THE STRUCTURE OF TAME MINIMAL DYNAMICAL SYSTEMS

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ABSTRACT. A dynamical version of the Bourgain-Fremlin-Talagrand dichotomy shows that the enveloping semigroup of a dynamical system is either very large and contains a topological copy of $\beta\mathbb{N}$, or it is a "tame" topological space whose topology is determined by the convergence of sequences. In the latter case the dynamical system is called tame. We use the structure theory of minimal dynamical systems to show that, when the acting group is Abelian, a tame metric minimal dynamical system (i) is almost automorphic (i.e. it is an almost 1-1 extension of an equicontinuous system), and (ii) admits a unique invariant probability measure such that the corresponding measure preserving system is measure theoretically isomorphic to the Haar measure system on the maximal equicontinuous factor.

CONTENTS

INTRODUCTION

In this work a dynamical system is a pair (X, Γ), where X is a compact Hausdorff space and Γ an abstract group acting as a group of homeomorphisms of the space X. That is we are given a homomorphism (not necessarily an isomorphism) of Γ into Homeo (X). For $\gamma \in \Gamma$ and $x \in X$ we write γx for the image of x under the homeomorphism which corresponds to γ. We will often abuse this notation and consider γ as a homeomorphism of X.

The *enveloping semigroup* $E(X, \Gamma)$ of the dynamical system (X, Γ) is defined as the closure of image of Γ in the product space X^X. It is not hard to check that, under composition of maps, $E(X, \Gamma)$ is a compact *right topological semigroup* , i.e. for each $q \in E(X, \Gamma)$ the map $R_q : p \mapsto pq$ is continuous. In fact the canonical homomorphism of Γ into $E(X, \Gamma)$ is a *right topological semigroup compactification* of Γ; i.e. it has a dense range and for each $\gamma \in \Gamma$ multiplication on the left $L_\gamma : p \mapsto \gamma p$ is continuous

on $E(X, \Gamma)$. This left multiplication by elements of Γ makes $(E(X, \Gamma), \Gamma)$ a dynamical system.

The enveloping semigroup was introduced by Robert Ellis in 1960 and became an indispensable tool in abstract topological dynamics. However explicit computations of enveloping semigroups are quite rare. One reason for this is that often $E(X, G)$ is non-metrizable.

Following an idea of A. Köller, [19], Glasner and Megrelishvily proved the following dynamical version of the Bourgain-Fremlin-Talagrand dichotomy theorem, [14].

0.1. **Theorem** (A dynamical BFT dichotomy)**.** *Let (X, G) be a metric dynamical system and let $E(X)$ be its enveloping semigroup. We have the following dichotomy. Either*

1. *$E(X)$ is separable Rosenthal compact, hence with cardinality* card *$E(X) \leq 2^{\aleph_0}$; or*
2. *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence* card *$E(X) = 2^{2^{\aleph_0}}$.*

A dynamical system is called *tame* if the first alternative occurs, i.e. $E(X)$ is Rosenthal compact. Recently dynamical characterizations of both tame dynamical systems and dynamical systems whose enveloping groups are metrizable were obtained by Glasner, Megrelishvili and Uspenskij in [14] and [15]:

0.2. **Theorem.** *A compact metric dynamical system (G, X) is tame if and only if every element of $E(X)$ is a Baire 1 function from X to itself.*

0.3. **Theorem.** *Let X be a compact metric G-space. The following conditions are equivalent:*

1. *the dynamical system (G, X) is hereditarily almost equicontinuous (HAE);*
2. *the dynamical system (G, X) is RN, that is, admits a proper representation on a Radon–Nikodým Banach space;*
3. *the enveloping semigroup $E(X)$ is metrizable.*

For the definitions of HAE (hereditarily almost equicontinuous) systems and the other undefined notions which appear in these theorems, as well as for some further motivation and examples we refer the reader to the papers [14], [13] and [15].

In [13] I have shown that a *minimal* metrizable tame dynamical system with a commutative acting group is PI and has zero topological entropy. Recently Huang [16], and independently Kerr and Li [18], improved these results to show that under the same conditions a minimal tame system is an almost 1-1 extension of its maximal equicontinuous factor and is uniquely ergodic (see also [17]). In these works the authors make a heavy use of the structure theory of minimal dynamical systems, as developed by R. Ellis, W. Veech, Ellis-Glasner-Shapiro, McMahon and van der Woude (see e.g. the survey [12] and the references thereof). However the main tool in both works (of Huang and Kerr and Li) is the combinatorial notion of independence and the various related notions of independence n-tuples. In fact, Kerr and Li in their work [18], use independence to unify the theory of these various notions and in particular they are able to characterize tame systems (which they call regular) as those systems that (in some precise sense) do not admit infinite independence sets ([18, Proposition

6.4.2]). In turn they use this characterization to define a notion of relative regularity and develop the whole theory in the relative setup.

In the present work, which can be regarded as a continuation of my work [13], I pursue purely structure theoretical methods to recover the results of Huang and Kerr and Li mentioned above, avoiding the combinatorial treatment altogether. The key tool used in the proof, here as well as in [13], is a proposition about diffused measures (Proposition 3.3 below), which first appeared in [11].

Section 1 is a brief review of the structure theory of minimal dynamical systems. In Section 2, I prove an analogue of an old theorem of Ditor and Eifler [4], which may have some independent interest. It shows that when a continuous surjection $\pi : X \to Y$, with X and Y compact metric, is semiopen then so is the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ on the spaces of probability measures equipped with the weak* topology. In Section 3, I pursue the idea of diffused measures, first used in [11], and prove the key Proposition 3.3. Section 4 develops the theory of tame systems using and extending results from [13]. In the final Section 5, the main theorem is proved.

Except for the introductory Section 1, the group Γ is assumed to be Abelian. For simplicity I handle only the case where the dynamical system is metrizable and treat only the absolute (and not the more general relative) case.

1. A BRIEF SURVEY OF ABSTRACT TOPOLOGICAL DYNAMICS

This section is a brief review of the structure theory of minimal dynamical systems. We will emphasize some aspects which will be relevant in the present work. For full details the reader is referred to the books [6], [10], [1] and [22] and the review articles [21] and [12].

A *topological dynamical system* or briefly a system is a pair (X, Γ), where X is a compact Hausdorff space and Γ an abstract infinite group which acts on X as a group of homeomorphisms. When there is no room for confusion we write X for the system (X, Γ). A *sub-system* of (X, Γ) is a closed invariant subset $Y \subset X$ with the restricted action. For a point $x \in X$, we let $\mathcal{O}_\Gamma(x) = \{\gamma x : \gamma \in \Gamma\}$, and $\overline{\mathcal{O}}_\Gamma(x) = \mathrm{cls}\,\{\gamma x : \gamma \in \Gamma\}$. These subsets of X are called the *orbit* and *orbit closure* of x respectively. We say that (X, Γ) is *point transitive* if there exists a point $x \in X$ with a dense orbit. In that case x is called a *transitive point*. If every point is transitive we say that (X, Γ) is a *minimal system*. We say that $x \in X$ is an *almost periodic* or a *minimal* point if $\overline{\mathcal{O}}_\Gamma(x)$ is a minimal system.

The dynamical system (X, Γ) is *topologically transitive* if for any two nonempty open subsets U and V of X there exists some $\gamma \in \Gamma$ with $\gamma U \cap V \neq \emptyset$. Clearly a point transitive system is topologically transitive and when X is metrizable the converse holds as well: in a metrizable topologically transitive system the set of transitive points is a dense G_δ subset of X.

The system (X, Γ) is *weakly mixing* if the product system $(X \times X, \Gamma)$ (where $\gamma(x, x') = (\gamma x, \gamma x'),\ x, x' \in X,\ \gamma \in \Gamma$) is topologically transitive.

If (Y, Γ) is another system then a continuous onto map $\pi : X \to Y$ satisfying $\gamma \circ \pi = \pi \circ \gamma$ for every $\gamma \in \Gamma$ is called a *homomorphism* of dynamical systems. In this case we say that (Y, Γ) is a *factor* of (X, Γ) and also that (X, Γ) is an *extension* of (Y, Γ). With the system (X, Γ) we associate the induced action (the *hyper system* associated with (X, Γ)) on the compact space 2^X of closed subsets of X equipped

with the Vietoris topology. A subsystem Y of $(2^X, \Gamma)$ is a *quasifactor* of (X, Γ) if $\bigcup\{A : A \in Y\} = X$.

