# MINIMAL ACTIONS OF THE GROUP $\mathbb{S}(\mathbb{Z})$ OF PERMUTATIONS OF THE INTEGERS

E. GLASNER AND B. WEISS

Abstract. Each topological group $G$ admits a unique universal minimal dynamical system $(M(G), G)$. For a locally compact non-compact group this is a nonmetrizable system with a very rich structure, on which $G$ acts effectively. However there are topological groups for which $M(G)$ is the trivial one point system (extremely amenable groups), as well as topological groups $G$ for which $M(G)$ is a metrizable space and for which one has an explicit description. One such group is the topological group $\mathbb{S}$ of all the permutations of the integers $\mathbb{Z}$, with the topology of pointwise convergence. In this paper we show that $(M(\mathbb{S}), \mathbb{S})$ is a symbolic dynamical system (hence in particular $M(\mathbb{S})$ is a Cantor set), and we give a full description of all its symbolic factors.

## 0. Introduction

Let $G$ be a topological group, $X$ a compact Hausdorff space. A dynamical system $(X, G)$ is given by a jointly continuous action of $G$ on $X$. If $(Y, G)$ is a second dynamical system then a continuous onto map $\pi : (X, G) \to (Y, G)$ which intertwines the $G$ actions is called *a homomorphism*. The dynamical system $(X, G)$ is *minimal* if every orbit is dense. The following fact is well known (see e.g. [4]).

*There exists a unique universal minimal dynamical system,* $(M(G), G)$*; i.e. for every minimal dynamical system* $(X, G)$ *there exists* $\pi : (M, G) \to (X, G)$*, and any two such universal systems are isomorphic. Moreover the dynamical system* $(M(G), G)$ *is coalescent; i.e. every endomorphism* $\phi : (M(G), G) \to (M(G), G)$ *(which is necessarily onto) is an isomorphism onto.*

The existence of a vast collection of non-isomorphic minimal $\mathbb{Z}$-systems means that $M(\mathbb{Z})$ is a huge space; for example one can show that it has the structure of a semigroup and with respect to this structure it has $2^{2^{\aleph_0}}$ distinct idempotents. This fact reflects the extremely complex nature of the proximal relation in some minimal dynamical systems.

Nonetheless the universal minimal system $M(\mathbb{Z})$ is restricted in the following strong sense. We say that a function $f \in \ell^\infty(\mathbb{Z})$ *comes* from the dynamical system $(X, \mathbb{Z})$ if there exists a point $x_0 \in X$ and a continuous function $F : X \to \mathbb{R}$ such that $f(n) = F(T^n x_0)$, where $T : X \to X$ generates the $\mathbb{Z}$-action. The map $F \mapsto f$ defines an isometric isomorphism of the Banach algebra $C(X)$ onto its image $\mathcal{A}_{x_0}$, a closed sub-algebra of $\ell^\infty(\mathbb{Z})$. If we choose another "base" point $x_1 \in X$ we get another sub-algebra $\mathcal{A}_{x_1}$ and in general sums and products of functions in $\mathcal{A}_{x_0}$ and $\mathcal{A}_{x_1}$ are no longer elements of either sub-algebra and, in fact, need not

*Date*: November 29, 2001.
1991 *Mathematics Subject Classification.* 22A05, 22A10, 54H20.

come from $(X,\mathbb{Z})$. Thus usually the closed algebra $\mathcal{A}(X,\mathbb{Z})$ which is generated by $\bigcup\{\mathcal{A}_x : x \in X\}$ — that is by all the functions coming from $(X,\mathbb{Z})$ — is very large.

If $\mathcal{A}$ is a translation invariant closed subalgebra of $\ell^\infty(\mathbb{Z})$ we say that a subset $A$ of $\mathbb{Z}$ is an $\mathcal{A}$-*interpolation set* if every bounded real valued function on $A$ can be extended to a function in $\mathcal{A}$. A subset $A$ of $\mathbb{Z}$ is called *small* if for every $k > 0$ there exists $N_k > 0$ such that every interval of length $N_k$ in $\mathbb{Z}$ contains an interval of length $k$ which does not meet $A$. The following theorem is proved in [9]. The last assertion of the theorem provides a negative answer to questions of H. Furstenberg and R. Ellis (see [8]).

0.1. **Theorem.** *Let $\mathcal{A}$ be the closed algebra of $\ell^\infty(\mathbb{Z})$ generated by functions coming from the dynamical system $M(\mathbb{Z})$. A subset $A$ of $\mathbb{Z}$ is small iff it is an $\mathcal{A}$-interpolation set. In particular $\mathcal{A} \subsetneqq \ell^\infty(\mathbb{Z})$.*

The topological group $G$ has the *fixed point on compacta property* (f.p.c.) (or is *extremely amenable*) if whenever it acts continuously on a compact space, it has a fixed point. Thus the group $G$ has the f.p.c. property iff its universal minimal dynamical system is the trivial one point system.

Let $(X, d, \mu)$ be a metric space with probability measure (such a triple is called an *mm-space*). For $A \subseteq X$, and $\epsilon > 0$ let $A_\epsilon$ be the set of all points whose distance from $A$ is at most $\epsilon$.

The *concentration function*:

$$\alpha(\epsilon) = 1 - \inf\{\mu(A_\epsilon) : A \subseteq X,\ \mu(A) \geq 1/2\},$$

is the least upper bound of measures of all the 'caps' $X \setminus A_\epsilon$ for $A \subseteq X$, $\mu(A) \geq 1/2$.

A family of *mm* spaces $(X_n, d_n, \mu_n)$ is called a *Lévy family* if for every $\epsilon$, $\alpha_n(\epsilon) \to 0$. When $G$ is a topological group having compact subgroups $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq G$, such that, with respect to a global metric on $G$ and with the sequence of Haar measures $\mu_n$, it is a Lévy family and moreover $\cup_{i=1}^\infty G_n$ is everywhere dense, then $G$ is called a *Lévy group*. We refer to [16] and [10] as good references to this theory. The classical example for a Lévy family is the family of unit spheres $\{S^n\}_{n\in\mathbb{N}}$ with rotation-invariant probability measures and Euclidean (or geodesic) distances.

The following theorem of Gromov and Milman, [11], is the source of many examples of topological groups with the f.p.c. property.

0.2. **Theorem** (Gromov–Milman). *A Lévy group $G$ has the f.p.c. property.*

The following list contains most of the known examples of extremely amenable groups.

- The unitary group $U(\infty) = \cup_{n=1}^\infty U(n)$ with the uniform operator topology (Gromov–Milman, [11]).
- The monothetic Polish group $L_m(\mathbb{I}, S^1)$, consisting of all (classes) of measurable maps from the unit interval $\mathbb{I}$ into the circle group $S^1$ with the topology of convergence in measure induced by, say, Lebesgue measure on $\mathbb{I}$ (Glasner, [8]; Furstenberg–Weiss). More generally, $L_m(\mathbb{I}, G)$, where $G$ is any locally compact amenable group (Pestov, [21]).
- The group of measurable automorphisms $\mathrm{Aut}\,(\mathrm{X}, \mu)$ of a standard sigma-finite measure space $(X, \mu)$, with respect to the weak topology (Giordano–Pestov [6]).

• Using Ramsey's theorem (rather than a concentration phenomenon) Pestov has shown that the group $\mathrm{Aut}\,(\mathbb{Q},<)$, of order automorphism of the rational numbers with pointwise convergence topology, is extremely amenable, [19].

The latter result was used by Pestov to demonstrate that the universal minimal dynamical system $(M(G),G)$ for the group $G=\mathrm{Homeo}_+(S^1)$, of orientation-preserving homeomorphisms of the circle with the compact open topology, coincides with the natural action of $G$ on $S^1$. This provided the first non-trivial example of a *metrizable* universal minimal system, [19]. In [22] V. Uspenskij shows that the action of a topological group $G$ on its universal minimal system $M(G)$ is never 3-transitive (in fact, can not satisfy the following property: for every three points $x,y,z\in M(G)$ there is $g\in G$ such that $gx=x, gy=z$ and $gz=y$). It thus follows that, e.g., for manifolds $X$ of dimension $>1$ as well as for $X=Q$, the Hilbert cube, with $G=\mathrm{Homeo}_+(X)$, $(M(G),G)$ does not coincide with the natural action of $G$ on $X$.

Let $\mathbb{S}=\mathbb{S}(\mathbb{Z})$ be the group of all permutations of the integers $\mathbb{Z}$. With respect to the topology of pointwise convergence $\mathbb{S}$ is a Polish topological group. The subgroup $\mathbb{S}_0\subset\mathbb{S}$ consisting of the permutations which fix all but a finite set in $\mathbb{Z}$ is an amenable dense subgroup (being the union of an increasing sequence of finite groups) and therefore $\mathbb{S}$ is amenable as well.

In [11] Gromov and Milman conjectured, in view of the concentration of measure on $S_n$ with Hamming distance, that $\mathbb{S}_0$ has the f.p.c? In [19] and [20] V. Pestov has shown that, on the contrary, $\mathbb{S}$ acts effectively on $M(\mathbb{S})$ and that, in fact, there is no Hausdorff topology making $\mathbb{S}_0$ a topological group with the f.p.c. property. He as well as A. Kechris (in private communication) asked for explicit examples of $\mathbb{S}$-minimal systems.

The main purpose of the present work is to provide an explicit description of the universal minimal system $(M(\mathbb{S}),\mathbb{S})$, which turns out to be the Cantor set. In fact we shall show that it is a 'symbolic' dynamical system, and we shall also provide concrete formulas for all of its symbolic factors.

For every integer $k\ge 2$ let

$$\mathbb{Z}^k_*=\{(i_1,i_2,\dots,i_k)\in\mathbb{Z}^k : i_1,i_2,\dots,i_k \ \text{ are distinct elements of } \mathbb{Z}\},$$

and set $\Omega^k=\{1,-1\}^{\mathbb{Z}^k_*}$. Consider the dynamical system $(\Omega^k,\mathbb{S})$, where for $\alpha\in\mathbb{S}$ and $\omega\in\Omega^k$ we let

$$(\alpha\omega)(i_1,i_2,\dots,i_k)=\omega(\alpha^{-1}i_1,\alpha^{-1}i_2,\dots,\alpha^{-1}i_k).$$

Let $\Omega^k_{alt}\subset\Omega^k$ consist of all the *alternating* configurations, that is those elements $\omega\in\Omega^k$ satisfying

$$\omega(\sigma(i_1),\sigma(i_2),\dots,\sigma(i_k))=\mathrm{sgn}\,(\sigma)\omega(i_1,i_2,\dots,i_k),$$

for all $\sigma\in S_k$ and $(i_1,i_2,\dots,i_k)\in\mathbb{Z}^k_*$. Clearly $\Omega^k_{alt}$is a closed and $\mathbb{S}$-invariant subset of $\Omega^k$.

