EVENTUAL NONSENSITIVITY AND TAME DYNAMICAL SYSTEMS

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Abstract. In this paper we characterize tame dynamical systems and functions in terms of eventual non-sensitivity and eventual fragmentability. As a notable application we obtain a neat characterization of tame subshifts $X \subset \{0,1\}^{\mathbb{Z}}$: for every infinite subset $L \subseteq \mathbb{Z}$ there exists an infinite subset $K \subseteq L$ such that $\pi_K(X)$ is a countable subset of $\{0,1\}^K$. The notion of eventual fragmentability is one of the properties we encounter which indicate some "smallness" of a family. We investigate a "smallness hierarchy" for families of continuous functions on compact dynamical systems, and link the existence of a "small" family which separates points of a dynamical system (G,X) to the representability of X on "good" Banach spaces. For example, for metric dynamical systems the property of admitting a separating family which is eventually fragmented is equivalent to being tame. We give some sufficient conditions for coding functions to be tame and, among other applications, show that certain multidimensional analogues of Sturmian sequences are tame. We also show that linearly ordered dynamical systems are tame and discuss examples where some universal dynamical systems associated with certain Polish groups are tame.

Contents

1. Introduction

Tame dynamical systems were introduced by A. Köhler [52] (under the name "regular systems") and their theory was later developed in a series of works by several authors (see e.g. [28], [31], [33], [44], [51] and [29]).

More recently connections to other areas of mathematics like coding theory, substitutions and tilings, and even model theory and logic were established (see e.g. [3], [46], and the survey [36] for more details). In the present work we introduce a new approach to the study of tame and other related dynamical systems in terms of "small families" of functions.

Given a topological semigroup S, one way to measure the complexity of a compact dynamical S-system X is to investigate its representability on "good" Banach spaces, [36, 35, 33]. Another is to ask whether the points of X can be separated by a norm bounded S-invariant family $F \subset C(X)$ of continuous functions on X, such that F is "small" in some sense or another. Usually being "small" means that the pointwise closure $\mathrm{cls}_p(F)$ of F (the *envelope of* F) in $\mathbb{R}^X$ is a "small" compactum.

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Two typical examples are (a) when $\operatorname{cls}_p(F) \subset C(X)$ and (b) when $\operatorname{cls}_p(F)$ consists of fragmented functions (Baire 1, when X is metrizable).

It turns out (Theorem 3.11) that the first case (a) characterizes the reflexively representable dynamical systems, (these are the dynamical analog of Eberlein compacta) or, for metric dynamical systems X, the class of WAP systems.

In the second case (b) we obtain a characterization of the class of Rosenthal representable dynamical systems, or, for metric dynamical systems X, the class of tame systems (a Banach space V is said to be *Rosenthal* [33, 35] if it does not contain an isomorphic copy of the Banach space l_1).

We also have a characterization of the intermediate class of Asplund representable, or hereditarily nonsensitive (HNS) systems. Namely, an S-system X is HNS iff there exists a separating bounded family $F \subset C(X)$ which is a fragmented family (Definition 2.1). See Theorem 3.12 below.

These subjects are treated in Sections 2, 3 and 4. In section 5 we apply these results to symbolic $\mathbb{Z}$-systems, and, in Section 6, relate them to entropy theory.

In Section 7 we give some sufficient conditions on certain coding functions which ensure that the associated dynamical systems are tame. In particular, we conclude that some multidimensional Sturmian-like functions are tame (Theorem 7.18).

In Section 8 we show that order preserving dynamical systems are tame. The same is true for the system $(H_+(\mathbb{T}), \mathbb{T})$, where $H_+(\mathbb{T})$ is the Polish group of orientation preserving homeomorphisms of the circle $\mathbb{T}$. Recall that for every topological group G there exists a universal minimal system $M(G)$ and also a universal irreducible affine G-system $IA(G)$. In Section 9 We discuss some examples, where $M(G)$ and $IA(G)$ are tame. In particular, we show that this is the case for the group $G = H_+(\mathbb{T})$, using a well known result of Pestov [71] which identifies $M(G)$ as the tautological action of G on the circle $\mathbb{T}$.

Our main results in this work are:

- A characterization of tame systems and functions in terms of eventual non-sensitivity (Theorems 4.8 and 4.9).
- A characterization of tame symbolic dynamical systems (Theorems 5.5 and 6.1.2).
- A combinatorial characterization of tame subsets $D \subset \mathbb{Z}$ (i.e., subsets D such that the associated subshift $X_D \subset \{0,1\}^{\mathbb{Z}}$ is tame), Theorem 5.8.
- In Theorem 3.11 we consider a hierarchy of smallness of bounded S-invariant families $F \subset C(X)$ on a compact dynamical system X. In particular, we investigate when the evaluation map $F \times X \to \mathbb{R}$ comes from the canonical bilinear evaluation map $V \times V^* \to \mathbb{R}$ for good classes of Banach spaces V. We then show that fragmentability and eventual fragmentability of a separating family F characterize Asplund and Rosenthal representability respectively (Theorem 3.12).
- Theorems 7.14 and 8.23 give some useful sufficient conditions for the tameness of coding functions.
- Theorem 8.9 shows that any linearly ordered dynamical system is tame. Moreover, such dynamical systems are representable on Rosenthal Banach spaces. By Theorem 8.16, the universal minimal G-system $M(G) = \mathbb{T}$ for $G = H_+(\mathbb{T})$ is tame.

1.1. **Preliminaries.** We use the notation of [35, 36]. By a topological space we mean a Tychonoff (completely regular Hausdorff) space. The closure operator in topological spaces will be denoted by cls, and "compact" will mean "compact and Hausdorff". A topological space X is called *hereditarily Baire* if every closed subspace of X is a Baire space. A function $f : X \to Y$ is *Baire class 1 function* if the inverse image $f^{-1}(O)$ of every open set is F_σ in X.

For a pair of topological spaces X and Y we let $C(X, Y)$ denote the space of continuous functions from X into Y. We will take $C(X)$ to be the Banach space of *bounded* continuous real valued functions even when X is not necessarily compact. For a compact space X we let

$$P(X) = \{\mu \in C(X)^* : \ \|\mu\| = \mu(\mathbf{1}) = 1\}$$

be the w^*-compact subset of $C(X)^*$ which, as usual, is identified with the space of *probability measures* on X.

By a *dynamical S-system* X we mean a topological space X with a separately continuous action $\pi : S \times X \to X$. We say also that X is an *S-space.* In most cases X will be compact.

All semigroups S are assumed to be monoids, i.e, semigroups with a neutral element which will be denoted by e. Also actions are *monoidal* (meaning $ex = x, \forall x \in X$).

By an (invertible) *cascade* on X we mean a continuous action $S \times X \to X$, where $S := \mathbb{N} \cup \{0\}$ is the additive semigroup of nonnegative integers (respectively, $S = (\mathbb{Z}, +)$). We write it sometimes as a pair (X, σ) where σ is the s-translation of X corresponding to $s = 1$ (0 acts as the identity). By $\bar{\mathcal{O}}_G(x_0)$ we denote the closure of the orbit Gx_0 in X.

For every compact dynamical S-system X we have a monoid homomorphism $j : S \to C(X,X)$, $j(s) = \tilde{s}$, where $\tilde{s} : X \to X, x \mapsto sx = \pi(s,x)$ is the *s-translation* ($s \in S$). The *enveloping semigroup* $E(S,X)$ (or just $E(X)$) is defined as the pointwise closure $E(S,X) = \mathrm{cls}_p(j(S))$ of $\tilde{S} = j(S)$ in X^X. It is always a right topological compact monoid. Algebraic and topological properties of the families $j(S)$ and $E(X)$ reflect the asymptotic dynamical behavior of (S,X).

More generally, for a family $F \subset C(X,Y)$ we define its *envelope* as the pointwise closure $\mathrm{cls}_p(F)$ of F in Y^X.

Let F, X, Y be topological spaces and $w : F \times X \to Y, w(f,x) := f(x)$ a function. We say that F has the *Double Limit Property* (DLP) on X if for every sequence $\{f_n\} \subset F$ and every sequence $\{x_m\} \subset X$ the limits

$$\lim_n \lim_m f_n(x_m) \quad \text{and} \quad \lim_m \lim_n f_n(x_m)$$

are equal whenever they both exist.

Given a function $f \in C(X)$ on a compact S-space X we consider its orbit $fS := \{f \circ \tilde{s} : s \in S\} \subset C(X)$. One can estimate the dynamical complexity of f by considering the pointwise closure

$$\mathrm{cls}_p(fS) = fE(S,X) := \{f \circ q : q \in E(S,X)\}$$

in $\mathbb{R}^X$. Various degrees of "smallness" of this compactum lead to a natural hierarchy. The classical examples are the almost periodic (AP) and weakly almost periodic (WAP) functions. The norm compactness of $\mathrm{cls}_p(fS)$ in $C(X)$ is the characteristic trait of a Bohr almost periodic function. For the latter property we have:

Definition 1.1. Let X be a compact S-system.

(1) $f \in C(X)$ is said to be WAP if one of the following equivalent conditions is satisfied:
   (a) fS is weakly precompact in $C(X)$;
   (b) $\mathrm{cls}_p(fS) \subset C(X)$;
   (c) fS has DLP on X.

(2) (S,X) is said to be WAP if one of the following equivalent conditions is satisfied:
   (a) every member $p \in E(S,X)$ is a continuous function $X \to X$;
   (b) $\mathrm{WAP}(X) = C(X)$.

The equivalences can be verified using Grothendieck's classical results. See for example, [9, Theorem A4] and [9, Theorem A5].

Remark 1.2. If (S,X) is WAP then it is easy to see that $S \times X \to X$ has DLP (assuming the contrary, choose $f \in C(X)$ which separates the corresponding double limits in X. Then $f \notin WAP(X)$). If the compactum X is metrizable (or, more generally, sequentially compact) then the converse is also true. To see this use Definition 1.1 and the diagonal arguments.

When V is a Banach space we denote by B, or B_V, the closed unit ball of V. $B^* = B_{V^*}$ and $B^{**} := B_{V^{**}}$ will denote the weak* compact unit balls in the dual V^* and second dual V^{**} of V respectively. The DLP characterizes reflexive spaces as the following theorem shows.

Theorem 1.3. *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is reflexive.*
(2) *B has DLP on B^*.*
(3) *every bounded subset $F \subset V$ has DLP on every bounded $X \subset V^*$.*
(4) *$B \subset V$ is weakly compact.*

Later we will recall and examine the definitions of Asplund and tame functions which are based on the concept of *fragmentability*. This concept has its roots in Banach space theory. We will also introduce various definitions of non-sensitivity for invariant families of functions. Such families are the main object of study in this paper. As some recent results show this approach is quite effective and provides the right level of generality. See for example the study of HNS systems in [31] and some applications to fixed point theorems in [34].

1.2. **Some classes of dynamical systems.** We next briefly describe the two main classes of dynamical systems which will be analyzed. We begin with a generalized version of the notion of dynamical sensitivity.

Definition 1.4. (See for example [4, 38, 31].) Let (X, τ) be a compact S dynamical system endowed with its unique compatible uniform structure μ.

(1) The dynamical S-system X has *sensitive dependence on initial conditions* (or, simply is *sensitive*) if there exists an $\varepsilon \in \mu$ such that for every open nonempty subset $O \subset X$ there exist $s \in S$ and $x, y \in O$ such that $(sx, sy) \notin \varepsilon$.
(2) Otherwise we say that (S, X) is *non-sensitive*, NS for short. This means that for every $\varepsilon \in \mu$ there exists an open nonempty subset O of X such that sO is ε-small in (X, μ) for all $s \in S$.

The following definition (for continuous group actions) originated in [31].

Definition 1.5. [31, 35] We say that a compact S-system X is *hereditarily non-sensitive* (HNS, in short) if for every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε-small for every $s \in S$.

We also have the following characterizations (for the definitions of fragmented, eventually fragmented and eventually weakly fragmented families see Definition 2.1 below).

Theorem 1.6.

(1) [33] *A dynamical system (S, X) is HNS iff $E(S, X)$ (equivalently, $\tilde{S}$), as an S-invariant family, is fragmented.*
(2) [37] *A metric dynamical system (S, X) is HNS iff $E(S, X)$ is metrizable.*

The second class of dynamical systems we will be interested in is the class of tame dynamical systems. For the history of this notion, which is originally due to Köhler [52], we refer to [28] and [33]. The following principal result is a dynamical analog of a well known BFT dichotomy [10, 82].

Theorem 1.7. [52, 31, 35] (A dynamical version of BFT dichotomy) *Let X be a compact metric dynamical S-system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative. Either*

(1) *E is a separable Rosenthal compact (hence E is Fréchet and card $E \leq 2^{\aleph_0}$); or*
(2) *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$ (hence card $E = 2^{2^{\aleph_0}}$).*

The first possibility holds iff X is a tame S-system.

