TOPOLOGICAL GROUPS WITH ROHLIN PROPERTIES

ELI GLASNER AND BENJAMIN WEISS

Abstract. In his classical paper [28] P. R. Halmos shows that weak mixing is generic in the measure preserving transformations. Later, in his book [29], he gave a more streamlined proof of this fact based on a fundamental lemma due to V. A. Rohlin. For this reason the name of Rohlin has been attached to a variety of results, old and new, relating to the density of conjugacy classes in topological groups. In this paper we will survey some of the new developments in this area.

Contents

Date: March 15, 2007.

2000 Mathematics Subject Classification. Primary 22A05, 22D05, 37A05, 54H20, Secondary 37E15, 54C40.

Research supported by ISF grant # 1333/04.

In the classical paper of P. R. Halmos [28] in which he shows that weak mixing is generic in the measure preserving transformations he writes in the opening paragraph:

> The principal new and quite surprising fact used in the proof is that for any almost nowhere periodic measure preserving transformation T (and *a fortiori* for any mixing T) the set of all conjugates of T (i.e. the set of all STS^{-1}) is everywhere dense. It is this possibility of a dense conjugate class in a comparatively well behaved topological group (a rather natural generalization of the finite symmetric groups) that is contrary to naive intuition.

In his book [29] Halmos gave a more streamlined proof of this new fact based on a fundamental lemma due to V. A. Rohlin, and For this reason the name of Rohlin has been attached to a variety of results, old and new, relating to the density of conjugacy classes in topological groups. In this paper we will survey some of the new developments in this area.

As this subject touches upon aspects of group theory, topology, ergodic theory and many other branches of mathematics, a survey can easily grow to a book size. This was not our intention and we have therefore concentrated on a few leading themes that were mainly determined by aspects of the theory that we were involved with.

We will begin with a brief discussion of the purely algebraic aspects of the question which pertain to countable groups in which any two elements that differ from the identity are conjugate. Next we take up the first topological version of the question and refer the readers to [25], where an example of a locally compact group in which there are dense conjugacy classes is exhibited. In general we will say that a topological group has the Rohlin property (RP) if it has a dense conjugacy class and the strong Rohlin (SRP) property if it has a co-meager conjugacy class.

Passing from locally compact to Polish groups we begin with the motivating example of $\mathrm{Aut}\,(X, \mathcal{X}, \mu)$, the group of measure preserving transformations of a standard Lebesgue space. We will present here a new proof due to G. Hjorth of the fact that this group does not have the SRP, i.e. all conjugacy classes in $\mathrm{Aut}\,(X, \mathcal{X}, \mu)$ are meager.

In the more abstract ergodic theory one considers actions of groups Γ more general than $\mathbb{Z}$ but retains the notion of conjugacy by elements of $G = \mathrm{Aut}\,(X, \mathcal{X}, \mu)$. We say that two Γ-actions $\mathbf{S}$ and $\mathbf{T}$ are *isomorphic* if there is $R \in G$ such that $\mathbf{T}_\gamma = R\mathbf{S}_\gamma R^{-1}$ for every $\gamma \in \Gamma$. In other words, if and only if **S** and **T** belong to the same orbit of the natural action of the Polish group G on the Polish space of actions, $\mathbb{A}_\Gamma$, by conjugation. We say that the group Γ has the weak Rohlin property (WRP) if this action of G on $\mathbb{A}_\Gamma$ is topologically transitive; i.e. for any two nonempty open sets U and V in $\mathbb{A}_\Gamma$ there is some $R \in G$ such that $RUR^{-1} \cap V \neq \emptyset$. It turns out that every countable group Γ does have the weak Rohlin property (see [24]) and so the associated $\mathbb{A}_\Gamma$ contains dense conjugacy classes. On the other hand, for there to exist a free ergodic action with a dense conjugacy class in $\mathbb{A}_\Gamma$ a necessary and sufficient condition is that the group Γ will not have Kazhdan's property T.

The next brief section takes up the group of unitary operators on a separable Hilbert space. Here the fact that it has the RP was established in [15]. We will show that it does not have the SRP and then make use of this understanding of the unitary group to extend the above results from Aut $(X, \mathcal{X}, \mu)$ to the group $NS(X, \mathcal{X}, \mu)$ of nonsingular automorphisms of Lebesgue space.

The next sections deal with groups of homeomorphisms of compact spaces ranging from the Cantor set through manifolds to the Hilbert cube. We also describe some of the recent results of [9] on the nature of generic homeomorphisms of compact manifolds.

In sections 8 to 10 we discuss the strong Rohlin property. Perhaps the simplest group that has the SRP is the group of all permutations of a countable set with the topology of point-wise convergence. In fact it is easy to describe the generic permutation, its cycle decomposition contains only finite cycles and for each natural number n there are infinitely many cycles of that length. The group of homeomorphisms of the Cantor set also has the SRP [41], but now the description of the generic element is more complicated [8]. However, it still can be made explicit and we do so in the following section.

The fact that the group of homeomorphisms of the Cantor set has the SRP was established by Kechris and Rosendal using model theory, [41]. We describe their work in section 11. In section 12 we present a proof, due to Kechris, of the fact that the group of isometries of the Urysohn space does not have the SRP. This proof is based on the idea of Hjorth mentioned above. We thank them both for the permission to publish here these results. The last two sections deal with the notion of ample generics (see [41]) and other related recent developments.

While working on this survey we received from A. Kechris a draft of a book [Kech-07], titled "Global aspects of ergodic group actions and equivalence relations" which he is in the process of writing. Of course it covers in great depth many of the subjects which are treated in our

survey. We thank Ethan Akin for a careful reading of the paper and for providing helpful suggestions.

1. Countable groups with two conjugacy classes

Is there a group G with just two conjugacy classes, $\{e\}$ and $G \setminus \{e\}$ (e of course is the identity element of G)? Certainly the group with two elements is such a group. Is there any other such finite group? The answer is no, as a beginner in group theory can prove as an exercise. How about countable groups? Here is a natural construction of a countable group with exactly two conjugacy classes due to G. Higman, B. H. Neumann and H. Neumann, [30] (see also [52, Exercise 12.63]). The main tool is the HNN extension theorem.

1.1. **Theorem.** *Let G be a torsion free group. There is a torsion free group H such that $G < H$ and all pairs of nonidentity elements g_1, g_2 in G are conjugate in H.*

Now start with an infinite cyclic group $G_0 = \{a^n : n \in \mathbb{Z}\}$ and by induction construct a chain of torsion free groups $G_0 < G_1 < G_2 < \cdots$ such that for every n every nonidentity elements g_1, g_2 in G_n are conjugate in G_{n+1}. Clearly $G = \bigcup_{n\in\mathbb{N}} G_n$ is a countable group with just two conjugacy classes.

The really difficult question is: are there infinite, *finitely generated* groups with exactly two conjugacy classes. The surprising and resoundingly positive answer is due to Osin, [45].

1.2. **Theorem.** *Any countable group G can be embedded into a 2-generated group C such that any two elements in C of the same order are conjugate in C. In particular if G is torsion free it can be embedded into 2-generated group C which has exactly two conjugacy classes.*

In fact Osin shows that there are uncountably many pairwise non-isomorphic torsion free 2-generated groups with exactly two conjugacy classes.

2. Locally compact groups with a dense conjugacy class

Is there a topological analogue to these kind of problems? In a recent work [8] the authors provide the following results.

2.1. **Theorem.** 1. *There exists a locally compact σ-compact topological group G with a dense conjugacy class.*

2. *Let G be a locally compact σ-compact topological group. Then every conjugacy class of G is either meager or open. If, in addition, G has a dense conjugacy class then either every conjugacy class is of first category or there is a unique open conjugacy class which is dense. In particular, if there is a dense comeager conjugacy class then this conjugacy class is open.*

These results naturally lead to the following problems.

2.2. **Problems.** 1. Is there a locally compact topological group with a comeager conjugacy class?

2. Is there a non-discrete, locally compact, topological group with exactly two conjugacy classes?

3. The group Aut $(X, \mathcal{X}, \mu)$

Let us consider next the non-locally-compact Polish topological group $G = \mathrm{Aut}\,(X, \mathcal{X}, \mu)$, where $(X, \mathcal{X}, \mu)$ is an atomless Lebesgue space; say, $X = [0, 1]$, $\mathcal{X}$ the σ-algebra of Borel sets, and μ Lebesgue measure. A countable algebra $\{A_k\}_{k=1}^{\infty}$ of sets, which separates points of X, gives rise to a complete metric which induces the **weak topology** on G,

$$d(S,T) = \sum_{k=1}^{\infty} 2^{-k} \left(\mu(SA_k \triangle TA_k) + \mu(S^{-1}A_k \triangle T^{-1}A_k) \right).$$

With this metric G is a Polish topological group. The **Koopman representation** $\kappa : T \mapsto U_T$, where $U_T(f) = f \circ T^{-1},\ f \in L^2(\mu)$, is a topological isomorphism of G onto its image in the unitary group $\mathcal{U}(L^2(\mu))$, where the latter is equipped with its strong operator topology. Either directly or via Lavrentiev's theorem we deduce that the image of G is a G_δ subset in $\mathcal{U}(L^2(\mu))$ and therefore closed. (In general a G_δ subgroup of a Polish group is closed. For the theorems of Alexandroff and Lavrentiev see e.g. [49, Section 12], or [40, page 16]). Of course the conjugacy classes in G are typically much smaller than the conjugacy classes under the bigger group $\mathcal{U}(L^2(\mu))$. The first is the isomorphism type of a transformation T, the latter its unitary equivalence class. Thus, for example, by a theorem of Kolmogorov all the K-automorphisms in G are unitarily equivalent (see e.g. [22], page 120). A crucial step in the development of modern ergodic theory was the introduction by Kolmogorov of the notion of entropy of an automorphism of (X, μ) as an invariant that can distinguish between two nonisomorphic automorphisms whose Koopman operators are unitarily equivalent. Subsequently one of the great achievements of ergodic theory was Ornstein's theorem which asserts that in the class of Bernoulli automorphisms (a subclass of the class of K-automorphisms) entropy is a complete invariant, [46]. In a later work Ornstein and Shields [47], show the existence of an uncountable family of pairwise nonisomorphic non-Bernoulli K-automorphisms.

A well known theorem of Halmos [28], asserts that the conjugacy class of each aperiodic transformation is dense in G. The standard proof (see [29, pages 69-74]) relies on a Rohlin type lemma and this is the motivation for our nomenclature.

3.1. **Theorem** (Rohlin's lemma)**.** *Let $(X, \mathcal{X}, \mu, T)$ be an aperiodic system (i.e., for all n, $\mu\{x \in X : T^n x = x\} = 0$), N a positive integer and $\epsilon > 0$, then there exists a subset $B \in \mathcal{X}$ such that the sets $B, TB, \dots, T^{N-1}B$ are pairwise disjoint and $\mu(\cup_{j=0}^{N-1} T^j B) > 1 - \epsilon$.*

By Rohlin's lemma any two aperiodic transformations, T and S, have arbitrarily large congruent Rohlin stacks. Thus some isomorphic copy RTR^{-1} is close to S, and so the set of such isomorphic copies of T is dense in the aperiodic transformations. The argument is finished by showing the aperiodics to be dense in G.

Halmos's book studies generic (i.e. residual) properties in the weak topology on G. For instance, it is shown there that "weak-mixing" is generic (due to Halmos), whereas "mixing" is meager, (due to Rohlin) [29], pp. 77, 78. The exploration of this notion of genericity became an active research area; see [16], [15] for results and extensive bibliographies.