A configuration $\omega \in \Omega^2$ *determines a linear order* on $\mathbb{Z}$ if it is alternating, and satisfies the conditions:

$$\omega(m,n) = 1 \ \wedge \ \omega(n,l) = 1 \qquad \Rightarrow \qquad \omega(m,l) = 1.$$

Let $<_\omega$ be the corresponding linear order on $\mathbb{Z}$, where $m <_\omega n$ iff $\omega(m,n) = 1$. Let $X = \Omega^2_{lo}$ be the subset of $\Omega^2$ consisting of all the configurations which determine a linear order. The correspondence $\omega \longleftrightarrow <_\omega$ is a surjective bijection between $\Omega^2_{lo}$ and the collection of linear orders on $\mathbb{Z}$. Clearly $X$ is a closed $\mathbb{S}$-invariant set and using Ramsey's theorem we shall show that $(X, \mathbb{S})$ is a minimal system.

Say that a configuration $\omega \in \Omega^3$ is *determined by a circular order* if there exists a sequence $\{z_m : m \in \mathbb{Z}\} \subset S^1$ with $m \neq n \quad \Rightarrow \quad z_m \neq z_n$ such that: $\omega(l,m,n) = 1$ for $(l,m,n) \in \mathbb{Z}^3_*$ iff the directed arc in $S^1$ defined by the ordered triple $(z_l, z_m, z_n)$ is oriented in the positive direction. Let $Y = \Omega^3_c$ denote the collection of all the configurations in $\Omega^3$ which are determined by a circular order. It follows that the set $Y = \Omega^3_c$ is closed and invariant and using Ramsey's theorem one can show that it is minimal.

If we go now to $\Omega^4_{alt}$, can one find a sequence of points $\{z_n\}$ on the sphere $S^2$ in general position such that the tetrahedron defined by any four points $z_{n_1}, z_{n_2}, z_{n_3}, z_{n_4}$ has positive orientation when $n_1 < n_2 < n_3 < n_4$?

Starting with any sequence $\{z_n\} \subset S^2$ in general position one can use Ramsey's theorem to find a subsequence with the required property. Another way to see this is to use the 'moment curve'

$$t \mapsto (t, t^2, t^3).$$

Again it turns out that the orbit closure in $\Omega^4_{alt}$ which is determined by such a sequence forms a minimal dynamical system.

It now seems as if going up to $\Omega^k_{alt}$ with larger and larger $k$'s we encounter more and more complicated minimal systems. However, as we shall see this is not the case and the entire story is already encoded in the simplest symbolic dynamical system $\Omega^2_{lo}$.

0.3. **Theorem.** *$\Omega^2_{lo}$ is the universal minimal $\mathbb{S}$-system.*

The key result needed for the proof of the universality theorem is the following:

0.4. **Theorem.** *Every minimal subsystem $\Sigma$ of the system $(\Omega^k, \mathbb{S})$ is a factor of the minimal system $(\Omega^2_{lo}, \mathbb{S})$.*

Once this is proven the rest follows from some general results about topological groups, and by running along various commutative diagrams of homomorphisms of dynamical systems.

In Section 8 we show that the dynamical system $(\Omega^2_{lo}, \mathbb{S})$ is in fact uniquely ergodic; thus showing that every minimal $\mathbb{S}$ system is uniquely ergodic. In the last section we consider a natural minimal action of the *discrete* group $\mathbb{S}$ (in fact $\mathbb{S}/\mathbb{S}_0$) and, using Uspenskij's 'maximal chains' construction, show that it is not the universal minimal action of $\mathbb{S}/\mathbb{S}_0$.

We would like to thank V. Pestov for his contribution to this paper. By providing crucial information, by asking the right questions and most of all by the clarity and beauty of his recent papers on the subject, he stimulated the writing of the present work. The first named author thanks the Marsden Fund of the Royal Society of New Zealand for supporting his visit to V. Pestov at Wellington, N.Z. during February 2001.

## 1. The topological group $\mathbb{S}$

Let $\mathbb{S} = \mathbb{S}(\mathbb{Z})$ be the group of all permutations of the integers $\mathbb{Z}$. With respect to the topology of pointwise convergence on $\mathbb{Z}$, $\mathbb{S}$ is a Polish topological group. The subgroup $\mathbb{S}_0 \subset \mathbb{S}$ consisting of permutations which fix all but a finite set in $\mathbb{Z}$ is an amenable dense subgroup (being the union of an increasing sequence of finite groups) and therefore $\mathbb{S}$ is amenable as well. The simplest minimal actions of a group are the equicontinuous ones. The existence of non-trivial such actions is equivalent to the existence of non-trivial continuous finite dimensional representations. When a group admits none it is called *minimally almost periodic* (MAP). These were first studied by von Neumann and Wigner [17] who showed that $PSL(2, \mathbb{Q})$ has this property. H. Dye [3] gave examples of countable amenable groups that are MAP. The next theorem shows that the topological group $\mathbb{S}$ is MAP.

1.1. **Theorem.** *The topological group $\mathbb{S}$ admits no non-trivial finite dimensional unitary representations; i.e. it is minimally almost periodic.*

*Proof.* By a theorem of Jordan (see e.g. [2], page 98) to every $n \geq 1$ corresponds a natural number $N = N(n)$ such that for every finite linear group $H \subset GL(n, \mathbb{C})$ there exists a normal abelian subgroup $A \lhd H$ with $[H : A] \leq N$. Assume now that $\psi : \mathbb{S} \to \mathbb{U}(n)$ is a continuous homomorphism where $\mathbb{U}(n)$ is the unitary group in dimension $n$. For any $k \geq 1$ we let $H_k = \psi(S_{2k+1})$, where $S_{2k+1}$ is the symmetric group of order $2k+1$, identified with the subgroup of elements $\sigma \in \mathbb{S}_0$ with $\sigma(j) = j$ for $|j| > k$. Now Jordan's theorem shows that for all sufficiently large $k$, $\psi \restriction S_{2k+1}$ can not be injective. Since $A_{2k+1} \lhd S_{2k+1}$ — the corresponding alternating group — is the only normal subgroup of $S_{2k+1}$ and since $A_{2k+1}$ is simple, we conclude that $A_{2k+1} \subset \ker \psi$ and that the image of $\psi \restriction S_{2k+1}$ consists of at most two points.

Since for every $k$, $S_{2k+1} \subset S_{2k+3}$ and since $\mathbb{S}_0 = \cup_{k \geq 1} S_{2k+1}$, it follows that on $\mathbb{S}_0$ the homomorphism $\psi$ is either trivial or it is the "signum" homomorphism $\psi(\pi) = \operatorname{sgn} \pi$. However the latter does not have a continuous extension to $\mathbb{S}$ (e.g. the sequence $\pi_n = (1, 2)(n, n+1)$ tends to the limit $\pi = (1, 2)$ whereas, $\operatorname{sgn} \pi_{\rm n} = 1$ but $\operatorname{sgn} \pi = -1$). Thus $\psi$ restricted to the dense subgroup $\mathbb{S}_0$ is the trivial homomorphism and therefore so is $\psi : \mathbb{S} \to \mathbb{U}(k)$.

It is well known that a topological group $G$ is minimally almost periodic iff any continuous homomorphism $\psi : G \to \mathbb{U}(n)$ is trivial, whence follows the last assertion of the theorem. ∎

1.2. **Remark.** An alternative proof of this theorem that does not rely on Jordan's theorem may be deduced from the main result of our paper, Theorem 7.2.

## 2. The minimal set $\Omega^2_{lo}$

Let $\mathbb{Z}^2_* = \{(m,n) \in \mathbb{Z}^2 : m \neq n\}$ and set $\Omega^2 = \{1,-1\}^{\mathbb{Z}^2_*}$. Endow $\Omega^2$ with the (compact metric) topology of pointwise convergence and consider the dynamical system $(\Omega^2, \mathbb{S})$, where for $\alpha \in \mathbb{S}$ and $\omega \in \Omega^2$ we let $(\alpha\omega)(m,n) = \omega(\alpha^{-1}(m), \alpha^{-1}(n))$. We call the elements of $\Omega^2$ *configurations*. Let $\Omega^2_{alt} \subset \Omega^2$ consist of all the *alternating* configurations, that is those elements $\omega \in \Omega^2$ satisfying $\omega(n,m) = -\omega(m,n)$ for all $(m,n) \in \mathbb{Z}^2_*$. Similarly let $\Omega^2_{sym} = \{\omega \in \Omega : \omega(n,m) = -\omega(m,n)\}$ be the set of all *symmetric* configurations. Clearly $\Omega^2_{alt}$ and $\Omega^2_{sym}$ are closed and $\mathbb{S}$-invariant subsets of $\Omega^2$. We say that a configuration $\omega \in \Omega^2$ *determines a linear order* on $\mathbb{Z}$ if it is alternating, and satisfies the conditions:

$$\omega(m,n) = 1 \ \wedge \ \omega(n,l) = 1 \qquad \Rightarrow \qquad \omega(m,l) = 1,$$

for all $m, n$ and $l$ distinct elements of $\mathbb{Z}$. We let $<_\omega$ be the corresponding linear order on $\mathbb{Z}$, where $m < n$ iff $\omega(m,n) = 1$. Let $X = \Omega^2_{lo}$ be the subset of $\Omega^2$ consisting of all the configurations which determine a linear order. Of course any linear order on $\mathbb{Z}$ also defines a configuration in $X$ and the correspondence $\omega \longleftrightarrow <_\omega$ is a surjective bijection between $\Omega^2_{lo}$ and collection of linear orders on $\mathbb{Z}$. Again it is easy to see that $X$ is a closed and $\mathbb{S}$-invariant subset of $\Omega^2_{alt}$. Finally let $\mathbf{1}$ and $-\mathbf{1}$ denote the constant configurations with values 1 and $-1$ respectively.

2.1. **Theorem.** *The subsystem* $(X, \mathbb{S})$ *is minimal and the subsystems* $\{\mathbf{1}\}, \{-\mathbf{1}\}$ *and* $(X, \mathbb{S})$ *are the only minimal subsystems of* $(\Omega^2, \mathbb{S})$.

*Proof.* 1. We shall show that for every $\omega \in \Omega^2$ the orbit closure of $\omega$, $\bar{\mathcal{O}}_{\mathbb{S}}(\omega) = \mathrm{cls}\,\{\alpha\omega : \alpha \in \mathbb{S}\}$, meets at least one of the closed invariant sets $\Omega^2_{sym}$ or $\Omega^2_{alt}$.