Thus, a metrizable dynamical system is tame iff $\mathrm{card}\,(\mathrm{E(X)}) = 2^{\aleph_0}$ iff $E(X)$ is a Rosenthal compactum (or a Fréchet space). Moreover, by [37] a metric S-system is tame iff every $p \in E(X)$ is a Baire class 1 map $X \to X$. This result led us to the following definition for general (not necessarily, metrizable) systems.

Definition 1.8. [33, 35] A compact S-system X is said to be *tame* if every $p \in E(X)$ is a fragmented map (equivalently, Baire 1, when X is metrizable).

There are several other well known characterizations of tameness and in the present work we will obtain two more: by Theorem 4.8 (S, X) is tame iff fS is an eventually fragmented family for every $f \in C(X)$, and, in Theorem 4.9 we show that (S, X) is tame iff $\tilde{S} \subset X^X$, as an S-invariant family, is eventually weakly fragmented (see Definitions 2.1 and 2.5).

Note that, as it directly follows from the definitions (when considering the enveloping semigroup characterizations), every WAP system is HNS and every HNS is tame.

1.3. **Some classes of functions.**

Definition 1.9. Let $f \in C(X)$ on a (not necessarily, compact) S-system X.

(1) We say that f *comes* from the S-compactification $q : X \to Y$ (where the action of S on Y is at least separately continuous) if there exists a continuous function $f' : Y \to \mathbb{R}$ such that $f = f' \circ q$.
(2) We say that $f \in C(X)$ is *RMC* (*right multiplicatively continuous*) if f comes from some S-compactification $q : X \to Y$. For every compact S-system X we have $\mathrm{RMC}(X) = C(X)$.
(3) If we consider only jointly continuous S-actions on Y then the functions $f : X \to \mathbb{R}$ which come from such G-compactifications $q : X \to Y$ are *right uniformly continuous*. Notation: $f \in \mathrm{RUC}(X)$.
(4) f is said to be: a) *WAP*; b) *Asplund*; c) *tame* if f comes from an S-compactification $q : X \to Y$ such that (S, Y) is: WAP, HNS or tame respectively. For the corresponding classes of functions we use the notation: $\mathrm{WAP}(X), \mathrm{Asp}(X), \mathrm{Tame}(X)$, respectively. Each of these is a norm closed S-invariant subalgebra of the S-algebra $\mathrm{RMC}(X) \subset C(X)$ and
$$\mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{Tame}(X).$$
For more details see [35, 36].
(5) Note that as a particular case of (3) we have defined the algebras $\mathrm{WAP}(S), \mathrm{Asp}(S), \mathrm{Tame}(S)$ corresponding to the left action of S on $X := S$.

Definition 1.10. [31, 35] We say that a compact dynamical S-system X is *cyclic* if there exists $f \in C(X)$ such that (S, X) is topologically S-isomorphic to the Gelfand space X_f of the S-invariant unital subalgebra $\mathcal{A}_f \subset C(X)$ generated by the orbit fS.

Remark 1.11. Let X be a (not necessarily compact) S-system and $f \in \mathrm{RMC}(X)$. Then, as was shown in [35], there exist: a cyclic S-system X_f, a continuous S-compactification $\pi_f : X \to X_f$, and a continuous function $\tilde{f} : X_f \to \mathbb{R}$ such that $f = \tilde{f} \circ \pi_f$; that is, f comes from the S-compactification $\pi_f : X \to X_f$. The collection of functions $\tilde{f}S$ separates points of X_f. Finally, $f \in \mathrm{RUC}(X)$ iff the action of S on X_f is jointly continuous.

2. Sensitivity and fragmentability of families

The following topological definitions were motivated by the notions of sensitivity and tameness in topological dynamics (Definitions 1.4 and 1.8 above), as well as by the fragmentability concepts which come from Banach space theory.

Definition 2.1. Let (X, τ) be a topological space, (Y, ξ) a uniform space and $\varepsilon \in \xi$ is an entourage. We say that a family of (not necessarily continuous) functions $F = \{f_i : X \to Y\}_{i \in I}$ is:

(ε-NS) ε-*Non-Sensitive* (ε-NS in short) if there exists a non-void open subset O in X such that $f_i(O)$ is ε-small for every $i \in I$.
(NS) *Non-Sensitive* if F is ε-NS for every $\varepsilon \in \xi$.
(E-NS) *Eventually Non-Sensitive* if for every infinite subfamily $L \subset F$ and every $\varepsilon \in \xi$ there exists an infinite subfamily $K \subset L$ which is ε-NS.

(ε-Fr) ε-*Fragmented* if for every nonempty closed subset A of X the restriction $F|_A := \{f|_A : A \to Y\}_{f \in F}$ is an ε-NS family.
(Fr=HNS) *Fragmented*, or *Hereditarily Non-Sensitive*, if F is ε-Fr for every $\varepsilon \in \xi$.
(E-wFr) *Eventually weakly fragmented* if for every infinite subfamily $L \subset F$ and every $\varepsilon \in \xi$ there exists an infinite ε-Fr subfamily $K \subset L$.
(E-Fr) *Eventually Fragmented* if for every infinite subfamily $L \subset F$ there exists an infinite fragmented subfamily $K \subset L$.

(HAE) *Hereditarily Almost Equicontinuous*, or *barely continuous*, if for every nonempty closed subset A of X the family $F|_A$ has a point of equicontinuity.
(E-HAE) *Eventually Hereditarily Almost Equicontinuous* if for every infinite subfamily $L \subset F$ there exists an infinite HAE subfamily $K \subset L$.

When the family $F = \{f\}$ consists a single function we retrieve the definitions of NS, fragmented, and barely continuous functions. The definition of a fragmented function (as in [45]) is a slight generalization of the original one (for the identity function $f := id_X : (X, \tau) \to (X, d)$ with a pseudometric d on X; the (τ, d)-*fragmentability*) which is due to Jayne and Rogers from Banach space theory. It appears implicitly in a work of Namioka and Phelps [67] which provides a characterization of Asplund Banach spaces V in terms of (weak*,norm)-fragmentability. The set of fragmented maps from X into $Y := \mathbb{R}$ is denoted by $\mathcal{F}(X)$. See [66, 59, 61, 31, 33] for more details. Barely continuous maps are well known also as the maps with the *point of continuity property* (i.e., for every closed nonempty $A \subset X$ the restriction $f_{|A} : A \to Y$ has a continuity point)

Eventually fragmented families were introduced in [33], where they yield a new characterization of Rosenthal Banach spaces (see below Theorem 2.12.4).

In Example 3.4 we present some simple examples illustrating the definitions of families of functions which are (or are not) fragmented, eventually fragmented, or satisfy DLP.

For some applications of the fragmentability concept for topological transformation groups, see [59, 60, 61, 31, 34, 33, 35]. For other research directions involving fragmentability see for example [49].

Lemma 2.2. [31, 33]

(1) *It is enough to check the conditions in Definition 2.1 (in all items excepting (HAE) and (E-HAE), where it is irrelevant) only for $\varepsilon \in \gamma$ from a subbase γ of ξ (that is, the finite intersections of the elements of γ form a base of the uniform structure ξ).*

(2) *If X is Polish and Y is a separable metric space then $f : X \to Y$ is fragmented iff f is a Baire class 1 function (i.e., the inverse image of every open set is F_σ).*

(3) *When X is hereditarily Baire and (Y, ρ) is a pseudometric space then $f : X \to Y$ is fragmented iff f has the point of continuity property.*

(4) *A topological space (X, τ) is* scattered *(i.e., every nonempty subspace has an isolated point) iff X is (τ, ξ)-fragmented, for arbitrary uniform structure ξ on the* set *X.*

(5) *Let (X, τ) be a separable metrizable space and (Y, ρ) a pseudometric space. Suppose that $f : X \to Y$ is a fragmented onto map. Then Y is separable.*

(6) *$F = \{f_i : X \to (Y, \xi)\}_{i \in I}$ is a fragmented family iff the induced map $X \to (Y^F, \xi_U)$ is fragmented, where ξ_U is the uniformity of uniform convergence on Y^F.*

(7) *Let $\alpha : X \to X'$ be a continuous onto map between compact spaces. Assume that (Y, ξ) is a uniform space, $F := \{f_i : X \to Y\}_{i \in I}$ and $F' := \{f'_i : X' \to Y\}_{i \in I}$ are families such that $f'_i \circ \alpha = f_i$ for every $i \in I$. Then F is a fragmented family iff F' is a fragmented family.*

(8) *(See* [10, Cor. 1D] *or* [21, Lemma 3.7]*) Let X be a hereditarily Baire space and $f : X \to \mathbb{R}$ an arbitrary function. The following are equivalent:*
   (a) *f has the point of continuity property (equivalently: fragmented, by (3)).*
   (b) *For every nonempty closed subset $K \subset X$ and $a < b$ in Y the sets $K \cap \{f \leq a\}$, $K \cap \{f \geq b\}$ are not both dense in K.*

(9) *Let $p : X \to Y$ be a map from a topological space X into a compact space Y. Suppose that $\{f_i : Y \to Z_i\}_{i \in I}$ is a system of continuous maps from Y into Hausdorff uniform spaces Z_i such that it separates points of Y and $f_i \circ p \in \mathcal{F}(X, Z_i)$ for every $i \in I$. Then $p \in \mathcal{F}(X, Y)$.*

(10) [33, Lemma 2.3.4] *Let (X, τ) and (X', τ') be compact spaces, and let (Y, μ) and (Y', μ') be uniform spaces. Suppose that: $\alpha : X \to X'$ is a continuous onto map, $\nu : (Y, \mu) \to (Y', \mu')$ is uniformly continuous, $\phi : X \to Y$ and $\phi' : X' \to Y'$ are maps such that the following diagram*

$$\begin{tikzcd} (X, \tau) \arrow[r, "\phi"] \arrow[d, "\alpha"'] & (Y, \mu) \arrow[d, "\nu"] \\ (X', \tau') \arrow[r, "\phi'"] & (Y', \mu') \end{tikzcd}$$

commutes. If X is fragmented by ϕ then X' is fragmented by ϕ'.

Lemma 2.3.

(1) *Always, HAE $\subset$ HNS $\subset$ NS and E-HAE $\subset$ E-Fr $\subset$ E-wFr $\subset$ E-NS.*

(2) *In the definitions (E-Fr), (E-HAE), (E-HAE) and (E-NS) one can assume that the infinite sets K (and L) are* countable.
(3) *When (Y,ξ) is a pseudometric uniformity and every $f_i \in F$ is a fragmented map, then E-Fr=E-wFr.*
(4) *When X is a hereditarily Baire space and (Y,ξ) is a pseudometric uniformity then HAE = Fr.*
(5) *If, in addition to the conditions in (4), every $f_i \in F$ is a fragmented map then E-HAE = E-Fr = E-wFr.*

Proof. (1) and (2) are trivial.

To get (3) use a diagonal argument. This is possible because the pseudometric uniformity ξ has a countable basis for the uniform structure. We have a decreasing sequence of entourages $\varepsilon_1 \supset \varepsilon_2 \supset \cdots$ which form a basis of the uniform structure ξ. Using the E-wFr condition for a given infinite sequence in F we extract a subsequence which is ε_1-Fr subsequence. Now for this subsequence one can extract an ε_2-Fr subsequence, and so on. Consider the diagonal sequence. This will be an ε-Fr family for all $\varepsilon \in \xi$. In the verification it is important to note that, by our assumption, every individual $f_i \in F$ is a fragmented map. This guarantees that every finite subfamily is fragmented. So, in particular, each initial finite segment of the diagonal sequence is fragmented. Finally note that the union of two fragmented families is a fragmented family.

For (4) we recall some observations from [31]. By Lemma 2.2.6, a family $F = \{f_i : X \to (Y,\xi)\}_{i\in I}$ is a fragmented family iff the induced map $X \to (Y^F, \xi_U)$ is fragmented. Now, when X is hereditarily Baire and ξ is pseudometrizable we obtain, using Lemma 2.2.3, that F is fragmented iff F is HAE.

For (5) combine parts (3) and (4) to get E-HAE = E-wFr. □

Of course the condition that (Y,ξ) be a pseudometric uniform space is satisfied when $Y = \mathbb{R}$, so that this assumption is automatically fulfilled for families in $C(X)$.