The system (X, Γ) can always be considered as a quasifactor of (X, Γ) by identifying x with $\{x\}$. Recall that if $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ is a homomorphism then in general $\pi^{-1} : Y \to 2^X$ is an upper-semi-continuous map and that $\pi : X \to Y$ is open iff $\pi^{-1} : Y \to 2^X$ is continuous, iff $\{\pi^{-1}(y) : y \in Y\}$ is a quasifactor of (X, Γ) .

We assume for simplicity that our acting group Γ is a discrete group. $\beta\Gamma$ will denote the Stone-Čech compactification of Γ. The universal properties of $\beta\Gamma$ make it

- a compact semigroup with right continuous multiplication (for a fixed $p \in \beta\Gamma$ the map $q \mapsto qp$, $q \in \beta\Gamma$ is continuous), and left continuous multiplication by elements of Γ, considered as elements of $\beta\Gamma$ (for a fixed $\gamma \in \Gamma$ the map $q \mapsto \gamma q$, $q \in \beta\Gamma$ is continuous).
- a dynamical system $(\beta\Gamma, \Gamma)$ under left multiplication by elements of Γ.

The system $(\beta\Gamma, \Gamma)$ is the universal point transitive Γ-system; i.e. for every point transitive system (X, Γ) and a point $x \in X$ with dense orbit, there exists a homomorphism of systems $(\beta\Gamma, \Gamma) \to (X, \Gamma)$ which sends e, the identity element of Γ, onto x. For $p \in \beta\Gamma$ we let px denote the image of p under this homomorphism. This defines an "action" of the semigroup $\beta\Gamma$ on every dynamical system. In fact, by universality there exists a unique homomorphism $(\beta\Gamma, \Gamma) \to (E(X, \Gamma), \Gamma)$ onto the enveloping semigroup $E(X, \Gamma)$ which is also a semigroup homomorphism and we can interpret, and often do, the $\beta\Gamma$ action on X via this homomorphism.

When dealing with the hyper system $(2^X, \Gamma)$ we write $p \circ A$ for the image of the closed subset $A \subset X$ under $p \in \beta\Gamma$ to distinguish it from the (usually non-closed) subset $pA = \{px : x \in A\}$. If p is the limit of a net γ_i in Γ then

$$p \circ A = \big\{x \in X : \text{there are a subnet } \gamma_{i_j} \text{ and a net } x_j \in A \text{ with } x = \lim_j \gamma_{i_j} x_j\big\}.$$

We always have $pA \subset p \circ A$.

The compact semigroup $\beta\Gamma$ has a rich algebraic structure. For instance for countable Γ there are 2^c minimal left (necessarily closed) ideals in $\beta\Gamma$ all isomorphic as systems and each serving as a universal minimal system. Each such minimal ideal, say M, has a subset J of 2^c idempotents such that $\{vM : v \in J\}$ is a partition of M into disjoint isomorphic (non-closed) subgroups. An idempotent in $\beta\Gamma$ is called *minimal* if it belongs to some minimal ideal. A point x in a dynamical system (X, Γ) is a minimal point iff there is some minimal idempotent v in $\beta\Gamma$ with $vx = x$, iff there exists some $v \in J$ with $vx = x$.

The group of dynamical system automorphisms of (M, Γ), $G = \text{Aut}(M, \Gamma)$ can be identified with any one of the groups vM as follows: with $\alpha \in vM$ we associate the automorphism $\hat{\alpha} : (M, \Gamma) \to (M, \Gamma)$ given by right multiplication $\hat{\alpha}(p) = p\alpha$, $p \in M$. The group G plays a central role in the algebraic theory. It carries a natural T_1 compact topology, called by Ellis the *τ-topology*, which is weaker than the relative topology induced on $G = uM$ as a subset of M. The τ-closure of a subset A of G consists of those $\beta \in G$ for which the set $\text{graph}(\beta) = \{(p, p\beta) : p \in M\}$ is a subset of the closure in $M \times M$ of the set $\bigcup\{\text{graph}(\alpha) : \alpha \in A\}$. Both right and left multiplication on G are τ continuous and so is inversion.

It is convenient to fix a minimal left ideal M in $\beta\Gamma$ and an idempotent $u \in M$. As explained above we identify G with uM and it follows that for any subset $A \subset G$,

$$\operatorname{cls}_\tau A = u(u \circ A) = G \cap (u \circ A).$$

Also in this way we can consider the "action" of G on every system (X, Γ) via the action of $\beta\Gamma$ on X. With every minimal system (X, Γ) and a point $x_0 \in uX = \{x \in X : ux = x\}$ we associate a τ-closed subgroup

$$\mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\},$$

the *Ellis group* of the pointed system (X, x_0). The quotient space $G/\mathfrak{G}(X, x_0)$ can be identified with the subset $uX \subset X$ via the map $\alpha \mapsto \alpha x_0$ and the induced quotient τ-topology is called the τ-*topology* on uX. Again the τ-topology is weaker than the relative topology induced on uX as a subset of X, it is T_1 and compact, and the closure operation is given by

$$\operatorname{cls}_\tau A = u(u \circ A) = uX \cap (u \circ A), \qquad A \subset uX.$$

For a homomorphism $\pi : X \to Y$ with $\pi(x_0) = y_0$ we have

$$\mathfrak{G}(X, x_0) \subset \mathfrak{G}(Y, y_0).$$

For a τ-closed subgroup F of G the *derived group* F' is given by:

(1.1) $$F' := \bigcap \{\operatorname{cls}_\tau O : O \text{ a } \tau\text{-open neighborhood of } u \text{ in } F\}.$$

F' is a τ-closed normal (in fact characteristic) subgroup of F and it is characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group. In particular, for an Abelian Γ, the topological group G/G' is the Bohr compactification of Γ.

A pair of points $(x, x') \in X \times X$ for a system (X, Γ) is called *proximal* if there exists a net $\gamma_i \in \Gamma$ and a point $z \in X$ such that $\lim t_i x = \lim t_i x' = z$ (iff there exists $p \in \beta\Gamma$ with $px = px'$). We denote by P the set of proximal pairs in $X \times X$. We have

$$P = \bigcap \{\Gamma V : V \text{ a neighborhood of the diagonal in } X \times X\}.$$

A system (X, Γ) is called *proximal* when $P = X \times X$ and *distal* when $P = \Delta$, the diagonal in $X \times X$. It is called *strongly proximal* when the following much stronger condition holds: the dynamical system $(\mathfrak{M}(X), \Gamma)$, induced on the compact space $\mathfrak{M}(X)$ of probability measures on X, is proximal. A minimal system (X, Γ) is called *point distal* if there exists a point $x \in X$ such that if x, x' is a proximal pair then $x = x'$.

The *regionally proximal relation* on X is defined by

$$Q = \bigcap \{\overline{\Gamma V} : V \text{ a neighborhood of } \Delta \text{ in } X \times X\}.$$

It is easy to verify that Q is trivial — i.e. equals Δ — iff the system is equicontinuous.