The dichotomy "generic" versus "meager" stems from the following general "zero-one law" (see e.g. [49], [40] or [23]). Recall that the action of a group G on a topological space X is **topologically transitive** if for any two nonempty open sets $U, V \subset X$ there is $g \in G$ with $gU \cap V \neq \emptyset$. When the space X is Polish this is equivalent to the existence of a point $x_0 \in X$ with a dense orbit: $\overline{Gx_0} = X$. And, if such a **transitive point** x_0 exists then the set of transitive points $X_0 \subset X$ is a dense G_δ (hence comeager) subset of X. A subset A of a Polish space X has the **Baire property** if it has the form $A = U \triangle M$ with U open and M meager. The collection of subsets of X with the Baire property is a σ-algebra which contains the analytic sets (see e.g. [40]). (Note that, following several different traditions, we use the words "comeager", "generic" and "residual" to denote one and the same notion. Of course, generic in this sense is not the same as Halmos' notion of genericity which is measure theoretical.)

3.2. **Theorem** (Zero one law)**.** *Let X be a Polish space and G a group of homeomorphisms of X. Suppose the action of G on X in topologically transitive. Then every G-invariant subset of X with the Baire property is either meager or comeager.*

3.3. **Definition.** A topological group has the **Rohlin property** (RP) if it has a dense conjugacy class. For a Polish G this is equivalent to the topological transitivity of the action of G on itself by conjugation. A topological group has the **strong Rohlin property** (SRP) if it has a comeager conjugacy class.

Thus, by Halmos' theorem the group $G = \mathrm{Aut}\,(X, \mathcal{X}, \mu)$ has the Rohlin property. Does it have the strong Rohlin property? Certainly not.

One way to see this is via a theorem of del Junco [37], according to which, the set $T^\perp$ of automorphisms in $G = \mathrm{Aut}\,(X, \mu)$ which are disjoint from a given element T is residual in G. It is easy to see that $T^\perp$ is also conjugation invariant. Thus if T has a comeager conjugacy class it should be disjoint from itself. Recall that, as defined by Furstenberg [21], two automorphisms $S, T \in G$ are **disjoint** if the only joining they admit is the product measure. (A probability measure λ on $X \times X$ is a **joining** of S and T if it is $S \times T$-invariant and projects onto μ in both coordinates). Since for any $T \in G$ the image of μ under the embedding $x \mapsto (x, x)$ of X into $X \times X$ is always a self-joining, an automorphism is never disjoint from itself.

Observe that $(1 \times R)_*\lambda$ is a joining between S and $RTR^{=1}$. From which it follows that $S^\perp$ is conjugacy invariant.

Recently G. Hjorth gave a proof of the fact that every conjugacy class in G is meager (see [Kech-07]) that is more direct and does not involve as much ergodic theory as the above proof. On the other hand, he uses two nontrivial but standard results from descriptive set theory. We will next present Hjorth' proof and will start by proving the first of these descriptive set theory results which is a version of the Jankov von Neumann theorem, [40, Theorem 29.9]. For an analytic set E, $\Sigma^1_1 = \Sigma^1_1(E)$ denotes the collection of analytic subsets of E, and $\sigma(\Sigma^1_1)$ is the σ-algebra generated by Σ^1_1.

3.4. **Theorem.** *Let X and Y be Polish spaces and $\phi : X \to Y$ a continuous map with $E := \phi(X)$. Then there is a $\sigma(\Sigma^1_1)$-measurable map $\psi : E \to X$ with $\phi \circ \psi = \mathrm{id}_E$.*

Proof. Let us first observe that we can assume that $X = \mathbb{N}^\mathbb{N}$ is the Baire space. In fact, since X is Polish there is a continuous surjection $\eta : \mathbb{N}^\mathbb{N} \to X$. Let $\phi_1 = \phi \circ \eta : \mathbb{N}^\mathbb{N} \to Y$ and observe that if $\psi_1 : E \to \mathbb{N}^\mathbb{N}$

is $\sigma(\Sigma^1_1)$-measurable with $\phi \circ \psi_1 = \mathrm{id}_E$, then $\psi = \eta \circ \psi_1$ is also $\sigma(\Sigma^1_1)$-measurable and $\phi \circ \psi = \mathrm{id}_E$. So we now assume that $X = \mathbb{N}^{\mathbb{N}}$ and use the usual notation for *cylinder sets*:

$$[i_1, \dots, i_k] = \{x \in X : x_1 = i_1, \dots, x_k = i_k\}.$$

For a closed subset $F \subset X$ we define $\alpha(F) = (a_1, a_2, \dots) \in F$ as follows.

$$\begin{aligned} a_1 &= \min\{i \in \mathbb{N} : [i] \cap F \neq \emptyset\}, \\ a_2 &= \min\{i \in \mathbb{N} : [a_1, i] \cap F \neq \emptyset\}, ..., \\ a_k &= \min\{i \in \mathbb{N} : [a_1, a_2, \dots, a_{k-1}, i] \cap F \neq \emptyset\}, ... \end{aligned}$$

Now set, for $y \in E$, $\psi(y) = \alpha(\phi^{-1}(y))$. Clearly $\phi(\psi(y)) = y$ for every $y \in E$ and it remains to show that ψ is $\sigma(\Sigma^1_1)$-measurable.

For this, it suffices to show that for each $\nu = (i_1, i_2, \dots, i_k) \in \bigcup_{j=1}^\infty \mathbb{N}^j$, the set $\psi^{-1}([\nu])$ is in $\sigma(\Sigma^1_1)$. Now a moment's reflection will show that, with the lexicographic order on each $\mathbb{N}^k$, we have

$$\psi^{-1}([i_1, i_2, \dots, i_k]) =$$
$$\phi([i_1, i_2, \dots, i_k]) \setminus \bigcup \{\phi([\nu']) : \nu' = (i'_1, i'_2, \dots, i'_k) < (i_1, i_2, \dots, i_k)\}.$$

Since for each ν, $\phi([\nu])$ is analytic our proof is now complete. ☐

Recall that a subset A of a topological space X has the Baire property if it has the form $A = U \triangle M$ with U open and M meager. It is well known that when X is a Baire space (e.g. when it is Polish) the collection $\mathcal{B}$ of sets with the Baire property is a σ-algebra. The second result we need from descriptive set theory is a theorem of Lusin and Sierpiński which asserts that in a Polish space every analytic subset has the Baire property, so that $\sigma(\Sigma^1_1) \subset \mathcal{B}$, [40, Theorem 21.6]. From this theorem we now deduce that *the map $\psi : E \to X$ in Theorem* 3.4 *is also Baire measurable.*

3.5. **Theorem.** *Every conjugacy class in $G = \mathrm{Aut}\,(X, \mu)$ is meager.*

Proof. (Hjorth) We will take $X = [0, 1]$ and μ as normalized Lebesgue measure. Assume that $T_0 \in G$ has a comeager conjugacy class. Applying Theorem 3.4 to the map $\phi : G \to G$, $\phi(R) = RT_0R^{-1}$, we obtain a $\sigma(\Sigma^1_1)$-measurable right inverse $\psi : C(T_0) = \{RT_0R^{-1} : R \in G\} \to G$, so that $\phi \circ \psi = \mathrm{id}_{C(T_0)}$. By the theorem of Lusin and Sierpiński, we know that ψ is also Baire measurable. Note that we have $T = \psi(T)T_0\psi(T)^{-1}$ and therefore also

$$\psi(T)T^N = T_0^N \psi(T) \tag{1}$$

for every $T \in C(T_0)$ and $N \in \mathbb{Z}$.

Next consider the Polish space $\mathcal{B}_{1/2}$ of all measurable subsets of $[0,1]$ with measure $1/2$, where the complete metric is given by $d(A,B) = \mu(A \triangle B)$. It will be convenient to choose a distinguished element, say $D_0 = [0,1/2]$, of $\mathcal{B}_{1/2}$. Choose a countable collection of sets $\{D_i\}_{i=1}^{\infty} \in \mathcal{B}_{1/2}$ such that the corresponding collection of sets $\mathcal{D}_i = \{D \in \mathcal{B}_{1/2} : \mu(D \triangle D_i) < 1/200\}$ is an open cover of $\mathcal{B}_{1/2}$. Then the collection $\{\mathcal{E}_i\}_{i=1}^{\infty}$, where

$$\mathcal{E}_i = \{T \in C(T_0) : \mu(\psi(T)D_0 \triangle D_i) < 1/200\},$$

is a countable cover of the comeager analytic subset $C(T_0)$ of G consisting of sets with the Baire property. If for each i, $\mathcal{E}_i = U_i \triangle M_i$ is a Baire representation with U_i open and M_i meager then, by Baire category theorem, there is at least one i with $U_i \neq \emptyset$. Choose one such i and set $U_i = U$ and $U_0 = U \cap \mathcal{E}_i$. For $T_1, T_2 \in U_0$ we have

$$\mu(\psi(T_1)D_0 \triangle \psi(T_2)D_0) < 1/100. \tag{2}$$

Since the collection of ergodic transformations is a dense G_δ subset of G we can choose an ergodic $T_1 \in U_0$. By ergodicity

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \mu(T_1^j D_0 \cap D_0) = 1/4,$$

hence there is a sequence $N_i \nearrow \infty$ with

$$\mu(T_1^{N_i} D_0 \cap D_0) < 1/3. \tag{3}$$

Now for a fixed N_i the set

$$A_i = \{T \in G : \mu(T^{N_i} D_0 \triangle D_0) < 1/100\}$$

is open and dense. It is clearly open and its density is a direct consequence of Rohlin's lemma, Theorem 3.1. Given any aperiodic $T \in G$ and $\epsilon > 0$ let $B, TB, \ldots, T^{N_i-1}B$ with $\mu(\cup_{j=0}^{N_i-1} T^j B) > 1-\epsilon$, be a Rohlin tower for T. Then the transformation S defined by: $S = T$ on $\cup_{j=0}^{N_i-2} T^j B$, $S = T^{N_i-1}$ on $T^{N_i-1}B$, and $S = \mathrm{id}$ elsewhere, is a periodic approximation to T with period N_i. Thus $A = \cap_{m>1} \cup_{i=m}^{\infty} A_i$ is a dense G_δ subset of G and we now pick an element $T_2 \in A \cap U_0$.

Since T_2 is in A there is an $N = N_i$ with $\mu(T_2^N D_0 \triangle D_0) < 1/100$. Since $\psi(T_2)$ is measure preserving we also have $\mu(\psi(T_2)T_2^N D_0 \triangle \psi(T_2)D_0) < 1/100$, and by (1)

$$\mu(T_0^N \psi(T_2)D_0 \triangle \psi(T_2)D_0) < 1/100. \tag{4}$$

Also by (3)

$$\mu(T_1^N D_0 \cap D_0) < 1/3,$$

hence

$$\mu(\psi(T_1)T_1^N D_0 \cap \psi(T_1)D_0) < 1/3,$$

and by (1)

$$\mu(T_0^N \psi(T_1)D_0 \cap \psi(T_1)D_0) < 1/3. \tag{5}$$

However, by (2) $\psi(T_1)D_0$ and $\psi(T_2)D_0$ differ by less than 1/100 so that (4) and (5) are in conflict. This contradiction shows that no conjugacy class in G is comeager and our proof is completed by the zero one law, Theorem 3.2. ☐

4. The weak RP

The "weak Rohlin property" was introduced in [23]. Let Γ be a discrete countable infinite group. We denote by $\mathbb{A}_\Gamma$ the collection of Γ measure preserving actions on X. Thus an element $\mathbf{T} \in \mathbb{A}_\Gamma$ is a representation $\mathbf{T} : \Gamma \to G$, $\gamma \mapsto \mathbf{T}_\gamma$, where $G = \mathrm{Aut}\,(X, \mathcal{X}, \mu)$.