Lemma 2.4. [33, 35]
(1) *Suppose F is a compact space, X is Čech-complete, Y is a uniform space and we are given a separately continuous map $w : F \times X \to Y$. Then the naturally associated family $\tilde{F} := \{\tilde{f} : X \to Y\}_{f\in F}$ (where $\tilde{f}(x) = w(f,x)$) is fragmented.*
(2) *Suppose F is a compact metrizable space, X a hereditarily Baire space, (e.g., Čech-complete, compact or Polish), and M separable and metrizable. Assume that we are given a map $w : F \times X \to M$ such that (i) $\tilde{x} : F \to M, f \mapsto w(f,x)$ is continuous for every $x \in X$, and (ii) $\tilde{f} : X \to M, x \mapsto w(f,x)$ is continuous for every $f \in Y$ for a dense subset Y of F. Then the family $\tilde{F}$ is HAE (hence, fragmented).*
(3) (A version of Osgood's theorem) *Let $f_n : X \to \mathbb{R}$ be a pointwise convergent sequence of continuous functions on a hereditarily Baire space X. Then $F := \{f_n\}_{n\in\mathbb{N}}$ is a fragmented family.*

Proof. (1): There exists a collection of uniformly continuous maps $\{\varphi_i : Y \to M_i\}_{i\in I}$ into metrizable uniform spaces M_i which generates the uniformity on Y. Now for every closed subset $A \subset X$ apply Namioka's joint continuity theorem to the separately continuous map $\varphi_i \circ w : F \times A \to M_i$ and take into account Lemma 2.2.1.

(2): Since every $\tilde{x} : F \to M$ is continuous, the natural map $j : X \to C(F,M)$, $j(x) = \tilde{x}$ is well defined. By assumption every closed nonempty subset $A \subset X$ is Baire. By [37, Proposition 2.4], $j|_A : A \to C(F,M)$ has a point of continuity, where $C(F,M)$ carries the sup-metric. Hence, $\tilde{F}_A = \{\tilde{f} \restriction_A : A \to M\}_{f\in F}$ is equicontinuous at some point $a \in A$. This implies that the family $\tilde{F}$ is HAE.

(3) : Follows from (2) applied to the evaluation map $w : F \times X \to \mathbb{R}$, where $F := \{f\} \cup \{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ with $f := \lim f_n$, the pointwise limit. □

For other properties of fragmented maps and fragmented families we refer to [61, 31, 33].

2.1. **Sensitivity conditions in dynamical systems.**

Definition 2.5. Let (X,τ) be a compact S-dynamical system endowed with its unique compatible uniform structure ξ. The set of translations $\tilde{S}$ can be treated as a family of functions $(X,\tau) \to (X,\xi)$.

We say that the S-system X is NS, HNS, HAE, E-wFr, E-Fr E-HAE whenever the family $\tilde{S}$ has the same property in the sense of Definition 2.1.

The notation for the corresponding classes is the same. This definition, in the case of HNS, gives exactly Definition 1.5. We will see below that tameness (Definition 1.8) is equivalent to E-wFr, and to E-Fr if X, in addition, is metrizable.

2.2. **Fragmentability and Banach spaces.**

2.2.1. *Asplund Banach spaces.* Recall that a Banach space V is an *Asplund space* if the dual of every separable linear subspace is separable.

In the following result the equivalence of (1), (2) and (3) is well known and (4) is a reformulation of (3) in terms of fragmented families.

Theorem 2.6. [67, 66] *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is an Asplund space.*
(2) *V^* has the Radon-Nikodým property.*
(3) *Every bounded subset A of the dual V^* is (weak*,norm)-fragmented.*
(4) *B is a fragmented family of real valued maps on the compactum B^*.*

For separable V, the assertion (4) can be derived from Lemma 2.4.2.

Reflexive spaces and spaces of the type $c_0(A)$ are Asplund. By [67] the Banach space $C(K)$ for compact K is Asplund iff K is a scattered compactum (see also Lemma 2.2.4). Namioka's joint continuity theorem implies that every weakly compact set in a Banach space is norm fragmented, [66]. This explains why every reflexive space is Asplund.

2.2.2. *Banach spaces not containing l_1.*

Definition 2.7. Let $f_n : X \to \mathbb{R}$ be a uniformly bounded sequence of functions on a *set* X. Following Rosenthal we say that this sequence is an l_1*-sequence* on X if there exists a real constant $a > 0$ such that for all $n \in \mathbb{N}$ and choices of real scalars $c_1, \dots, c_n$ we have

$$a \cdot \sum_{i=1}^{n} |c_i| \leq || \sum_{i=1}^{n} c_i f_i ||.$$

This is the same as requiring that the closed linear span in $l_\infty(X)$ of the sequence f_n be linearly homeomorphic to the Banach space l_1. In fact, in this case the map

$$l_1 \to l_\infty(X), \quad (c_n) \to \sum_{n \in \mathbb{N}} c_n f_n$$

is a linear homeomorphic embedding.

Definition 2.8. A sequence f_n of real valued functions on a set X is said to be *independent* if there exist real numbers $a < b$ such that

$$\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n \in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets P, M of $\mathbb{N}$.

Clearly every subsequence of an independent sequence is again independent.

Definition 2.9. A Banach space V is said to be *Rosenthal* if it does not contain an isomorphic copy of l_1.

Every Asplund space is Rosenthal (because $l_1^* = l_\infty$ is nonseparable).

Definition 2.10. Let X be a topological space. We say that a subset $F \subset C(X)$ is a *Rosenthal family* (for X) if F is norm bounded and the pointwise closure $\mathrm{cls}_p(F)$ of F in $\mathbb{R}^X$ consists of fragmented maps, that is, $\mathrm{cls}_p(F) \subset \mathcal{F}(X)$.

The following useful result synthesizes some known results. It is based on results of Rosenthal [77], Talagrand [81, Theorem 14.1.7] and van Dulst [21]. In [33, Prop. 4.6] we show why eventual fragmentability of F can be included in this list.

Theorem 2.11. *Let X be a compact space and $F \subset C(X)$ a bounded subset. The following conditions are equivalent:*

(1) *F does not contain an independent subsequence.*
(2) *F does not contain a subsequence equivalent to the unit basis of l_1.*
(3) *Each sequence in F has a pointwise convergent subsequence in $\mathbb{R}^X$.*
(4) *F is a Rosenthal family for X.*
(5) *F is an eventually fragmented family.*

We will also need some characterizations of Rosenthal spaces.

Theorem 2.12. *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is a Rosenthal Banach space.*
(2) (E. Saab and P. Saab [79]) *Each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B^*. Equivalently, $B^{**} \subset \mathcal{F}(B^*)$.*
(3) *B is a* Rosenthal family *for the weak* compact unit ball B^*.*
(4) *B is an eventually fragmented family of maps on B^*.*

Condition (2) is a reformulation (in terms of fragmented maps) of a criterion from [79] which was originally stated in terms of the point of continuity property. The equivalence of (1), (3) and (4) follows from Theorem 2.11.

2.3. **More properties of fragmented families.** Here we demonstrate a general principle: the fragmentability of a family of continuous maps defined on a compact space is "countably-determined". The following theorem is inspired by a result of Namioka [66, Theorem 3.4].

Theorem 2.13. *Let $F = \{f_i : X \to Y\}_{i \in I}$ be a bounded family of **continuous** maps from a compact (not necessarily metrizable) space (X, τ) into a pseudometric space (Y, d). The following conditions are equivalent:*

(1) *F is a fragmented family of functions on X.*
(2) *Every* countable *subfamily K of F is fragmented.*
(3) *For every countable subfamily K of F the pseudometric space $(X, \rho_{K,d})$ is separable, where*

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} d(f(x_1), f(x_2)).$$

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Let K be a countable subfamily of F. Consider the natural map

$$\pi : X \to Y^K, \pi(x)(f) := f(x).$$

By (2), K is a fragmented family. Thus by Lemma 2.2.6 the map π is (τ, μ_K)-fragmented, where μ_K is the uniformity of d-uniform convergence on $Y^K := \{f : K \to (Y, d)\}$. Then the map π is also (τ, d_K)-fragmented, where d_K is the pseudometric on Y^K defined by

$$d_K(z_1, z_2) := \sup_{f \in K} d(z_1(f), z_2(f)).$$

Since d is bounded, $d_K(z_1, z_2)$ is finite and d_K is well-defined. Denote by (X_K, τ_p) the subspace $\pi(X) \subset Y^K$ in pointwise topology. Since $K \subset C(X)$, the induced map $\pi_0 : X \to X_K$ is a continuous map onto the compact space (X_K, τ_p). Denote by $i : (X_K, \tau_p) \to (Y^K, d_K)$ the inclusion map. So, $\pi = i \circ \pi_0$, where the map π is (τ, d_K)-fragmented. Then by Lemma 2.2.7 we obtain that i is (τ_p, d_K)-fragmented. It immediately follows that the identity map $id : (X_K, \tau_p) \to (X_K, d_K)$ is (τ_p, d_K)-fragmented.

Since K is countable, $(X_K, \tau_p) \subset Y^K$ is metrizable. Therefore, (X_K, τ_p) is second countable (being a metrizable compactum). Now, since d_K is a pseudometric on Y^K, and $id : (X_K, \tau_p) \to (X_K, d_K)$ is (τ_p, d_K)-fragmented, we can apply Lemma 2.2.5. It directly implies that the set X_K is a separable subset of (Y^K, d_K). This means that $(X, \rho_{K,d})$ is separable.

(3) $\Rightarrow$ (1) : Suppose that F is not fragmented. Thus, there exists a non-empty closed subset $A \subset X$ and an $\varepsilon > 0$ such that for each non-empty open subset $O \subset X$ with $O \cap A \neq \emptyset$ there is some $f \in F$ such that $f(O \cap A)$ is not ε-small in (Y, d). Let V_1 be an arbitrary non-empty relatively open subset in A. There are $a, b \in V_1$ and $f_1 \in F$ such that $d(f_1(a), f_1(b)) > \varepsilon$. Since f_1 is continuous we

can choose relatively open subsets V_2, V_3 in A with $\mathrm{cls}\,(V_2 \cup V_3) \subset V_1$ such that $d(f_1(x), f_1(y)) > \varepsilon$ for every $(x,y) \in V_2 \times V_3$.

By induction we can construct a sequence $\{V_n\}_{n\in\mathbb{N}}$ of non-empty relatively open subsets in A and a sequence $K := \{f_n\}_{n\in\mathbb{N}}$ in F such that:

(i) $V_{2n} \cup V_{2n+1} \subset V_n$ for each $n \in \mathbb{N}$;
(ii) $d(f_n(x), f_n(y)) > \varepsilon$ for every $(x,y) \in V_{2n} \times V_{2n+1}$.

We claim that $(X, \rho_{K,d})$ is not separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f\in K} d(f(x_1), f(x_2)).$$

In fact, for each *branch*

$$\alpha := V_1 \supset V_{n_1} \supset V_{n_2} \supset \cdots$$

where for each $i, n_{i+1} = 2n_i$ or $2n_i + 1$, by compactness of X one can choose an element

$$x_\alpha \in \bigcap_{i\in\mathbb{N}} \mathrm{cls}\,(V_{n_i}).$$

If $x = x_\alpha$ and $y = x_\beta$ come from different branches, then there is an $n \in \mathbb{N}$ such that $x \in \mathrm{cls}\,(V_{2n})$ and $y \in \mathrm{cls}\,(V_{2n+1})$ or (vice versa). In any case it follows from (ii) and the continuity of f_n that $d(f_n(x), f_n(y)) \geq \varepsilon$, hence $\rho_{K,d}(x,y) \geq \varepsilon$. Since there are uncountably many branches we conclude that A and hence also X are not $\rho_{K,d}$-separable. □

Definition 2.14. [17, 61] Let X be a compact space and $F \subset C(X)$ a norm bounded family of continuous real valued functions on X. Then F is said to be an *Asplund family for X* if for every countable subfamily K of F the pseudometric space $(X, \rho_{K,d})$ is separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f\in K} |f(x_1) - f(x_2)|.$$

An Asplund family for a compact space X can be viewed, in view of [17, Lemma 1.5.3], as a particular case of the more general concept of an *Asplund set* in the Banach space $C(X)$.

Corollary 2.15. *Let X be a compact space and $F \subset C(X)$ a norm bounded family of continuous real valued functions on X. Then F is fragmented if and only if F is an Asplund family for X.*

Theorem 2.16. *Let $F = \{f_i : X \to Y\}_{i\in I}$ be a family of continuous maps from a compact (not necessarily metrizable) space (X, τ) into a pseudouniform space (Y, μ). Then F is fragmented if and only if every countable subfamily $A \subset F$ is fragmented.*

Proof. The proof can be reduced to Theorem 2.13. Every pseudouniform space can be uniformly approximated by pseudometric spaces. Using Lemma 2.2.1 we can suppose that (Y, μ) is pseudometrizable; i.e. there exists a pseudometric d such that $\mathrm{unif}(d) = \mu$. Moreover, replacing d by the uniformly equivalent pseudometric $\frac{d}{1+d}$ we can suppose that $d \leq 1$. □

3. Banach representations of dynamical systems and of functions

A *representation* of a semigroup S (with identity element e) on a Banach space V is a co-homomorphism $h : S \to \Theta(V)$, where $\Theta(V) := \{T \in L(V) : \ ||T|| \leq 1\}$, with $h(e) = id_V$. Here $L(V)$ is the space of continuous linear operators $V \to V$ and id_V is the identity operator. This is equivalent to the requirement that $h : S \to \Theta(V)^{op}$ be a monoid homomorphism, where $\Theta(V)^{op}$ is the opposite semigroup of $\Theta(V)$. If $S = G$ is a group then $h(G) \subset \mathrm{Iso}(V)$, where $\mathrm{Iso}(V)$ is the group of all linear isometries from V onto V.