In fact fixing $\omega \in \Omega^2$ we define a coloring of the set $P = \{\{m,n\} : (m,n) \in \mathbb{Z}^2_*\}$ of unordered pairs of distinct integers as follows. The pair $\{m,n\}$ is colored black if $\omega(m,n) = \omega(n,m)$ and otherwise it is colored red. An application of Ramsey's theorem yields an infinite subset $A$ of $\mathbb{Z}$ such that the subset $P_A = \{\{m,n\} : m, n$ are distinct elements of $A\}$ of $P$ is mono-chromatic, either black or red.

Let $m \leftrightarrow a_m$ be an arbitrary 1-1 correspondence of $A$ with $\mathbb{Z}$, $A = \{a_m : m \in \mathbb{Z}\}$ and choose a sequence of permutations $\alpha_n$ in $\mathbb{S}$ such that $\alpha_n^{-1}(m) = a_m$ for $m \in [-n, n]$. We have $(\alpha_n\omega)(k,l) = \omega(a_k, a_l)$ for all $k, l$ distinct members of the interval $[-n, n]$, so that clearly $\lim_{n\to\infty} \alpha_n\omega$ exists and is either in $\Omega^2_s$ or in $\Omega^2_{alt}$ according to whether $P_A$ is black or red.

2. Let $\omega$ be an element of $\Omega^2_{sym}$. Again we use $\omega$ to define a coloring of $P$ as follows. The pair $\{m,n\}$ is colored black if $\omega(m,n) = \omega(n,m) = 1$, otherwise it is colored red. Again we deduce that there exists an infinite $A \subset \mathbb{Z}$ for which $P_A$ is mono-chromatic and as above we conclude that $\omega' = \lim_{n\to\infty} \alpha_n\omega$ exists for a suitable sequence of permutations $\alpha_n \in \mathbb{S}$, where now $\omega'$ is either $\mathbf{1}$ or $-\mathbf{1}$.

3. Let $\xi_0 \in X$ be the configuration that determines the natural order on $\mathbb{Z}$; that is $\xi_0(m,n) = 1$ iff $m < n$. Let $\omega$ be an element of $\Omega_{alt}$. This time the pair $\{m,n\}$ is colored black if $\omega(m,n) = -\omega(n,m) = 1$ and red otherwise. As above we see that either $\xi_0$ or $-\xi_0$ is in $\bar{\mathcal{O}}_{\mathbb{S}}(\omega)$.

4. Define a sequence of permutations $\alpha_n \in \mathbb{S}_0$ by

$$\alpha_n(j) = -j, \qquad \text{for } j \in [-n,n], \quad \text{and} \quad \alpha_n(j) = j \text{ otherwise};$$

then clearly $\lim_{n\to\infty} \alpha_n(\xi_0) = -\xi_0$.

5. Our proof will be complete when we show that $X$ is indeed minimal. From steps 3. and 4. it follows that $\xi_0 \in \bar{\mathcal{O}}_{\mathbb{S}}(\omega)$ for every $\omega \in X$ and thus it suffices to verify that every $\xi \in X$ is in $\bar{\mathcal{O}}_{\mathbb{S}}(\xi_0)$. This however is easy to see since if $\xi$ determines a linear order (i.e. $\xi$ is an element of $X$) then for every $n$ we can find a permutation $\alpha_n \in \mathbb{S}$ for which $\xi(k,l) = x_0(\alpha_n^{-1}(k), \alpha_n^{-1}(l))$ for all $k,l$ distinct members of the interval $[-n,n]$ (all the linear orders on a finite set are equivalent). Now $\xi = \lim_{n\to\infty} \alpha_n \xi_0$ and the proof is complete. □

2.2. **Remark.** Let $\omega \in X = \Omega^2_{lo}$ be determined by a linear order $<_\omega$. Observe that the stability subgroup

$$H_\omega = \{\alpha \in \mathbb{S} : \alpha\omega = \omega\},$$

is exactly the subgroup $\mathrm{Aut}\,(\mathbb{Z}, <_\omega)$ of all the permutations in $\mathbb{S}$ which preserve the order $<_\omega$. E.g. for the element $\xi_0 \in \Omega^2_{lo}$ which is determined by the usual order on $\mathbb{Z}$ we have

$$T := H_{\xi_0} = \{\alpha \in \mathbb{S} : \alpha\xi_0 = \xi_0\} = \{\alpha_t : t \in \mathbb{Z}\} \cong \mathbb{Z},$$

where $\alpha_t \in \mathbb{S}$ is defined by $\alpha_t(n) = n+t$. Another example is obtained by fixing a 1-1 correspondence of $\mathbb{Z}$ with the set of rational numbers $\mathbb{Q}$ and considering the element $\eta \in \Omega^2_{lo}$ which is determined, via this correspondence, by the usual order on $\mathbb{Q}$. We then have $H_\eta = \mathrm{Aut}\,(\mathbb{Q}, <_\eta)$. By a theorem of Pestov [18] the topological group $G = \mathrm{Aut}\,(\mathbb{Q}, <) \cong \mathrm{Aut}\,(\mathbb{Q}, <_\eta) \subset \mathbb{S}$ (with the topology of pointwise convergence on $\mathbb{Q}$) has the 'fixed point on compacta' property and thus the existence of the $G$-fixed point $\eta \in X$ is a manifestation of this property for the system $(X, \mathrm{Aut}\,(\mathbb{Q}, <_\eta))$.

The next proposition shows that, as far as minimal sets are concerned, on $\mathbb{Z}^2_*$ there is no point in considering an alphabet larger than $\{\pm 1\}$.

2.3. **Proposition.** *A minimal subset $M \subset \{a,b,c\}^{\mathbb{Z}^2_*}$ is a set of configurations on at most two symbols (e.g. $M \subset \{a,b\}^{\mathbb{Z}^2_*}$ ).*

*Proof.* Let $\pi : \{a,b,c\}^{\mathbb{Z}^2_*} \to \{1,-1\}^{\mathbb{Z}^2_*}$ be the map defined by $a \mapsto 1, b \mapsto -1, c \mapsto -1$. Then $\pi(M)$ is a minimal subset of $\{1,-1\}^{\mathbb{Z}^2_*}$ hence is either the singleton $\{\mathbf{1}\}$, the singleton $\{-\mathbf{1}\}$, or the unique alternating minimal set. In the first two cases we are done. In the third let $\omega \in M$ be an element with $\pi(\omega) = \xi_0$ where $\xi_0 \in \Omega^2_{lo}$ is defined by $\xi_0(i,j) = 1$ iff $i < j$. Then for every $i < j$ either $\omega(j,i) = b$ or $\omega(j,i) = c$. Ramsey shows that there is a two-symbol element in the orbit closure of $\omega$. □

Similar results can be obtained for larger alphabets.

## 3. The symbolic system $\Omega^3$

Next let

$$\mathbb{Z}^3_* = \{(k,m,n) \in \mathbb{Z}^3 : k, m \text{ and } n \text{ are distinct elements of } \mathbb{Z}\},$$

and set $\Omega^3 = \{1,-1\}^{\mathbb{Z}^3_*}$. A configuration $\omega \in \Omega^3$ is called *alternating* if it satisfies

$$(\alpha\omega)(l,m,n) = (-1)^{\mathrm{sgn}\,(\alpha)}\omega(l,m,n), \qquad \forall\ (l,m,n) \in \mathbb{Z}^3_*,$$

and we let $\Omega^3_{alt}$ be the collection of all such configurations in $\Omega^3$. Clearly $\Omega^3_{alt}$ is a closed $\mathbb{S}$-invariant set.

We say that a configuration $\omega \in \Omega^3$ is *determined by a circular order* if there exists a sequence $\{z_m : m \in \mathbb{Z}\} \subset S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with $m \neq n \ \Rightarrow \ z_m \neq z_n$ such that $\omega(l,m,n) = 1$ for $(l,m,n) \in \mathbb{Z}^3_*$ iff the directed arc in $S^1$ defined by the ordered triple $(z_l, z_m, z_n)$ is oriented in the positive (i.e. counterclockwise) direction. We let $Y = \Omega^3_c$ denote the collection of all the configurations in $\Omega^3$ which are determined by a circular order. Clearly $\Omega^3_c \subset \Omega^3_{alt}$.

3.1. **Lemma.** *A configuration $\omega \in \Omega^3_{alt}$ is determined by a circular order iff it satisfies the following equations:*

$$(3.1) \quad \omega(l,m,n) = 1 \ \wedge \ \omega(n,k,l) = 1 \quad \Rightarrow \quad \omega(k,l,m) = 1 \ \wedge \ \omega(k,m,n) = 1$$

*It follows that the set $Y = \Omega^3_c$ is closed and invariant.*

*Proof.* The necessity is clear. Suppose $\omega \in \Omega^3_{alt}$ satisfies the equations (3.1). We shall construct inductively a sequence $\{z_n\}_{n \in \mathbb{Z}} \subset S^1$ whose circular order determines $\omega$.

**Step 1:** We first show that for every four elements $(k,l,m,n)$ (for simplicity $(0,1,2,3)$) the finite pattern $\omega \upharpoonright \{0,1,2,3\}^3_*$ is determined by the circular order of four points $z_0, z_1, z_2, z_3$ on $S^1$. Suppose e.g. that $\omega(0,1,2) = 1$ and put down $z_0, z_1, z_2$ in $S^1$ so that the arc $(z_0, z_1, z_2)$ is positively oriented. We shall show that exactly one of the relations

$$\omega(0,3,1) = 1, \quad \omega(1,3,2) = 1, \quad \omega(2,3,0) = 1$$

holds. It follows then that if we choose $z_3$ in the corresponding arc ($\overrightarrow{z_0, z_1}$, if $\omega(0,3,1) = 1$ etc.), all relations will be given by the circular order.

If one of these holds — say $\omega(2,3,0) = 1$ — then by (3.1) the other two fail to hold. It remains to see what happens when

$$\omega(0,3,1) = -1, \quad \omega(1,3,2) = -1, \quad \omega(2,3,0) = -1.$$

In that case we get, by the alternating property, $\omega(1,2,3) = 1 \wedge \ \omega(3,0,1) = 1$ and (3.1) implies $\omega(0,2,3) = 1$, hence $\omega(2,3,0) = 1$. This however contradicts our assumption that the relation $\omega(2,3,0) = -1$ holds and the proof of step 1. is complete.

**Step 2:** Assume, by induction, that for some $N \geq 4$ the restriction of $\omega$ to each $N$-set can be represented by a circular order. We must show the same for $N+1$ sets. Take $0, 1, 2, \dots, N$. Find $z_1, z_2, \dots, z_N$ to represent $1, 2, \dots, N$ and assume with no loss in generality that the arc $\overrightarrow{z_1, z_2, \dots, z_N}$ is positively oriented. Also find $z_0, z'_2, \dots, z'_N$ to represent $0, 2, \dots, N$. We may assume that $z_2 = z'_2, z_3 = z'_3, \dots, z_N = z'_N$ since the circular order is unique. We now consider two cases:

Case I: The points $z_0$ and $z_1$ fall in different intervals defined by $\{z_2, z_3, \dots, z_N\}$. In this case it is easy to see that we have found a circular imbedding consistent with the data. In fact if $z_0$ is in the arc $\overrightarrow{z_j, z_{j+1}}$ and $1 \leq i \leq N$, we have two subcases to consider. The first is $\omega(1,i,j) = 1$. This together with $\omega(j,0,1) = 1$ yields $\omega(1,i,0) = 1$, as required. The other subcase, namely $\omega(j+1,i,1) = 1$, is treated similarly. (We assumed, e.g. that $z_j \neq z_1$ and $z_j \neq z_i$, but the cases where equality occurs are similar).