As we have seen above (Section 3), a countable algebra $\{A_k\}_{k=1}^\infty$ of sets which separates points of X, gives rise to a metric on G. We now define a metric on the space $\mathbb{A}_\Gamma$ of Γ-actions as follows. Set

$$D(\mathbf{S}, \mathbf{T}) = \sum_{i=1}^{\infty} 2^{-i} d(\mathbf{S}_{\gamma_i}, \mathbf{T}_{\gamma_i}),$$

where $\{\gamma_i : i = 1, 2, \dots\}$ is some enumeration of Γ and d is the complete metric defined above for G. Again with this metric $\mathbb{A}_\Gamma$ is a Polish space.

We say that two Γ-actions $\mathbf{S}$ and $\mathbf{T}$ are *isomorphic* if there is $R \in G$ such that $\mathbf{T}_\gamma = R\mathbf{S}_\gamma R^{-1}$ for every $\gamma \in \Gamma$. In other words, if and only if $\mathbf{S}$ and $\mathbf{T}$ belong to the same orbit of the natural action of the Polish group G on the Polish space $\mathbb{A}_\Gamma$ by conjugation. We say that the group Γ has the **weak Rohlin property** (WRP) if this action of G on $\mathbb{A}_\Gamma$ is topologically transitive; i.e. for any two nonempty open sets U and V in $\mathbb{A}_\Gamma$ there is some $R \in G$ such that $RUR^{-1} \cap V \neq \emptyset$. An equivalent condition is that there is a dense G_δ subset $\mathcal{A}_0$ of $\mathbb{A}_\Gamma$ such that for every $\mathbf{T} \in \mathcal{A}_0$ the G-orbit, $\{R\mathbf{T}R^{-1} : R \in G\}$ is dense in $\mathbb{A}_\Gamma$. It was shown in [23] that every amenable Γ has the WRP. Now the results of [23] apply to groups having the weak Rohlin property and the question as to which groups have that property was left open. In the recent work [24] the authors show that in fact every discrete countable group has the WRP. (Hjorth (unpublished) had also independently proved that every countable discrete group has the WRP; see 10.7 and the preceding paragraph in [Kech-07]).

4.1. **Theorem.** *Every infinite countable group* Γ *has the weak Rohlin property.*

Combining this result with an earlier work [26] they obtain the following characterization.

4.2. **Theorem.** *The infinite countable group* Γ *admits an ergodic action* $\mathbf{T} \in \mathbb{A}_\Gamma$ *whose* G*-orbit* $\{R\mathbf{T}R^{-1} : R \in G\}$ *is dense in* $\mathbb{A}_\Gamma$ *if and only if* Γ *does not have the Kazhdan property. Thus for a non-Kazhdan group the set of ergodic actions* $\mathbf{T} \in \mathbb{A}_\Gamma$ *with a dense* G*-orbit is a dense* G_δ*, while for a Kazhdan group* Γ*, the set of ergodic actions forms a meager subset of* $\mathbb{A}_\Gamma$ *and for every ergodic* $\mathbf{T} \in \mathbb{A}_\Gamma$*,* $\mathrm{cls}\,\{R\mathbf{T}R^{-1} : R \in G\}$ *has an empty interior in* $\mathbb{A}_\Gamma$*.*

Actually the set of ergodic actions of a Kazhdan group is a closed set with empty interior in the space of actions (see, e.g., 12.2 (i) in [Kech-07]).

In a recent work, Kerr and Pichot [42] show that a result similar to the above theorem holds even for weak mixing. They prove that if G is a locally compact σ-compact group which does not have property T (in particular a countable amenable group) then the weakly mixing actions are a dense G_δ in the space of all actions. It then follows (see [26], especially 3.3) that there exist weakly mixing actions whose orbit is dense. They also show that when G has property T the set of weakly mixing actions is closed with empty interior.

The results of [24] already found some applications in a recent work of Ageev, [2], [3] where he proves that every finite or countable group Γ has **spectral rigidity**; i.e. for every $\gamma \in \Gamma$, on a residual subset of $\mathbb{A}_\Gamma$ the set of essential values of the multiplicity function $M(\hat{T}_\gamma) : \mathbb{T} \to \mathbb{N} \cup \infty$, associated with the unitary Koopman operator $\hat{T}_\gamma$, is a constant (see also [1]).

For amenable groups Foreman and Weiss show that the action of G on the free ergodic actions in $\mathbb{A}_\Gamma$ is **turbulent** in the sense of Hjorth, [18]. In particular this means that every free ergodic action has a dense conjugacy class which is meager. In Kechris' forthcoming book [Kech-07, Proposition 13.2], he shows that conversely if every free ergodic action has a dense conjugacy class in $\mathbb{A}_\Gamma$ then the group Γ must be amenable.

4.3. **Theorem.** *The following conditions on an infinite countable group* Γ *are equivalent:*

1. Γ *is amenable.*

2. *The conjugacy class of the "shift" Γ-action s_Γ on the product space $X = \{0,1\}^\Gamma$ equipped with the $\{1/2, 1/2\}$ product measure is dense in $\mathbb{A}_\Gamma$.*
3. *Every free ergodic action in $\mathbb{A}_\Gamma$ has a dense conjugacy class in $\mathbb{A}_\Gamma$.*

For more details on Hjorth' notion of turbulence see [31], [32]. Also see the forthcoming work of Foreman, Rudolph and Weiss [19], where it is shown that the conjugacy relation on the set G_{erg} of ergodic measure preserving automorphisms is not a Borel subset of $G_{\mathrm{erg}} \times G_{\mathrm{erg}}$.

4.4. **Remark.** In connection with Theorem 4.3 three further conditions come to mind: (We let $\mathbb{A}_\Gamma^{erg}$ denote the subset of ergodic actions in $\mathbb{A}_\Gamma$.)

4. Every free ergodic action in $\mathbb{A}_\Gamma$ has a dense conjugacy class in $\mathbb{A}_\Gamma^{erg}$.
5. The conjugacy class of s_Γ is dense in $\mathbb{A}_\Gamma^{erg}$.
6. There exists a free ergodic action with a dense conjugacy class in $\mathbb{A}_\Gamma^{erg}$.

We do not know whether either one of the conditions 4. and 5. can be added to the list of equivalent conditions in Theorem 4.3. On the other hand, by considering the ergodic decomposition of the measure λ constructed in the proof of Theorem 1.1 in [24], one can use essentially the same proof to see that, like the WRP, condition 6. holds for every Γ. See also the remark following Proposition 13.2 in [Kech-07].

5. The unitary group $\mathcal{U}(H)$

The Koopman representation $\kappa : T \mapsto U_T$ embeds $\mathrm{Aut}\,(X, \mathcal{X}, \mu)$ in the unitary group $\mathcal{U}(L^2(\mu))$, equipped with the strong operator topology. With this topology $\mathcal{U}(L^2(\mu))$ is itself a Polish topological group and it is known that this group has the RP. In fact every $U \in \mathcal{U}(L^2(\mu))$ with maximal spectral measure with full support has a dense conjugacy class, [15]. However, as we will next show, it does not have the SRP. We will work in the abstract setup where $G = \mathcal{U}(H)$ is the unitary group of a separable infinite dimensional Hilbert space H. With $U \in G$ we associate its **maximal spectral measure** which is of course only a measure class. However, choosing an orthonormal basis for H, say, $\{x_n : n = 1, 2, \dots\}$ we define the map

$$\mathrm{Spec} : U \mapsto \sigma_U := \sum_{n=1}^{\infty} 2^{-n} \sigma_{U, x_n}$$

— where $\hat{\sigma}_{U,x_n}(k) = \langle U^k x_n, x_n \rangle$ — which picks a concrete representative for the maximal spectral type of U. It is not hard to check that the map $U \mapsto \sigma_U$ is continuous map from G into the space $M(\mathbb{T})$ of probability measures on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1], \pmod 1$ with its weak* topology. Following [38] we have:

5.1. **Lemma.** *For a fixed $\mu \in M(\mathbb{T})$ the set*

$$\mu^{\perp} := \{\nu \in M_1(\mathbb{T}) : \nu \perp \mu\}$$

is a G_δ subset of $M(\mathbb{T})$.

Proof. Here of course $\perp$ means mutually singular. Let $f_i \in C(\mathbb{T})$ be a norm dense sequence in the set $\{f \in C(\mathbb{T}) : 0 \leq f \leq 1\}$ and as in [38] one shows that

$$\{\nu \in M(\mathbb{T}) : \nu \perp \mu\} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \{\nu : \nu(f_i) < 1/n, \ \mu(1 - f_i) < 1/n\}.$$

$\square$

We conclude that for any fixed $U \in G$ the set

$$U^{\perp} := \{V \in G : \sigma_V \perp \sigma_U\} = \mathrm{Spec}^{-1}(\sigma_U{}^{\perp})$$

is a G_δ subset of G. We will say that the elements of $U^{\perp}$ are **spectrally disjoint** from U.

Next we show that $U^{\perp}$ is dense in G. Any $V \in G$ can be approximated, in the strong operator topology, by operators which are the identity on the orthogonal complement of a finite dimensional subspace of H. Of course the spectral measure of such operators is a finite set of the unit circle which includes 1. If we replace the identity operator on this complement by an operator which moves every nonzero vector slightly, also the eigenvalue 1 can be avoided while preserving a good approximation. Thus we have a complete freedom in the choice of the countable purely discrete spectrum of the approximating operator. These considerations prove the density of $U^{\perp}$.

To complete the proof that G does not have the SRP we now assume that $U \in G$ has a residual conjugacy class U^G, and then observe that any V in the residual, hence nonempty, intersection $U^G \cap U^{\perp}$ has a spectral measure which is singular to itself. Of course this conflict completes the proof.

The following theorem is well known, although a proof is hard to find (see e.g. [22, page 366]). It shows that the group $NS(X, \mathcal{X}, \mu)$ of nonsingular automorphisms of Lebesgue space embeds naturally into $\mathcal{U}(L^2(\mu))$ as the subgroup of positive unitary operators. In particular $NS(X, \mathcal{X}, \mu)$ is a closed subgroup of $\mathcal{U}(L^2(\mu))$.

5.2. **Theorem.** *Let $(\mathbb{T}, \lambda)$ be the circle equipped with Lebesgue's measure and $V : L^2(\lambda) \to L^2(\lambda)$ a unitary operator which is also positive (i.e. $Vf \geq 0$ for $f \geq 0$). Then V has the form: $Vf(t) = \sqrt{\frac{dT\lambda}{d\lambda}(t)}f(Tt)$, where $T : \mathbb{T} \to \mathbb{T}$ is an invertible measurable nonsingular map.*

The group $NS(X, \mathcal{X}, \mu)$ has the Rohlin property. E.g., using the method of [29], Friedman shows that every aperiodic transformation in $NS(X, \mathcal{X}, \mu)$ has a dense orbit (see [20, Theorem 7.13]). Again it is not hard to see that $NS(X, \mathcal{X}, \mu)$ does not have the SRP. In fact as above, for each nonsingular $T \in NS(X, \mathcal{X}, \mu)$ the collection $T^\dagger$ of **spectrally disjoint** $S \in NS(X, \mathcal{X}, \mu)$; i.e. those $S \in NS(X, \mathcal{X}, \mu)$ for which V_S is spectrally disjoint from V_T, is again a G_δ subset of $NS(X, \mathcal{X}, \mu)$. Its density can be proved by choosing an irrational rotation R_α of the circle whose (discrete) spectral measure is disjoint from the spectral measure of T and then using the fact that the conjugacy class of R_α is dense in $NS(X, \mathcal{X}, \mu)$.