Since $\Theta(V)^{op}$ acts from the right on V and from the left on V^* we sometimes write vs for $h(s)(v)$ and $s\varphi$ for $h(s)^*(\varphi)$, where $h(s)^* : V^* \to V^*$ is the adjoint of $h(s) : V \to V$. Then $\langle vs, \varphi\rangle = \langle v, s\varphi\rangle$. In this way we get the *dual action* (induced by h)

$$S \times V^* \to V^*, \ (s\varphi)(v) := \varphi(vs) = \langle vs, \varphi\rangle.$$

Definition 3.1. [61, 31] Let X be an S-space.

(1) A *representation* of (S,X) on a Banach space V is a pair
$$h : S \to \Theta(V), \quad \alpha : X \to V^*$$
where $h : S \to \Theta(V)$ is a weakly continuous representation (co-homomorphism) of semigroups and $\alpha : X \to V^*$ is a weak* continuous bounded S-mapping with respect to the dual action $S \times V^* \to V^*,\ (s\varphi)(v) := \varphi(vs)$.

As usual, α is an "S-mapping" means that $\alpha(sx) = s\alpha(x)$ for every $(s,x) \in S \times X$. That is, the following diagram commutes

$$\begin{tikzcd} S \times X \arrow[r] \arrow[d, "h"', shift right=3ex] \arrow[d, "\alpha", shift left=2ex] & X \arrow[d, "\alpha"] \\ \Theta^{op} \times V^* \arrow[r] & V^* \end{tikzcd}$$

We say that the representation (h,α) is *strongly continuous* if h is strongly continuous. *Faithful* will mean that α is a topological embedding.

(2) In particular, if in (1) $S := G$ is a group then, necessarily, $h(G)$ is a subgroup of $\mathrm{Iso}(V)$.

(3) If $\mathcal{K}$ is a subclass of the class of Banach spaces, we say that a dynamical system (S,X) is *(strongly) $\mathcal{K}$-representable* if there exists a weakly (respectively, strongly) continuous faithful representation of (S,X) on a Banach space $V \in \mathcal{K}$.

(4) A dynamical system (S,X) is said to be *(strongly) $\mathcal{K}$-approximable* if (S,X) can be embedded in a product of (strongly) $\mathcal{K}$-representable S-spaces.

(5) For a topological group G *$\mathcal{K}$-representability* will mean that there exists an embedding (equivalently, a co-embedding) of G into the group $\mathrm{Iso}(V)$ where $V \in \mathcal{K}$ and $\mathrm{Iso}(V)$ is endowed with the strong operator topology.

Note that when X is compact then every weak-star continuous $\alpha : X \to V^*$ is necessarily bounded.

Remark 3.2. The notion of a reflexively (Asplund) representable compact dynamical system is a dynamical version of the purely topological notion of an *Eberlein* (respectively, *Radon-Nikodým* (RN)) compactum, in the sense of Amir and Lindenstrauss (respectively, in the sense of Namioka).

Definition 3.3. We say that a dynamical system (S,X) is: (i) *Eberlein* when it is reflexively representable; (ii) Radon-Nikodým (RN) when it is Asplund representable; and, (iii) *Weakly Radon-Nikodým* (WRN), when it is Rosenthal representable.

As a word of warning note that this does not mean that X, as a compactum, has the same property (Eberlein, RN or WRN). In fact *every* compact metrizable space is obviously (uniformly) Eberlein as it can be embedded in a separable Hilbert space. However, it is not hard to find metric dynamical systems which distinguish the classes of dynamical systems mentioned above.

Example 3.4.

(1) Let $X = [0,1]$ be the unit interval. Consider the cascade $(\mathbb{Z},X)$ generated by the homeomorphism $\sigma(x) = x^2$. Then $(\mathbb{Z},X)$, as a dynamical system, is RN and not Eberlein. To see this observe that the pair of sequences $x_n = 1 - \frac{1}{n}$ in $X = [0,1]$ and $\sigma^m \in G$ with $\sigma^m(x) = x^{2^m}$ does not satisfy DLP. The corresponding limits are 0 and 1. This means that $(\mathbb{Z},X)$ is not Eberlein (Remark 1.2 and Theorem 3.6). The enveloping semigroup $E(\mathbb{Z},X)$ is metrizable, being homeomorphic to the two-point compactification of $\mathbb{Z}$. Hence, by [37], $(\mathbb{Z},X)$ is RN. The sequence $\{\sigma^m : [0,1] \to [0,1]\}_{m\in\mathbb{N}}$ is a fragmented family which does not satisfy DLP.

(2) The Sturmian symbolic dynamical system (X,σ) from Example 7.1.3 is WRN but not RN. The sequence $\{\sigma^n : X \to X\}_{n\in\mathbb{N}}$ of (positive) iterations is an eventually fragmented but not a fragmented family.

(3) The natural action of the Polish group $H_+[0,1]$ of increasing homeomorphisms of $[0,1]$ on $[0,1]$ is tame but not HNS; this system is WRN but not RN. See for example [35] or Section 8.1 below. The family $H_+[0,1]$, as a family of functions, (or any dense subsequence in $H_+[0,1]$) is eventually fragmented but not fragmented.

(4) The Bernoulli shift $(\mathbb{Z}, \{0,1\}^{\mathbb{Z}})$ is not WRN (equivalently, not tame). In fact, the enveloping semigroup of this system can be identified with $\beta\mathbb{Z}$. Now use the dynamical version of BFT dichotomy (Theorem 1.7). Another way to see that the shift system is not tame is by the well known fact (see for example [82]) that the sequence of projections on the Cantor cube

$$\{\pi_m : \{0,1\}^{\mathbb{N}} \to \{0,1\}\}_{m \in \mathbb{N}}$$

is independent. Hence by Theorem 2.11 this family fails to be eventually fragmented.

For a compact space X we denote by $H(X)$ the topological group of self-homeomorphisms of X endowed with the compact open topology.

Lemma 3.5. *Let X be a compact G-space, where G is a topological subgroup of $H(X)$. Assume that (h, α) is a strongly continuous faithful representation of (G, X) on a Banach space V (that is, $h : G \to \mathrm{Iso}(V)$ is strongly continuous and $\alpha : X \to (V^*, w^*)$ is an embedding, see Definition 3.1). Then $h : G \to \mathrm{Iso}(V)$ is a topological group embedding.*

Proof. Recall that the strong operator topology on $\mathrm{Iso}(V)^{op}$ is identical with the compact open topology inherited from the action of this group on the weak-star compact unit ball (B^*, w^*). □

We recall the following theorems.

Theorem 3.6. ([35, Theorem 11] and [61]) *Let X be a compact S-system.*

(1) *(S, X) is a tame system iff (S, X) is Rosenthal-approximable (that is, approximable by WRN systems).*
(2) *(S, X) is a HNS system iff (S, X) is Asplund-approximable (that is, approximable by RN systems).*
(3) *(S, X) is a WAP (continuous) system iff (S, X) is reflexively-approximable (that is, approximable by Eberlein systems).*

(*) If X is metrizable then in (1), (2) and (3) "approximable" can be replaced by "representable" and the corresponding Banach space can be assumed to be separable.

Remark 3.7. If the given action $S \times X \to X$ is jointly continuous then the representations in Theorem 3.6 can be assumed to be strongly continuous. The same is true for the results below: Theorems 3.11, 3.12 4.5, 4.6, 4.8, 4.9 and 8.9.1.

Of course not every $\mathcal{K}$-approximable system is $\mathcal{K}$-representable. For example, (S, X) with $S := \{e\}$ and $X := [0,1]^{\mathbb{R}}$ is clearly reflexively-approximable but not reflexively-representable (because X, as a compactum, is not Eberlein).

Remark 3.8. (subrepresentations)

(1) Let (h, α) be a representation of an S-space X on V. For every S-invariant closed linear subspace $V_0 \subset V$ we have a natural induced representation (h_0, α_0) on V_0. Indeed, $h_0 : S \to \Theta(V_0)^{op}$ is uniquely defined using the induced action $V_0 \times S \to V_0$. Define α_0 as the composition $\alpha_0 := i^* \circ \alpha : X \to V_0^*$, where $i^* : V^* \to V_0^*$ is the adjoint of the embedding $i : V_0 \hookrightarrow V$.
(2) If in (1), V_0 separates the points of $\alpha(X)$ then α_0 is an injection iff α is injective. So, if X is compact then we get a faithful representation (in the sense of Definition 3.1). This argument implies that if a compact *metrizable* S-system X is strongly representable on V then it is strongly representable on a *separable* Banach subspace V_0 of V.

Next we deal with the representability of families of real-valued functions on compact systems. This topic is closely related to the "smallness" of the family F in terms of its pointwise closure in the spirit of Theorem 2.11.

Definition 3.9. Let $\mathcal{K} \subset \mathbf{Ban}$ be a subclass of Banach spaces.

(1) Let X be an S-space and (h, α) a representation of (S, X) on a Banach space V. Let $F \subset C(X)$ be a bounded S-invariant family of continuous functions on X and $\nu : F \to V$ a

bounded mapping. We say that (ν, h, α) is an *F-representation* of the triple (F, S, X) if ν is an S-mapping (i.e., $\nu(fs) = \nu(f)s$ for every $(f, s) \in F \times S$) and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall\, f \in F, \quad \forall\, x \in X.$$

In other words, the following diagram commutes

$$\begin{tikzcd} F \times X \arrow[r] \arrow[d, "\nu"'] \arrow[d, "\alpha"] & \mathbb{R} \arrow[d, "id_{\mathbb{R}}"] \\ V \times V^* \arrow[r] & \mathbb{R} \end{tikzcd} \tag{3.1}$$

(2) We say that a family $F \subset C(X)$ is *$\mathcal{K}$-representable* if there exists a Banach space $V \in \mathcal{K}$ and a representation (ν, h, α) of the triple (F, S, X). A function $f \in C(X)$ is said to be *$\mathcal{K}$-representable* if the orbit fS is $\mathcal{K}$-representable.
Note that we do not assume in (1) or (2) that α is injective. However, when the family F separates points on X it follows that the map α is necessarily an injection.
(3) In particular, we obtain the definitions of reflexively, Asplund and Rosenthal representable families of functions on dynamical systems.

Clearly, every bounded S-invariant family $F \subset C(X)$ on an S-system X is Banach representable via the canonical representation on $V = C(X)$.

Remark 3.10. In some particular cases $\mathcal{K}$-approximability and $\mathcal{K}$-representability are equivalent. This happens for example in the following important cases:

(1) X is metrizable and $\mathcal{K}$ is closed under countable l_2-sums;
(2) (S, X_f) is a cyclic system, $\mathcal{K}$ is closed under subspaces and f is $\mathcal{K}$-representable.
(3) (S, X_f) is a cyclic system and $\mathcal{K}$ is one of the following classes: reflexive, Asplund, Rosenthal.
The assertions (1) and (2) are quite straightforward. In order to verify (3) (by reducing it to the case (2)) we note that every WAP (Asplund, tame) function $f \in C(X)$ is reflexive (Asplund, Rosenthal) representable. See Theorems 4.5 and 4.8 below.

The following result is very close to [35, Theorem 12]. It serves as a central ingredient in the proof of Theorem 3.6. We give a sketch of the proof just to illustrate the definitions.

Theorem 3.11. (Small families of functions)
Let X be a compact S-space and $F \subset C(X)$ a norm bounded S-invariant subset of $C(X)$.

(1) *(F, S, X) admits a Rosenthal representation iff F is an eventually fragmented family iff* $\mathrm{cls}_p(F) \subset \mathcal{F}(X)$.
(2) *(F, S, X) admits an Asplund representation iff F is a fragmented family iff the envelope* $\mathrm{cls}_p(F)$ *of F is a fragmented family.*
(3) *(F, S, X) admits a reflexive representation iff* $\mathrm{cls}_p(F) \subset C(X)$ *iff F has DLP on X.*

Proof. The "if part" in the proof of (1) and (2) is based on a dynamical version of the well known DFJP construction, [11]. See the proof of Theorem 12 in [35, Sect. 8]. The "only if part" is a direct consequence of the characterizations of Asplund and Rosenthal spaces in terms of fragmented and eventually fragmented families, Theorems 2.6.4 and 2.12.4.