Case II: Both $z_0$ and $z_1$ fall in $\overrightarrow{z_N, z_2}$. By step 1. (the case $N = 4$) there is an imbedding $\{z_N, z_0, z_1, z_2\}$, say with $\overrightarrow{z_N, z_0, z_1, z_2}$. Again we must verify that $\omega(0,1,i) = 1$ for all $3 \leq i \leq N-1$. Since $z_i$ is in the arc $\overrightarrow{z_1, z_N}$ and since $z_0$ is in the arc $\overrightarrow{z_N, z_1}$ we have $\omega(1,i,N) \ \wedge \ \omega(N,0,1)$ and (3.1) implies $\omega(0,1,i) = 1$ as required.

This concludes the proof by induction and shows that (3.1) characterizes the configurations in $Y$. The last assertion of the lemma follows and the proof of the lemma is complete. □

3.2. **Remark.** It was pointed out to us by the referee that this lemma may be deduced from the work of E. V. Huntington on postulate systems for cyclical orders cf. [13].

3.3. **Theorem.** $(Y, \mathbb{S})$ *is a minimal subsystem of* $(\Omega^3, \mathbb{S})$.

*Proof.* As in step 4 of the proof of Theorem 2.1 we observe that our assertion follows from the fact that all finite circular orders are equivalent. □

3.4. **Remark.** Using the notations of Remark 2.2 it is easy to see that there are exactly two fixed points in $X \subset \Omega^2$ for the action of the subgroup $T \subset \mathbb{S}$, namely $\xi_0$ and $-\xi_0$. On the other hand there are infinitely many fixed points for $T$ in the system $Y \subset \Omega^3$. E.g. given any irrational number $\xi \in \mathbb{R}$ let $z_n = e^{2\pi i n\xi}$, $(n \in \mathbb{Z})$ and define $\omega_\xi \in Y$ by the requirement that $\omega_\xi(l, m, n) = 1$ iff $(z_l, z_m, z_n)$ defines a positively oriented arc. Conversely, the order type of the sequence $\{z_n\}_{n\in\mathbb{Z}}$ determines the irrational number $\xi$ (mod 1), as is shown in the study of rotation numbers for circle homeomorphisms. For every irrational $\xi$ and $t \in \mathbb{Z}$ we clearly have $\alpha_t\omega_\xi = \omega_\xi$ and it follows that there are $2^{\aleph_0}$ distinct elements of the form $\omega_\xi$ in $Y$. We conclude that the minimal systems $(X, \mathbb{S})$ and $(Y, \mathbb{S})$ are not isomorphic. However $(Y, S)$ is a factor of $(X, S)$, as the following theorem shows.

3.5. **Theorem.** *The map* $\pi : \Omega^2_{lo} \to \Omega^3_c$ *defined by*

$$(\pi\xi)(i, j, k) = -\xi(i, j)\xi(j, k)\xi(k, i), \qquad (\xi \in X)$$

*is a homomorphism of* $X$ *onto* $Y$.

*Proof.* The continuity of $\pi$ is clear and it is easy to check that $\pi\xi$ is alternating when $\xi$ is. In view of the minimality of $X$ and $Y$, all we need to check is that $\pi$ intertwines the actions of $\mathbb{S}$ on $X$ and $Y$ and that $\eta_0 := \pi\xi_0 \in Y$; i.e. that $\eta_0$ is determined by a circular ordering. In fact for $\alpha \in \mathbb{S}$, $\xi \in X$ and $\eta := \alpha\xi$ we have

$$\begin{aligned}(\alpha(\pi\xi))(i, j, k) &= -\xi(\alpha^{-1}i, \alpha^{-1}j)\xi(\alpha^{-1}j, \alpha^{-1}k)\xi(\alpha^{-1}k, \alpha^{-1}i)\\ &= -(\alpha\xi)(i, j)(\alpha\xi)(j, k)(\alpha\xi)(k, i)\\ &= (\pi(\alpha\xi))(i, j, k),\end{aligned}$$

hence $\pi$ is a homomorphism (of $\Omega^2$ into $\Omega^3$). It remains to check that $\eta_0 = \pi\xi_0$ is determined by a circular order.

Let $z_n = e^{\pi i(\frac{n}{n+1})}$ for $n \geq 0$ and $z_n = \overline{z}_{-n}$ for $n < 0$. The fact that the circular order of the sequence $\{z_n\}_{n\in\mathbb{Z}}$ determines the configuration $\eta_0$ is easy to check. □

The analysis of the next section will enable us to determine all minimal subsets of $\Omega^3$.

## 4. The symbolic systems $\Omega^k$ and their minimal subsets

For every integer $k \geq 2$ let

$$\mathbb{Z}^k_* = \{(i_1, i_2, \dots, i_k) \in \mathbb{Z}^k : i_1, i_2, \dots, i_k \ \text{ are distinct elements of } \mathbb{Z}\}.$$

We write $\mathbb{Z}^{(k)}$ for the collection of *un-ordered* subsets with $k$ distinct elements of $\mathbb{Z}$. Thus the map

$$(i_1, i_2, \dots, i_k) \mapsto \{i_1, i_2, \dots, i_k\},$$

is $k!$ to 1 from $\mathbb{Z}^k_*$ onto $\mathbb{Z}^{(k)}$. Let $\Omega^k = \{1, -1\}^{\mathbb{Z}^k_*}$, and endow it with the (compact metric) topology of pointwise convergence. Consider the dynamical system $(\Omega^k, \mathbb{S})$, where for $\alpha \in \mathbb{S}$ and $\omega \in \Omega^k$ we let

$$(\alpha\omega)(i_1, i_2, \dots, i_k) = \omega(\alpha^{-1} i_1, \alpha^{-1} i_2, \dots, \alpha^{-1} i_k).$$

We call the elements of $\Omega^k$ *configurations*. Let $\Omega^k_{alt} \subset \Omega^k$ consist of all the *alternating* configurations, that is those elements $\omega \in \Omega^k$ satisfying

$$\omega(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)) = \operatorname{sgn}(\sigma)\omega(\mathrm{i}_1, \mathrm{i}_2, \dots, \mathrm{i}_k),$$

for all $\sigma \in S_k$ and $(i_1, i_2, \dots, i_k) \in \mathbb{Z}^k_*$. Clearly $\Omega^k_{alt}$ is a closed and $\mathbb{S}$-invariant subset of $\Omega^k$.

We denote the involution that sends an element $\omega \in \Omega^k$ to $-\omega$ by $\jmath$. Clearly the map $\jmath : \Omega^k \to \Omega^k$ commutes with the action of $\mathbb{S}$; i.e. $\jmath$ is an automorphism of the system $(\Omega^k, \mathbb{S})$. Denoting by $\tilde{\omega}$ the set $\{\omega, -\omega\}$ and by $\tilde{\Omega}^k = \Omega^k / \{\mathrm{id}, \jmath\}$, the corresponding quotient space, we observe that the map $\omega \mapsto \tilde{\omega}$ from $\Omega^k$ to $\tilde{\Omega}^k$ is a group extension.

Set $\mathcal{T} = \{1, -1\}^{S_k}$ and consider the collection $\mathcal{T}^{\mathbb{Z}^{(k)}}$ of all maps from $\mathbb{Z}^{(k)}$ to $\mathcal{T}$. We refer to elements of $\mathcal{T}$ as *tables*. For each $\omega \in \Omega^k$ define an element $\hat{\omega}$ in $\mathcal{T}^{\mathbb{Z}^{(k)}}$ as follows. For $\{i_1, i_2, \dots, i_k\} \in \mathbb{Z}^{(k)}$, with $i_1 < i_2 < \dots < i_k$ and $\sigma \in S_k$ set

$$\hat{\omega}(i_1, i_2, \dots, i_k)(\sigma) = \omega(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(k)}).$$

Clearly this is a continuous correspondence and, in fact, the configuration $\omega \in \Omega^k$ is completely determined by $\hat{\omega} \in \mathcal{T}^{\mathbb{Z}^{(k)}}$ so that $\hat{\omega}$ is just a different way of looking at $\omega$.

We let $S_k$ act on $\mathcal{T}$ by multiplication on the left:

$$(L_\tau T)(\sigma) = T(\tau^{-1}\sigma), \qquad (\tau, \sigma \in S_k).$$

Given $T \in \mathcal{T}$, we set

$$H_T = \{\pi \in S_k : L_\pi T = T\} = \{\pi \in S_k : T(\pi^{-1}\sigma) = T(\sigma), \text{ for every } \sigma \in S_k\}.$$

Clearly $H$ is a subgroup of $S_k$. Note that

$$H_{L_\tau T} = \tau H_T \tau^{-1}.$$

4.1. **Lemma.** *For $\omega \in \Omega^k$, $i_1 < i_2 < \dots < i_k$, $\alpha \in \mathbb{S}$ let $\{j_1, j_2, \dots, j_k\} = \alpha^{-1}\{i_1, i_2, \dots, i_k\}$, with $j_1 < j_2 < \dots < j_k$. Set $T = \hat{\omega}(j_1, j_2, \dots, j_k)$. Then there exists $\tau = \tau(\alpha\,; i_1 < i_2 < \dots < i_k) \in S_k$ such that*

$$\widehat{\alpha\omega}(i_1, i_2, \dots, i_k) = L_{\tau^{-1}} T$$

*or more explicitly:*

$$\widehat{\alpha\omega}(i_1, i_2, \dots, i_k)(\sigma) = T(\tau\sigma) = L_{\tau^{-1}}\hat{\omega}(j_1, j_2, \dots, j_k)(\sigma),$$

*for every* $\sigma \in S_k$.