Note that the same proof works also for $\mathrm{Aut}\,(X, \mathcal{X}, \mu)$. Since spectral disjointness implies disjointness (see e.g. [22, Theorem 6.28]) we have $T^\dagger \subset T^\perp$, so that residuality of $T^\dagger$ implies the residuality of $T^\perp$. This provides a strengthening of del Junco's theorem (see Section 3 above).

6. Groups of homeomorphisms with the RP

Recall that a Polish topological group G has the topological Rohlin property when it acts topologically transitively on itself by conjugation. Let us say that a compact topological space X has the **Rohlin Property** when $G = H(X)$ — the topological group of homeomorphisms of X equipped with the topology of uniform convergence — has the Rohlin Property; i.e., $H(X)$ is the closure of a single conjugacy class. For some connected spaces like spheres the existence of orientation of a homeomorphism, which is clearly preserved under conjugation, means that $H(S^d)$ can not have the Rohlin property; therefore we say that a sphere satisfies the Rohlin property when the group $H_0(S^d)$ — the connected component of the identity in $H(S^d)$ — has the Rohlin property. With this definitions it is shown in [27] that the Hilbert cube, the Cantor set and the even dimensional spheres have the Rohlin property. (The result for the Cantor set was independently obtained by Akin, Hurley and Kennedy in [9]).

6.1. **Theorem.** 1. *The Hilbert cube $Q = [-1, 1]^{\mathbb{N}}$ has the Rohlin property.*

2. *The Cantor set has the Rohlin property.*
3. *The group G of of homeomorphisms of the cube I^d which fix each point of the boundary ∂I^d, has the Rohlin property.*
4. *Every even dimensional sphere S^{2d} has the Rohlin property.*

On the other hand it appears that for general compact manifolds of positive finite dimension the answer is rather different. For circle homeomorphisms, Poincaré's rotation number, $\tau : H^+(S^1) \to \mathbb{R}/\mathbb{Z},\ h \mapsto \tau(h)$, where $H^+(S^1) = H_0(S^1)$ is the subgroup of index 2 of orientation preserving homeomorphisms, is a continuous conjugation invariant mapping and thus there is a continuum of different closed disjoint conjugation invariant subsets. We refer the reader to the work [9], by Akin, Hurley and Kennedy for a detailed discussion of circle homeomorphisms.

A related problem is concerned with the generic behavior of entropy in these spaces. What is the topological entropy of a typical homeomorphism in $H(X)$? The machinery developed in [27] for dealing with the Rohlin property, enables the authors to answer the entropy problem as follows. For the Hilbert cube, and spheres $S^d,\ d \geq 2$, the set of homeomorphisms with infinite entropy is residual while for the Cantor set it is the set of zero entropy which is a dense G_δ subset of $H(X)$.

7. The dynamics of topologically generic homeomorphisms

For the Cantor set X, as we will see in Section 10 below, the group Homeo (X) has the SRP and if $T_0 \in$ Homeo (X) has a dense G_δ conjugacy class than the dynamical properties of T_0 are, by definition, the dynamical properties of the generic homeomorphism of X. Of course typically the group of homeomorphisms of a manifold does not have the SRP, that is, there is no comeager conjugacy class in Homeo (X). It is nonetheless natural to enquire what are the dynamical properties held by a generic set of homeomorphisms, or as the title of the fundamental work of Akin, Hurley and Kennedy [9] suggests, to ask what is the typical dynamics of a generic homeomorphism. This section is a brief survey of the results obtained in [9].

We already mentioned some of the results obtained in [9] about Homeo (X) with X being the Cantor set or the circle. In this section we will describe the results of [9] which concern the case where X is a compact manifold. Thus in the sequel we will assume that X is a compact piecewise linear manifold of dimension at least 2 (e.g.

a smooth manifold with no boundary of dimension ≥ 2). We fix a compatible metric on X.

Unlike many classical works which treat the general diffeomorphism of a manifold, in [9] the objects one deals with are merely homeomorphisms and the dynamical properties which are considered are strictly topological. Many of these were first introduced and studied by Conley, [17]. We follow the notation of Akin's monograph [4].

Given $f \in \mathrm{Homeo}\,(\mathrm{X})$ and $\epsilon > 0$ an **ϵ-chain** between two points $x, y \in X$ is a finite sequence $\{x_j\}_{j=0}^n$ with $x_0 = x$, $x_n = y$ and $n \geq 1$ such that each x_{j+1} is within ϵ of $f(x_j), j = 0, ..., n-1$. Set

$$\mathcal{C}f = \{(x, y) \in X \times X : \text{for every } \epsilon > 0 \text{ there exits an } \epsilon\text{-chain} \\ \text{connecting } x \text{ and } y\ \}$$

and for $x \in X$

$$\mathcal{C}f(x) = \{y : (x, y) \in \mathcal{C}f\}$$

The relation $\mathcal{C}f$ is a closed transitive relation with $\mathcal{C}(f^{-1}) = \{(y, x) : (x, y) \in \mathcal{C}f\}$. It is reflexive when restricted to the set of chain recurrent points. A point $x \in X$ is **chain recurrent** when $x \in \mathcal{C}f(x)$ or, equivalently, when $(x, x) \in \mathcal{C}f$. Denoted $|\mathcal{C}f|$ the set of chain recurrent points is a closed f-invariant set with $|\mathcal{C}f| = |\mathcal{C}f^{-1}|$. The relation $\mathcal{C}f \cap \mathcal{C}f^{-1}$ is a closed, f-invariant equivalence relation on $|\mathcal{C}f|$. The equivalence classes are called the **chain components** of f. The chain components are closed, f-invariant subsets and the chain components of f are the same as the chain components of f^{-1}.

A subset $D \subset X$ is called **f-chain invariant** when $\mathcal{C}f(D) \subset D$, i.e. $\mathcal{C}f(x) \subset D$ for all $x \in D$. Equivalently, for every $\epsilon > 0$ there exists a $\delta > 0$ such that no δ-chain beginning in D can leave an ϵ neighborhood of D. A closed set U is called **inward** for f if $f(U) \subset \mathrm{int}\, f(U)$. An inward set is f-chain invariant. To an inward set corresponds the **attractor** A which is defined as $A = \bigcap_{n=1}^{\infty} f^n(U)$. The open set $W = \bigcup_{n=0}^{\infty} f^{-n}(U)$ is the **basin of attraction** of A and the closed set $X \setminus W$ is the associated **repellor** (= attractor for f^{-1}) with $X \setminus A$ the **basin of repulsion** for R. An **attractor-repellor pair** (A, R) is characterized as a pair of disjoint closed, f-invariant sets such that (i) $|\mathcal{C}f| \subset A \cup R$ and (ii) $\mathcal{C}f(A) = A$ and $\mathcal{C}f^{-1}(R) = R$.

On the space of chain components the relation $\mathcal{C}f$ induces a partial order by $B_1 \rightsquigarrow B_2$ if $(x, y) \in \mathcal{C}f$ for $x \in B_1$ and $y \in B_2$ (this relation does not depend upon the choice of x and y.) A chain component is called **terminal** if $B \rightsquigarrow B_1$ implies $B = B_1$. Equivalently, B is a $\mathcal{C}f$-invariant chain component. B is called **initial** if it is terminal for f^{-1}. If A is a closed set such that $\mathcal{C}f(A) = A$, or equivalently, if $f(A) = A$

and A is f-chain invariant, then A is called a **quasi-attractor**. A quasi-attractor A is the intersection of a monotone sequence of attractors and the inward set neighborhoods of A form a base for the neighborhood system of A. For example a chain component is a quasi-attractor iff it is terminal.

Using Zorn's lemma one shows that every nonempty quasi-attractor contains a terminal chain component. Finally a point $x \in X$ is a **chain continuity point** (see Akin [5]) if for each $\epsilon > 0$ there is a $\delta > 0$ with the property that if $\{x_n\}$ is a δ-chain with $x_0 = x$, then the distance from x_n to $f^n(x)$ is less than ϵ for every $n > 0$.

We are now ready to state the main results of [9]. In the sequel when we say that "a generic homeomorphism of X" has a certain property, we mean that the collection of $f \in \mathrm{Homeo}\,(\mathrm{X})$ with that property is residual. Recall that we assume here that X is a compact piecewise linear manifold of dimension at least 2.

1. Let A be an attractor of a generic $f \in \mathrm{Homeo}\,(\mathrm{X})$ then:
   (a) A contains infinitely many repellors for f.
   (b) $\mathrm{int}\, A \neq \emptyset$ and it is the union of the basins of repulsion for the repellors contained in A.
   (c) ∂A is a quasi-attractor (but not an attractor, having an empty interior).
   (d) Thus there are uncountably many distinct sequences $A = A_1 \supset R_1 \supset A_2 \supset R_2 \supset \cdots$, with the A_i attractors and the R_i repellors.
2. For a generic $f \in \mathrm{Homeo}\,(\mathrm{X})$:
   (a) $|\mathcal{C}(f)|$ is a Cantor set.
   (b) In $|\mathcal{C}f|$, the set of periodic points is dense and meager in $|\mathcal{C}f|$. Hence $|\mathcal{C}f|$ coincides with the sets of non-wandering points, and also with the closure of the set of recurrent points for f.
3. For a generic $f \in \mathrm{Homeo}\,(\mathrm{X})$:
   (a) There are uncountably many chain components.
   (b) The union of chain components which admit a **subshift of finite type** as a factor is dense in $|\mathcal{C}f|$.
   (c) The restriction of f to each terminal chain component is either a finite periodic orbit or an adding machine.
   (d) The union of the terminal chain components which are not periodic is a residual subset of $|\mathcal{C}f|$. Of course the same is true for the initial sets and thus the set of "dynamically isolated points", which lie in chain components which are both initial and terminal, form a residual subset of $|\mathcal{C}f|$.

4. For a generic $f \in \mathrm{Homeo}\,(\mathrm{X})$ the set of points $x \in X$ whose ω-limit set is an adding machine terminal chain component and whose α-limit set is a distinct adding machine initial chain component is residual in X. Note that such a point can not be one of the "dynamically isolated" point as in 3 (d).
5. For a generic $f \in \mathrm{Homeo}\,(\mathrm{X})$ the points which are chain continuous for both f and f^{-1} form a residual subset of X whose intersection with $|\mathcal{C}f|$ is residual in $|\mathcal{C}f|$. In particular the generic f is **almost equicontinuous** in the sense that the set of equicontinuity points for $\{f^n : n \in \mathbb{Z}\}$ is a residual set in X (see [25] and [7]).

Some of these generic properties carry over when the dimension is 1 or 0, but some do not. As we will see in Section 10 below, for the "Special Homeomorphism of the Cantor Space" $T = T(D, C)$, whose conjugacy class is comeager, the chain recurrence set $|\mathcal{C}(T)|$ is a disjoint union of an uncountable collection of universal adding machines, it coincides with the set of non-wandering points and, in this case, also with the set of "dynamically isolated" points.

8. Groups with the strong Rohlin property

Is there any Polish topological group with the strong Rohlin property? The answer is: Yes there are many. Here is perhaps the simplest example.

8.1. **Theorem.** *The Polish group $S_\infty = S(\mathbb{N})$ of all permutations of a countable set, with the topology of pointwise convergence has the strong Rohlin property.*

Proof. Let $\mathbb{N} = \bigcup\{A_{n,k} : n = 1, 2, 3, \dots,\ k = 1, 2, 3, \dots\}$ be a disjoint decomposition of $\mathbb{N}$ such that $\mathrm{card}\, A_{n,k} = n$. Choose a linear order on each $A_{n,k}$ and let $\pi \in S_\infty$ be defined by the requirement that its restriction to each $A_{n,k}$ is a cyclic permutation. Since clearly a conjugacy class in S_∞ is uniquely determined by a cycle structure, our claim will follow by showing that the set of permutations in S_∞ having the same cycle structure as π, forms a dense G_δ subset of S_∞.