An extension $(X, \Gamma) \overset{\pi}{\to} (Y, \Gamma)$ of minimal systems is called a *proximal extension* if the relation $R_\pi = \{(x, x') : \pi(x) = \pi(x')\}$ satisfies $R_\pi \subset P$ and a *distal extension* when $R_\pi \cap P = \Delta$. One can show that every distal extension is open. π is a *highly proximal (HP) extension* if for every closed subset A of X with $\pi(A) = Y$, necessarily $A = X$. It is easy to see that a HP extension is proximal. In the metric case an

extension $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ of minimal systems is HP iff it is an *almost 1-1 extension*, that is the set $\{y \in Y : \text{ with } \pi^{-1}(y) \text{ is a singleton}\}$ is a dense G_δ subset of Y. The map π is *strongly proximal* if for every $y \in Y$ and every probability measure ν with $\operatorname{supp} \nu \subset \pi^{-1}(y)$, there exists a net $\gamma_i \in \Gamma$ and a point $x \in X$ such that $\lim_i \gamma_i \nu = \delta_x$ in the weak* topology on the space $\mathfrak{M}(X)$ of probability measures on X. The extension π is called an *equicontinuous extension* if for every ϵ, a neighborhood of the diagonal $\Delta = \{(x,x) : x \in X\} \subset X \times X$, there exists a neighborhood of the diagonal δ such that $\gamma(\delta \cap R_\pi) \subset \epsilon$ for every $\gamma \in \Gamma$. In the metric case an equicontinuous extension is also called an *isometric extension*. The extension π is a *weakly mixing extension* when R_π as a subsystem of the product system $(X \times X, \Gamma)$ is topologically transitive.
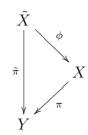
The algebraic language is particularly suitable for dealing with such notions. For example an extension $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ of minimal systems is a proximal extension iff the Ellis groups $\mathfrak{G}(X, x_0) = A$ and $\mathfrak{G}(Y, y_0) = F$ coincide. It is distal iff for every $y \in Y$, and $x \in \pi^{-1}(y)$, $\pi^{-1}(y) = \mathfrak{G}(Y, y)x$; iff:

> for every $y = py_0 \in Y$, p an element of M, $\pi^{-1}(y) = p\pi^{-1}(y_0) = pFx_0$, where $F = \mathfrak{G}(Y, y_0)$.

In particular (X, Γ) is distal iff $Gx = X$ for some (hence every) $x \in X$. The extension π is an equicontinuous extension iff it is a distal extension and, denoting $\mathfrak{G}(X, x_0) = A$ and $\mathfrak{G}(Y, y_0) = F$,

$$F' \subset A.$$

In this case, setting $A_0 = \bigcap_{g \in F} gAg^{-1}$, the group F/A_0 is the group of the *group extension* $\tilde{\pi}$ associated with the equicontinuous extension π. More precisely, there exists a minimal dynamical system $(\tilde{X}, \Gamma)$, with $\mathfrak{G}(\tilde{X}, \tilde{x}_0) = A_0$, on which the compact Hausdorff topological group $K = F/A_0$ acts as a group of automorphisms and we have the following commutative diagram
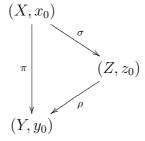

where $\tilde{\pi} : \tilde{X} \to Y \cong \tilde{X}/K$ is a group extension and so is the extension $\phi : \tilde{X} \to X \cong \tilde{X}/L$ with $L = A/A_0 \subset F/A_0 = K$. ($\tilde{X} = X$ iff A is a normal subgroup of F.)

A minimal system (X, Γ) is called *incontractible* if the union of minimal subsets is dense in every product system (X^n, Γ). This is the case iff $p \circ Gx = X$ for some (hence every) $x \in X$ and $p \in M$. When Γ is Abelian Gx is always dense in X so that every minimal system is incontractible. However the following relative notion is an important tool even when Γ is Abelian.
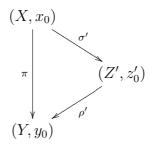
We say that $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ is a RIC (*relatively incontractible*) *extension* if:

> for every $y = py_0 \in Y$, p an element of M, $\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0$, where $F = \mathfrak{G}(Y, y_0)$.

One can show that every RIC extension is open and that every distal extension is RIC. It then follows that every distal extension is open.

We have the following theorem from [7] about the interpolation of equicontinuous extensions. For a proof see [11], Theorem X.2.1.

1.1. **Theorem.** *Let $\pi : X \to Y$ be a RIC extension of minimal systems. Fix a point $x_0 \in X$ with $ux_0 = x_0$ and let $y_0 = \pi(x_0)$. Let $A = \mathfrak{G}(X, x_0)$ and $F = \mathfrak{G}(Y, y_0)$. Then there exists a commutative diagram of pointed systems*

$$
\begin{array}{ccc}
(X, x_0) & & \\
\ \ \downarrow \pi & \searrow^{\sigma} & \\
 & & (Z, z_0) \\
\ \ \downarrow & \swarrow_{\rho} & \\
(Y, y_0) & &
\end{array}
$$

such that ρ is an equicontinuous extension with Ellis group $\mathfrak{G}(Z, z_0) = AF'$ and the extension ρ is an isomorphism iff $AF' = F$. Moreover if

$$
\begin{array}{ccc}
(X, x_0) & & \\
\ \ \downarrow \pi & \searrow^{\sigma'} & \\
 & & (Z', z_0') \\
\ \ \downarrow & \swarrow_{\rho'} & \\
(Y, y_0) & &
\end{array}
$$

is another such diagram with ρ' an equicontinuous extension then there exists a homomorphism $(Z, z_0) \to (Z', z_0')$.

Given a homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$ of minimal metric systems, there are several standard constructions of associated "shadow diagrams". In the *O shadow diagram*

$$
\begin{array}{ccc}
X & \xleftarrow{\theta^*} & X^* = X \vee Y^* \\
\downarrow{\pi} & & \downarrow{\pi^*} \\
Y & \xleftarrow{\theta} & Y^*
\end{array}
$$

the map π^* is open and the maps θ and θ^* are almost 1-1. The explicit constructions is as follows. The set valued map $\pi^{-1} : Y \to 2^X$ (where the latter is the compact space of closed subsets of X, equipped with the Hausdorff, or Vietoris, topology) is uppersemicontinuous and we let $Y_0 \subset Y$ be the set of continuity points of this map. Set $Y^* = \mathrm{cls}\{\pi^{-1}(y) : y \in Y_0\} \subset 2^X$, and $X^* = X \vee Y^* = \mathrm{cls}\{(x, \pi^{-1}(y)) : y \in Y_0, \ \pi(x) = y\} \subset X \times Y^*$. By the uppersemicontinuity of π^{-1} every $y^* \in Y^*$ is contained in a fiber $\pi^{-1}(y)$ for some $y \in Y$ and we let $\theta(y^*) = y$. The maps π^* and θ^* are the restriction to X^* of the coordinate projections on X and Y^* respectively. One then shows that $X^* = \{(x, y^*) : x \in y^* \in Y^*\}$ and that indeed, π^* is open and the maps θ and θ^* are highly proximal. The O shadow diagram collapses, i.e. $Y = Y^*$,

$X = X^*$ and $\pi = \pi^*$ iff $\pi : X \to Y$ is an open map; iff the map $\pi^{-1} : Y \to 2^X$ is continuous.

In the *RIC-shadow diagram*

$$
\begin{array}{ccc}
X & \xleftarrow{\;\theta^*\;} & X^* = X \vee Y^* \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi^*} \\
Y & \xleftarrow{\;\theta\;} & Y^*
\end{array}
$$

π^* is RIC and θ, θ^* are proximal (thus we still have $A = \mathfrak{G}(X, x_0) = \mathfrak{G}(X^*, x_0^*)$ and $F = \mathfrak{G}(Y, y_0) = \mathfrak{G}(Y^*, y_0^*)$). The concrete description of these objects uses quasifactors and the circle operation:

$$ Y^* = \{ p \circ F x_0 : p \in M \} \subset 2^X, \qquad X^* = \{ (x, y^*) : x \in y^* \in Y^* \} \subset X \times Y^* $$

and

$$ \theta(p \circ F x_0) = p y_0, \quad \theta^*(x, y^*) = x, \quad \pi^*(x, y^*) = y^*, \quad (p \in M), $$

where $F = \mathfrak{G}(Y, y_0)$. The map θ is an isomorphisms (hence $\pi = \pi^*$) when and only when π is already RIC.

Finally we say that $\pi : (X, \Gamma) \to (Y, \Gamma)$ has a *relatively invariant measure* (RIM), if there exists a projection $P : C(X) \to C(Y)$ such that

1. $P(f) \geq 0$ for $f \geq 0$ in $C(X)$.
2. $P(\mathbf{1}) = 1$.
3. $P(h \circ \pi) = h$ for every $h \in C(Y)$.
4. $P(f \circ \gamma) = P(f) \circ \gamma$ for every $f \in C(X)$ and $\gamma \in \Gamma$.