*Proof.* By definition, for $i_1 < i_2 < \cdots < i_k$,

$$\begin{aligned} \widehat{\alpha\omega}(i_1, i_2, \dots, i_k)(\sigma) &= (\alpha\omega)(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(k)}) \\ &= \omega(\alpha^{-1} i_{\sigma(1)}, \alpha^{-1} i_{\sigma(2)}, \dots, \alpha^{-1} i_{\sigma(k)}) \\ &= \omega(j_{\tau_\sigma(1)}, j_{\tau_\sigma(2)}, \dots, j_{\tau_\sigma(k)}) \\ &= T(\tau_\sigma), \end{aligned}$$

where the permutation $\tau_\sigma = \tau_\sigma(\alpha\,;\, i_1 < i_2 < \cdots < i_k) \in S_k$ is defined by

$$j_{\tau_\sigma(p)} = \alpha^{-1} i_{\sigma(p)}, \qquad p = 1, 2, \dots, k.$$

Since for $\sigma = e$ we have

$$\alpha(j_{\tau_e(p)}) = i_p,$$

it follows that

$$\alpha(j_{\tau_\sigma(p)}) = i_{\sigma(p)} = \alpha(j_{\tau_e \circ \sigma(p)}).$$

Thus we get the identity $\tau_\sigma = \tau_e \circ \sigma$. Taking $\tau = \tau_e$ we obtain the formula claimed in the lemma. $\square$

For a subgroup $H \subset S_k$, we denote the conjugacy class of $H$ by $[H]$ and set

$$\begin{aligned} \Omega^k_{[H]} = \{\omega \in \Omega^k : \forall\, \{i_1, i_2, \dots, i_k\} \in \mathbb{Z}^{(k)}, \\ H_T \in [H],\ \text{where } T = \hat{\omega}(i_1, i_2, \dots, i_k)\}. \end{aligned}$$

4.2. **Corollary.** *The subset* $\Omega^k_{[H]} \subset \Omega^k$ *is closed and* $\mathbb{S}$*-invariant.*

Given an element $\tau \in S_k$ we define an automorphism $\tilde{\tau} : \Omega^k \to \Omega^k$ as follows:

$$(\tilde{\tau}(\omega))(i_1, i_2, \dots, i_k) = \omega(i_{\tau 1}, i_{\tau 2}, \dots, i_{\tau k}), \qquad \forall\, (i_1, i_2, \dots, i_k) \in \mathbb{Z}^k_*$$

In terms of the representation $\hat{\omega}$ we have

$$\widehat{(\tilde{\tau}(\omega))}(\sigma) = \widehat{\omega}(\sigma \circ \tau).$$

It is easy to check that $\tilde{\tau}$ is indeed an automorphism of the system $(\Omega^k, \mathbb{S})$. It is also clear that each $\Omega^k_{[H]}$ is $\tilde{\tau}$-invariant.

We denote by $\mathrm{gr}\,(\tilde{\tau})$ the graph $\{(\omega, \tilde{\tau}(\omega)) : \omega \in \Omega^k\}$ of the automorphism $\tilde{\tau}$.

4.3. **Theorem.** (1) *For every minimal set* $\Sigma \subset \Omega^k$ *there exists a subgroup* $H \subset S_k$ *with* $\Sigma \subset \Omega^k_{[H]}$ *and there exists a configuration* $\omega_0 \in \Sigma$ *for which* $\hat{\omega}_0 : \mathbb{Z}^k_* \to \mathcal{T}$ *has a constant value* $T \in \mathcal{T}$ *independent of the set of indices* $\{i_1, i_2, \dots, i_k\}$ *and such that* $H_T = H$.

(2) *There is only a finite number* ($\le 2^{k!}$) *of minimal sets in each* $\Omega^k_{[H]}$*, and hence in* $\Omega^k$.

(3) *Let* $H$ *be a subgroup of* $S_k$ *and* $\Sigma \subset \Omega^k_{[H]}$ *a minimal subset. For every pair* $(\omega, \omega') \in \Sigma \times \Sigma$ *there exists a permutation* $\tau \in S_k$ *with* $\bar{\mathcal{O}}_{\mathbb{S}}(\omega, \omega') \cap \mathrm{gr}\,(\tilde{\tau}) \neq \emptyset$.

(4) *Set*

$$K = \{\tilde{\tau} \in S_k : \tau \in S_k \text{ and } \tilde{\tau}\Sigma \cap \Sigma \neq \emptyset\},$$

*Then* $K = \mathrm{Aut}\,(\Sigma, \mathbb{S})$*, the automorphism group of the minimal system* $(\Sigma, \mathbb{S})$ *and* $\Sigma \to \tilde{\Sigma} := \Sigma/K$ *is a group extension with* $\tilde{\Sigma}$ *a proximal system. (In fact, as we shall see later, this group extension is either two-to-one or trivial one-to-one).*

*Proof.* 1. Given $\omega \in \Omega^k$ we consider $\hat{\omega} : \mathbb{Z}^{(k)} \to \mathcal{T}$ as coloring of $\mathbb{Z}^{(k)}$ and then Ramsey's theorem yields an infinite set $J \subset \mathbb{Z}$ and a fixed element $T \in \mathcal{T}$ with $\hat{\omega}(i_1, i_2, \dots, i_k)(\sigma) = T(\sigma)$, for all $\sigma \in S_k$ and all $\{i_1, i_2, \dots, i_k\} \subset J$. Using an appropriate sequence of permutations $\alpha_n \in \mathbb{S}$ we obtain an element $\omega_0 = \lim_{n\to\infty} \alpha_n \omega$ with

$$\hat{\omega}_0(i_1, i_2, \dots, i_k)(\sigma) = T(\sigma)$$
$$\text{for all } \sigma \in S_k \text{ and } \{i_1, i_2, \dots, i_k\} \in \mathbb{Z}^{(k)}.$$

If $\Sigma$ is a minimal subset of $\Omega^k$ and we start with a configuration $\omega \in \Sigma$ we obtain this way a configuration $\omega_0 \in \Sigma$ for which the corresponding $\hat{\omega}_0 : \mathbb{Z}^k_* \to \mathcal{T}$ has a constant value $T \in \mathcal{T}$ independent of the set of indices $\{i_1, i_2, \dots, i_k\}$. From the first part of the proof we deduce that there exists a subgroup $H \subset S_k$, namely $H = H_T$, with $\Sigma \subset \Omega^k_{[H]}$.

2. There are $2^{k!}$ tables.

3. Given $(\omega, \omega') \in \Sigma \times \Sigma$ we note that, since $\omega' = \lim_{n\to\infty} \alpha_n \omega$ for some sequence $\alpha_n \in \mathbb{S}$, for a fixed $k$-tuple $(i_1, i_2, \dots, i_k) \in \mathbb{Z}^k_*$, eventually $\omega'(i_1, i_2, \dots, i_k) = \omega(\alpha_n^{-1} i_1, \alpha_n^{-1} i_2, \dots, \alpha_n^{-1} i_k)$ is independent of $n$ and, by Lemma 4.1, there exists $\tau \in S_k$ with $\widehat{\omega'} = L_{\tau^{-1}} \hat{\omega}$. Define a map $c : \mathbb{Z}^{(k)} \to S_k$ as follows: for $i_1 < i_2 < \cdots < i_k$

$$c(i_1, i_2, \dots, i_k) = \tau, \quad \text{where} \quad \widehat{\omega'} = L_{\tau^{-1}} \hat{\omega},$$

i.e.

$$\widehat{\omega'}(i_1, i_2, \dots, i_k)(\sigma) = \hat{\omega}(i_1, i_2, \dots, i_k)(\tau\sigma), \quad (\sigma \in S_k).$$

By Ramsey's theorem there exist $\tau \in S_k$ and an infinite $J \subset \mathbb{Z}$ with $c(i_1, i_2, \dots, i_k) = \tau$ for every $\{i_1, i_2, \dots, i_k\} \subset J$. This implies that for some sequence $\alpha_n \in \mathbb{S}$ we have $(\rho, \rho') = \lim_{n\to\infty} \alpha_n(\omega, \omega')$ with

$$\widehat{\rho'}(i_1, i_2, \dots, i_k)(\sigma) = \hat{\rho}(i_1, i_2, \dots, i_k)(\tau\sigma), \quad (\sigma \in S_k).$$

for every $(i_1, i_2, \dots, i_k) \in \mathbb{Z}^k_*$; i.e. $(\rho, \rho') \in \mathrm{gr}\,(\tilde{\tau})$.

4. By the minimality of $\Sigma$, $\tilde{\tau}\Sigma \cap \Sigma \neq \emptyset$ for $\tau \in S_k$ implies $\tilde{\tau}\Sigma = \Sigma$ and therefore $\tilde{\tau} : \Sigma \to \Sigma$ is an automorphism. From the previous step we conclude that $\mathrm{gr}\,(\tilde{\tau})$, for $\tilde{\tau} \in K$ are the only minimal subsets of $\Sigma \times \Sigma$. It follows that $K$ is the automorphism group of the minimal system $(\Sigma, \mathbb{S})$ and $\Sigma \to \tilde{\Sigma} := \Sigma/K$ is a group extension with $\tilde{\Sigma}$ a proximal system. □

## 5. The symbolic factors of $X = \Omega^2_{lo}$

In this section we show that all the minimal sets constructed so far are factors of $\Omega^2_{lo}$.

5.1**. Theorem.** *Every minimal subset of the system* $(\Omega^k, \mathbb{S})$ *is a factor of the minimal system* $(\Omega^2_{lo}, \mathbb{S})$*.*

*Proof.* Fix a minimal subset $\Sigma \subset \Omega^k$. We shall construct a homomorphism $\phi : \Omega^2_{lo} \to \Sigma$.

1. By Theorem 4.3 there exists a subgroup $H \subset S_k$ such that $\Sigma \subset \Omega^k_{[H]}$ and there exits a configuration $\omega_0 \in \Sigma$ for which the corresponding $\hat{\omega}_0 : \mathbb{Z}^k_* \to \mathcal{T}$ has a constant value $T \in \mathcal{T}$ independent of the set of indices $\{i_1, i_2, \ldots, i_k\}$ and such that $H_T = H$.

2. Define the map $\phi : \Omega^2_{lo} \to \Sigma$ as follows. For $\xi \in \Omega^2_{lo}$ and $(u_1, u_2, \ldots, u_k) \in \mathbb{Z}^k_*$ we recall that $\xi$ determines an order $\underset{\xi}{<}$ and set

$$(\phi\xi)(u_1, u_2, \ldots, u_k) = T(\sigma),$$

where $\sigma \in S_k$ is defined by:

$$u_{\sigma^{-1}(1)} \underset{\xi}{<} u_{\sigma^{-1}(2)} \underset{\xi}{<} \cdots \underset{\xi}{<} u_{\sigma^{-1}(k)}.$$

Note that $\phi\xi_0 = \omega_0$, where $\xi_0 \in \Omega^2_{lo}$ is the configuration determined by the natural order on $\mathbb{Z}$.

3. We need to check that $\phi$ commutes with the $\mathbb{S}$-actions on $\Omega^2_{lo}$ and $\Sigma$. This will establish that $\pi : \Omega^2_{lo} \to \Sigma$ is a homomorphism with $\phi(\xi_0) = \omega_0$. Since both $\Omega^2_{lo}$ and $\Sigma$ are minimal this will complete the proof.