Now, clearly for each n and k the collection of permutations having at least k disjoint n-cycles, is a dense open set. Thus, the intersection A of these collections, namely the set of permutations which admit infinitely many n-cycles for each n, is a dense G_δ subset of S_∞. On the other hand so is the set

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{\sigma : \sigma^k(n) = n\}$$

of permutations with no infinite orbits. Since $A \cap B$ is exactly the conjugacy class of π our proof is complete. $\square$

9. The group $H_+[0,1]$ has SRP

The group $G = H_+[0,1]$ of order preserving homeomorphisms of the unit interval $I = [0,1]$ is a Polish group when equipped with the topology of uniform convergence.

We will next show that G has the SRP. For $f \in G$ let $\mathrm{Fix}(f) := \{x \in I : f(x) = x\}$. Let us say that an order preserving homeomorphism of an interval $[a,b]$ is of *type* $\pm$ if $f(x) > x$ (respectively $f(x) > x$) for every $x \in [a,b]$. If $\mathrm{Fix}(f) = A$ is a Cantor subset of I, then $\mathcal{J}(A)$, the countable collection of components of $I \setminus A$, has the order type of the rational numbers, and we say that it is *typical* if for any distinct $j_1, j_2 \in \mathcal{J}(A)$ there are i_1, i_2 in $\mathcal{J}(A)$ such that the restrictions of f to i_1 and i_2 have opposite signs. It is easy to show that any two typical homeomorphisms are conjugate in G and we denote the conjugacy class of typical homeomorphisms by by $\mathcal{T}$. It is also clear that $\mathcal{T}$ is dense in G. Thus it only remains to show that $\mathcal{T}$ is G_δ.

It is not hard to check that the map $f \mapsto \mathrm{Fix}(F)$, from G to the metric space 2^I, with the Hausdorff metric, is upper-semi-continuous, and it follows that the set O_1 of elements $g \in G$ where it is actually continuous, forms a dense G_δ set in G.

By a well known result of Kuratowski the collection of Cantor subsets of I forms a dense G_δ subset, say $\mathcal{C}$, of 2^I and it follows that $O_2 = \Phi^{-1}(\mathcal{C})$, where Φ is the restriction of Fix to O_1, is a G_δ subset of O_1, hence also of G.

For $0 \le a < b \le 1$ in $\mathbb{Q}$, let $V_{a,b}$ be the set of $g \in O_2$ such that, if a and b lie in two distinct elements of $\mathcal{J}(\mathrm{Fix}(g))$, say j_1 and j_2, then there are i_1, i_2 in $\mathcal{J}(\mathrm{Fix}(g))$ that lie between j_1 and j_2 and such that the restrictions of g to i_1 and i_2 have opposite types. Clearly each $V_{a,b}$ is open and their intersection is $\mathcal{T}$. This completes the proof that $\mathcal{T}$ is a dense G_δ conjugacy class in G.

The SRP for $G = H_+[0,1]$ was first shown by Kuske and Truss in [44]. They also show that H does not have the stronger property of ample generics (see Section 13 below).

10. The Cantor group has the SRP

As was mentioned above, in [27] it was shown that the Polish group $H(X)$ of homeomorphisms of the Cantor set X has the Rohlin property and the same result was independently obtained in [9]. In the latter work the authors posed the question whether a much stronger property holds for $H(X)$, namely that there exists a conjugacy class which is a dense G_δ subset of G; i.e. whether $H(X)$ has the strong Rohlin property.

In [6] it was shown that the subgroup G_μ of the Polish group $G = H(X)$ of all homeomorphisms of the Cantor set X which preserve a special kind of a probability measure μ on X has SRP. Recently this was shown by Kechris and Rosendal in [41] to be the case for many other closed subgroups of $G = H(X)$, including G itself. The authors of [41] use abstract model theoretical arguments in their proof and they present it as an open problem to give an explicit description of the generic homeomorphism.

In [8] the authors provide a new and more constructive proof of the fact that the group $H(X)$ of homeomorphism the Cantor set has the SRP. Moreover their proof relies on a detailed description of the generic homeomorphism of X. In the sequel we provide a detailed picture of this "Special Homeomorphism of the Cantor Space". For full details the reader is referred to [8].

Let $\mathbb{Z}$ denote the ring of integers and Θ_m denote the quotient ring $\mathbb{Z}/m\mathbb{Z}$ of integers modulo m for $m = 1, 2, ...$ Let $\Pi : \mathbb{Z} \to \Theta_m$ denote the canonical projection. If m divides n then this factors to define the projection $\pi : \Theta_n \to \Theta_m$. The positive integers are directed with respect to the divisibility relation. We denote the inverse limit of the associated inverse system of finite rings by Θ. This is a topological ring with a monothetic additive group on a Cantor space having projections $\pi : \Theta \to \Theta_m$ for positive integers m. We denote by Π the induced map from $\mathbb{Z}$ to Θ which is an injective ring homomorphism. We also use Π for the maps $\pi \circ \Pi : \mathbb{Z} \to \Theta_m$. Notice that we use π for the maps with compact domain and Π for the maps with domain $\mathbb{Z}$. We can obtain Θ by using any cofinal sequence in the directed set of positive integers. We will usually use the sequence $k!$.

On each of these topological rings we denote by τ the homeomorphism which is translation by the identity element, i.e. $\tau(t) = t + 1$. The dynamical system (Θ, τ) is called the **universal adding machine**. The adjective "universal" is used because it has as factors periodic orbits of every period.

Let $\mathbb{Z}_*$ denote the two point compactification with limit points $\pm\infty$. Let τ be the homeomorphism of $\mathbb{Z}_*$ which extends the translation map by fixing the points at infinity. The points of $\mathbb{Z}$ form a single orbit of τ which tends to the fixed points in the positive and negative directions. We construct an alternative compactification Σ of $\mathbb{Z}$ with copies of Θ at each end. Σ is the closed subset of $\mathbb{Z}_* \times \Theta$ given by

$$\Sigma \quad =_{def} \quad \{(x,t) : x = \pm\infty \quad \text{or} \quad x \in \mathbb{Z} \text{ and } t = \Pi(x)\}.$$

Σ is invariant with respect to $\tau \times \tau$ and we denote its restriction by $\tau : \Sigma \to \Sigma$. A **spiral** is any dynamical system isomorphic to (Σ, τ). We will also refer to the underlying space as a spiral.

The points of $\{\pm\infty\} \times \Theta$ are the **recurrent points** of the spiral. The remaining points, i.e. $\{(x, \Pi(x)) : x \in \mathbb{Z}\}$ are the **wandering points** of the spiral.

We define the map ζ which collapses the spiral and identifies the ends:

$$\zeta : \Sigma \to \Theta \qquad \text{by} \qquad \zeta(x,t) = t.$$

That is, ζ is just the projection onto the second, Θ, coordinate. Clearly, $\zeta : (\Sigma, \tau) \to (\Theta, \tau)$ is an action map.

Next we describe the construction of a "Cantor set of spirals". Let $I = [0,1]$ be the unit interval and C be the classical Cantor set in I consisting of those points a which admit a ternary expansion $.a_0a_1a_2...$ with no $a_i = 2$. Let D consist of those points a which admit a ternary expansion $.a_0a_1a_2...$ such that the smallest index $i = 0, 1, ...$ with $a_i = 2$ — if any a_i does equal 2 — is even. That is, for the Cantor set C we eliminate all the middle third open intervals, first one of length $1/3$, then two of length $1/9$, then four of length $1/27$ and so forth. For D we retain the interval of length $1/3$, eliminate the two of length $1/9$, keep the four of length $1/27$, eliminate the eight of length $1/81$ and so forth. The boundary of D is the Cantor set C. $\mathcal{J}(D \setminus C)$ consists of the open intervals of length $1/3^{2k+1}$ which we retained in D whereas $\mathcal{J}(I \setminus D)$ consists of the open intervals of length $1/3^{2k}$ which we eliminated from D.

The set $D \setminus C$ is an open subset of $\mathbb{R}$. It is the union of the countable set $\mathcal{J}(D \setminus C)$ of the disjoint open intervals which are the components of $D \setminus C$. If $j \in \mathcal{J}(D \setminus C)$ then $j = (j_-, j_+)$ with endpoints $j_-, j_+ \in C$.

Notice that between any two subintervals in $\mathcal{J}(D \setminus C) \cup \mathcal{J}(I \setminus D) = \mathcal{J}(I \setminus C)$ there occur infinitely many intervals of $\mathcal{J}(D \setminus C)$ and of $\mathcal{J}(I \setminus D)$. We let $[D]$ denote the set of components of D. A component of D is either a closed interval $\bar{j}$ for $j \in \mathcal{J}(D \setminus C)$ or a point a of C which is not the endpoint of an interval in $\mathcal{J}(D \setminus C)$.

We obtain the compact, zero-dimensional space $Z(D,C)$ from the disjoint union

$$\mathcal{J}(D \setminus C) \times \Sigma \quad \bigcup \quad C \times \Theta$$

by identifications so that in $Z = Z(D,C)$

$$(j, -\infty, t) = (j_-, t) \qquad \text{and} \qquad (j, +\infty, t) = (j_+, t)$$

for all $j \in \mathcal{J}(D \setminus C)$ and $t \in \Theta$. That is, after taking the product of D with the group Θ we replace each interval $j \times \Theta$ by a copy of the spiral Σ. The homeomorphism $1_{\mathcal{J}} \times \tau \cup 1_C \times \tau$ factors through the identifications to define the dynamical system $(Z, \tau) = (Z(D,C), \tau_{(D,C)})$.

For each $r \in C$, the subset $\{r\} \times \Theta$ is an invariant set for $\tau_{(D,C)}$ on which $\tau_{(D,C)}$ is simply the adding machine translation τ on the Θ factor. For each $j \in \mathcal{J}(D \setminus C)$ the subset $\{j\} \times \Sigma$ is an invariant set for $\tau_{(D,C)}$ on which $\tau_{(D,C)}$ is the spiral τ on the Σ factor. That is, we have a collection of adding machines indexed by the closed nowhere dense set C with a countable number of gap pairs $j_- < j_+$ of C spanned by spirals.

The space $Z(D,C)$ is compact and zero-dimensional, but the wandering points within the spirals are discrete. Now define

$$\begin{aligned} X(D,C) &=_{def} Z(D,C) \times C \\ T(D,C) &=_{def} \tau_{(D,C)} \times 1_C. \end{aligned}$$

Thus, $T(D,C)$ is a homeomorphism on the Cantor space $X(D,C)$.

The projection map $C \times \Theta \to C$ which collapses each adding machine to a point extends to a continuous map $q : Z(D,C) \to D$ by embedding the orbit of wandering points of $\{j\} \times \Sigma$ in an order preserving manner to a bi-infinite sequence $\{q(j,(x,\Pi(x))) : x \in \mathbb{Z}\}$ in the interval j which converges to $j_\pm$ as $x \in \mathbb{Z}$ tends to $\pm\infty$.