This property is equivalent to the existence of a continuous *section*, i.e. a continuous Γ equivariant map $y \mapsto \lambda_y$ from Y into $\mathfrak{M}(X)$ such that $\pi(\lambda_y) = \delta_y$ for every $y \in Y$. Here and in the sequel we use the same letter π to denote the induced map $\pi : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ on the spaces of probability measures. Sometimes though we will write π_* for the induced map.

In the *RIM shadow diagram*

$$
\begin{array}{ccc}
X & \xleftarrow{\;\tilde{\theta}\;} & \tilde{X} = X \vee \tilde{Y} \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} \\
Y & \xleftarrow{\;\theta\;} & \tilde{Y}
\end{array}
$$

the map $\tilde{\pi}$ has a RIM and the maps θ and $\tilde{\theta}$ are strongly proximal. It can be shown that every isometric extension has a RIM and is open. See [9] for more details, also a treatment of SPI systems can be found in [11].

We say that a minimal system (X, Γ) is a *strictly PI system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \leq \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota < \nu}$, and (iv) $W_\eta = X$. We say that (X, Γ) is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by HP extensions (almost 1-1 extensions in the metric case) we get the notion of *HPI* (*AI-systems* in the metric case). If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have *I-systems*. In this terminology the structure theorem for distal systems (Furstenberg [8], 1963) can be stated as follows:

1.2. **Theorem.** *A metric minimal system is distal iff it is an I-system.*

And, the Veech-Ellis structure theorem for point distal systems (Veech [21], 1970 and Ellis [5], 1973):

1.3. **Theorem.** *A metric minimal dynamical system is point distal iff it is an AI-system.*

The structure theorem for the general minimal system is proved in [7] and [20] (see also [21]) and asserts that every minimal system admits a canonically defined proximal extension which is a weakly mixing RIC extension of a strictly PI system. Both the Furstenberg and the Veech-Ellis structure theorems are corollaries of this general structure theorem.

1.4. **Theorem** (Structure theorem for minimal systems). *Given a minimal system (X, Γ), there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*

$$
\begin{array}{ccccccccccc}
X & \xleftarrow{\theta_0^*} & X_0 & \xleftarrow{\theta_1^*} & X_1 & \cdots & X_\nu & \xleftarrow{\theta_{\nu+1}^*} & X_{\nu+1} & \cdots & X_\eta = X_\infty \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\pi_0}\downarrow & {\scriptstyle\sigma_1}\searrow & {\scriptstyle\pi_1}\downarrow & & {\scriptstyle\pi_\nu}\downarrow & {\scriptstyle\sigma_{\nu+1}}\searrow & {\scriptstyle\pi_{\nu+1}}\downarrow & & {\scriptstyle\pi_\infty}\downarrow \\
pt & \xleftarrow{\theta_0} & Y_0 & \xleftarrow{\rho_1} Z_1 \xleftarrow{\theta_1} & Y_1 & \cdots & Y_\nu & \xleftarrow{\rho_{\nu+1}} Z_{\nu+1} \xleftarrow{\theta_{\nu+1}} & Y_{\nu+1} & \cdots & Y_\eta = Y_\infty
\end{array}
$$

where for each $\nu \leq \eta$, π_ν is RIC, ρ_ν is isometric, θ_ν, θ_ν^* are proximal and π_∞ is RIC and weakly mixing. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$. Thus X_∞ is a proximal extension of X and a RIC weakly mixing extension of the strictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) iff X is a PI-system.

Two further corollaries of this theorem are the theorems of Bronstein on the structure of PI systems, [3] and of van der Woude on HPI systems, [23]. Here we will use the latter which I now proceed to describe.

A homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$ is called *semiopen* if the interior of $\pi(U)$ is nonempty for every nonempty open subset U of X. When X is minimal every $\pi : (X, \Gamma) \to (Y, \Gamma)$ is semiopen (see Lemma 2.4 below). We will say that a subset $W \subset X \times X$ is a *S-set* if it is closed invariant topologically transitive and the restriction to W of the projection maps are semiopen.

1.5. **Theorem** (van der Woude). *A minimal system (X, Γ) is HPI iff every S-set in $X \times X$ is minimal.*

2. On semiopen maps

A result of Ditor and Eifler from 1972, [4] asserts that a continuous surjection $\pi : X \to Y$ between compact Hausdorff spaces X and Y is open iff the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is an open surjection. In the course of the proof of our main theorem (Theorem 5.1) we will need an analogous result (in the metric case) for semiopen maps (Theorem 2.3), which we now proceed to establish. First we need two preliminary lemmas.

2.1. Lemma. *Let $\pi : X \to Y$ be a continuous surjection between compact Hausdorff spaces. The conditions 1 and 2 below are equivalent. If X is metrizable then the three conditions are equivalent:*

1. *π is semiopen.*
2. *The preimage of every dense subset in Y is dense in X.*
3. *The set*

   $$X_0 = \{x \in X : \text{ the set valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } \pi(x)\}$$

   is dense in X.

Proof. The equivalence of 1 and 2 is straightforward. For any continuous surjection $\pi : X \to Y$ the corresponding set map $\pi^{-1} : Y \to 2^X$ is uppersemicontinuous and, when X is metrizable, this implies that it has a dense G_δ subset $Y_0 \subset Y$ of continuity points. Assuming 2 we conclude that $X_0 = \pi^{-1}(Y_0)$ is a dense G_δ subset of X. Conversely if 3 is valid and $U \subset X$ is open and nonempty, then $U \cap X_0 \neq \emptyset$ and if x_0 is any point in this intersection then π is open at x_0, so that $\pi(U)$ is a neighborhood of $\pi(x_0)$ and we conclude that int $(\pi(U)) \neq \emptyset$.   □

2.2. Lemma. *Let $\pi : X \to Y$ be a continuous surjection between compact metric spaces. Let $f : X \to \mathbb{R}$ be a continuous function and define $f^* : Y \to \mathbb{R}$ by*

$$f^*(y) = \sup\{f(x) : \pi(x) = y\}.$$

Then f^* is continuous at every point of the set

$$Y_0 = \{y \in Y : \text{ the set valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } y\}.$$

Proof. Fix $y \in Y_0$ and suppose $y_n \to y$ is a convergent sequence. For each n let $x_n \in X$ satisfy $\pi(x_n) = y_n$ and $f^*(y_n) = f(x_n)$. Let $x_{n_k} \to x$ be a convergent subsequence. Then $\pi(x) = y$ hence

$$f^*(y) \geq f(x) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f^*(y_{n_k}).$$

Since this is true for every partial limit of $f^*(y_n)$ we conclude that

$$f^*(y) \geq \limsup_{n \to \infty} f^*(y_n).$$

On the other hand if $f^*(y) = f(x)$ with $\pi(x) = y$ then, since π is open at x, we can find a sequence x_n with $\pi(x_n) = y_n$ and $x_n \to x$, so that

$$f^*(y) = f(x) = \lim_{n \to \infty} f(x_n) \leq \liminf_{n \to \infty} f^*(y_n).$$

□

2.3. Theorem. *Let $\pi : X \to Y$ be a continuous surjection between compact metric spaces which is semiopen. Then the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is a semiopen surjection.*

Proof. Let

$$Y_0 = \{y \in Y : \text{ the set valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } y\}$$

and

$$X_0 = \{x \in X : \text{ the set valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } \pi(x)\}.$$

Then $X_0 = \pi^{-1}(Y_0)$ and by Lemma 2.1 Y_0 and X_0 are dense G_δ subsets of Y and X respectively. Let $\mathfrak{M}_0(X)$ be the collection of measures of the form $\sum_{i=1}^{n} c_i \delta_{x_i}$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{n} c_i = 1$. If in addition we require that each x_i is in X_0 we obtain the smaller collection $\mathfrak{M}_{00}(X)$. Clearly $\mathfrak{M}_{00}(X)$, and hence also $\mathfrak{M}_0(X)$, are dense in $\mathfrak{M}(X)$. We define $\mathfrak{M}_0(Y)$ analogously as the collection of measures in $\mathfrak{M}(Y)$ of the form $\nu = \sum_{i=1}^{n} c_i \delta_{y_i}$. Again $\mathfrak{M}_0(Y)$ is dense in $\mathfrak{M}(Y)$.

