AN ENVELOPING SEMIGROUP PROOF OF THE FACT THAT $RP^{[d]}$ IS AN EQUIVALENCE RELATION

ELI GLASNER

Let T be a countable abelian group and let (X,T) be a minimal dynamical system; i.e. X is a compact Hausdorff space and T acts on it as a group of homeomorphisms. Following [4] and [7] we introduce the following notations (generalizing from the case $T=\mathbb{Z}$ to the case of a general T action). For an integer $d \geq 1$ let $X^{[d]} = X^{2^d}$. We index the coordinates of an element $x \in X^{[d]}$ by subsets $\epsilon \subset \{1,\dots,d\}$. Thus $x = (x_\epsilon : \epsilon \subset \{1,\dots,d\})$, where for each ϵ, $x_\epsilon \in X_\epsilon = X$. E.g. for $d=2$ we have $x = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}})$. We let $X_*^{[d]} = X^{2^d-1} = \prod\{X_\epsilon : \epsilon \neq \emptyset\}$ and for $x \in X^{[d]}$ we let $x_* \in X_*^{[d]}$ denote its projection; i.e. x_* is obtained by omitting the $\emptyset$-coordinate of x. For each $\epsilon \subset \{1,\dots,d\}$ we denote by π_ϵ the projection map from $X^{[d]}$ onto $X_\epsilon = X$. For a point $x \in X$ we let $x^{[d]} \in X^{[d]}$ and $x_*^{[d]} \in X_*^{[d]}$ be the *diagonal points* all of whose coordinates are x. $\Delta^{[d]} = \{x^{[d]} : x \in X\}$ is the *diagonal* of $X^{[d]}$ and $\Delta_*^{[d]} = \{x_*^{[d]} : x \in X\}$ the *diagonal* of $X_*^{[d]}$. Another convenient representation of $X^{[d]}$ is as a product space $X^{[d]} = X^{[d-1]} \times X^{[d-1]}$ (with $X^{[0]} = X$). When using this decomposition we write $x = (x', x'')$.

We next define two group actions on $X^{[d]}$, the *face group action* $\mathcal{F}^{[d]}$ and the *total group action* $\mathcal{G}^{[d]}$. These actions are representations of $T^d = T \times T \times \cdots \times T$ (d times) and T^{d+1}, respectively, as subgroups of $\mathrm{Homeo}\,(X^{[d]})$. For the $\mathcal{F}^{[d]}$ action $\mathcal{F}^{[d]} \times X^{[d]} \to X^{[d]}$,

$$((t_1,\dots,t_d),\ (x_\epsilon : \epsilon \subset \{1,\dots,d\})) \mapsto (t_\epsilon x_\epsilon : \epsilon \subset \{1,\dots,d\}),$$

where $t_\epsilon x_\epsilon = t_{n_1}\cdots t_{n_j} x_\epsilon$, if $\epsilon = \{n_1,\dots,n_j\}$ and $t_\emptyset x_\emptyset = x_\emptyset$. We can then represent the homeomorphism $\tau \in \mathcal{F}^{[d]}$ which corresponds to $(t_1,\dots,t_d) \in T^d$ as

$$\tau = \tau^d_{(t_1,\dots,t_d)} = (t_\epsilon : \epsilon \subset \{1,\dots,d\}).$$

We will also consider the restriction of the $\mathcal{F}^{[d]}$ action to $X_*^{[d]}$ which is defined by omitting the first coordinate.

For example, if we consider a minimal cascade (X,f), taking $T = \mathbb{Z} = \{f^n : n \in \mathbb{Z}\}, d = 3$ and $\tau = \tau^3_{(2,5,11)} \in \mathcal{F}^{[3]} \cong \mathbb{Z}^3$, we have:

$$\tau(x) = (x_\emptyset, f^2 x_{\{1\}}, f^5 x_{\{2\}}, f^{2+5} x_{\{1,2\}}, f^{11} x_{\{3\}}, f^{2+11} x_{\{1,3\}}, f^{2+5+11} x_{\{1,2,3\}}),$$

and

$$\tau(x_*) = (f^2 x_{\{1\}}, f^5 x_{\{2\}}, f^{2+5} x_{\{1,2\}}, f^{11} x_{\{3\}}, f^{2+11} x_{\{1,3\}}, f^{2+5+11} x_{\{1,2,3\}}).$$

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Note that the fact that the $\mathcal{F}^{[d]}$ action is well defined depends on the commutativity of the group T.