Via q we can pull back the ordering on $D \subset \mathbb{R}$ to obtain a total quasi-order on $Z(D,C)$. On the other hand, the collapsing map ζ on each spiral defines

$$\begin{aligned} &\zeta : Z(D,C) \to \Theta \\ &\zeta(j,(x,t)) = t \qquad \text{for } (j,(x,t)) \in \mathcal{J}(D \setminus C) \times \Sigma. \\ &\zeta(a,t) = t \qquad \text{for } (a,t) \in C \times \Theta. \end{aligned}$$

Clearly, $q \times \zeta : Z(D,C) \to I \times \Theta$ and $q \times \zeta \times \pi_C : X(D,C) \to I \times \Theta \times C$ are embeddings.

We call $(Z(D,C), \tau_{(I,C)})$ a **Cantor set of spirals**. Of course, there are only countably many spirals in $Z(D,C)$, and the ordering on the set of spirals is order dense. However $q : Z(D,C) \to D$ induces a much larger order than the chain relation. If x_0 is on a spiral and x_1 is not on the same spiral then $q(x_0)$ and $q(x_1)$ are separated by a gap in $I \setminus D$ of length greater than ϵ provided ϵ is sufficiently small. This gap cannot be crossed by an ϵ chain for $\tau_{(D,C)}$. It follows that the chain relation is exactly the orbit closure relation for $\tau_{(D,C)}$.

We call $T(D, C)$ the **Special Homeomorphism of the Cantor Space** $X(D, C)$ and the main result of [8] is to show that the collection of homeomorphisms $h \in H(C)$ which are topologically conjugate to $T(D, C)$ forms a dense G_δ conjugacy class of the Polish group $C(X)$.

11. Fraïssé structures and their automorphism groups

In this section we will describe parts of the work of Kechris and Rosendal [41] which deals with various Rohlin properties of groups of automorphisms of certain countable model theoretical structures. For the, not too heavy, model theory that is used we refer the reader to [34] and [41].

Briefly, a (countable) *signature* L consists of two (finite or countable) collections of symbols, the relation symbols $\{R_i : i \in I\}$ and the function symbols $\{f_j : j \in J\}$ (one of these may be empty). Each symbol has its *arity* — a positive integer — $n(i)$ for R_i and $m(j)$ for f_j. A *structure*

$$\mathbf{A} = \langle A, \{R_i^{\mathbf{A}} : i \in I\}, \{f_j^{\mathbf{A}} : j \in J\}\rangle$$

in a given signature L, is a nonempty set A and two collections, of relations $R_i^{\mathbf{A}} \subset A^{n(i)},\ i \in I$ and of functions $f_j^{\mathbf{A}} : A^{m(j)} \to A,\ j \in J$. An *embedding* of a structure $\mathbf{A}$ into a structure $\mathbf{B}$ is a map $\pi : A \to B$ such that

$$\begin{aligned} R_i^{\mathbf{A}}(a_1, \ldots, a_{n(i)}) &\Leftrightarrow R_i^{\mathbf{B}}(\pi(a_1), \ldots, \pi(a_{n(i)})) \quad \text{and} \\ \pi(f_j^{\mathbf{A}}(a_1, \ldots, a_{m(j)})) &= f_j^{\mathbf{B}}(\pi(a_1), \ldots, \pi(a_{m(j)})). \end{aligned}$$

A simple example of a class of structures is the class of graphs. Here the signature L consists of a single binary relation (the edge relation). A graph $\mathbf{A}$ is then a set of vertices A together with a subset $R^{\mathbf{A}}$ of $A \times A$ which is irreflexive and symmetric.

Let $\mathcal{K}$ be a class of finite structures in a fixed countable signature L. We say that $\mathcal{K}$ is a **Fraïssé class** if it satisfies the following properties:

1. (HP) $\mathcal{K}$ is **hereditary**, i.e., $\mathbf{A} \le \mathbf{B}$ and $\mathbf{B} \in \mathcal{K}$ implies $\mathbf{A} \in \mathcal{K}$ (where $\mathbf{A} \le \mathbf{B}$ means $\mathbf{A}$ can be embedded into $\mathbf{B}$).
2. (JEP) The **joint embedding property**, i.e., if $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ then there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{A} \le \mathbf{C}$ and $\mathbf{B} \le \mathbf{C}$.

3. (AP) The **amalgamation property**, i.e., if $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ are embeddings with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ then there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $r : \mathbf{B} \to \mathbf{D}$, $s : \mathbf{C} \to \mathbf{D}$ with $r \circ f = s \circ g$.
4. $\mathcal{K}$ contains, up to isomorphism, only countably many structures; and contains structures of arbitrarily large (finite) cardinality.

For any Fraïssé class $\mathcal{K}$ there is a corresponding **Fraïssé limit**

$$\mathbf{K} = \mathrm{Flim}(\mathcal{K}),$$

which is the unique countably infinite structure with the properties:

(a) $\mathbf{K}$ is **locally finite**, i.e., finitely generated substructures of $\mathbf{K}$ are finite.
(b) $\mathbf{K}$ is **ultrahomogeneous**, i.e., any isomorphism between finite substructures extends to an automorphism of $\mathbf{K}$.
(c) $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$, where $\mathrm{Age}(\mathbf{K})$ is the class of all finite structures that can be embedded in $\mathbf{K}$.

A countably infinite structure $\mathbf{K}$ satisfying properties (a) and (b), is called a **Fraïssé structure** and the correspondence

$$\mathcal{K} \mapsto \mathbf{K} = \mathrm{Flim}(\mathcal{K}) \qquad \mathbf{K} \mapsto \mathrm{Age}(\mathbf{K})$$

is a canonical bijection between Fraïssé classes and Fraïssé structures. Every closed subgroup $G \le S_\infty$ is of the form $G = \mathrm{Aut}\,(\mathbf{K})$ for some Fraïssé structure $\mathbf{K}$.

Examples of Fraïssé structures include the trivial structure $(\mathbb{N}, =)$, $\mathbf{R}$ the random graph, $(\mathbb{Q}, <)$ the order type of the rational numbers, $\mathbf{B}_\infty$ the countable atomless Boolean algebra, and $\mathbf{U}_0$ the rational Urysohn space. The latter is the Fraïssé limit of the class of finite metric spaces with rational distances.

With a Fraïssé class $\mathcal{K}$ Truss [55] associates the class $\mathcal{K}_p$ of all systems $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\mathbf{B}, \mathbf{C} \le \mathbf{A}$ and $\psi : \mathbf{B} \to \mathbf{C}$ is an isomorphism. An **embedding** of a second system $\mathcal{T} = \langle \mathbf{D}, \phi : \mathbf{E} \to \mathbf{F} \rangle$ in $\mathcal{S}$ is an isomorphism $f : \mathbf{A} \to \mathbf{D}$ such that f embeds $\mathbf{B}$ into $\mathbf{E}$, $\mathbf{C}$ into $\mathbf{F}$ and $f \circ \psi \subset \phi \circ f$.

Kechris and Rosendal show that the property of a Fraïssé structure $\mathbf{K}$ that ensures the Rohlin property of $G = \mathrm{Aut}\,(\mathbf{K})$ is the JEP for $\mathcal{K}_p$.

11.1. **Theorem.** *Let $G = \mathrm{Aut}\,(\mathbf{K})$ where $\mathcal{K}$ is a Fraïssé structure with Fraïssé limit $\mathbf{K}$. The following are equivalent:*

1. *G has a dense conjugacy class.*
2. *The class $\mathcal{K}_p$ satisfies the JEP.*

As corollaries they deduce, e.g., that each of the groups $\mathrm{Aut}\,(\mathbf{B}_\infty)$, $\mathrm{Aut}\,(\mathbf{F}, \lambda)$ and $\mathrm{Aut}\,(\mathbf{U}_0)$, which correspond to the Fraïssé classes:

1. $\mathcal{K}$ = finite Boolean algebras,
2. $\mathcal{K}$ = finite measure Boolean algebras with rational measure,
3. $\mathcal{K}$ = finite metric spaces with rational distances,

respectively, has the Rohlin property.

Since Aut$(\mathbf{B}_\infty)$ is canonically isomorphic to $H(X)$, the homeomorphism group of the Cantor set, the first case retrieves the results of Glasner-Weiss [27] and Akin-Hurly-Kennedy [9]. The Polish group Aut$(\mathbf{F}, \lambda)$ embeds densely into the Polish group Aut$(X, \mathcal{X}, \mu)$, where $(X, \mathcal{X})$ is the standard Borel space and μ is an atomless Borel probability measure on X. The image is a dense subgroup and the RP of Aut$(X, \mathcal{X}, \mu)$ follows (retrieving the classical Halmos-Rohlin theorem). Similarly, Iso$(\mathbf{U}_0)$ is the group of isometries of the universal rational Urysohn space $\mathbf{U}_0$ and as this Polish group embeds densely into the group Iso$(\mathbf{U})$ of isometries of the universal Urysohn space $\mathbf{U}$, it follows that the latter group also has the RP. This latter fact was proven also by Glasner and Pestov. In the next section we will present a new proof, due to A. Kechris, of the fact that Iso$(\mathbf{U})$ does not have the SRP (Theorem 12.5).

Similarly by considering the diagonal action of Aut$(\mathbf{K})$ on Aut$(\mathbf{K})^n$ for every $n \in \mathbb{N}$ Kechris and Rosendal obtain the following results.

11.2. **Theorem.** *Each of the following Polish groups has the RP:*

$$H(2^{\mathbb{N}})^{\mathbb{N}}, H(2^{\mathbb{N}}, \sigma)^{\mathbb{N}}, \mathrm{Aut}\,(X, \mu)^{\mathbb{N}}, \mathrm{Aut}\,(\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}, \mathrm{Aut}\,(\mathbf{U}_0)^{\mathbb{N}}, \mathrm{Iso}\,(\mathbf{U})^{\mathbb{N}}.$$

We now turn to the strong Rohlin property (having a "generic automorphism" in the terminology of Kechris and Rosendal). It turns out that here the relevant properties of $\mathcal{K}_p$ are JEP and WAP (see also Truss [55] and Ivanov [36]). A class $\mathcal{K}_p$ satisfies the **weak amalgamation property** (WAP) if for any $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \in \mathcal{K}_p$ there is $\mathcal{T} = \langle \mathbf{D}, \phi : \mathbf{E} \to \mathbf{F} \rangle$ and an embedding $e : \mathcal{S} \to \mathcal{T}$, such that for any embeddings $f : \mathcal{T} \to \mathcal{T}_0$, $g : \mathcal{T} \to \mathcal{T}_1$, where $\mathcal{T}_0, \mathcal{T}_1 \in \mathcal{K}_p$, there is $\mathcal{U} \in \mathcal{K}_p$ and embeddings $r : \mathcal{T}_0 \to \mathcal{U}$, $s : \mathcal{T}_1 \to \mathcal{U}$ with $r \circ f \circ e = s \circ g \circ e$.

11.3. **Theorem.** *Let* $G = \mathrm{Aut}\,(\mathbf{K})$ *where* $\mathcal{K}$ *is a Fraïssé structure with Fraïssé limit* $\mathbf{K}$. *The following are equivalent:*

1. *G has the SRP.*
2. *The class $\mathcal{K}_p$ satisfies the JEP and WAP.*

Kechris and Rosendal then show that for $\mathcal{K} = \mathcal{BA}$, the class of finite Boolean algebras, $\mathcal{K}_p$ satisfies both JEP and WAP and deduce the SRP for the group $H(X)$ of homeomorphisms of the Cantor set.

12. Iso $(\mathbb{U})$ does not have the SRP

In this section we present a new theorem of A. Kechris which asserts that every conjugacy class in the group Iso $(\mathbb{U})$ of isometries of the universal Urysohn space $\mathbb{U}$, is meager; that is, Iso $(\mathbb{U})$ does not have the SRP. The proof is built on the same lines as Hjorth' proof of Theorem 3.5, which asserts that the same holds for the group Aut (X,μ). However the basic fact that is used here, in lieu of Rohlin's lemma, is a deep theorem of Solecki. This result was also, independently, obtained by A. Vershik (yet unpublished).