Given $\alpha \in \mathbb{S}$ and $\xi \in \Omega^2_{lo}$, denote $\eta = \alpha\xi$. Now

$$(\phi\alpha\xi)(i_1, i_2, \ldots, i_k) = (\phi\eta)(i_1, i_2, \ldots, i_k) = T(\sigma),$$

where

$$i_{\sigma^{-1}(1)} \underset{\eta}{<} i_{\sigma^{-1}(2)} \underset{\eta}{<} \cdots \underset{\eta}{<} i_{\sigma^{-1}(k)}.$$

The latter conditions hold iff

$$\alpha^{-1} i_{\sigma^{-1}(1)} \underset{\xi}{<} \alpha^{-1} i_{\sigma^{-1}(2)} \underset{\xi}{<} \cdots \underset{\xi}{<} \alpha^{-1} i_{\sigma^{-1}(k)}. \tag{5.1}$$

On the other hand, denoting $\alpha^{-1} i_p = j_p,\ p = 1, 2, \ldots, k$,

$$(\phi\xi)(j_1, j_2, \ldots, j_k) = T(\rho)$$

means

$$j_{\rho^{-1}(1)} \underset{\xi}{<} j_{\rho^{-1}(2)} \underset{\xi}{<} \cdots \underset{\xi}{<} j_{\rho^{-1}(k)}$$

hence we find that if

$$\begin{aligned}(\alpha\phi\xi)(i_1, i_2, \ldots, i_k) &= (\phi\xi)(\alpha^{-1} i_1, \alpha^{-1} i_2, \ldots, \alpha^{-1} i_k)\\ &= (\phi\xi)(j_1, j_2, \ldots, j_k)\\ &= T(\rho)\end{aligned}$$

then

$$j_{\rho^{-1}(1)} \underset{\xi}{<} j_{\rho^{-1}(2)} \underset{\xi}{<} \cdots \underset{\xi}{<} j_{\rho^{-1}(k)}.$$

Since $j_{\rho^{-1}(q)} = \alpha^{-1} i_{\rho^{-1}(q)},\ q = 1, 2, \ldots, k$, we get

$$\alpha^{-1} i_{\rho^{-1}(1)} \underset{\xi}{<} \alpha^{-1} i_{\rho^{-1}(2)} \underset{\xi}{<} \cdots \underset{\xi}{<} \alpha^{-1} i_{\rho^{-1}(k)}. \tag{5.2}$$

Comparing equations (5.1) with (5.2) we get $\rho = \sigma$, hence $(\phi\alpha)(\xi) = (\alpha\phi)(\xi)$ and we conclude that indeed $\phi \circ \alpha = \alpha \circ \phi$. $\square$

5.2. **Corollary.** *Let $\Sigma \subset \Omega^k$ be a minimal subset of the system $(\Omega^k, \mathbb{S})$. Either $(\Sigma, \mathbb{S})$ is proximal or there is an automorphism $\theta \in \mathrm{Aut}\,(\Sigma, \mathbb{S})$ of order two such that $\pi \circ \jmath = \theta \circ \pi$, in which case $\Sigma \to \tilde{\Sigma} := \Sigma/\{\mathrm{id}, \theta\}$ is a two-to-one group extension and $(\tilde{\Sigma}, \mathbb{S})$ is proximal.*

*Proof.* Let $\pi : \Omega^2_{lo} \to \Sigma$ be the homomorphism whose existence is established in Theorem 5.1. Let $\xi$ be an arbitrary point in $\Omega^2_{lo}$ and consider the pair $(\pi(\xi), \pi(\jmath\xi))$. Since $(\xi, \jmath\xi)$ is an almost periodic point of $\Omega^2_{lo} \times \Omega^2_{lo}$ (i.e. it belongs to a minimal subset) it follows that $(\pi(\xi), \pi(\jmath\xi))$ is an almost periodic point of $\Sigma \times \Sigma$ and by Theorem 4.3 we conclude that $\pi(\jmath\xi)) = \tilde{\tau}(\pi(\xi))$ for some $\tilde{\tau} \in K = \mathrm{Aut}\,(\Sigma, \mathbb{S})$. Denote $\theta = \tilde{\tau}$ and observe that, by minimality of $\Sigma$, $\pi \circ \jmath = \theta \circ \pi$. Since $\jmath^2 = \jmath$ it follows that $\theta^2 = \theta$ and the last assertion of the corollary follows. ☐

In section 7 we shall show how the information we have gained about $\Omega^2_{lo}$ suffices to show that all minimal systems are factors of it.

## 6. The universal transitive and minimal systems of a topological group $T$

Let $T$ be a topological group. We write $\mathcal{L}(T)$ for the commutative $C^*$-algebra of bounded left uniformly continuous $\mathbb{C}$-valued functions on $T$ with the norm $\|f\| = \sup_{t\in T} |f(t)|$, and with $f^*(t) = \overline{f(t)}$. Recall that a function $f : T \to \mathbb{C}$ is in $\mathcal{L}(T)$ iff it is bounded and for every $\epsilon > 0$ there exists a symmetric neighborhood $V = V^{-1}$ of the unit element $e \in T$ with

$$st^{-1} \in V \quad \Rightarrow \quad |f(s) - f(t)| < \epsilon.$$

An equivalent condition is $\|L_r f - f\| < \epsilon$ for every $r \in V$, where $L_r f(t) = f(r^{-1}t)$. It is easy to see that $\mathcal{L}(T)$ is right and left $T$-invariant; that is, $f \in \mathcal{L}(T) \Rightarrow R_s f \in \mathcal{L}(T)$ and $L_s f \in \mathcal{L}(T)$, where $R_s f(t) = f(ts)$ and $L_s f(t) = f(s^{-1}t)$. The next lemma is well known and its proof is straightforward.

6.1. **Lemma.** *Let $(X, x_0, T)$ be a pointed $T$-dynamical system (i.e. $X$ is a compact Hausdorff space and the action $(t, x) \mapsto tx$, $T \times X \to X$ is jointly continuous; $x_0 \in X$ is a distinguished point with $\bar{\mathcal{O}}_T(x_0) = X$). Let $F \in C(X)$. Then the function $f = f_{x_0}$ defined by the equation $f_{x_0}(t) = F(tx_0)$ is an element of $\mathcal{L}(T)$. In fact the map*

$$\Phi : F \mapsto f_{x_0}, \qquad \Phi : C(X) \to \mathcal{L}(T),$$

*is a linear isometry of $C^*$-algebras such that for every $s \in T$*

$$\Phi \circ L_s = L_s \circ \Phi.$$

In the situation described in the lemma, we say that the function $f$ is *coming from the pointed system* $(X, x_0, T)$.

Let $\mathbf{L}$ be the Gelfand space corresponding to the $C^*$-algebra $\mathcal{L}(T)$ and $l_0 \in \mathbf{L}$ the multiplicative functional $\ell_0 : f \mapsto f(e)$ corresponding to the evaluation of a function in $\mathcal{L}(T)$ at the identity element $e \in T$.

6.2. **Corollary.** *With the natural action of the group $T$ on the Gelfand space $\mathbf{L}$ and the distinguished point $\ell_0$, the pointed dynamical system $(\mathbf{L}, x_0, T)$ is the universal* *point transitive $T$-system. I.e. to every point transitive $T$-system $(X, x_0, T)$ with distinguished transitive point $x_0$ there exists a unique homomorphism*

$$\phi : (\mathbf{L}, \ell_0, T) \to (X, x_0, T).$$

*Proof.* The map $\phi$ is realized by the dual of the isometric isomorphism $\Phi : C(X) \to \mathfrak{L}(T)$ on the corresponding Gelfand spaces. □

We shall use the notation $|\mathcal{A}|$ for the Gelfand space of a closed $T$-invariant subalgebra $\mathcal{A} \subset \mathfrak{L}(T)$ (i.e. $L_s\mathcal{A} = \mathcal{A}$ for every $s \in T$). Thus with this notation $\mathbf{L} = |\mathfrak{L}(T)|$ and in the above corollary $|\mathcal{A}| \cong X$ where $\mathcal{A} = \Phi(C(X))$.

Let now $\mathbf{M} \subset \mathbf{L}$ be any minimal subset. If $(X, T)$ is a minimal system then the restriction of the map $\phi : \mathbf{L} \to X$ to $\mathbf{M}$ is a homomorphism $\phi : \mathbf{M} \to X$. So in this sense $\mathbf{M}$ is a universal minimal system. It turns out that in fact any two minimal sets $\mathbf{M}_1$ and $\mathbf{M}_2$ of $\mathbf{L}$ are isomorphic as dynamical systems (we shall not prove this fact, see for example [4]). Thus $(\mathbf{M}, T)$ is the unique universal minimal system (although not as a pointed system; fixing a distinguished point $m_0 \in \mathbf{M}$ and given a pointed minimal system $(X, x_0, T)$, a homomorphism $\phi : \mathbf{L} \to X$ with $\phi(m_0) = x_0$ may not exist). The next theorem is due to Pestov [19] (see also [14]).

6.3. **Theorem.** *If the topology of $T$ admits a basis for neighborhoods at $e$ consisting of clopen subgroups, then the topological space $\mathbf{L}$ (and hence also $\mathbf{M}$) is zero dimensional.*

*Proof.* Given a clopen subgroup $H \subset T$ let

$$\mathfrak{L}_H = \{f \in \mathfrak{L}(T) : L_s f = f, \ \forall\ s \in H\}.$$

If $\tilde{f}$ is *any* bounded function on the discrete space $H\backslash T$ then the corresponding lift $f(t) = \tilde{f}(Ht)$ is an element of $\mathfrak{L}_H$ and conversely every element of $\mathfrak{L}_H$ defines a function in $l^\infty(H\backslash T)$. Thus $|\mathfrak{L}_H| \cong \beta(H\backslash T)$ where $\beta(H\backslash T)$ is the Stone-Čech compactification of the discrete space $H\backslash T$; in particular $|\mathfrak{L}_H|$ is zero-dimensional (in fact extremely disconnected when $H\backslash T$ is infinite; see [5]).

Given $f \in \mathfrak{L}(T)$ and $\epsilon > 0$ there exists a clopen subgroup $H \subset T$ with $\sup_{t \in T} \operatorname{diam} \mathrm{f(Ht)} < \epsilon$. The function $\tilde{g} \in l^\infty(H\backslash T)$ defined by $\tilde{g}(Ht) = \inf\{f(rt) : r \in H\}$ lifts to a function $g(t) = \tilde{g}(Ht)$ in $\mathfrak{L}_H$ with $\|g - f\| \le \epsilon$.