For the "if part" in (3) we use [61, Theorem 4.11] which we reformulate here in terms of Definition 3.9. Let $w : F \times X \to \mathbb{R}$ be a separately continuous bounded map with compact spaces F and X such that S acts on X (from left) and on F (from right) via separately continuous actions such that $w(f, sx) = w(fs, x)$. Assume that F (regarded as a (bounded) family of maps $X \to \mathbb{R}$) separates the points of X. Then according to [61, Theorem 4.11] there exists a reflexive Banach space V and a faithful representation (ν, h, α) of the triple (F, S, X).

Another important part of the proof of (3) (which is close to Grothendieck's double limit theorem) is Theorem A.4 in [9]. It asserts that for every compact space X a bounded family $F \subset C(X)$ has the Double Limit Property on X iff $\mathrm{cls}_p(F) \subset C(X)$.

For the "only if" part in (3) observe that if V is a reflexive Banach space then every bounded subset F of the dual V^* has DLP on every bounded subset $X \subset V$ (Theorem 1.3). $\square$

Note that, in Definition 3.9, when the family F separates points on X it follows that the map α is necessarily an injection. In view of this remark, Theorem 3.11 implies Theorem 3.6 and also the essential part in the following useful result.

Theorem 3.12. *A compact S-system X is RN (WRN, Eberlein) iff there exists a bounded S-invariant X-separating family $F \subset C(X)$ which is fragmented (resp.: eventually fragmented, DLP).*

Proof. The "if part" follows from Theorem 3.11.

The "only if part" follows by considering the family F of functions on $X \subset V^*$ induced by B_V taking into account the characterization properties of reflexive, Asplund and Rosenthal Banach spaces. See Theorems 1.3.3, 2.6.4 and 2.12.4, respectively. □

Lemma 3.13. *Let X be a compact S-system and $f \in C(X)$.*

(1) *$f \in \mathrm{WAP}(X)$ if and only if fS has DLP.*
(2) *$f \in \mathrm{Asp}(X)$ if and only if fS is fragmented.*
(3) *$f \in \mathrm{Tame}(X)$ if and only if fS is eventually fragmented.*

Proof. (1) This was mentioned in Definition 1.1.

(2) If $f \in \mathrm{Asp}(X)$ then it comes from a HNS factor $q : X \to Y$ and $f' \circ q = f$ for some $f' \in C(Y)$. Since (S, Y) is HNS, the family of translations $\{\tilde{s} : Y \to Y\}$ is fragmented. It follows that $f'S$ and hence also fS are fragmented.

(3) If $f \in \mathrm{Tame}(X)$ then it comes from a tame factor $q : X \to Y$ and $f' \circ q = f$ for some $f' \in C(Y)$. Since (S, Y) is tame, every $p \in E(S, Y)$ is a tame function. Therefore, $f'E(S, Y) = \mathrm{cls}_p(f'S) \subset \mathcal{F}(Y)$. That is, $f'S$ is a Rosenthal family. Then $f'S$ is eventually fragmented by Theorem 2.11. Now, Lemma 2.2.7 guarantees that fS is also eventually fragmented.

The "only if" parts of (2) and (3) come from Theorems 4.5 and 4.8. □

3.1. **The purely topological case.** Note that the definitions and results of Section 3 (for instance, Theorem 3.11) make sense in the purely topological setting, for trivial $S = \{e\}$ actions, yielding characterizations of "small families" of functions, and of RN, WRN and Eberlein compact spaces.

The "only if" parts of these results, in the cases of Eberlein and RN compact spaces (with trivial actions), are consequences of known characterizations of reflexive and Asplund Banach spaces. The Eberlein case yields a well-known result: a compact space X is Eberlein iff there exits a pointwise compact subset $Y \subset C(X)$ which separates the points of X. The RN case is very close to results of Namioka [66] (up to some reformulations). The case of WRN spaces seems to be new.

Recall that by a classical result of Benjamini-Rudin-Wage [6], (which answered a question posed by Lindenstrauss) continuous surjective maps preserve the class of Eberlein compact spaces. The same is true for uniformly Eberlein (that is, Hilbert representable) compacta. Recently, Aviles and Koszmider [2] proved that this is not the case for the class RN of Asplund representable compacta, answering a long standing open problem posed by Namioka [66]. In view of these results the following question seems to be interesting.

Question 3.14. Is the class of WRN compact spaces closed under continuous onto maps (in the realm of Hausdorff spaces) ?

Note that the class of WRN compact spaces is closed under countable products (see [35]).

Remark 3.15.

(1) $\beta\mathbb{N}$ is an example of a compact space which is not WRN. We thank Stevo Todorčević for communicating to us a beautiful proof of this fact which is presented as an appendix to this work. See Theorem 10.1 in Section 10.
(2) Below, in Corollary 8.8, we show that the two arrows space is WRN. This space is not RN by a result of Namioka [66, Example 5.9].

Theorem 3.16. *A metric dynamical S-system X whose enveloping semigroup $E(S, X)$ is a WRN compactum is tame.*

Proof. If $E(S, X)$ is a WRN compactum, then, by Todorčević's Theorem 10.1 X can not contain a copy of $\beta\mathbb{N}$. Hence, by the dynamical version of the BFT Theorem 1.7, the system (S, X) is tame. □

Question 3.17. Is the converse true ? I.e. is it true that the enveloping semigroup of a metric tame system is necessarily a WRN compactum ?

One can also derive the following corollary of Theorem 3.11.

Corollary 3.18. *Let X be a compact space and $F \subset C(X)$ a bounded family. If F has DLP on X then F is a fragmented family.*

Proof. Theorem 3.11.3 guarantees that there exists a representation of $(F, \{e\}, X)$ on a reflexive space V. Since V is Asplund we easily obtain by Theorem 2.6.4 that F is fragmented. Alternatively, one can derive the latter statement from Lemma 2.4.1. □

Remark 3.19. For trivial actions, Theorem 3.11.3 yields the following corollary. Let $w : F \times X \to \mathbb{R}$ be a separately continuous bounded function with compact spaces F and X. Then there exists a reflexive Banach space V and weakly continuous maps $\nu : F \to V$, $\alpha : X \to V^*$ such that $\langle \nu(f), \alpha(x) \rangle = w(f, x)$. This is a result of Raynaud [74, Prop. 1.1] who generalized an earlier result by Krivine and Maurey [55, Theorem II.3] which dealt with metrizable X and F. One can refine these results (even in the general action setting) as follows. The fundamental DFJP-factorization construction from [11] has an "isometric modification" [56]. Taking into account this modification (which is compatible with our S-space setting) note that in Theorem 3.11 we can prove more. Namely, if the given family $F \subset C(X)$ is *bounded by the constant* 1, then we can assume that $\nu(F) \subset B$ and $\alpha(X) \subset B^*$. Hence the following sharper (than the diagram 3.1 in Definition 3.9) diagram commutes:

$$\begin{tikzcd} F \times X \arrow[r] \arrow[d, "\nu"', shift right] \arrow[d, "\alpha", shift left] & {[-1,1]} \arrow[d, "id"] \\ B \times B^* \arrow[r] & {[-1,1]} \end{tikzcd}$$

For more details see [35].

4. WAP, HNS and tame systems

4.1. **WAP systems and functions.** Besides the three equivalent conditions in Definition 1.1.1 (for being a WAP function) we can now say a bit more. In the proof below we twice use the following observation: if a continuous function on a compact S-system X comes from an S-factor $q : X \to Y$ with $\tilde{f} \in C(X)$, $f = \tilde{f} \circ q$, then fS has DLP on X iff $\tilde{f}S$ has DLP on Y. This means, by Definition 1.1.1, that $f \in \mathrm{WAP}(X)$ iff $\tilde{f} \in \mathrm{WAP}(Y)$.

Lemma 4.1. *Let X be a compact S-space and $f \in C(X)$. The following conditions are equivalent:*

1. *$f \in \mathrm{WAP}(X)$.*
2. *f is reflexively representable.*
3. *The cyclic S-space X_f is reflexively representable (i.e., Eberlein).*
4. *f comes from an Eberlein factor.*
5. *f comes from a WAP factor.*

Proof. (1) ⇔ (2): Use Theorem 3.11.3.

The implications (2) ⇒ (3) and (5) ⇒ (1) follow from the above observation. In the first case use also the facts that because $\tilde{f}S$ has DLP on X_f and $\tilde{f}S$ separates the points of X_f, we can apply Theorem 3.6.

(3) ⇒ (4): Is trivial because f comes from the factor $X \to X_f$.

(4) ⇒ (5): Every Eberlein system is WAP by Theorem 3.6.3. □

4.2. HNS systems and functions.

Definition 4.2. [31, 35] A compact S-system X is *hereditarily non-sensitive* (HNS, in short) if one of the following equivalent conditions are satisfied:

(1) For every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε-small for every $s \in S$.
(2) The family of translations $\widetilde{S} := \{\tilde{s} : X \to X\}_{s \in S}$ is a fragmented family of maps.
(3) $E(S, X)$ is a fragmented family of maps from X into itself.

The equivalence of (1) and (2) is evident from the definitions. Clearly, (3) implies (2) because $\widetilde{S} \subset E(S, X)$. As to the implication (2) $\Rightarrow$ (3), observe that the pointwise closure of a fragmented family is again a fragmented family, [33, Lemma 2.8].

Remark 4.3. Note that if $S = G$ is a group then in Definition 4.2.1 one can consider only *G-invariant* closed subsets A (see the proof of [31, Lemma 9.4]).

Lemma 4.4.
(1) *The class of HNS S-dynamical systems is closed under subsystems, products and factors.*
(2) *A compact dynamical S-system X is HNS iff* $\mathrm{Asp}(X) = C(X)$.

Proof. For group actions this was proved in [31, Sect. 2]. The same method works for general semigroup actions. □

We collect here some characterizations of Asplund functions and HNS systems. For continuous group actions they can be found in [31, 37, 33].

Theorem 4.5. *Let X be a compact S-space and $f \in C(X)$. The following conditions are equivalent:*
(1) $f \in \mathrm{Asp}(X)$ *(that is, f comes from a HNS-factor).*
(2) *fS is a fragmented family.*
(3) *f is Asplund representable.*
(4) *The cyclic S-system X_f is RN.*
(5) *f comes from an RN-factor.*

Proof. (1) $\Rightarrow$ (2): This is Lemma 3.13.1.

(2) $\Leftrightarrow$ (3): By Theorem 3.11.2.

(2) $\Rightarrow$ (4): Let $\tilde{f}$ be the function on the cyclic S-factor X_f such that $f = \tilde{f} \circ \pi_f$ (see Remark 1.11). By Lemma 2.2.7, fS is a fragmented family on X iff $\tilde{f}S$ is a fragmented family on X_f. In this case, since $\tilde{f}S$ separates the points of X_f, we can conclude by Theorem 3.12 that X_f is Asplund-representable.

(4) $\Rightarrow$ (5): This is trivial because f comes from the factor $X \to X_f$.

(5) $\Rightarrow$ (1): An Asplund-representable (that is, RN) system is Asplund-approximable, and therefore it is HNS by Theorem 3.6.2. □

Theorem 4.6. [31, 37, 35] *Let X be a compact S-space. The following conditions are equivalent:*
(1) (S, X) *is* HNS.
(2) The family $\tilde{S} \subset X^X$ of translations (equivalently, $E(S, X)$) is a fragmented family.
(3) For every countable subset $A \subset S$ the family $\tilde{A}$ is fragmented.
(4) (S, X) has sufficiently many representations on Asplund Banach spaces (that is, (S, X) is RN-approximable).
(5) $\mathrm{Asp}(X) = C(X)$.

If X is metrizable then each of the conditions above is equivalent also to any of the following conditions:

(6) The enveloping semigroup $E(X)$ of (S, X) is metrizable.
(7) X is RN (that is, (S, X) admits a faithful representation on a (separable) Asplund space).
(8) X is HAE.
(9) The pseudometric
$$d_S(x, y) := \sup\{d(sx, sy) : \quad s \in S\}$$
on X is separable (where d is a compatible metric on X).
(10) For every $f \in C(X)$ the pointwise closure $\mathrm{cls}_p(fS)$ is a metrizable subset of $\mathbb{R}^X$.

Proof. (1) $\Leftrightarrow$ (2): Directly from Definition 4.2.
(2) $\Leftrightarrow$ (3): Directly from Theorem 2.16.
(1) $\Leftrightarrow$ (4): Theorem 3.6.2.
(1) $\Leftrightarrow$ (5): Lemma 4.4.2.