The action of T^{d+1} on $X^{[d]}$, denoted by $\mathcal{G}^{[d]}$, is the action generated by the face group action $\mathcal{F}^{[d]}$ and the *diagonal θ-action* of T, $T \times X^{[d]} \to X^{[d]}$, defined by

$$(t, x) \mapsto \theta^d_t x = (tx_\epsilon : \epsilon \subset \{1, \dots, d\}).$$

Thus for the $\mathcal{G}^{[d]}$ action $\mathcal{G}^{[d]} \times X^{[d]} \to X^{[d]}$,

$$((t_1, \dots, t_d, t_{d+1}),\ (x_\epsilon : \epsilon \subset \{1, \dots, d\})) \mapsto (t_{d+1} t_\epsilon x_\epsilon : \epsilon \subset \{1, \dots, d\}),$$

where $t_\epsilon x_\epsilon = t_{n_1} \cdots t_{n_j} x_\epsilon$, if $\epsilon = \{n_1, \dots, n_j\}$ and $t_\emptyset x_\emptyset = x_\emptyset$. In other words, the $\mathcal{G}^{[d]}$ action on $X^{[d]}$ is given by the representation:

$$T^{d+1} \to \mathrm{Homeo}\,(X^{[d]}), \qquad (t_1, \dots, t_d, t_{d+1}) \mapsto \theta^d_{t_{d+1}} \tau^d_{(t_1, \dots, t_d)}.$$

Notice that

$$\tau^d_{(t_1,\dots,t_d)}(x', x'') = (\tau^{d-1}_{(t_1,\dots,t_{d-1})} x', \theta^{d-1}_{t_d} \tau^{d-1}_{(t_1,\dots,t_{d-1})} x''). \tag{1}$$

In their paper [7] Shao and Ye prove that $RP^{[d]}$, the generalized regionally proximal relation of order d, is always an equivalence relation for a minimal cascade (X, T). Their proof is based on the detailed analysis of the $\mathcal{G}^{[d]}$ action provided by Host Kra and Mass in [4]. In turn, the results obtained in [4] are based on the profound ergodic theoretical results obtained by Host and Kra in [3]. The main tool used by Shao and Ye is a theorem which asserts that for each $x \in X$ the face action $\mathcal{F}^{[d]}$ is minimal on the orbit closure $\mathrm{cls}\,\mathcal{F}^{[d]} x^{[d]}_*$. Their proof of this theorem is based on the general structure theory of minimal flows due to Ellis-Glasner-Shapiro [2], McMahon [6] and Veech [8]. But in fact, unknown to them, I have already shown, a few years earlier, to Bernard Host and Bryna Kra (in a private conversation) a direct proof of this fact which is very similar to the proof by Ellis and Glasner given in, [5, page 46]. The possibility of applying the Ellis Glasner proof as a shortcut to Shao and Ye's proof was also discovered by Ethan Akin. In the next section I present this short proof, established for a general commutative group. For the interested reader I will, in a subsequent section, briefly reproduce the Shao-Ye proof of the fact that for each $d \geq 1$, $RP^{[d]}$ is an equivalence relation.

1. The minimality of the face action on $Q^{[d]}_{x*}$

Let (X, T) be a minimal flow with T abelian. Let

$$Q^{[d]} = \mathrm{cls}\,\{g x^{[d]} : x \in X,\ g \in \mathcal{G}^{[d]}\} = \overline{\mathcal{F}^{[d]} \Delta^{[d]}}.$$

For $x \in X$ let $Q^{[d]}_x = Q^{[d]} \cap \{x\} \times X^{2^d - 1}$ and let $Y^{[d]}_x = \mathcal{F}^{[d]}(x^{[d]}$ be the orbit closure of $x^{[d]}$ under $\mathcal{F}^{[d]}$.

1.1. **Theorem** (Shaw and Ye). 1. *The flow $(Q^{[d]}, \mathcal{G}^{[d]})$ is minimal.*
2. *For each $x \in X$, the flow $(Y^{[d]}_x, \mathcal{F}^{[d]})$ is minimal.*
3. *For each $x \in X$ the flow $Y^{[d]}_x$ is the unique minimal subflow of the flow $(Q^{[d]}, \mathcal{F}^d)$.*

Proof. 1. Let us denote $N := Q^{[d]}$ and $\mathcal{T} := \mathcal{G}^{[d]}$. Let $E = E(N, \mathcal{T})$ be the enveloping semigroup of $(N, \mathcal{T})$. Let $\pi_\epsilon : N \to X_\epsilon = X$ be the projection of N on the ϵ coordinate, where $\epsilon \subset \{1, ..., d\}$. We consider the action of the group $\mathcal{T}$ on the ϵ coordinate via the projection π_ϵ, that is, for $\epsilon \subset \{1, \dots, d\}, (t_1, \dots, t_d, t_{d+1}) \in T^{d+1}$ and $x \in X_\epsilon = X$,

$$\mathcal{T} \times X_\epsilon \to X_\epsilon, \quad (\theta^d_{t_{d+1}} \tau^d_{(t_1,\dots,t_d)}, x) \mapsto t_{d+1} t_\epsilon x.$$