Let (A,d) be a finite metric space. An isometry $p : D \to E$ from D onto E, with $D, E \subset A$, is called a **partial isometry**. A point $x \in A$ is a **cyclic point** if $p^n(x) \in D$ for all $n \in \mathbb{N}$. Otherwise x is called **acyclic**. Let $Z(p)$ denote the set of cyclic points. For a cyclic $x \in A$ let m_x be the smallest $n > 0$ with $p^n(x) = x$. If $x \in A$ is acyclic, let

$$n_x = \min\{n \geq 0 : p^n(x) \notin D\} + \max\{n \geq 0 : p^{-n}(x) \in D\},$$

where $\max \emptyset = 0$.

Although we will only need the case when all the points in A are acyclic (i.e. $Z(p) = \emptyset$) we cite below the general statement of the theorem.

12.1. **Theorem** (Solecki [54], Theorem 3.2)**.** *Let (A,d) be a finite metric space, $D, E \subset A$ and $p : D \to E$, a partial isometry from D onto E. There is then a finite metric space (B,ρ) with $A \subset B$ as metric spaces (i.e. $\rho \restriction A \times A = d$), an isometry $q : B \to B$ which extends p, and a natural number M such that:*

1. $q^{2M} = \mathrm{id}_B$.
2. *If $x \in A$ is acyclic then $q^j(x) \neq x$ for all $0 < j < 2M$.*
3. *$A \cup q^M(A)$ is the amalgam of A and $q^M(A)$ over $Z(p)$. I.e. $A \cap q^M(A) = Z(p)$, $q^M \restriction Z(p) = \mathrm{id}_{Z(p)}$, and for $a_1, a_2 \in A$,*

$$\rho(a_1, q^M(a_2)) = \begin{cases} 2 \ \mathrm{diam}\,(A) & \text{if } Z(p) = \emptyset \\ \min\{d(a_1,z) + d(a_2,z) : z \in Z(p)\} & \text{otherwise.} \end{cases}$$

In fact, denoting $\Delta = \mathrm{diam}\,(A)$, $\delta = \min\{d(x,y) : x, y \in A,\ x \neq y\}$, and $N = \max\{n_x : x \in A \setminus Z(p)\}$ (where $\max\emptyset = 0$), any natural number M divisible by all the m_x for $x \in Z(p)$ and satisfying

$$(M - N)\delta > 2N\Delta,$$

will serve in the above statement.

12.2. **Lemma.** *Given distinct* $z_1, \ldots, z_m \in \mathbb{U}$ *and* $\delta > 0$ *such that* $\delta < \min d(z_i, z_j)$*, there are* $z_1^\delta, \ldots, z_m^\delta \in \mathbb{U}$ *with* $d(z_i, z_i^\delta) = \delta$ *and* $d(z_i^\delta, z_j^\delta) = d(z_i, z_j)$.

Proof. It is enough to show there is an abstract metric space $(\{z_1, \ldots, z_m, v_1, \ldots, v_m\}, \rho)$ with

$$\rho(z_i, v_i) = \delta, \quad \rho(v_i, v_j) = d(z_i, z_j).$$

For this simply take the the ℓ_1 direct sum of $(\{z_1, \ldots, z_m\}, d)$ with a two point space with distance δ between the two points. Thus

$$\rho(v_i, v_j) = d(z_i, z_j), \qquad \rho(z_i, v_j) = d(z_i, z_j) + \delta.$$

□

12.3. **Lemma.** *Given* $f \in \mathrm{Iso}(\mathbb{U}),\ y_1, \ldots, y_n \in \mathbb{U},\ \epsilon > 0$*, one can find* $g \in \mathrm{Iso}(\mathbb{U})$ *such that*

$$d(g(y_i), f(y_i)) < \epsilon \quad \textit{and} \quad g(\{y_1, \ldots, y_n\}) \cap \{y_1, \ldots, y_n\} = \emptyset.$$

Proof. Let $\{y_1, \ldots, y_n, f(y_1), \ldots, f(y_n)\} = \{z_1, \ldots, z_m\}$, where $z_i \neq z_j$ for $i \neq j$ (so that $n \leq m \leq 2n$). With $0 < \delta < \min_{i \neq j} d(z_i, z_j), \delta < \epsilon$ apply Lemma 12.2 to define $\{z_1^\delta, \ldots, z_m^\delta\} \subset \mathbb{U}$. Now extend the isometry $h : \{z_1, \ldots, z_m\} \to \{z_1^\delta, \ldots, z_m^\delta\}$ to an isometry $h \in \mathrm{Iso}(\mathbb{U})$ and set $g = h \circ f$. □

12.4. **Lemma.** *Fix* $x \in \mathbb{U}$*. The set*

$$\{(f, g) \in \mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U}) : \exists\, n\ [d(f^n(x), x) < 1/2 \textit{ and } d(g^n(x), x) > 1]\}$$

is open and dense in $\mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U})$.

Proof. Fix $f, g \in \mathrm{Iso}(\mathbb{U}),\ x_0 = x, x_1, \ldots, x_k \in \mathbb{U},\ \epsilon > 0$. We need to find $f_0, g_0 \in \mathrm{Iso}(\mathbb{U})$ such that

$$d(f_0(x_i), f(x_i)) < \epsilon, \qquad d(g_0(x_i), g(x_i)) < \epsilon, \qquad i = 0, \ldots, k$$

and for some n

$$d(f_0^n(x), x) < 1/2, \qquad d(g_0^n(x), x) > 1.$$

We can assume (by adding more points to $x_1, \ldots, x_k$) that $d(x_i, x_j) > 1/2$ for some i and j. By Solecki's theorem the isometry

$$x_i \mapsto f(x_i), \qquad i = 0, \ldots, n$$

extends to an isometry $\phi : A \to A$ of some finite $A \supset \{x_0, \ldots, x_k, f(x_0), \ldots, f(x_k)\}$. Let $f_0 \supset \phi$ be an element of $\mathrm{Iso}(\mathbb{U})$.

Since A is finite, there exists $\ell > 0$ such that $\phi^\ell(x) = x$ and so $f_0^\ell(x) = x$, hence $f_0^{\ell m}(x) = x$ for all m.

Next apply Lemma 12.3 to choose $g' \in \mathrm{Iso}\,(\mathbb{U})$ such that $d(g'(x_i), g(x_i)) < \epsilon$ and

$$g'(\{x_0, \dots, x_k\}) \cap \{x_0, \dots, x_k\} = \emptyset. \tag{6}$$

By Solecki's Theorem 12.1 the partial isometry $x_i \mapsto g'(x_i)$ $i = 0, 1, \dots, k$ extends, for arbitrarily large M, to an isometry ϕ_M of some finite set such that

$$d(x_i, \phi_M^M(x_i)) = 2 \ \mathrm{diam}\,\{x_0, \dots, x_k\} > 1.$$

(Note that by (6) every point of $\{x_0, \dots, x_k\}$ is acyclic, so that there are no divisibility conditions on M).

Extend ϕ_M to $g_{0,M} \in \mathrm{Iso}\,(\mathbb{U})$. Then

$$d(x, g_{0,M}^M(x)) > 1. \tag{7}$$

Also $g_{0,M} \supset g' \restriction \{x_0, \dots, x_k\}$ so $d(g_{0,M}(x_i), g(x_i)) < \epsilon$.

Now choose $M = n = \ell m$ (with m sufficiently large) so that if $g_0 = g_{0,n}$,

$$d(x, g_0^n(x)) > 1 \text{ and also } d(x, f_0^n(x)) = 0 < 1/2.$$

$\square$

We are now ready for the proof of Kechris' theorem. We will make use of the following useful notation. If Z is a topological space with the Baire property the formula " $\forall^* z \in Z$ " reads " for a comeager set of $z \in Z$ ".

12.5. **Theorem** (Kechris). *Every conjugacy class in* $\mathrm{Iso}\,(\mathbb{U})$ *is meager.*

Proof. Suppose the conjugacy class of $f_0 \in \mathrm{Iso}\,(\mathbb{U})$ is comeager. As in the proof of Theorem 3.5 we use the Jankov von Neumann theorem (Theorem 3.4) to find a Borel map $F : \mathrm{Iso}\,(\mathbb{U}) \to \mathrm{Iso}\,(\mathbb{U})$ such that

$$\forall^* \ f \in \mathbb{U} \ [F(f) f F(f)^{-1} = f_0].$$

We then have

$$\forall^* \ f \in \mathbb{U} \ [F(f) f^n = f_0^n F(f)]$$

for every $n \in \mathbb{Z}$.

We chose a suitable sequence $\{y_i\} \subset \mathbb{U}$ and then cover $\mathrm{Iso}\,(\mathbb{U})$ by a countable collection of sets $\mathcal{V}_i$ such that $d(F(f)(x), y_i) < 1/8$ for every $f \in \mathcal{V}_i$. Since the map $f \mapsto F(f)(x)$ is Baire measurable each $\mathcal{V}_i = \mathcal{U}_i \bigtriangleup M_i$ with $\mathcal{U}_i$ open and M_i meager. By Baire's theorem there exists at least one i with $\mathcal{U}_i \neq \emptyset$. With $\mathcal{U} = \mathcal{U}_i$ and $y = y_i$ we have for the open nonempty $\mathcal{U} \subset \mathrm{Iso}\,(\mathbb{U})$ and $y \in \mathbb{U}$

$$\forall^* \ f \in \mathcal{U} \ [d(F(f)(x), y) < 1/8].$$

By Lemma 12.4 there are $f_1, f_2 \in \mathcal{U}$ and n with

$$F(f_i)f_iF(f_i)^{-1} = f_0,$$
$$d(f_1^n(x), x) < 1/2, \quad d(f_2^n(x), x) > 1$$
$$d(F(f_i)(x), y) < 1/8$$
$$d(f_0^nF(f_1)(x)), F(f_1)(x)) < 1/2, \quad d(f_0^nF(f_2)(x), F(f_2)(x)) > 1,$$
$$\text{and} \quad d(F(f_i)(x), y) < 1/8.$$

So

$$d(f_0^n(y), y) \leq$$
$$d(f_0^n(y), f_0^nF(f_1)(x)) + d(f_0^nF(f_1)(x), F(f_1)(x)) + d(F(f_1)(x), y) \leq$$
$$1/8 + 1/2 + 1/8 = 3/4.$$

And

$$d(f_0^n(y), y) + d(f_0^n(y), f_0^nF(f_2)(x)) + d(F(f_2)(x), y) \geq$$
$$d(f_0^nF(f_2)(x), F(f_2)(x)) > 1$$

So

$$d(f_0^n(y), y) > 1 - 1/8 - 1/8 = 3/4.$$

This conflict completes the proof. □

13. Groups with ample generic elements

In their paper [41] Kechris and Rosendal define an even stronger property than SRP (see also Hodges et. al. [35]). A Polish group G has **ample generic elements** (or has **ample generics** for short) if for each finite n there is a comeager orbit for the (diagonal) conjugacy action of G on G^n:

$$g \cdot (g_1, g_2, \dots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \dots, gg_ng^{-1}).$$

(In the nomenclature of Ergodic Theory this property would be called **SRP of all finite orders**. Kechris and Rosendal show, with a clever short argument, that no Polish group can have the infinite version of the SRP, see [41] the second Remark after Proposition 5.1.)

