of Math, **158**(2007), 225-247.

[93] S. Shelah, *A combinatorial problem; stability and order for models and theories in infinitary languages*, Pacific J. Math., **41** (1972), 247-261.

[94] S. Shao, X. Ye and R. Zhang, *Sensitivity and regionally proximal relation in minimal systems*, Science in China, to appear.

[95] B. Song and X. Ye, *Minimal c.p.e. not u.p.e. systems*, J. Diff. Equa. and Appli. (a special issue on Combinatorial and Topological Dynamics), to appear.

[96] S. Todorčević, *Topics in topology*, Springer-Verlag, Lecture Notes in Mathematics, **1652**, 1997.

[97] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York, 1982.

[98] B. Weiss, *Countable generators in dynamics - Universal minimal models*, AMS Contemporary Mathematics, Proc. of Conf. honoring Dorothy Maharam Stone. **94** (1989), 321-326.
[99] B. Weiss, *Strictly ergodic models for dynamical systems*, Bulletin of the Amer. Math. Soc., **13** (1985), 143-146.
[100] B. Weiss, *Single Orbit Dynamics*, AMS Regional Conference Series in Mathematics, vol. **95**, 2000.
[101] B. Weiss, *Multiple recurrence and doubly minimal systems*, Contemporary Mathematics **215**, (1998), 189-196.
[102] J. Xiong, *Chaos in topological transitive systems*, Science in China, **48**(2005), 929-939.
[103] X. Ye and G. H. Zhang, *Entropy points and applications*, Trans. Amer. Math. Soc., to appear.
[104] X. Ye and R. F. Zhang, *On sensitive sets in topological dynamics*, preprint, 2006.
[105] G. H. Zhang, *Relative entropy, asymptotic pairs and chaos*, J. of Lon. Math. Soc., **73** (2006), 157-172.
[106] G. H. Zhang, *Relativization of complexity and sensitivity*, Erg. Th. and Dynam. Sys., **27** (2007), 1349-1371.

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel
E-mail address: glasner@math.tau.ac.il

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China
E-mail address: yexd@ustc.edu.cn