Let $\mathcal{U} \subset \mathfrak{M}(X)$ be a closed set with nonempty interior. We have to show that $\text{int}\,(\pi_*(\mathcal{U})) \neq \emptyset$. Suppose to the contrary that $\text{int}\,(\pi_*(\mathcal{U})) = \emptyset$. Fix $\mu_0 = \sum_{i=1}^{m} c_i \delta_{x_i} \in \mathfrak{M}_{00}(X) \cap \text{int}\,(\mathcal{U})$ and let $\nu_0 = \pi_*(\mu_0) = \sum_{i=1}^{m} c_i \delta_{y_i}$, where $y_i = \pi(x_i)$. Let $\nu_j \in \mathfrak{M}_0(Y) \setminus \pi_*(\mathcal{U})$ be a sequence which converges to ν_0. Set $Q_j = \pi_*^{-1}(\nu_j)$. Each Q_j is a closed and convex subset of $\mathfrak{M}(X)$ and with no loss of generality we assume that $Q = \lim_{j\to\infty} Q_j$ exists in $2^{\mathfrak{M}(X)}$. Then Q is a compact convex subset of $\mathfrak{M}(X)$ with $\pi_*(\theta) = \nu_0$ for every $\theta \in Q$.

If $\mu_0 \in Q$ then eventually $Q_j \cap \mathcal{U} \neq \emptyset$, hence $\nu_j \in \pi_*(\mathcal{U})$, contradicting our assumption. Thus we have $\mu_0 \notin Q$ and by the separation theorem there exist a function $f \in C(X)$ and $\epsilon > 0$ with

(2.1) $$\mu_0(f) \geq \theta(f) + \epsilon \text{ for every } \theta \in Q.$$

Define the associated function

$$f^*(y) = \sup\{f(x) : \pi(x) = y\}.$$

Each measure ν_j has the form $\sum_{i=1}^{m_j} c_{j,i} \delta_{y_{j,i}}$ and we choose points $x_{j,i}$ with $\pi(x_{j,i}) = y_{j,i}$ such that $f^*(y_{j,i}) = f(x_{j,i})$. Now form the measures $\mu_j = \sum_{i=1}^{m_j} c_{j,i} \delta_{x_{j,i}}$ and assume, with no loss of generality, that $\mu = \lim_{j\to\infty} \mu_j$ exists in $\mathfrak{M}(X)$. Since $\mu_j \in Q_j$ for each j, we have $\mu \in Q$.

Note that by our construction $\nu_j(f^*) = \mu_j(f)$ for every j. By assumption the set $\text{supp}\,\nu_0 = \{y_1, \ldots, y_m\}$ is a subset of Y_0 and therefore, by Lemma 2.2, each y_i is a continuity point for f^*. From this fact it is easy to deduce that

$$\lim_{j\to\infty} \nu_j(f^*) = \nu_0(f^*).$$

It then follows that

$$\mu(f) = \lim_{j\to\infty} \mu_j(f) = \lim_{j\to\infty} \nu_j(f^*) = \nu_0(f^*) = \sum_{i=1}^{m} c_i f^*(y_i) = \mu_0(f^* \circ \pi) \geq \mu_0(f).$$

This contradicts (2.1) and our proof is complete. $\qquad\square$

Recall the following well known result; for completeness we include a proof.

2.4. **Lemma.** *Let $\pi : (X, \Gamma) \to (Y, \Gamma)$ be a homomorphism between minimal systems. Then π is semiopen.*

Proof. Let $W \subset X$ be a closed set with nonempty interior. By minimality of (X, Γ) there is a finite set $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ with $X = \bigcup_{i=1}^n \gamma_i W$. Therefore $Y = \bigcup_{i=1}^n \pi(\gamma_i W)$ and it follows that for some i the interior of the closed set $\pi(\gamma_i W) = \gamma_i \pi(W)$ is nonempty. Thus, as required, also $\mathrm{int}\,(\pi(\mathrm{W})) \neq \emptyset$. $\square$

3. A KEY PROPOSITION ON DIFFUSED MEASURES

As explained above we fix a minimal ideal M in $\beta\Gamma$ and let u be an idempotent in M. We denote the subgroup uM of M by G and identify it with the group of automorphisms of the universal Γ-minimal system (M, Γ), where for $\alpha \in G$ the corresponding automorphism $R_\alpha : M \to M$ is given by right multiplication $p \mapsto p\alpha$. For an Abelian Γ each subgroup $vM \subset M$, where v is an idempotent in M, is dense in M and it follows that the G-dynamical system (M, G) (where G acts by right multiplication) is minimal. From now on we always assume that Γ is an Abelian group.

3.1. **Lemma.** *Let (Y^*, Γ) be a metric minimal system and $\theta : (Y^*, \Gamma) \to (Y, \Gamma)$ its maximal equicontinuous factor. Suppose further that θ is an almost 1-1 extension. Let O be a nonempty open subset of Y^*.*
   1. *There is a nonempty open subset $V \subset O$ such that $\theta^{-1}(\theta(V)) \subset O$.*
   2. *There is a nonempty open subset $W \subset O$ such that $\mathrm{cls}\,_\tau(W \cap uY^*) \subset O$.*

Proof. 1. Suppose $O \subset Y^*$ is a nonempty open set for which the statement of the lemma fails. Choose a point $y^* \in O$ such that $\theta^{-1}(\theta(y^*)) = \{y^*\}$ and let $V_n \subset O$ be a sequence of open balls centered at y^* with $\mathrm{diam}\,(V_n) \searrow 0$. By assumption there are pairs of points $z_n \in V_n$ and $z_n' \notin O$ with $\theta(z_n) = \theta(z_n')$. However, as $\lim_{n \to \infty} \theta^{-1}(\theta(z_n)) = \{y^*\}$, we have $z_n' \to y^*$ in contradiction of the fact that O is a neighborhood of y^*.

2. Since Y is equicontinuous its τ-topology coincides with its compact Hausdorff group topology. Since θ is an almost 1-1 map, the restriction $\theta \restriction uY^* : uY^* \to Y$ is a homeomorphism of uY^*, equipped with the τ topology, onto Y. Let $O \subset Y^*$ be a nonempty open set. Let $V \subset O$ be as in part 1, and let W be a nonempty open subset such that $\overline{W} \subset V$. Now

$$
\begin{aligned}
\mathrm{cls}\,_\tau(W \cap uY^*) &= \theta^{-1}(\overline{\theta(W)}) \cap uY^* \\
&\subset \theta^{-1}(\overline{\theta(W)}) = \theta^{-1}(\theta(\overline{W})) \\
&\subset \theta^{-1}(\theta(V)) \subset O.
\end{aligned}
$$

$\square$

3.2. **Lemma.** *Let (X, Γ) be a minimal metric system and let $\phi : (X, \Gamma) \to (Y, \Gamma)$ be its maximal equicontinuous factor. Suppose further that Y is infinite and that we have the following diagram*

$$(X, \Gamma) \xrightarrow{\pi} (Y^*, \Gamma) \xrightarrow{\theta} (Y, \Gamma),$$

where π is an isometric extension, θ is an almost 1-1 extension and $\phi = \theta \circ \pi$. Let U be a nonempty open subset of X. Fix a minimal ideal $M \subset \beta\Gamma$ and an idempotent $u \in M$. Then

$$\mathrm{cls}_\tau(U \cap uX) \supset \pi^{-1}(\pi(U)) \cap uX.$$

Proof. Fix a point $x_0 \in X$ with $ux_0 = x_0$ and let y_0 and y_0^* be its images in Y and Y^* respectively. Let

$$A = \mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}, \text{ and}$$

$$F = \mathfrak{G}(Y, y_0) = \mathfrak{G}(Y^*, y_0^*) = \{\alpha \in G : \alpha y_0 = y_0\} = \{\alpha \in G : \alpha y_0^* = y_0^*\}.$$

The assumption that π is an isometric extension implies that $B = F/A$ is a homogeneous space of a Hausdorff compact topological group. The fact that (Y, Γ) is the maximal equicontinuous factor of (X, Γ) implies that $F \supset G'$ and that $G'A = AG' = F$. Let $\sigma : M \to X$ denote the evaluation map $p \mapsto px_0$.