With respect to this action of $\mathcal{T}$ on $X_\epsilon = X$ the map $\pi_\epsilon : (N, \mathcal{T}) \to (X_\epsilon, \mathcal{T})$ is a flow homomorphism. Let $\pi^*_\epsilon : E(N, \mathcal{T}) \to E(X_\epsilon, \mathcal{T})$ be the corresponding homomorphism of enveloping semigroups. Notice that for the action of $\mathcal{T}$ on X_ϵ, $E(X_\epsilon, \mathcal{T}) = E(X, T)$ as subsets of X^X (as $t_{d+1} t_\epsilon \in T$).

Let now $u \in E(X, T)$ be any minimal idempotent. Then $\tilde{u} = (u, u, ..., u) \in E(N, \mathcal{T})$. Choose v a minimal idempotent in the closed left ideal $E(N, \mathcal{T})\tilde{u}$. Then $v\tilde{u} = v$. We want to show that $\tilde{u}v = \tilde{u}$. Set, for $\epsilon \subset \{1, \dots, d\}$, $v_\epsilon = \pi^*_\epsilon v$. Note that, as an element of $E(N, \mathcal{T})$ is determined by its projections, it suffices to show that for each ϵ, $uv_\epsilon = u$. Since for each ϵ the map π^*_ϵ is a semigroup homomorphism, we have that $v_\epsilon u = v_\epsilon$ as $v\tilde{u} = v$. In particular we deduce that v_ϵ is an element of the minimal left ideal $E(X_\epsilon, T)u = E(X, T)u$ which contains u. This implies (see [5, Exercise 1.23.2.(b)]) that

$$uv_\epsilon = uv_\epsilon u = u;$$

and it follows that indeed $\tilde{u}v = \tilde{u}$. Thus, $\tilde{u}$ is an element of the minimal left ideal $E(N, \mathcal{T})v$ which contains v, and therefore $\tilde{u}$ is a minimal idempotent of $E(N, \mathcal{T})$.

Now let $x \in X$ and let u be a minimal idempotent in $E(X, T)$ with $ux = x$ (since (X, T) is minimal there always exists such an idempotent). By the above argument, $\tilde{u}$ is also a minimal idempotent of $(N, \mathcal{T})$ which implies that $N = Q^{[d]}_{x_*}$, the orbit closure of $x^{[d]}_* = \tilde{u} x^{[d]}_*$, is $\mathcal{T}$ minimal (see [5, Exercise 1.26.2]).

2. Given $x \in X$ we now let $N := Q^{[d]}_{x_*}$ and $\mathcal{T} := \mathcal{F}^{[d]}$. The proof of the minimality of the flow $(Q^{[d]}_{x_*}, \mathcal{F}^{[d]})$ is almost verbatim the same, except that here the claim that for u a minimal idempotent in $E(X, T)$, the map $\tilde{u} = (u, u, ..., u)$ is in $E(Q^{[d]}_{x_*}, \mathcal{F}^{[d]})$, is not that evident. However, as u is an idempotent this fact follows from the following lemma (with $p_1 = \cdots = p_d = u$).

1.2. **Lemma.** *Let $p_1, \dots, p_d \in E(X, T)$ and for $\epsilon = \{n_1, \dots, n_k\} \subset \{1, \dots, d\}$, with $n_1 < \cdots < n_k$, let $q_\epsilon = p_{n_1} \cdots p_{n_k}$. Then the map $(q_\epsilon : \epsilon \subset \{1, \dots, d\},\ \epsilon \neq \emptyset)$ is an element of $E(Q^{[d]}_{x_*}, \mathcal{F}^{[d]})$.*

Proof. By induction on d, using the identity (1), or more specifically

$$\tau^d_{(e,\dots,e,t_d)}(x', x'') = (x', \theta^{d-1}_{t_d} x''),$$

and the fact that right multiplication in $E(X, T)$ is continuous. □

□

2. $RP^{[d]}$ IS AN EQUIVALENCE RELATION

In this section we outline the Shao-Ye proof that $RP^{[d]}$ is an equivalence relation. We assume that (X, T) is a minimal compact *metrizable* T-flow, where T is an abelian group. We fix a compatible metric ρ on X.