It follows that the algebra

$$\bigcup \{\mathfrak{L}_H : H \text{ is a clopen subgroup of } T\},$$

is dense in $\mathfrak{L}(T)$ and by the Stone-Weierstrass theorem, its closure is all of $\mathfrak{L}(T)$. We conclude that

$$\mathbf{L} = \lim_{\leftarrow} |\mathfrak{L}_H|$$

is the inverse limit of the zero dimensional spaces $|\mathfrak{L}_H|$ over the directed system of clopen subgroups $H \subset T$. In particular we conclude that $\mathbf{L}$ and therefore also its subset $\mathbf{M}$ are zero dimensional. □

Again let $(X, T)$ be a $T$ dynamical system and let $F \in C(X)$ be a real valued function. Let $I$ be the interval $[-\|F\|, \|F\|]$, and consider the compact space $I^T$ of all maps from $T$ to $I$ (with the topology of pointwise convergence). We define a map $\psi : X \to I^T$ by $\psi(x) = f_x$; i.e. $f_x(t) = F(tx)$, $(t \in T)$. Let $Y = \psi(X)$. As observed in Lemma 6.1 each $f_x$ is in $\mathfrak{L}$. It is easy to check that $\psi$ is a continuous map and if we let $T$ act on $Y$ according to the formula $tf_x(s) = f(st)$, then $t\psi(x) = tf_x = f_{tx} = \psi(tx)$. In fact we have:

6.4. **Lemma.** *The action of $T$ on $Y$ is jointly continuous and the map $\psi : (X, T) \to (Y, T)$ is a homomorphism of $T$-systems.*

## 7. The universal minimal $\mathbb{S}$-system

We recall that the system of clopen subgroups $H_k = \{\alpha \in \mathbb{S} : \alpha(i) = i, \ \forall\, |i| \le k\}$, $k = 1, 2, \dots$, forms a basis for the topology of $\mathbb{S}$ at the identity $e \in \mathbb{S}$. By Theorem 6.3 it follows that the universal minimal dynamical system $\mathbf{M}$ is zero-dimensional. Let $D \subset \mathbf{M}$ be a clopen subset and $F_D = 2\mathbf{1}_D - \mathbf{1} \in C(\mathbf{M})$, with $\mathbf{1}_D$ the indicator function of $D$. Let $\psi_D$ be the map constructed above (see the paragraph preceding Lemma 6.4) using $\mathbf{1}_D$ as $F$. Fix a point $m_0 \in \mathbf{M}$, set $Y_D = \psi_D(\mathbf{M}), y_D = \psi_D(m_0)$ and let $(Y_D, y_D, \mathbb{S})$ be the corresponding pointed dynamical system as in Lemma 6.4. Of course $(Y_D, y_D, \mathbb{S})$ as a factor of $(\mathbf{M}, m_0, \mathbb{S})$ is minimal.

7.1. **Lemma.** *There exists an integer $k \ge 1$ such that $(Y_D, \mathbb{S})$ is isomorphic to a minimal symbolic system $\Sigma \subset \Omega^{2k+1}$.*

*Proof.* There exists a neighborhood $V$ of $e \in \mathbb{S}$ such that $\gamma D = D$ for every $\gamma \in V$ — for otherwise we would have nets $\gamma_i \to e$ and $x_i \in D$ with $\gamma_i x_i \to y \notin D$, contradicting the continuity of $\mathbf{1}_D$. For some $k \ge 1$ we have $H_k \subset V$ and it follows that $\gamma D = D$ for every $\gamma \in H_k$. Thus $\gamma \in H_k, \alpha \in \mathbb{S}$ imply

$$y_D(\gamma\alpha) = F_D(\gamma\alpha m_0) = F_{\gamma^{-1}D}(\alpha m_0) = F_D(\alpha m_0) = y_D(\alpha),$$

i.e. $y_D$ can be viewed as a function on $H_k\backslash\mathbb{S}$. Now the function

$$H_k\alpha \mapsto (\alpha^{-1}(-k), \dots, \alpha^{-1}(-1), \alpha(0), \alpha^{-1}(1), \dots, \alpha^{-1}(k)),$$

is a bijection of $H_k\backslash\mathbb{S}$ onto $\mathbb{Z}_*^{2k+1}$. This bijection is clearly equivariant with respect to the $\mathbb{S}$ actions and thus we can view $Y_D$ as a subset of $\Omega^{2k+1}$ as claimed. ∎

7.2. **Theorem.** *$\Omega^2_{lo}$ is the universal minimal $\mathbb{S}$-system.*

*Proof.* 1. We denote $\Omega^2_{lo} = X$ and choose $\xi_0 \in X$ as the distinguished point. Since $\mathbf{M}$ is the universal $\mathbb{S}$-minimal system, there exists a homomorphism $\pi : \mathbf{M} \to X$ and we pick some $m_0 \in \mathbf{M}$ with $\pi(m_0) = \xi_0$ as the distinguished point for $\mathbf{M}$.

2. Given a clopen subset $D \subset \mathbf{M}$ consider the following diagram:

$$\begin{array}{ccc} (\mathbf{M}, m_0) & \xrightarrow{\ \pi\ } & (X, \xi_0) \\ {\scriptstyle \psi_D}\downarrow & & \downarrow{\scriptstyle \phi_D} \\ (Y_D, y_D) & & (Y_D, y'_D). \end{array}$$

The homomorphism $\psi_D$ was defined above (see the first paragraph of the section) and $y_D = \psi_D(m_0)$. In view of Lemma 7.1 we can apply Theorem 5.1 to define the homomorphism $\phi_D$, with $y'_D$ defined to be $\phi_D(\xi_0)$.

Now the image $(\psi_D \times (\phi_D \circ \pi))(\mathbf{M}, m_0) = (W, (y_D, y'_D))$, with $W \subset Y_D \times Y_D$, is a minimal subset of the product system $(Y_D \times Y_D, \mathbb{S})$ and we conclude from Corollary 5.2 that either $y_D = y'_D$ or $y'_D = \theta y_D$. In the latter case we replace $\phi_D$ by $\phi_D \circ \jmath$ and it follows that in either case the above diagram can be replaced by

$$\begin{array}{ccc} (\mathbf{M}, m_0) & \xrightarrow{\ \pi\ } & (X, \xi_0) \\ & {\scriptstyle \psi_D} \searrow \quad \swarrow {\scriptstyle \phi_D} & \\ & (Y_D, y_D) & \end{array} .$$

Next form the product space

$$\Pi = \prod \{Y_D : D \text{ a clopen subset of } \mathbf{M}\},$$

and let $\psi : \mathbf{M} \to \Pi$ be the map whose $D$-projection is $\psi_D$ (i.e. $(\psi(m))_D = \psi_D(m)$). We set $Y = \psi(\mathbf{M})$ and observe that, since clearly the maps $\psi_D$ separate points on $\mathbf{M}$, the map $\psi : \mathbf{M} \to Y$ is an isomorphism, with $\psi(m_0) = y_0$ where $y_0 \in Y$ is defined by $(y_0)_D = y_D$. Likewise define $\phi : X \to Y$ by $(\phi(m))_D = \phi_D(m)$, so that also $\phi(\xi_0) = y_0$. These equations force the identity $\psi = \phi \circ \pi$ in the diagram

$$\begin{array}{ccc} (\mathbf{M}, m_0) & \xrightarrow{\ \pi\ } & (X, \xi_0) \\ & {\scriptstyle \psi} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & (Y, y_0) & \end{array} .$$

Since $\psi$ is a bijection it follows that so are $\pi$ and $\phi$ and the proof is complete. □

## 8. Unique ergodicity

8.1. **Theorem.** *The universal minimal system* $(\Omega^2_{lo}, \mathbb{S})$ *is uniquely ergodic and therefore so is every minimal* $\mathbb{S}$*-system.*

*Proof.* Denote $X = \Omega^2_{lo}$ and let $M_{\mathbb{S}}(X)$ be the set of $\mathbb{S}$-invariant probability measures on $X$. Note that this is the same as the set of $\mathbb{S}_0$-invariant measures. Since $\mathbb{S}_0$ is amenable $M_{\mathbb{S}}(X)$ is non-empty. We fix $\mu \in M_{\mathbb{S}}(X)$. For an array $\{a_{ij} : i < j,\ |i|, |j| \le n_0\}$ with $a_{ij} \in \{1, -1\}$ set

$$A = \{\xi \in X : \xi(i,j) = a_{ij}\},$$

the corresponding cylinder set in $X$. Clearly the sequence $S_{2k+1} \subset \mathbb{S}_0$ is a Følner sequence for the amenable group $\mathbb{S}_0$ and the mean ergodic theorem implies that, for every function $f \in L^2(\mu)$, the ergodic averages

$$\mathbb{A}_k f = \frac{1}{(2k+1)!} \sum_{\sigma \in S_{2k+1}} f(\sigma\xi)$$

converge in $L^2(\mu)$ to the function $\mathbb{E}^{\mathfrak{I}}(f)$ — the conditional expectation of $f$ with respect to the $\sigma$-algebra $\mathfrak{I}$ of Borel $\mathbb{S}_0$-invariant subsets of $X$. In particular this is true for the function $f = \mathbf{1}_A$. However, a moment's reflection will show that for $k > n_0$ the function $\mathbb{A}_k \mathbf{1}_A(\xi)$ is a constant and we conclude that $\mathbb{A}_k \mathbf{1}_A(\xi) = \int_X \mathbf{1}_A \, d\mu = \mu(A)$. Since this holds for every cylinder set $A$ and every $\mu \in M_{\mathbb{S}}(X)$, we conclude that $M_{\mathbb{S}}(X) = \{\mu\}$ is a singleton; i.e. $(X, \mathbb{S})$ is uniquely ergodic. The second assertion of the theorem now follows by the universality of $(X, \mathbb{S})$. □

As described in Remark 2.2 fix a 1-1 correspondence of $\mathbb{Z}$ with the set of rational numbers $\mathbb{Q}$ and consider the element $\eta \in \Omega^2_{lo}$ which is determined, via this correspondence, by the usual order on $\mathbb{Q}$.