Let U be a nonempty open subset of X. Set $\tilde{U} = \sigma^{-1}(U) = \{p \in M : px_0 \in U\}$. Then $\tilde{U}$ is a nonempty open subset of M and by minimality of the G-system (M, G), the collection $\{\tilde{U}\alpha : \alpha \in G\}$ is an open cover of M. Choose a finite subcover, say $\{\tilde{U}\alpha_i : i = 1, 2, \ldots, n\}$. Now

$$\bigcup_{i=1}^n \mathrm{cls}_\tau(\tilde{U}\alpha_i \cap G) = \bigcup_{i=1}^n \mathrm{cls}_\tau(\tilde{U} \cap G)\alpha_i = G;$$

hence $\mathrm{cls}_\tau(\tilde{U} \cap G)$ has a nonempty τ-interior. Since $\mathrm{cls}_\tau(\tilde{U} \cap G)$ is also τ-closed, it must contain a left translate of G', say $\beta G'$ for some $\beta \in G$ (this follows from the definition of G', see equation (1.1)). Projecting back to X via σ we get

$$\mathrm{cls}_\tau(U \cap uX) = \sigma(\mathrm{cls}_\tau(\tilde{U} \cap G))$$
$$\supset \beta G'x_0 = \beta G'Ax_0 = \beta F x_0$$
$$= (\beta F \beta^{-1})\beta x_0 = \pi^{-1}(\pi(\beta x_0)).$$

Let $O = \pi(U)$, then O is a nonempty open subset of Y^* and by Lemma 3.1 there is a nonempty open subset $W \subset O$ such that $\mathrm{cls}_\tau(W \cap uY^*) \subset O$. Set $U_1 = \pi^{-1}(W) \cap U$. Then U_1 is a nonempty open subset of X and by the above argument there exists $\beta_1 \in G$ with $\pi^{-1}(\pi(\beta_1 x_0)) \subset \mathrm{cls}_\tau(U_1 \cap uX) \subset \mathrm{cls}_\tau(U \cap uX)$. Now

$$\pi(\beta_1 x_0) \in \pi(\mathrm{cls}_\tau(U_1 \cap uX))$$
$$= \mathrm{cls}_\tau(\pi(U_1) \cap uY^*)$$
$$= \mathrm{cls}_\tau(W \cap uY^*) \subset O = \pi(U).$$

Thus we have shown that for every nonempty open subset $U \subset X$, the set $\mathrm{cls}_\tau(U \cap uX)$ contains a full fiber $\pi^{-1}(y^*)$ for some $y^* \in \pi(U)$. Since π is an open map we conclude that

$$\mathrm{cls}_\tau(U \cap uX) \supset \pi^{-1}(\pi(U)) \cap uX$$

as required. □

3.3. **Proposition.** *Let (X, Γ) be a minimal metric system and let $\phi : (X, \Gamma) \to (Y, \Gamma)$ be its maximal equicontinuous factor. Suppose further that Y is infinite and that we have the following diagram*

$$(X, \Gamma) \xrightarrow{\pi} (Y^*, \Gamma) \xrightarrow{\theta} (Y, \Gamma),$$

where π is an isometric extension, θ is an almost 1-1 extension and $\phi = \theta \circ \pi$. For every $y^* \in Y^*$ the fiber $\pi^{-1}(y^*)$ has the structure of a homogeneous space of a compact Hausdorff topological group and we let λ_{y^*} be the corresponding Haar measure on this fiber. Thus π is a RIM and open extension and $y^* \mapsto \lambda_{y^*}$, $Y^* \to \mathfrak{M}(X)$, is the corresponding section. Let $\Lambda : \mathfrak{M}(Y^*) \to \mathfrak{M}(X)$, defined by

$$\Lambda(\nu) = \int_{Y^*} \lambda_{y^*} \, d\nu(y^*),$$

be the associated affine injection. Set $\mathfrak{M}_m(X) = \{\Lambda(\nu) : \nu \in \mathfrak{M}(Y^*)\}$. Then the set

$$R = \{\nu \in \mathfrak{M}(X) : \text{the orbit closure of } \nu \text{ meets } \mathfrak{M}_m(X)\}$$

is a dense G_δ subset of $\mathfrak{M}(X)$.

Proof. Fix a compatible metric d on $\mathfrak{M}(X)$ and let $\kappa \in \mathfrak{M}(X)$ and $\epsilon, \eta > 0$ be given. Find an atomic measure $\lambda = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$ such that $d(\kappa, \lambda) < \epsilon/2$. Since Y is infinite and π is an open map we can also assume that $\pi(x_i) \neq \pi(x_j)$ for $i \neq j$. Choose open disjoint neighborhoods U_i of x_i, so small that every measure of the form $\mu = \frac{1}{n} \sum_{i=1}^{n} \mu_i$, $\mu_i \in \mathfrak{M}(X)$ with $\operatorname{supp} \mu_i \subset U_i$, will satisfy $d(\mu, \lambda) < \epsilon/2$, and hence also $d(\mu, \kappa) < \epsilon$.

Set $\nu = \frac{1}{n} \sum_{i=1}^{n} \lambda_{y_i^*}$ with $y_i^* = \pi(x_i)$. For each y_i^* choose points $\{x_{i,j}'\}_{j=1}^{k} \in \pi^{-1}(y_i^*)$ so that $d(\mu', \nu) < \epsilon/2$, where

$$\mu' = \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_{x_{i,j}'}.$$

By Lemma 3.2

$$u \circ \left(\bigcup_{i=1}^{n} U_i \right) \supset uX \cap u \circ \left(uX \cap \bigcup_{i=1}^{n} U_i \right)$$

$$= \operatorname{cls}_\tau \left(uX \cap \bigcup_{i=1}^{n} U_i \right)$$

$$\supset uX \cap \left(\bigcup_{i=1}^{n} \pi^{-1}(\pi(U_i)) \right).$$

Therefore there exist an element $\gamma \in \Gamma$ and for each i a set $\{x_{i,j}\}_{j=1}^{k} \subset U_i$, such that $d(\gamma x_{i,j}, x_{i,j}')$ is so small that the inequality $d(\gamma \mu, \mu') < \eta/2$ is satisfied, with

$$\mu = \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_{x_{i,j}}.$$

Thus $d(\gamma \mu, \nu) < \eta$. Since we also have $d(\mu, \kappa) < \epsilon$ and $\epsilon > 0$ is arbitrary, we have shown that the open set

$$R_\eta = \{\mu \in \mathfrak{M}(X) : \text{ there exists } \gamma \in \Gamma \text{ with } d(\gamma \mu, \mathfrak{M}_m(X)) < \eta\}$$

is dense in $\mathfrak{M}(X)$. Clearly $R = \bigcap \{R_\eta : \eta > 0\}$ is the required dense G_δ subset of $\mathfrak{M}(X)$. $\square$

4. Some properties of tame minimal systems

4.1. Theorem. *(See* [13]*) Let* (X, Γ) *be a metric tame dynamical system. Let* $\mathfrak{M}(X)$ *denote the compact convex set of probability measures on* X *(with the weak* topology). Then each element* $p \in E(X, \Gamma)$ *defines an element* $p_* \in E(\mathfrak{M}(X), \Gamma)$ *and the map* $p \mapsto p_*$ *is both a dynamical system and a semigroup isomorphism of* $E(X, \Gamma)$ *onto* $E(\mathfrak{M}(X), \Gamma)$.

Proof. Since $E(X, \Gamma)$ is Fréchet we have for every $p \in E$ a sequence $\gamma_i \to p$ of elements of Γ converging to p. Now for every $f \in C(X)$ and every probability measure $\nu \in \mathfrak{M}(X)$ we get by the Riesz representation theorem and Lebesgue's dominated convergence theorem

$$\gamma_i \nu(f) = \nu(f \circ \gamma_i) \to \nu(f \circ p) := p_* \nu(f).$$

Since the Baire class 1 function $f \circ p$ is well defined and does not depend upon the choice of the convergent sequence $\gamma_i \to p$, this defines the map $p \mapsto p_*$ uniquely. It is easy to see that this map is an isomorphism of dynamical systems, whence a semigroup isomorphism. Finally as Γ is dense in both enveloping semigroups, it follows that this isomorphism is onto. $\qquad\square$

As we have seen, when (X, Γ) is a metrizable tame system the enveloping semigroup $E(X, \Gamma)$ is a separable Fréchet space. Therefore each element $p \in E$ is a limit of a sequence of elements of Γ, $p = \lim_{n \to \infty} \gamma_n$. It follows that the subset $C(p)$ of continuity points of each $p \in E$ is a dense G_δ subset of X. More generally, if $Y \subset X$ is any closed subset then the set $C_Y(p)$ of continuity points of the map $p \restriction Y : Y \to X$ is a dense G_δ subset of Y. For an idempotent $v = v^2 \in E$ we write C_v for $C_{\overline{vX}}(v)$.