2.1. **Definition.** The *regionally proximal relation of order d* is the relation $RP^{[d]} \subset X^{[d]} \times X^{[d]}$ defined by the following condition: $(x, y) \in RP^{[d]}$ iff for every $\delta > 0$ there is a pair $x', y' \in X$ and $(t_1, \ldots, t_d) \in T^d$ such that:

1. $\rho(x, x') < \delta$ and $\rho(y, y') < \delta$.
2. For every $\emptyset \neq \epsilon \subset \{1, \ldots, d\}$,

$$\rho^{[d]}(\tau^{[d]}_{(t_1,\ldots,t_d)} x'^{[d]}_*, \tau^{[d]}_{(t_1,\ldots,t_d)} y'^{[d]}_*) := \sup\{\rho(t_\epsilon x', t_\epsilon y') : \epsilon \subset \{1, \ldots, d\},\ \epsilon \neq \emptyset\} < \delta.$$

For $d = 1$ this relation is the classical *regionally proximal relation*, see e.g. [1].

2.2. **Lemma.** *Let (X, T) be a minimal system. Let $d \geq 1$ and $x, y \in X$. Then $(x, y) \in RP^{[d]}$ if and only if there is some $a_* \in X^{[d]}_*$ such that $(x, a_*, y, a_*) \in Q^{[d+1]}$.*

With the help of Theorem 1.1, we can prove that $RP^{[d]}$ is an equivalence relation. First we have the following equivalent conditions for $RP^{[d]}$.

2.3. **Theorem.** *Let (X, T) be a minimal flow and $d \geq 1$. The following conditions are equivalent:*

1. $(x, y) \in RP^{[d]}$.
2. $(x, y, y, \ldots, y) = (x, y^{[d+1]}_*) \in Q^{[d+1]}$.
3. $(x, y, y, \ldots, y) = (x, y^{[d+1]}_*) \in \overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$.

Proof. (3) ⇒ (2) is obvious. (2) ⇒ (1) follows from Lemma 2.2. Hence it suffices to show (1) ⇒ (3). Let $(x, y) \in RP^{[d]}$. Then by Lemma 2.2 there is some $a \in X^{[d]}$ such that $(x, a_*, y, a_*) \in Q^{[d+1]}$. Observe that $(y, a_*) \in Q^d$. By Theorem 3.1-(2), there is a sequence $\{F_k\} \subset \mathcal{F}^{[d]}$ such that $F_k(y, a_*) \to y^{[d]}$. Hence

$$F_k \times F_k(x, a_*, y, a_*) \to (x, y^{[d]}_*, y, y^{[d]}_*) = (x, y^{[d+1]}_*).$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $(x, a_*, y, a_*) \in Q^{[d+1]}$, we have that $(x, y^{[d+1]}_*) \in Q^{[d+1]}$. By Theorem 3.1-(1), $y^{[d+1]}$ is $\mathcal{F}^{[d+1]}$-minimal. It follows that $(x, y^{[d+1]}_*)$ is also $\mathcal{F}^{[d+1]}$-minimal. Now $(x, y^{[d+1]}_*) \in Q^{[d+1]}[x]$ is $\mathcal{F}^{[d+1]}$-minimal and by Theorem 3.1-(2), $\overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$ is the unique $\mathcal{F}^{[d+1]}$-minimal subset in $Q^{[d+1]}[x]$. Hence we have that $(x, y^{[d+1]}_*) \in \overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$, and the proof is completed. □

By Theorem 2.3, we have the following theorem immediately.

2.4. **Theorem.** *Let (X, T) be a minimal system and $d \geq 1$. Then $RP^{[d]}$ is an equivalence relation.*

Proof. It suffices to show the transitivity, i.e. if $(x, y), (y, z) \in RP^{[d]}$, then $(x, z) \in RP^{[d]}(X)$. Since $(x, y), (y, z) \in RP^{[d]}(X)$, by Theorem 2.3 we have

$$(y, x, x, \ldots, x), (y, z, z, \ldots, z) \in \overline{\mathcal{F}^{[d+1]}(y^{[d+1]})}.$$

By Theorem 1.1 $(\overline{\mathcal{F}^{[d+1]}(y^{[d+1]})}, \mathcal{F}^{[d+1]})$ is minimal, it follows that $(y, z, z, \ldots, z) \in \overline{\mathcal{F}^{[d+1]}(y, x, x, \ldots, x)}$. It follows that $(x, z, z, \ldots, z) \in \overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$. By Theorem 2.3 again, $(x, z) \in RP^{[d]}$. □

2.5. **Remark.** By Theorem 3.4 we know that in the definition of regionally proximal relation of d, x' can be replaced by x. More precisely, $(x, y) \in RP^{[d]}$ if and only if for any $\delta > 0$ there exist $y' \in X$ and a vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that for any nonempty $\epsilon \subset \{1, \ldots, d\}$, $\rho(y, y') < \delta$ and $\rho(T^{n \cdot \epsilon} x, T^{n \cdot \epsilon} y') < \delta$. This conclusion is first given in [23] for a minimal distal system.

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Department of Mathematics, Tel Aviv University, Tel Aviv, Israel
E-mail address: `glasner@math.tau.ac.il`