If X is metrizable then
(1) $\Leftrightarrow$ (6): By [35, Theorem 7] (for group actions see [37]).
(1) $\Leftrightarrow$ (7): By Theorem 3.6.
(2) $\Leftrightarrow$ (8): Lemma 2.3.4.
(2) $\Leftrightarrow$ (9): The family $\tilde{S}$ of the corresponding translation maps is a fragmented family. This means that the natural map $X \to X^S$ is fragmented, where X^S carries the uniformity of d-uniform convergence. It then follows, by Lemma 2.2.5, that the image of X into X^S is separable. This exactly means that (X, d_S) is separable.
(2) $\Leftrightarrow$ (10): Apply [33, Lemma 4.4] to the family fS for every $f \in C(X)$. $\square$

4.3. **Tame systems and functions.** Recall that, by Definition 1.8, a compact dynamical S-system X is said to be tame if every $p \in E(X)$ is a fragmented map. For HNS systems, $E(X)$ is a fragmented family, hence, every HNS system is tame.

Lemma 4.7.
(1) *The class of tame S-systems is closed under subsystems, arbitrary products and factors.*
(2) *A compact S-dynamical system X is tame iff* $\mathrm{Tame}(X) = C(X)$.

Proof. For group actions this was proved in [33]. The same method works for general semigroup actions. $\square$

Theorem 4.8. *Let X be a compact S-space and $f \in C(X)$. The following conditions are equivalent:*
(1) $f \in \mathrm{Tame}(X)$.
(2) *fS does not contain an independent sequence.*
(3) $\mathrm{cls}_p(fS) \subset \mathcal{F}(X)$ *(i.e. the orbit fS is a Rosenthal family for X).*
(4) *fS is an eventually fragmented family.*
(5) *For every countable infinite subset $A \subset S$ there exists a countable infinite subset $A' \subset A$ such that the corresponding pseudometric*
$$\rho_{f,A'}(x,y) := \sup\{|f(gx) - f(gy)| : \quad g \in A'\}$$
on X is separable.
(6) *f is Rosenthal representable.*
(7) *The cyclic S-space X_f is Rosenthal representable (i.e., WRN).*
(8) *f comes from a WRN-factor.*

Proof. (1) $\Rightarrow$ (4): Lemma 3.13.3.
The equivalence of (2), (3) and (4) follows from Theorem 2.11.
(4) $\Leftrightarrow$ (5): Use Theorem 2.13.
(4) $\Leftrightarrow$ (6): By Theorem 3.11.1.
(4) $\Rightarrow$ (7): Let $\tilde{f}$ be the function on the cyclic S-factor X_f such that $f = \tilde{f} \circ \pi_f$ (Remark 1.11). By Lemma 2.2.7, fS is an eventually fragmented family on X iff $\tilde{f}S$ is an eventually fragmented family on X_f. In this case, since $\tilde{f}S$ separates the points of X_f we can conclude by Theorem 3.12 that X_f is Asplund-representable.
(7) $\Rightarrow$ (8): Is trivial because f comes from the factor $X \to X_f$.
(8) $\Rightarrow$ (1): By Theorem 3.6.1 every WRN system is tame. $\square$

Theorem 4.9. [31, 37, 35] *Let X be a compact dynamical S-system. The following conditions are equivalent:*
(1) *(S, X) is tame (that is, every $p \in E(X)$ is a fragmented map).*
(2) *(S, X) is E-wFr.*
(3) *(S, X) has sufficiently many representations on Rosenthal Banach spaces (that is, it is WRN-approximable).*

(4) $\mathrm{Tame}(X) = C(X)$.

If X is metrizable then each of the conditions above is equivalent also to any of the following conditions:

(5) *Every $p \in E(X)$ is a Baire 1 map.*
(6) *(S,X) is WRN (that is, (S,X) admits a faithful representation on a (separable) Rosenthal space).*
(7) *(S,X) is E-HAE.*
(8) *(S,X) is E-Fr.*
(9) *For every infinite subset $A \subset S$ there exists a (countable) infinite subset $A' \subset A$ such that the corresponding pseudometric*

$$\rho_{A'}(x,y) := sup\{d(sx,sy) : \ s \in A'\}$$

on X is separable, where d is a compatible metric on X.

Proof. (1) ⇔ (3): Theorem 3.6.1.

(1) ⇔ (4): Lemma 4.7.2.

(1) ⇒ (2): We modify a proof from [31, Prop. 4.1] where we dealt with jointly continuous group actions. By Definitions 2.1, 2.5 and Lemma 2.3.2, it suffices to show that (S_1, X) is E-wFr for every *countable* subsemigroup $S_1 \subset S$. Clearly, (S_1, X) remains tame. Since S_1 is countable, metric factors separate points on X (and therefore (S_1, X) can be embedded in a product of metrizable S_1-systems). Now we use *cyclic compactifications* (Remark 1.11). Observe that since S_1 is countable, every cyclic S_1-factor X_f of X is metrizable for every $f \in C(X)$.

It is easy to see that the class of E-wFr systems is closed under products and passing to subsystems. So, it suffices to show that every metrizable S_1-factor M of X is E-wFr. By Lemma 4.7.1, (S_1, M) is also tame. Therefore $E(S_1, M)$ is a Rosenthal compactum. Hence, the topological space $E(S_1, M)$ has the Fréchet property, [10]. Let $j : S_1 \to E(S_1, M), s \mapsto \tilde{s}$ be the canonical Ellis compactification. Then, given an infinite subset $j(L) \subset j(S_1)$, there exists a countable subset $\{t_n\}_{n \in \mathbb{N}}$ in L such that $K := \{j(t_n)\}_{n \in \mathbb{N}}$ is infinite and the sequence $j(t_n)$ converges in $E(S_1, M)$. We apply Lemma 2.4.3 to conclude that K, as a family of maps from M into itself, is fragmented, so that (S_1, M) is indeed E-wFr.

(2) ⇒ (1): Let (S, X) be E-wFr. In order to show that X is tame it suffices to prove, by Lemma 4.7.2, that $f \in \mathrm{Tame}(X)$ for every $f \in C(X)$. By our assumption (2) $\tilde{S}$ is E-Fr. Since $f : (X, d) \to \mathbb{R}$ is uniformly continuous we obtain that fS is E-wFr. By Lemma 2.3.3 we conclude that fS is eventually fragmented. Thus, f is tame by Theorem 4.8.

If X is metrizable then

(1) ⇔ (5): Lemma 2.2.2.

(1) ⇔ (6): By Theorem 3.6.

(2) ⇔ (7) ⇔ (8): By Lemma 2.3.5.

(8) ⇒ (9): For every countable infinite subset $A \subset S$ there exists a countable infinite subset $A' \subset A$ such that A' is a fragmented family. This means, by Lemma 2.2.6, that the induced map $X \to X^{A'}$ is fragmented, where $X^{A'}$ carries the uniformity of uniform convergence. Since X is second countable, Lemma 2.2.5 implies that the image of X into $X^{A'}$ is separable. This exactly means that $(X, \rho_{A'})$ is separable.

(9) ⇒ (4): Let $f \in C(X)$. Since $f : (X, d) \to \mathbb{R}$ is uniformly continuous it is easy to see that the map $1_X : (X, \rho_{A'}) \to (X, \rho_{f,A'})$ is uniformly continuous. This implies that $(X, \rho_{f,A'})$ is also separable. By Theorem 4.8 we conclude that $f \in \mathrm{Tame}(X)$. □

Since every E-Fr is E-wFr, the implication (8) ⇒ (1) holds for every (not necessarily metric) compact system.

5. A characterization of tame symbolic systems

5.1. **Symbolic systems.** The classical *Bernoulli shift system* is defined as the cascade $(\mathbb{Z}, \Omega)$, where $\Omega := \{0,1\}^{\mathbb{Z}}$. We have the natural $\mathbb{Z}$-action on Ω induced by the left T-shift:

$$\mathbb{Z} \times \Omega \to \Omega, \quad T^m(\omega_i)_{i \in \mathbb{Z}} = (\omega_{i+m})_{i \in \mathbb{Z}} \quad \forall (\omega_i)_{i \in \mathbb{Z}} \in \Omega, \quad \forall m \in \mathbb{Z}.$$

We equip the compact metric space Ω with the standard metric

$$d((x_i),(y_i)) = \frac{1}{1 + min\{|k| : x_k \neq y_k, \quad k \in \mathbb{Z}\}}.$$

More generally, for a discrete monoid S and a finite alphabet $A := \{0, 1, \dots, n\}$ the compact space A^S is an S-space under the action

$$S \times \Omega \to \Omega, \quad (s\omega)(t) = \omega(ts), \ \omega \in A^S, \quad s, t \in S.$$

A closed S-invariant subset $X \subset A^S$ defines a subsystem (S, X). Such systems are called *subshifts* or *symbolic dynamical systems*. For a nonempty $L \subseteq S$ define the natural projection

$$\pi_L : A^S \to A^L.$$

The compact zero-dimensional space A^S is metrizable iff S is countable (and, in this case, A^S is homeomorphic to the Cantor set).

It is easy to see that the full shift system $\Omega = A^S$ (hence also every subshift) is *uniformly expansive*. This means that there exists an entourage $\varepsilon_0 \in \mu$ in the natural uniform structure of A^S such that for every distinct $\omega_1 \neq \omega_2$ in Ω one can find $s \in S$ with $(s\omega_1, s\omega_2) \notin \varepsilon_0$. Indeed, take

$$\varepsilon_0 := \{(u, v) \in \Omega \times \Omega : \ u(e) = v(e)\},$$

where e, as usual, is the neutral element of S.

Lemma 5.1. *Every symbolic dynamical S-system $X \subset \Omega = A^S$ is cyclic (Definition 1.10).*

Proof. It suffices to find $f \in C(X)$ such that the orbit fS separates the points of X since then, by the Stone-Weierstrass theorem, (S, X) is isomorphic to its cyclic S-factor (S, X_f). The family

$$\{\pi_s : X \to A = \{0, 1, \dots, n\} \subset \mathbb{R}\}_{s \in S}$$

of basic projections clearly separates points on X and we let $f := \pi_e : X \to \mathbb{R}$. Now observe that $fS = \{\pi_s\}_{s \in S}$. □

Proposition 5.2. [61, Prop. 7.15] *Every scattered compact jointly continuous S-space X is RN.*

Proof. A compactum X is scattered iff $C(X)$ is Asplund, [67]. Now use the canonical S-representation $S \to \Theta(V)_s, \alpha : X \hookrightarrow B^*$ of (S, X) on the Asplund space $V := C(X)$. □

The following result recovers and extends [31, Sect. 10] and [61, Sect. 7].

Theorem 5.3. *For a discrete monoid S and a finite alphabet A let $X \subset A^S$ be a subshift. The following conditions are equivalent:*

1. *(S, X) is Asplund representable (that is, RN).*
2. *(S, X) is HNS.*
3. *X is scattered.*

If, in addition, X is metrizable (e.g., if S is countable) then each of the conditions above is equivalent also to:

4. *X is countable.*

Proof. (1) $\Rightarrow$ (2): Follows directly from Theorem 3.6.2.

(2) $\Rightarrow$ (3): Let μ be the natural uniformity on X and μ_S the (finer) uniformity of uniform convergence on $X \subset X^S$ (we can treat X as a subset of X^S under the assignment $x \mapsto \hat{x}$, where $\hat{x}(s) = sx$). If X is HNS then the family $\tilde{S}$ is fragmented. This means that X is μ_S-fragmented. As we already mentioned, every subshift X is uniformly S-expansive. Therefore, μ_S coincides with the discrete uniformity μ_Δ on X (the largest possible uniformity on the *set* X). Hence, X is also μ_Δ-fragmented. This means, by Lemma 2.2.4, that X is a scattered compactum.

(3) $\Rightarrow$ (1): Use Proposition 5.2.

If X is metrizable then

(4) $\Leftrightarrow$ (3): A scattered compactum is metrizable iff it is countable. □

Every zero-dimensional compact $\mathbb{Z}$-system X can be embedded into a product $\prod X_f$ of (cyclic) subshifts (X_f (where, one may consider only continuous functions $f : X \to \{0,1\}$) of the Bernoulli system $\{0,1\}^{\mathbb{Z}}$.

For more information about countable (that is, HNS) subshifts see e.g. [80] and [12].

Problem 5.4. Find a nice characterization for WAP (necessarily, countable) $\mathbb{Z}$-subshifts.

Next we consider tame subshifts.