Of course "ample generics" implies SRP but there are Polish groups with SRP which do not have ample generic elements. One such group is the group $\mathrm{Aut}\,(\mathbb{Q}, <)$ of order preserving bijections of the rational numbers (this is due to Hodkinson (unpublished) see the paper of Truss, *On notions of genericity and mutual genericity*, University of Leeds preprint, 18, 2005, in his web page).

The list of Polish groups known to have ample generics includes the automorphism groups of ω-stable $\aleph_0$-categorical structures, the automorphism group of the random graph, and the automorphism group of the rational Urysohn space. To this list the authors of [41] add the group of Haar measure preserving homeomorphisms of the Cantor space, $H(2^{\mathbb{N}}, \sigma)$, and the group of Lipschitz homeomorphisms of the Baire space $\mathbb{N}^{\mathbb{N}}$.

As far as we know, the question whether the group $H(X)$ of homeomorphisms of the Cantor set X has ample generics is still open. Also note that the case of the dyadic (Haar) measure σ on $2^{\mathbb{N}}$ is not included in the type of measures handled by Akin in [6].

Having ample generic elements is a very powerful property. Let us mention two of the many consequences proven in [41].

13.1. **Theorem.** *A Polish group with ample generics has also the* **small index property***; i.e., any subgroup of index* $< 2^{\aleph_0}$ *is open.*

13.2. **Theorem.** *Let G be a Polish group with ample generics. Then any homomorphism $\pi : G \to H$ of G into a separable topological group is necessarily continuous.*

Regarding the latter result, see also the recent work of Rosendal and Solecki [51].

14. Further related works

A famous result of Oxtoby and Ulam [50] asserts that ergodicity is residual for Lebesgue measure-preserving homeomorphisms of the cube. The book by Alpern and Prassad [10] is devoted to generalizations of this classical theorem in the context of groups of measure-preserving homeomorphisms of cubes and compact connected manifolds.

In a series of papers Bezuglyi, Dooley, Kwiatkowski and Medynets, [11], [12], [13], [14], introduce several topologies, on the group $H(X)$ of homeomorphisms of the Cantor set and establish categorical statements concerning various naturally defined subsets of $H(X)$ with respect to these topologies.

The main theme in Glasner and King [23] as well as in Rudolph's paper [53] is a **correspondence principle** which asserts that two, seemingly completely different "settings", are in fact "generically" related in the sense that a dynamical property is meager/comeager in one if and only if it is meager/comeager in in the other. On the one hand we have the classical setting of $\mathrm{Aut}\,(X, \mathcal{X}, \mu)$, and on the other the, no less classical, setting of the space of shift invariant measures on the infinite dimensional torus or the Hilbert cube.

Is there a topological analogue to this correspondence principle? A natural candidate for a topological setting is the group $H(X)$ of homeomorphisms of the Cantor set. However, by the Rosendal-Kechris result, namely the fact that $H(X)$ has a dense G_δ conjugacy class, we see that the discussion of generic properties in $H(X)$ is trivial. Notwithstanding, in a recent work Mike Hochman, [33], establishes a correspondence principle between the setting $H(X)$ on the one hand and the setting of closed invariant subsets of, say the Hilbert cube, on the other, which becomes meaningful when it is restricted to some naturally defined subspaces of both settings. Using coding arguments he proves various facts in the space of shift-invariant subsets setting and then transports them to the $H(X)$ setting. As a striking example we mention the fact that in the subset of $H(X)$ consisting of the totally transitive homeomorphisms, being prime is a generic property.

References

[1] O. Ageev, *On homogeneous spectrum problem in ergodic theory*, Inventiones Mathematicae, **160**, (2005), 417-446.

[2] O. Ageev, *On spectral invariants in modern ergodic theory*, Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2005, 417-446.

[3] O. Ageev, *Spectral rigidity of group actions and Kazhdan' groups*, preprint.

[4] E. Akin, *The general topology of dynamical systems*, Amer. Math. Soc., Providence, 1993.

[5] E. Akin, *On chain continuity*, Discrete and Cont. Dynam. Sys., **2**, (1996), 111-120.

[6] E. Akin, *Good measures on Cantor space*, Trans. Amer. Math. Soc. **357**, (2005), 2681-2722.

[7] E. Akin, J. Auslander, and K. Berg, *When is a transitive map chaotic* , Convergence in Ergodic Theory and Probability, Walter de Gruyter & Co. 1996, pp. 25-40.

[8] E. Akin, E. Glasner and B. Weiss, *Generically there is but one self homeomorphism of the Cantor set*, to appear in Trans. Amer. Math. Soc.

[9] E. Akin, M. Hurley and J. Kennedy, *Dynamics of topologically generic homeomorphisms*, Mem. Amer. Math. Soc. **164**, (2003), no. 783.

[10] S. Alpern and V. S. Prasad, *Typical dynamics of volume preserving homeomorphisms*, Cambridge University Press, 2000.

[11] S. Bezuglyi, A.H. Dooley and J. Kwiatkowski, *Topologies on the group of homeomorphisms of a Cantor set*, Topol. Methods Nonlinear Anal. **27**, (2006), 299-331.

[12] S. Bezuglyi, A.H. Dooley and J. Kwiatkowski, *Topologies on the group of Borel automorphisms of a standard Borel space*, Topol. Methods Nonlinear Anal. **27**, (2006), 333-385.

[13] S. Bezuglyi, A.H. Dooley and K. Medynets, *The Rokhlin lemma for homeomorphisms of a Cantor set*, Proc. Amer. Math. Soc. **134**, (2005), 2957-2964.

[14] S. Bezuglyi, A.H. Dooley and K. Medynets, *Approximation in ergodic theory, Borel, and Cantor dynamics*, Algebraic and topological dynamics, 39-64, Contemp. Math., 385, Amer. Math. Soc., Providence, RI, 2005.

[15] J.R. Choksi and M. G. Nadkarni, *Baire category in space of measures, unitary operators, and transformations*, Invariant subspaces and allied topics, Narosa Publishing House, New Delhi, 1990, pp. 147-163.

[16] J.R. Choksi and V. S. Prasad, *Approximation and Baire category theorems in ergodic theory*, Measure theory and its applications, Springer Lecture Notes in Math. **1033**, Springer Verlag, 1983, pp. 94-113.

[17] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Reg. Conf. Ser. in Math., **38**, Amer. Math. Soc., Providence, 1978.

[18] M. Foreman and B. Weiss, *An anti-classification theorem for ergodic measure preserving transformations*, J. Eur. Math. Soc. **6**, (2004), 277-292.

[19] M. Foreman, D. J. Rudolph and B. Weiss, *On the conjugacy relation in ergodic theory*, To appear.

[20] N. A. Friedman, *Introduction to ergodic theory*, van Nostrand Reinhold, New York, 1970.

[21] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. System Theory **1**, (1967), 1-49.

[22] E. Glasner, *Ergodic Theory via joinings*, Math. Surveys and Monographs, AMS, **101**, 2003.

[23] E. Glasner and J. King, *A zero-one law for dynamical properties*, Topological dynamics and applications (Minneapolis, MN, 1995), Contemporary Math. **215**, Amer. Math. Soc., (1998), 231-242.

[24] E. Glasner, J.-P. Thouvenot and B. Weiss, *Every countable group has the weak Rohlin property*, Bulletin of the London Math. Soc., **38**, (2006), 932-936.

[25] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity **6**, (1993), 1067-1075.

[26] E. Glasner and B. Weiss, *Kazhdan's property T and the geometry of the collection of invariant measures*, GAFA, Geom. funct. anal. **7**, (1997), 917-935.

[27] E. Glasner and B. Weiss, *The topological Rohlin property and topological entropy*, Amer. J. Math. **123**, (2001), 1055-1070.

[28] P. R. Halmos, *In General a Measure Preserving Transformation is Mixing* Annals of Mathematics, **45**, (1944), 786-792.

[29] P. R. Halmos, *Ergodic Theory*, Chelsea Publishing Co., New York, 1956.

[30] G. Higman, B. H. Neumann and H. Neumann, *Embedding theorems for groups* J. London Math. Soc. **24**, (1949), 247-254.

[31] G. Hjorth, *Classification and orbit equivalencerelation*, Math. Surveys Monographs **75**, Amer. Math. Soc., Providence, RI (2000).

[32] G. Hjorth, *On invariants for measure preserving transformations*, Fund. Math. **169**, (2001), 51-84.

[33] M. Hochman, *Genericity in topological dynamics*, To appear. arXiv:Math.DS/0605386, May 2006.

[34] W. Hodges, *A shorter model theory*, Cambridge University Press, 1977.

[35] W. Hodges, I. Hodkinson, D. Lascar and S. Shelah, *The small index property for ω-stable ω-categorical structures and for the random graph*, J. London Math. Soc. **48**, (1993), 204-218.

[36] A. A. Ivanov, *Generic expansions of ω-categorical structures and semantics of generalized quantifiers*, J. Symbolic Logic **64**, (1999), 775-789.

[37] A. del Junco, *Disjointness of measure-preserving transformations, minimal self-joinings and category*, In *Ergodic theory and dynamical systems, I (College Park, Md., 1979-80)*, volume 10 of *Prog. Math.*, pages 81-89. Birkhauser Boston, Mass., 1981.

[38] A. del Junco and M. Lemanczyk, *Generic spectral properties of measure-preserving maps and applications*, Proc. Amer. Math. Soc. **115**, (1992), 725-736.

[39] A. del Junco and D. Rudolph, *Residual behavior of induced maps*, Israel. J. Math. **93**, (1996), 387-398.

[40] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[Kech-07] A. S. Kechris, *Global aspects of ergodic group actions and equivalence relations*. http://www.math.caltech.edu/people/kechris.html

[41] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation and generic automorphisms of homogeneous structures*, arXiv:math.LO/0409567

[42] D. Kerr and M. Pichot, *Asymptotic abeliannes, weak mixing, and property T*, a preprint.

[43] K. Kuratowski, *Topology* volume 1, Academic Press, New York & London, 1966.

[44] D. Kuske and J. K. Truss, *Generic automorphisms of the universal partial order*, Proc. Amer. Math. Soc. **129**, (2000), 1939-1948.

[45] D.V. Osin *Small cancellations over hyperbolic groups and embedding theorems*, arXiv:math.GR/0411039 2 Nov 2004

[46] D. S. Ornstein, *Bernoulli shifts with the same entropy are isomorphic*, Advances in Math. **4**, (1970), 337-352.

[47] D.S. Ornstein and P. Shields, *An uncountable family of K-auotomorphisms*, Advances in Math. **10**, (1973), 63-88.

[48] D.S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. d'Analyse Math. **48**, (1987), 1-141.

[49] J. Oxtoby *Measure and category* (Second Ed.), (1980), Springer-Verlag, Berlin.

[50] J. Oxtoby and S. M. Ulam *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. **42**, (1941), 874-920.

[51] C. Rosendal and S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*, arXiv: math.LO/0604575 26 April 2006.

[52] J. J. Rotman (1980) *Theory of Groups* (third edition) W. C. Brown Publishers, Dubuque, Iowa.

[53] D. Rudolph, *Residuality and orbit equivalence*, Topological dynamics and applications (Minneapolis, MN, 1995), Contemporary Math. **215**, Amer. Math. Soc., (1998), 243-254.

[54] S. Solecki, *Extending partial isometries*, Israel J. Math. **150**, (2005), 315-332.

[55] J. K. Truss, *Generic automorphisms of homogeneous structures*, Proc. London Math. Soc. **65**, (1992), 121-141.

[56] B. Weiss, *Actions of amenable groups*, Topics in dynamics and ergodic theory, LMS Lecture notes series **310**, (2003), 226-262.

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel
E-mail address: glasner@math.tau.ac.il

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel
E-mail address: weiss@math.huji.ac.il