8.2**. Theorem.** *The unique $\mathbb{S}$-invariant measure $\mu$ on $\Omega^2_{lo}$ is supported by a single $\mathbb{S}$-orbit; namely $\mu(\mathbb{S}\eta) = 1$.*

*Proof.* The orbit $\mathbb{S}\eta$ consists of all the linear orders on $\mathbb{Z}$ whose order type is that of the rational numbers $\mathbb{Q}$; in other words $\omega \in \mathbb{S}\eta$ iff

$$\forall i, j,\ \omega(i,j) = 1 \quad \Rightarrow \quad \exists k,\ \omega(i,k) = \omega(k,j) = 1, \tag{8.1}$$

and

$$\forall i\ \exists k, l,\ \omega(i,k) = \omega(l,i) = 1. \tag{8.2}$$

Now, given $i \neq j \in \mathbb{Z}$, the measure of the "event"

$$A_{i,j} = \{\omega \in \Omega^2_{lo} : \omega(i,j) = 1\ \wedge\ \exists k,\ \omega(i,k) = \omega(k,j) = 1\}$$

can be computed as

$$\lim_{n\to\infty} \mathbb{A}_n f = \lim_{n\to\infty} \frac{1}{(2n+1)!} \sum_{\sigma \in S_{2n+1}} \mathbf{1}_A(\sigma\xi).$$

Since, no matter which configuration $\xi$ we start from, for large $k$, for most $\sigma \in S_{2k+1}$ we shall have $\sigma\xi \in A$, it follows that this limit is equal to one. Similarly every event of the form (8.2) will have measure one and we conclude that the intersection of all these events, namely the set $\mathbb{S}\eta$, has $\mu$ measure one. □

8.3**. Remark.** The measure $\mu$ and its "factor" measures $\pi_\Sigma(\mu)$, where $\pi_\Sigma : \Omega^2_{lo} \to \Sigma$ are the homomorphisms of $\Omega^2_{lo}$ onto the various minimal sets $\Sigma \subset \Omega^k$, are by no means the only invariant probability measures of the dynamical systems $(\Omega^k, \mathbb{S})$. For other such measures see Theorem 5.1 in [1].

## 9. A minimal action of the discrete group $G = \mathbb{S}/\mathbb{S}_0$

The discrete group $\mathbb{S}$ admits the following interesting minimal dynamical system. Recall that $\beta\mathbb{Z}$, the *Stone-Čech compactification* of the integers $\mathbb{Z}$, is the space of ultrafilters on $\mathbb{Z}$ and that the collection $\mathcal{U} = \{U_A : A \subset \mathbb{Z}\}$, where for each $A \subset \mathbb{Z}$ the set $U_A$ is the set of ultrafilters in $\beta\mathbb{Z}$ containing $A$ ($U_A = \{p \in \beta\mathbb{Z} : A \in p\}$), forms a basis for a compact Hausdorff topology on $\beta\mathbb{Z}$. The collection of *fixed* ultrafilters; i.e. ultrafilters of the form $p_n = \{A \subset \mathbb{Z} : n \in A\}$ for $n \in \mathbb{Z}$, forms an open, discrete, dense subset of $\beta\mathbb{Z}$. When this collection is identified with $\mathbb{Z}$, the space $\beta\mathbb{Z}$ becomes the *universal compactification* of the discrete topological space $\mathbb{Z}$ in the sense that any map $\phi : \mathbb{Z} \to K$, where $K$ is a compact Hausdorff space, can be extended uniquely to a continuous map $\tilde{\phi} : \beta\mathbb{Z} \to K$. For more details see [5], [4] and [12].

Consider the compact subspace $X = \beta\mathbb{Z} \setminus \mathbb{Z}$ of $\beta\mathbb{Z}$, sometimes called the *corona*. The universal property of $\beta\mathbb{Z}$ enables one to extend the natural action of $\mathbb{S}$ on $\mathbb{Z}$ to an action on $\beta\mathbb{Z}$. Evidently this action leaves $X$ invariant and it is also clear that each element of the normal subgroup $\mathbb{S}_0$ restricts to the identity map on $X$. We can therefore view this action as a $G = \mathbb{S}/\mathbb{S}_0$ dynamical system $(X, G)$. As we shall see, this action is minimal and Uspenskij's idea will serve us in showing that it is not the universal one (see [22]).

9.1. **Theorem.** (1) *The system $(X, G)$ is minimal.*

(2) *The system $(X, G)$ is extremely proximal; i.e. for every closed set $\emptyset \neq F \subsetneq X$ there exists a net $\{g_i\}_{i \in I}$ in $G$ such that, in the Vietoris topology on the compact space $2^X$ of closed subsets of $X$, we have $\lim_{i \in I} g_i F = \{x_0\}$ for some point $x_0 \in X$ (see [7]).*

(3) *The minimal system $(X, G)$ is not isomorphic to the universal minimal system $(M(G), G)$.*

*Proof.* 1. Note that the collection $\mathcal{U} = \{\mathrm{cls}_{\beta\mathbb{Z}} \mathrm{A} \cap \mathrm{X} : \mathrm{A}$ an infinite subset of $\mathbb{Z}\}$ is a basis for the topology on $X$ consisting of clopen sets. Since clearly $G$ acts transitively on this collection, it follows that for every $U \in \mathcal{U}$ we have $\cup\{\alpha(U) : \alpha \in G\} = X$. This property is equivalent to the minimality of the system $(X, G)$.

2. Fix some $x_0$ in $X$ such that $x_0 \notin F$. For an arbitrary basic clopen neighborhood $U = \mathrm{cls}_{\beta\mathbb{Z}} \mathrm{A} \cap \mathrm{X}$ of $x_0$ which is disjoint from $F$ choose $\alpha_U \in G$ such that $\alpha_U(A^c) = A$. Then $\alpha$ satisfies $\alpha_U(F) \subset U$. Clearly now $\{\alpha_U : U$ a neighborhood of $x_0\}$ is the required net.

3. Using Uspenskij's notation, let $\Phi \subset 2^{2^X}$ be the collection of maximal chains on $X$. Recall that a *chain* is a nonempty family $c = \{F_t\}$ (with $t$ running over some parameter set) of closed subsets of $X$ such that for every $F_1, F_2 \in c$ either $F_1 \subset F_2$ or $F_2 \subset F_1$. A chain $c$ is *maximal* if it is not properly contained in another chain. Uspenskij shows that for any dynamical system $(X, G)$ the collection $\Phi$ is a closed invariant subset of the compact space $2^{2^X}$. It is easy to see that every $c \in \Phi$ has a first element $F$ which is necessarily of the form $F = \{x\}$. Moreover, calling $x$ the *root* of the chain $c$, it is clear that the map $\pi : \Phi \to X$, sending a chain to its root, is a homomorphism of dynamical systems.

Suppose $(X, G)$ is isomorphic to the universal minimal $G$ system. Let $Y \subset \Phi$ be a minimal subset of $\Phi$. Then, by the coalescence of the universal minimal system (see the first paragraph of the introduction), the restriction $\pi : Y \to X$ is an isomorphism. Fix $c_0 \in Y$ and let $p_0 \in X$ be its root; i.e. $\pi(c_0) = p_0$. Let $H = \{\alpha \in G : \alpha p_0 = p_0\}$, the stability group of $p_0$. Since $\pi$ is an isomorphism we also have $H = \{\alpha \in G : \alpha c_0 = c_0\}$. Choose $F \in c_0$ such that $\{p_0\} \subsetneq F \subsetneq X$ and let $p_0 \neq a \in F$. There exists an infinite $B \subset \mathbb{Z}$ such that $\mathrm{cls}_{\beta\mathbb{Z}} \mathrm{B} \cap \mathrm{X}$ is disjoint from $F$. Choose subsets $P$ and $A$ of $\mathbb{Z}$ with the following properties: $P \cap A = \emptyset$, $P \cup A = B^c$, $a \in \mathrm{cls}_{\beta\mathbb{Z}} \mathrm{A}$ and $p_0 \in \mathrm{cls}_{\beta\mathbb{Z}} \mathrm{P}$. Thus $\{P, A, B\}$ is a partition of $\mathbb{Z}$.

Next choose a permutation $\alpha \in G$ such that $\alpha \restriction P = \mathrm{id}$, $\alpha(A) = B$ and $\alpha^2 = \mathrm{id}$, and set $b = \alpha(a)$. We now have $\alpha(p_0) = p_0$, $b = \alpha(a) \in \alpha(F) \setminus F$ and $a = \alpha(b) \in F \setminus \alpha(F)$, so that $F$ and $\alpha(F)$ are not comparable. On the other hand $\alpha(p_0) = p_0$ means $\alpha \in H$ whence also $\alpha(c_0) = c_0$. In particular $\alpha(F) \in c_0$ and as $c_0$ is a chain one of the inclusions $F \subset \alpha(F)$ or $\alpha(F) \subset F$ must hold. This contradiction shows that $(X, G)$ cannot be the universal minimal $G$ system. □

## References


[1] D. J. Aldous, *Representations for partially exchangeable arrays of ranom variables*, J. Multivariate Anal. **11**, (1981), 581-598.

[2] J. D. Dixon, *The structure of linear groups*, Van Nostrand Reinhold Company, Mathematical studies **37**, New York, 1971.

[3] H. A. Dye, *On the ergodic mixing theorem*, Trans. Amer. Math. Soc. **118**, (1965), 123-130.
[4] R. Ellis, *Lectures on Topological Dynamics*, W. A. Benjamin, Inc. , New York, 1969.
[5] L. Gillman and M. Jerison, *Rings of continuous functions*, van Nostrand, Princeton, 1960.
[6] T. Giordano and V. Pestov, *Some extremely amenable groups*, To appear, C.r. Acad. Sci. Paris.
[7] E. Glasner, *Topological dynamics and group theory*, Trans. Amer. Math. Soc. **187**, (1974), 327-334.
[8] ———, *On minimal actions of Polish groups*, Topology and its applications **85**, (1998), 119-125.
[9] E. Glasner and B. Weiss, *Interpolation sets for subalgebras of $\ell^\infty(\mathbb{Z})$*, Israel J. of Math. **44**, (1983), 345-360.
[10] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics **152**, Birkhäuser Boston, Inc., Boston, MA, 1999.
[11] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. **105**, (1983), 843-854.
[12] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification*, de Gruyter, Berlin, New York, 1998.
[13] E. V. Huntington, *Inter-relations among the four principal types of order*, Trans. Amer. Math. Soc. **38**, (1935), 1-9.
[14] M. G. Megrelishvili and T. Scarr, *The equivariant universality and couniversality of the Cantor cube*, Fund. Math. **167**, (2001), 269-275.
[15] V. D. Milman, *The heritage of P. Lévy in geometrical functional analysis*, Astérisque **157-158**, (1998), 273-301.
[16] V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer, Berlin, 1979.
[17] J. von Neumann and E. P. Wigner, *Minimally almost periodic groups*, Ann. of Math. **41**, (1940), 746-750.
[18] V. G. Pestov, *Some universal constructions in abstract topological dynamics*, Contemporary Mathematics **215**, (1998), 83-99.
[19] ———, *On free actions, minimal flows, and a problem by Ellis*, Trans. Amer. Math. Soc. **350**, (1998), 4149-4165.
[20] ———, *Amenable representations and dynamics of the unit sphere in an infinite-dimensional Hilbert space*, GAFA **10**, (2000), 1171-1201.
[21] ———, *Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, to appear in the Israel J. of Math.
[22] V. Uspenskij, *On universal minimal compact G-spaces*, to appear.

Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel
*E-mail address*: glasner@math.tau.ac.il

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel
*E-mail address*: weiss@math.huji.ac.il