Theorem 5.5. *Let X be a subshift of $\Omega = A^S$. The following conditions are equivalent:*

1. *(S,X) is a tame system.*
2. *For every infinite subset $L \subseteq S$ there exists an infinite subset $K \subseteq L$ and a countable subset $Y \subseteq X$ such that*
$$\pi_K(X) = \pi_K(Y).$$
That is,
$$\forall x = (x_s)_{s\in S} \in X,\ \exists y = (y_s)_{s\in S} \in Y \quad \textit{with} \quad x_k = y_k \ \ \forall k \in K.$$
3. *For every infinite subset $L \subseteq S$ there exists an infinite subset $K \subseteq L$ such that $\pi_K(X)$ is a countable subset of A^K.*
4. *(S,X) is Rosenthal representable (that is, WRN).*

Proof. (1) $\Leftrightarrow$ (2): As in the proof of Lemma 5.1 define $f := \pi_e \in C(X)$. Then X is isomorphic to the cyclic S-space X_f. By Theorem 4.9, (S,X) is a tame system iff $C(X) = \mathrm{Tame}(X)$. By Lemma 5.1, $C(X) = \mathcal{A}_f$, so we have only to show that $f \in \mathrm{Tame}(X)$.

By Theorem 4.8, $f := \pi_e : X \to \mathbb{R}$ is a tame function iff for every infinite subset $L \subset \mathbb{Z}$ there exists an infinite subset $K \subset L$ such that the corresponding pseudometric

$$\rho_{f,K}(x,y) := \sup_{k\in K}\{|(\pi_e)(kx) - (\pi_e)(ky)|\} = \sup_{k\in K}\{|x_k - y_k|\}$$

on X is separable. The latter assertion means that there exists a countable subset Y which is $\rho_{f,K}$-dense in X. Thus for every $x \in X$ there is a point $y \in Y$ with $\rho_{f,K}(x,y) < 1/2$. As the values of the function $f = \pi_0$ are in the set A, we conclude that $\pi_K(x) = \pi_K(y)$, whence

$$\pi_K(X) = \pi_K(Y).$$

The equivalence of (2) and (3) is obvious.

(1) $\Rightarrow$ (4): (S,X) is Rosenthal-approximable (Theorem 3.6.1). On the other hand, (S,X) is cyclic (Lemma 5.1). By Theorem 4.8.7 we can conclude that (S,X) is WRN.

(4) $\Rightarrow$ (1): Follows directly by Theorem 3.6.1. ☐

Remark 5.6. From Theorem 5.5 we can deduce the following peculiar fact. If X is a tame subshift of $\Omega = \{0,1\}^{\mathbb{Z}}$ and $L \subset \mathbb{Z}$ an infinite set, then there exist an infinite subset $K \subset L$, $k \geq 1$, and $a \in \{0,1\}^{2k+1}$ such that $X \cap [a] \neq \emptyset$ and $\forall x, x' \in X \cap [a]$ we have $x|_K = x'|_K$. Here $[a] = \{z \in \{0,1\}^{\mathbb{Z}} : z(j) = a(j),\ \forall |j| \leq k\}$. In fact, since $\pi_K(X)$ is a countable closed set it contains an isolated point, say w, and then the open set $\pi_K^{-1}(w)$ contains a subset $[a] \cap X$ as required.

5.2. **Tame and HNS subsets of $\mathbb{Z}$.** We say that a subset $D \subset \mathbb{Z}$ is *tame* if the characteristic function $\chi_D : \mathbb{Z} \to \mathbb{R}$ is a tame function on the group $\mathbb{Z}$. That is, when this function *comes* from a pointed compact tame $\mathbb{Z}$-system (X, x_0). Analogously, we say that D is *HNS* (or *Asplund*), *WAP*, or *Hilbert* if $\chi_D : \mathbb{Z} \to \mathbb{R}$ is an Asplund, WAP or Hilbert function on $\mathbb{Z}$, respectively. By basic properties of the *cyclic system* $X_D := \mathrm{cls}\{\chi_D \circ T^n : n \in \mathbb{Z}\} \subset \{0,1\}^{\mathbb{Z}}$ (see Remark 1.11), the subset $D \subset \mathbb{Z}$ is tame (Asplund, WAP) iff the associated subshift X_D is tame (Asplund, WAP).

Surprisingly it is not known whether $X_f := \mathrm{cls}\{f \circ T^n : n \in \mathbb{Z}\} \subset \mathbb{R}^{\mathbb{Z}}$ is a Hilbert system when $f : \mathbb{Z} \to \mathbb{R}$ is a Hilbert function (see [40]). The following closely related question from [62] is also open: Is it true that Hilbert representable compact metric $\mathbb{Z}$-spaces are closed under factors?

Remark 5.7. The definition of WAP sets was introduced by Ruppert [78]. He has the following characterisation ([78, Theorem 4]):

$D \subset \mathbb{Z}$ is a WAP subset if and only if every infinite subset $B \subset \mathbb{Z}$ contains a finite subset $F \subset B$ such that the set

$$\cap_{b\in F}(b+D) \setminus \cap_{b\in B\setminus F}(b+D)$$

is finite. See also [30].

Theorem 5.8. *The following conditions are equivalent:*

(1) *$D \subset \mathbb{Z}$ is a tame subset (i.e., the associated subshift $X_D \subset \{0,1\}^{\mathbb{Z}}$ is tame).*
(2) *For every infinite subset $L \subseteq \mathbb{Z}$ there exists an infinite subset $K \subseteq L$ and a countable subset $Y \subseteq \beta\mathbb{Z}$ such that for every $x \in \beta\mathbb{Z}$ there exists $y \in Y$ such that*

$$n + D \in x \Longleftrightarrow n + D \in y \quad \forall n \in K$$

(treating x and y as ultrafilters on the set $\mathbb{Z}$).

Proof. By the universality of the greatest ambit $(\mathbb{Z}, \beta\mathbb{Z})$ it suffices to check when the function

$$f = \chi_{\overline{D}} : \beta\mathbb{Z} \to \{0,1\}, \ f(x) = 1 \Leftrightarrow x \in \overline{D},$$

the natural extension function of $\chi_D : \mathbb{Z} \to \{0,1\}$, is tame (in the usual sense, as a function on the compact cascade $\beta\mathbb{Z}$), where we denote by $\overline{D}$ the closure of D in $\beta\mathbb{Z}$ (a clopen subset). Applying Theorem 4.8 to f we see that the following condition is both necessary and sufficient: For every infinite subset $L \subseteq \mathbb{Z}$ there exists an infinite subset $K \subseteq L$ and a countable subset $Y \subseteq \beta\mathbb{Z}$ which is dense in the pseudometric space $(\beta\mathbb{Z}, \rho_{f,K})$. Now saying that Y is dense is the same as the requiring that Y be ε-dense for every $0 < \varepsilon < 1$. However, as f has values in $\{0,1\}$ and $0 < \varepsilon < 1$ we conclude that for every $x \in \beta\mathbb{Z}$ there is $y \in Y$ with

$$x \in n + \overline{D} \Longleftrightarrow y \in n + \overline{D} \quad \forall n \in K,$$

and the latter is equivalent to

$$n + D \in x \Longleftrightarrow n + D \in y \quad \forall n \in K.$$

□

Theorem 5.9. *The following conditions are equivalent:*

(1) *$D \subset \mathbb{Z}$ is an Asplund set (i.e., the associated subshift $X_D \subset \{0,1\}^{\mathbb{Z}}$ is Asplund).*
(2) *There exists a **countable** subset $Y \subseteq \beta\mathbb{Z}$ such that for every $x \in \beta\mathbb{Z}$ there exists $y \in Y$ such that*

$$n + D \in x \Longleftrightarrow n + D \in y \quad \forall n \in \mathbb{Z}.$$

Proof. Similar to Theorem 5.8 using Theorem 4.6. □

Example 5.10. $\mathbb{N}$ is an Asplund subset of $\mathbb{Z}$ which is not a WAP subset. In fact, let $X_{\mathbb{N}}$ be the corresponding subshift. Clearly $X_{\mathbb{N}}$ is homeomorphic to the two-point compactification of $\mathbb{Z}$, with $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$ as minimal subsets. Since a transitive WAP system admits a unique minimal set, we conclude that $X_{\mathbb{N}}$ is not WAP (see e.g. [27]). On the other hand, since $X_{\mathbb{N}}$ is countable we can apply Theorem 5.3 to show that it is HNS. Alternatively, using Theorem 5.9, we can take Y to be $\mathbb{Z} \cup \{p, q\}$, where we choose p and q to be any two non-principal ultrafilters such that p contains $\mathbb{N}$ and q contains $-\mathbb{N}$.

6. Entropy and null systems

We begin by recalling the basic definitions of topological (sequence) entropy. Let (X, T) be a cascade, i.e., a $\mathbb{Z}$-dynamical system, and $A = \{a_0 < a_1 < \ldots\}$ a sequence of integers. Given an open cover $\mathcal{U}$ define

$$h^A_{top}(T, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} N(\bigvee_{i=0}^{n-1} T^{-a_i}(\mathcal{U}))$$

The *topological entropy along the sequence* A is then defined by

$$h^A_{top}(T) = \sup\{h^A_{top}(T, \mathcal{U}) : \mathcal{U} \text{ an open cover of } X\}.$$

When the phase space X is zero-dimensional, one can replace open covers by clopen partitions. We recall that a dynamical system (T, X) is called *null* if $h^A_{top}(T) = 0$ for every infinite $A \subset \mathbb{Z}$. Finally when $Y \subset \{0,1\}^{\mathbb{Z}}$, and $A \subset \mathbb{Z}$ is a given subset of $\mathbb{Z}$, we say that Y *is free on* A or that A *is an interpolation set for* Y, if $\{y|_A : y \in Y\} = \{0,1\}^A$.

By a theorem of Kerr and Li [50] every null $\mathbb{Z}$-system is tame. From results of Glasner-Weiss [39] (for (1)) and Kerr-Li [51] (for (2) and (3)), the following results can be easily deduced. (See Propositions 3.9.2, 6.4.2 and 5.4.2 of [51] for the positive topological entropy, the untame, and the nonnull claims, respectively.)

Theorem 6.1.

(1) *A subshift $X \subset \{0,1\}^{\mathbb{Z}}$ has positive topological entropy iff there is a subset $A \subset \mathbb{Z}$ of positive density such that X is free on A.*
(2) *A subshift $X \subset \{0,1\}^{\mathbb{Z}}$ is not tame iff there is an infinite subset $A \subset \mathbb{Z}$ such that X is free on A.*
(3) *A subshift $X \subset \{0,1\}^{\mathbb{Z}}$ is not null iff for every $n \in \mathbb{N}$ there is a finite subset $A_n \subset \mathbb{Z}$ with $|A_n| \geq n$ such that X is free on A_n.*

Proof. We consider the second claim; the other claims are similar.

Certainly if there is an infinite $A \subset \mathbb{Z}$ on which X is free then X is not tame (e.g. use Theorem 5.5). Conversely, if X is not tame then, by Propositions 6.4.2 of [51], there exists a non diagonal IT pair (x, y). As x and y are distinct there is an n with, say, $x(n) = 0, y(n) = 1$. Since $T^n(x, y)$ is also an IT pair we can assume that $n = 0$. Thus $x \in U_0$ and $y \in U_1$, where these are the cylinder sets $U_i = \{z \in X : z(0) = i\}, i = 0, 1$. Now by the definition of an IT pair there is an infinite set $A \subset \mathbb{Z}$ such that the pair (U_0, U_1) has A as an independence set. This is exactly the claim that X is free on A. $\square$

The following theorem was proved (independently) by Huang [44], Kerr and Li [51], and Glasner [29]. See also Remark 9.5 below.

Theorem 6.2. (A structure theorem for minimal tame dynamical systems) *Let (G, X) be a tame minimal metrizable dynamical system with G an abelian group. Then:*

(1) *(G, X) is an almost one to one extension $\pi : X \to Y$ of a minimal equicontinuous system (G, Y).*
(2) *(G, X) is uniquely ergodic and the factor map π is, measure theoretically, an isomorphism of the corresponding measure preserving system on X with the Haar measure on the equicontinuous factor Y.*

Examples 6.3.