4.2. Lemma. *Let* (X, Γ) *be a metrizable tame dynamical system,* $E = E(X, \Gamma)$ *its enveloping semigroup.*

1. *For every* $p \in E$ *the set* $C(p) \subset X$ *is a dense* G_δ *subset of* X.
2. *For every idempotent* $v \in E$, *we have* $C_v \subset vX$.
3. *If* $v \in E$ *is an idempotent such that* $\overline{vX} = X$ *then* $C(v) \subset vX$.
4. *When* Γ *is commutative* (X, Γ) *minimal we have* $C(v) \subset vX$ *for every idempotent* $v \in E$.

Proof. 1. See the remark above.

2. Given $x \in C_v$ choose a sequence $x_n \in vX$ with $\lim_{n \to \infty} x_n = x$. We then have $vx = \lim_{n \to \infty} vx_n = \lim_{n \to \infty} x_n = x$, hence $C_v \subset vX$.

3. For such v we have $C(v) = C_v \subset vX$ by part 2.

4. When Γ is Abelian $\gamma p = p \gamma$ for every $\gamma \in \Gamma$ and $p \in E$. In particular the subset vX is Γ invariant hence dense in X. $\qquad\square$

4.3. Proposition. *Let* (X, Γ) *be a metric tame dynamical system. Then* $p_* \mu = \mu$ *for every* $p \in E(X, \Gamma)$ *and every* Γ *invariant measure* $\mu \in \mathfrak{M}(X)$.

Proof. Let $p = \lim_{n\to\infty} \gamma_n$, then for every $f \in C(X)$ and $x \in X$ we have $\lim_{n\to\infty} f \circ \gamma_n(x) = f(px)$, hence by Lebesgue dominated convergence theorem

$$
\begin{aligned}
\int f(x)\, d\mu(x) &= \lim_{n\to\infty} \int f(\gamma_n x)\, d\mu(x) \\
&= \int f(px)\, d\mu(x) \\
&= \int f(x)\, dp_*\mu(x).
\end{aligned}
$$

$\square$

4.4. Proposition. *Let Γ be an Abelian group. Then any metric tame minimal system (X, Γ) is point distal.*

Proof. We will prove that the condition in Theorem 1.5 holds; i.e. that every S-set in $X \times X$ is minimal. So let $W \subset X \times X$ be an S-set. Let $v = v^2$ be some minimal idempotent in $E(X, \Gamma)$. By Theorem 4.2.3 the set $C(v)$ of continuity points of the map $v : X \to X$ is a dense G_δ subset of X and moreover $C(v) \subset vX$, so that vX is residual in X. Since by assumption the projection maps $\pi_i : W \to X$ are semiopen, it follows that the sets $\pi_i^{-1}(vX)$ are residual in W. Since W_{tr}, the set of transitive points in W, is a dense G_δ subset of W we conclude that the set $(\pi_1^{-1}(vX) \cap \pi_2^{-1}(vX)) \cap W_{tr} = (vX \times vX) \cap W_{tr}$ is residual in W and in particular it is nonempty. Now if (x, x') is any point in this intersection then $v(x, x') = (vx, vx') = (x, x')$ and (x, x') is a minimal point. Therefore $W = \overline{\mathcal{O}}_\Gamma(x, x')$ is minimal. $\square$

5. Minimal tame systems are almost automorphic and uniquely ergodic

5.1. Theorem. *Let Γ be an Abelian group and (X, Γ) a metric tame minimal system. Then:*

1. *The system (X, Γ) is almost automorphic. Thus there exist:*
   (a) *A compact topological group Y with Haar measure η, and a group homomorphism $\kappa : \Gamma \to Y$ with dense image.*
   (b) *A homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$, where the Γ action on Y is via κ.*
   (c) *The sets $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ and $Y_0 = \pi(X_0)$ are dense G_δ subsets of X and Y respectively.*
2. *The system (X, Γ) is uniquely ergodic with unique invariant measure μ such that $\pi_*(\mu) = \eta$, and $\pi : (X, \mu, \Gamma) \to (Y, \eta, \Gamma)$ is a measure theoretical isomorphism of the corresponding measure preserving systems.*

Proof. 1. By Proposition 4.4 the system (X, Γ) is point distal. By the general theory of minimal point distal systems, if (X, Γ) is nontrivial, it admits a unique, nontrivial, maximal equicontinuous factor $\pi : (X, \Gamma) \to (Y, \Gamma)$. The system (Y, Γ) is thus of the form stated in the theorem. If Y is a finite group it follows from structure theory that $X = Y$. Thus, we now assume that Y is infinite.

Let

$$X \xleftarrow{\theta^*} X^* = X \vee Y^*$$

with vertical maps π and π^*, and

$$Y \xleftarrow{\theta} Y^*$$

be the associated O shadow diagram. Then (X^*, Γ) is also a minimal point distal system and (Y, Γ) is its maximal equicontinuous factor. Again by the theory of point distal systems the extension π^* is RIC. Now, either π^* is an isomorphism — in which case $X = X^* = Y^*$ and X is an almost 1-1 extension of Y, hence almost automorphic — or there exists a maximal intermediate isometric extension

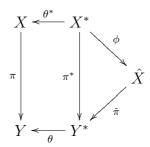$$X^* \xrightarrow{\phi} \hat{X} \xrightarrow{\hat{\pi}} Y^* \xrightarrow{\theta} Y,$$

with $\hat{\pi} : \hat{X} \to Y^*$ a nontrivial isometric extension. Thus we now assume that $\hat{\pi}$ is a nontrivial isometric extension. As was shown in Proposition 3.3 (see also [11]) $\hat{\pi}$ has a RIM and we let $y^* \mapsto \lambda_{y^*}$ be the corresponding section. Associated with this section we also have the affine injection $\Lambda : \mathfrak{M}(Y^*) \to \mathfrak{M}(\hat{X})$ defined by

$$\Lambda(\nu) = \int_{Y^*} \lambda_{y^*} \, d\nu(y^*),$$

and we set $\mathfrak{M}_m(\hat{X}) := \{\Lambda(\nu) : \nu \in \mathfrak{M}(Y^*)\}$.

Being isometric, $\hat{\pi}$ is also open and thus Proposition 3.3 applies (with $\hat{X}$ and $\hat{\pi}$ in the roles of X and π respectively). We conclude that the set $\hat{R}$ of measures in $\mathfrak{M}(\hat{X})$ whose orbit closure meets $\mathfrak{M}_m(\hat{X}) = \Lambda(\mathfrak{M}(Y^*))$ is a dense G_δ subset of $\mathfrak{M}(\hat{X})$.

2. We now have the following diagram

$$X \xleftarrow{\theta^*} X^* \xrightarrow{\phi} \hat{X}$$

with vertical maps π, π^*, the map $\hat{\pi}$ from $\hat{X}$, and

$$Y \xleftarrow{\theta} Y^*$$

and we set

$$\mathfrak{M}_m(X^*) := \phi^{-1}(\mathfrak{M}_m(\hat{X})) \text{ and}$$

$$\mathfrak{M}_m(X) := \theta^*(\mathfrak{M}_m(X^*)) = \theta^*(\phi^{-1}(\mathfrak{M}_m(\hat{X}))).$$

Let

$$R^* := \phi^{-1}(\hat{R}) = \{\xi \in \mathfrak{M}(\hat{X}) : \text{the orbit closure of } \xi \text{ meets } \mathfrak{M}_m(\hat{X})\}$$

and

(5.1) $\quad R' := \theta^*(R^*) \subset R = \{\xi \in \mathfrak{M}(X) : \text{the orbit closure of } \xi \text{ meets } \mathfrak{M}_m(X)\}.$

By Lemma 2.4 the map $\phi : X^* \to \hat{X}$ is semiopen and, as $\hat{R}$ is a dense subset of $\mathfrak{M}(\hat{X})$, Theorem 2.3 implies that R^* is a dense subset of $\mathfrak{M}(X^*)$. Therefore R' is a dense subset of $\mathfrak{M}(X)$. From the definition of R (5.1) it is easy to deduce that it is a G_δ set and because it contains R', it is in fact a dense G_δ subset of $\mathfrak{M}(X)$.

3. Recall that the system (X, Γ) is tame and, by Theorem 4.1 so is $(\mathfrak{M}(X), \Gamma)$. Moreover we have $E(X, \Gamma) = E(\mathfrak{M}(X), \Gamma)$. In particular $u \in E(\mathfrak{M}(X), \Gamma)$, as a Baire class 1 function, has a dense G_δ set of continuity points, $C_{\mathfrak{M}(X)}(u) \subset \mathfrak{M}(X)$. Therefore $S := C_{\mathfrak{M}(X)}(u) \cap R$ is a dense G_δ subset of $\mathfrak{M}(X)$.

Since uX is a dense subset of X it follows that the collection of finite convex combinations of point masses picked from uX forms a dense subset of $\mathfrak{M}(X)$. This implies that $u\mathfrak{M}(X)$ is dense in $\mathfrak{M}(X)$ and by Lemma 4.2.3 we have $C_{\mathfrak{M}(X)}(u) \subset u\mathfrak{M}(X)$. Thus also $S \subset u\mathfrak{M}(X)$. Now if $\nu \in S$ then $u\nu = \nu$ and, u being a minimal idempotent, the closure of the Γ orbit of ν in $\mathfrak{M}(X)$ is a minimal set, whence this entire orbit closure is contained in $\mathfrak{M}_m(X)$. In particular $\nu \in \mathfrak{M}_m(X)$ and we conclude that

$$S = C_{\mathfrak{M}(X)}(u) \cap R \subset \mathfrak{M}_m(X).$$

Therefore S is dense in $\mathfrak{M}_m(X)$ and in turn, this implies the equality:

(5.2) $$\mathfrak{M}_m(X) = \mathfrak{M}(X).$$

4. Given a point $x \in X$, the corresponding point mass $\delta_x \in \mathfrak{M}(X)$ must have, by (5.2), a preimage in $\mathfrak{M}_m(X^*)$, say $\theta^*(\xi) = \delta_x$ with $\xi \in \mathfrak{M}_m(X^*)$. In particular, for x with $\theta^{*-1}(x) = \{x^*\}$ a singleton, we must have $\xi = \delta_{x^*} \in \mathfrak{M}_m(X^*)$ and therefore $\phi_*(\delta_{x^*}) = \delta_{\hat{x}} \in \mathfrak{M}_m(\hat{X})$ with $\hat{x} = \phi(x^*)$. By the definition of $\mathfrak{M}_m(\hat{X})$ there exists a measure $\rho \in \mathfrak{M}(Y^*)$ with

$$\delta_{\hat{x}} = \Lambda(\rho) = \int_{Y^*} \lambda_{y^*} \, d\rho(y^*).$$

This clearly implies that the measure ρ is a point mass, say $\rho = \delta_{y^*}$ and that the measure λ_{y^*} — which is the Haar measure on the homogeneous space which forms the fiber $\hat{\pi}^{-1}(y^*) \subset \hat{X}$ — is also a degenerate point mass. That is, the isometric extension $\hat{\pi}$ is in fact an isomorphism. Now the collapse of $\hat{\pi}$, implies the collapse of the entire AI tower, so that in fact $Y^* = X^* = X$, and we have shown that X is indeed an almost 1-1 extension of Y.

5. Suppose that μ_1 and μ_2 are two invariant probability measures on X. Then, (X, Γ) being tame, by Proposition 4.3, $u_* \mu_i = \mu_i$ and we conclude that $\mu_i(uX) = 1$, for $i = 1, 2$. Since π is a proximal extension, for every $y \in Y = uY$ the fiber $\pi^{-1}(y)$ intersects uX at exactly one point: $\pi^{-1}(y) \cap uX = \{x\}$. Now by disintegrating each μ_i over η, inside the set uX, we conclude that $\mu_1 = \mu_2$. This proves the unique ergodicity of (X, Γ). It is also clear from the proof that the map $\pi : (X, \mu, \Gamma) \to (Y, \eta, \Gamma)$, where μ is the unique invariant measure on X, is an isomorphism of measure preserving systems. $\square$

5.2. **Remark.** The set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is a dense G_δ and Γ-invariant subset of X and thus has μ measure either zero or one. In [18, Section 11] Kerr and Li construct a minimal Toeplitz system which is tame and not null. Since in this construction the growth of the sequence $\{n_1 < n_2 < \cdots\}$ is arbitrary it follows that the resulting Toeplitz system can be made not regular in the sense that the densities of the periodic parts converge to $d < 1$. For such nonregular systems $\mu(X_0) = 0$. This shows that the unique invariant measure of a minimal tame system need not be supported by the set X_0 where π is 1-1.

References

1. J. Auslander, *Minimal Flows and their Extensions*, Mathematics Studies 153, Notas de Matemática, 1988.

2. J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math., **100** (1978), 845-886.

3. I. U. Bronstein, *A characteristic property of PD-extensions*, Bul. Akad. Stiimce RSS Moldoven, **3**, (1977), 11-15 (Russian).

4. S. Z. Ditor and L. Q. Eifler, *Some open mapping theorems for measures*, Trans. Amer. Math. Soc., **164**, (1972), 287-293.

5. R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc., **94**, (1960), 272-281.

6. ———, *Lectures on Topological Dynamics*, W. A. Benjamin, Inc. , New York, 1969.

7. R. Ellis, E. Glasner and L. Shapiro, *Proximal-Isometric flows*, Advances in Math., **17**, (1975), 213-260.

8. H. Furstenberg, *The structure of distal flows*, American J. of Math., **85**, (1963), 477-515.

9. E. Glasner, *Relatively invariant measures*, Pacific J. of Math., **58**, (1975), 393-410.

10. ———, *Proximal flows*, Lecture Notes in Math., **517**, Springer-Verlag, 1976.

11. ———, *Distal and semisimple affine flows*, Amer. J. Math., **109**, (1987), 115-131.

12. ———, *Structure theory as a tool in topological dynamics*, Descriptive set theory and dynamical systems, LMS Lecture note Series **277**, Cambridge University Press, Cambridge, 2000, 173-209.

13. ———, *On tame dynamical systems*, Colloq. Math., **105** (2006), 283-295.

14. E. Glasner and M. Megrelishvili, *Hereditarily non-sensitive dynamical systems and linear representations*, Colloq. Math., **104** (2006), no. 2, 223-283.

15. E. Glasner, M. Megrelishvili and V. V. Uspenskij, *On metrizable enveloping semigroups*, ArXiv: math.DS/0606373.

16. W. Huang, *Tame systems and scrambled pairs under an abelian group action*, Ergod. Th. Dynam. Sys., **26**, (2006), 1549-1567.

17. W. Huang, S. M. Li, S. Shao, and X. Ye, *Null systems and sequence entropy pairs*, Ergod. Th. Dynam. Sys., **23**, (2003), 1505-1523.

18. D. Kerr and H. Li, *Independence in topological and C^*-dynamics*, To appear.

19. A. Köhler, *Enveloping semigroups for flows*, Proceedings of the Royal Irish Academy, **95A**, (1995), 179-191.

20. D. C. McMahon, *Weak mixing and a note on the structure theorem for minimal transformation groups*, Illinois J. of Math., **20**, (1976), 186-197.

21. W. A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc., **83**, (1977), 775-830.

22. J. de Vries, *Elements of Topological Dynamics*, Kluwer Academic Publishers, 1993.

23. J. van der Woude, *Characterizations of H(PI) extensions*, Pacific J. Math., **120**, (1985), 453-467.

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