(1) According to Theorem 6.2 the Morse minimal system, which is uniquely ergodic and has zero entropy, is nevertheless not tame as it fails to be an almost 1-1 extension of its adding machine factor. We can therefore deduce that, a fortiori, it is not null.
(2) Let $L = IP\{10^t\}_{t=1}^{\infty} \subset \mathbb{N}$ be the IP-sequence generated by the powers of ten, i.e.
$$L = \{10^{a_1} + 10^{a_2} + \cdots + 10^{a_k} : 1 \leq a_1 < a_2 < \cdots < a_k\}.$$
Let $f = 1_L$ and let $X = \bar{\mathcal{O}}_T(f) \subset \{0,1\}^{\mathbb{Z}}$, where T is the shift on $\Omega = \{0,1\}^{\mathbb{Z}}$. The subshift (T, X) is not tame. In fact it can be shown that L is an interpolation set for X.
(3) Take u_n to be the concatenation of the words $a_{n,i}0^n$, where $a_{n,i},\ i = 1, 2, 3, \ldots, 2^n$ runs over $\{0,1\}^n$. Let $v_n = 0^{|u_n|}$, $w_n = u_n v_n$ and w_∞ the infinite concatenation $\{0,1\}^{\mathbb{N}} \ni w_\infty = w_1 w_2 w_3 \cdots$. Finally define $w \in \{0,1\}^{\mathbb{Z}}$ by $w(n) = 0$ for $n \leq 0$ and $w(n) = w_\infty(n)$. Then $X = \bar{\mathcal{O}}_T(w) \subset \{0,1\}^{\mathbb{Z}}$ is a countable subshift, hence HNS and a fortiori tame, but for an appropriately chosen sequence the sequence entropy of X is $\log 2$. Hence, X is not null. Another example with countable nonnull subshift can be found in [44, Example 5.12].
(4) In [43, Theorem 13.9] the authors show that for interval maps being tame is the same as being null.

Remark 6.4. Let $\sigma : [0,1] \to [0,1]$ be a continuous self-map on the closed interval. In an unpublished paper [53] the authors show that the enveloping semigroup $E(X)$ of the cascade ($\mathbb{N} \cup \{0\}$-system) $X = [0,1]$ is either metrizable or it contains a topological copy of $\beta\mathbb{N}$. The metrizable enveloping semigroup case occurs exactly when the system is HNS. This was proved in [37] for group actions but it remains true for semigroup actions, [35]. The other case occurs iff σ is Li-Yorke chaotic. Combining this result with Example 6.3.4 one gets: HNS = null = tame, for any cascade $([0,1],\sigma)$.

7. Some examples of tame functions and systems

The class of tame dynamical systems is quite large and contains the class of HNS (hence also of WAP) systems. Also, as was mentioned above, every null $\mathbb{Z}$-system is tame.

Example 7.1.

(1) In his paper [15] Ellis, following Furstenberg's classical work [22], investigates the projective action of $GL(n,\mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$. It follows from his results that the corresponding enveloping semigroup is not first countable. However, in a later work [1], Akin studies the action of $G = GL(n,\mathbb{R})$ on the sphere $\mathbb{S}^{n-1}$ and shows that here the enveloping semigroup is first countable (but not metrizable). It follows that the dynamical systems $D_1 = (G,\mathbb{P}^{n-1})$ and $D_2 = (G,\mathbb{S}^{n-1})$ are tame but not HNS. Note that $E(D_1)$ is Fréchet, being a continuous image of a first countable compact space, namely $E(D_2)$.

(2) (Huang [44]) An almost 1-1 extension $\pi : X \to Y$ of a minimal equicontinuous metric $\mathbb{Z}$-system Y with $X \setminus X_0$ countable, where $X_0 = \{x \in X : |\pi^{-1}\pi(x)| = 1\}$, is tame.

(3) (See [31]) Consider an irrational rotation $(\mathbb{T},R_\alpha)$. Choose $x_0 \in \mathbb{T}$ and split each point of the orbit $x_n = x_0 + n\alpha$ into two points $x_n^{\pm}$. This procedure results is a *Sturmian* (symbolic) dynamical system (X,σ) which is a minimal almost 1-1 extension of $(\mathbb{T},R_\alpha)$. Then $E(X,\sigma)\setminus \{\sigma^n\}_{n\in\mathbb{Z}}$ is homeomorphic to the two arrows space, a basic example of a non-metrizable Rosenthal compactum. It follows that $E(X,\sigma)$ is also a Rosenthal compactum. Hence, (X,σ) is tame but not HNS (by Theorem 1.7).

(4) Let P_0 be the set $[0,c)$ and P_1 the set $[c,1)$; let z be a point in $[0,1)$ (identified with $\mathbb{T}$) via the rotation R_α we get the binary bisequence u_n, $n \in \mathbb{Z}$ defined by $u_n = 0$ when $R_\alpha^n(z) \in P_0, u_n = 1$ otherwise. These are called *Sturmian like codings*. With $c = 1-\alpha$ we retrieve the previous example. For example, when $\alpha := \frac{\sqrt{5}-1}{2}$ and $c = 1-\alpha$ the corresponding sequence, computed at $z = 0$, is called the *Fibonacci bisequence*.

Motivated by the Example 7.1.4 we next present a new class of generalized Sturmian systems.

Example 7.2 (A class of generalized Sturmian systems). Let $\alpha = (\alpha_1,\dots,\alpha_d)$ be a vector in $\mathbb{R}^d$, $d \ge 2$ with $1,\alpha_1,\dots,\alpha_d$ independent over $\mathbb{Q}$. Consider the minimal equicontinuous dynamical system (R_α, Y), where $Y = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (the d-torus) and $R_\alpha y = y + \alpha$. Let D be a small closed d-dimensional ball in $\mathbb{T}^d$ and let $C = \partial D$ be its boundary, a $d-1$-sphere. Fix $y_0 \in \mathrm{int} D$ and let $X = X(D,y_0)$ be the symbolic system generated by the function

$$x_0 \in \{0,1\}^{\mathbb{Z}} \text{ defined by } x_0(n) = \chi_D(R_\alpha^n y_0), \qquad X = \overline{\mathcal{O}_\sigma x_0} \subset \{0,1\}^{\mathbb{Z}},$$

where σ denotes the shift transformation. This is a well known construction and it is not hard to check that the system (σ,X) is minimal and admits (R_α,Y) as an almost 1-1 factor:

$$\pi : (\sigma,X) \to (R_\alpha,Y).$$

Theorem 7.3. *There exists a ball $D \subset \mathbb{T}^d$ as above such that the corresponding symbolic dynamical system (σ,X) is tame.*

Proof. **1.** First we show that a sphere $C \subset [0,1)^d \cong \mathbb{T}^d$ can be chosen so that for every $y \in \mathbb{T}^d$ the set $(y + \{n\alpha : n \in Z\}) \cap C$ is finite. We thank Benjamin Weiss for providing the following proof of this fact.

(1) For the case $d = 2$ the argument is easy. If A is any countable subset of the square $[0,1)\times[0,1)$ there are only a countable number of circles that contain three points of A. These circles have some countable collection of radii. Take any circle with a radius which is different from all of them and no translate of it will contain more than two points from the set A. Taking $A = \{n\alpha : n \in \mathbb{Z}\}$ we obtain the required circle.

(2) We next consider the case $d = 3$, which easily generalizes to the general case $d \ge 3$. What we have to show is that there can not be infinitely many points in

$$A = \{(n\alpha_1 - [n\alpha_1], \alpha_2 - [n\alpha_2], \alpha_3 - [n\alpha_3]) : n \in \mathbb{Z}\}$$

that lie on a plane. For if that is the case, we consider all 4-tuples of elements from the set A that do not lie on a plane to get a countable set of radii for the spheres that they determine. Then taking a sphere with radius different from that collection we obtain our required sphere. In fact, if a sphere contains infinitely many points of A and no 4-tuple from A determines it then they all lie on a single plane.

So suppose that there are infinitely many points in A whose inner product with a vector $v = (z, x, y)$ is always equal to 1. This means that there are infinitely many equations of the form:

$$(*) \qquad z\alpha_1 + x\alpha_2 + y\alpha_3 = 1/n + z[n\alpha_1]/n + x[n\alpha_2]/n + y[n\alpha_3]/n.$$

Subtract two such equations with the second using m much bigger than n so that the coefficient of y cannot vanish. We can express $y = rz + sx + t$ with r, s and t rational. This means that we can replace (*) by

$$(**) \qquad z\alpha_1 + x\alpha_2 + y\alpha_3 = 1/n + t[n\alpha_3]/n + z([n\alpha_1]/n + r[n\alpha_3]/n) + x([n\alpha_2]/n + s[n\alpha_3]/n).$$

Now r, s and t have some fixed denominators and (having infinitely many choices) we can take another equation like (**) where n (and the corresponding r, s, t) is replaced by some much bigger k, then subtract again to obtain an equation of the form $x = pz + q$ with p and q rational. Finally one more step will show that z itself is rational. However, in view of (*), this contradicts the independence of $1, \alpha_1, \alpha_2, \alpha_3$ over $\mathbb{Q}$ and our proof is complete.

2. Next we show that for C as above

> for every converging sequence $n_i\alpha$, say $n_i\alpha \to \beta \in \mathbb{T}^d \cong E(R_\alpha, \mathbb{T}^d)$, there exists a subsequence $\{n_{i_j}\}$ such that for every $y \in \mathbb{T}^d$, $y + n_{i_j}\alpha$ is either eventually in the interior of D or eventually in its exterior.

Clearly we only need to consider points $y \in C - \beta$. Renaming we can now assume that $n_i\alpha \to 0$ and that $y \in C$. Passing to a subsequence if necessary we can further assume that the sequence of unit vectors $\frac{n_i\alpha}{\|n_i\alpha\|}$ converges,

$$\frac{n_i\alpha}{\|n_i\alpha\|} \to v_0 \in \mathbb{S}^{d-1}.$$

In order to simplify the notation we now assume that C is centered at the origin. For every point $y \in C$ where $\langle y, v_0\rangle \neq 0$ we have that $y + n_i\alpha$ is either eventually in the interior of D or eventually in its exterior. On the other hand, for the points $y \in C$ with $\langle y, v_0\rangle = 0$ this is not necessarily the case. In order to deal with these points we need a more detailed information on the convergence of $n_i\alpha$ to β. At this stage we consider the sequence of orthogonal projections of the vectors $n_i\alpha$ onto the subspace $V_1 = \{u \in \mathbb{R}^d : \langle u, v_0\rangle = 0\}$, say $u_i = \mathrm{proj}_{v_0}(n_i\alpha) \to u = \mathrm{proj}_{v_0}(\beta)$. If it happens that eventually $u_i = 0$ this means that all but a finite number of the $n_i\alpha$'s are on the line defined by v_0 and our required property is certainly satisfied [1]. Otherwise we choose a subsequence (again using the same index) so that

$$\frac{u_i}{\|u_i\|} \to v_1 \in \mathbb{S}^{d-2}.$$

Again (as we will soon explain) it is not hard to see that for points $y \in C \cap V_1$ with $\langle y, v_1\rangle \neq 0$ we have that $y + n_i\alpha$ is either eventually in the interior of D or eventually in its exterior. For points $y \in C \cap V_1$ with $\langle y, v\rangle = 0$ we have to repeat this procedure. Considering the subspace $V_2 = \{u \in V_1 : \langle u, v_1\rangle = 0\}$, we define the sequence of projections $u'_i = \mathrm{proj}_{v_1}(u_i) \in V_2$ and pass to a further subsequence which converges to a vector v_2

$$\frac{u'_i}{\|u'_i\|} \to v_2 \in \mathbb{S}^{d-3}.$$

Inductively this procedure will produce an **ordered orthonormal basis** $\{v_0, v_1, v_2, \ldots, v_{d-1}\}$ for $\mathbb{R}^d$ and a final subsequence (which for simplicity we still denote as n_i) such that

> for each $y \in \mathbb{T}^d$, $y + n_i\alpha$ is either eventually in the interior of D or it is eventually in its exterior.

[1] Actually this possibility can not occur, as is shown in the first step of the proof.

This is clear for points $y \in \mathbb{T}^d$ such that $y + \beta \notin C$. Now suppose we are given a point y with $y + \beta \in C$. We let k be the first index with $\langle y + \beta, v_k \rangle \neq 0$. As $\{v_0, v_1, v_2, \dots, v_{d-1}\}$ is a basis for $\mathbb{R}^d$ such k exists. We claim that the sequence $y + n_i\alpha$ is either eventually in the interior of D or it is eventually in its exterior. To see this consider the affine hyperplane which is tangent to C at $y + \beta$ (which contains the vectors $\{v_0, \dots, v_{k-1}\}$). Our assumption implies that the sequence $y + n_i\alpha$ is either eventually on the opposite side of this hyperplaine from the sphere, in which case it certainly lies in the exterior of D, or it eventually lies on the same side as the sphere. However in this latter case it can not be squeezed in between the sphere and the tangent hyperplane, as this would imply $\langle y + \beta, v_k \rangle = 0$, contradicting our assumption. Thus it follows that in this case the sequence $y + n_i\alpha$ is eventually in the interior of D.

3. Let now p be an element of $E(\sigma, X)$. We choose a **net** $\{n_\nu\} \subset \mathbb{Z}$ with $\sigma^{n_\nu} \to p$. It defines uniquely an element $\beta \in E(Y) \cong \mathbb{T}^d$ so that $\pi(px) = \pi(x) + \beta$ for every $x \in X$. Taking a subnet if necessary we can assume that the net $\frac{\beta - n_\nu\alpha}{\|\beta - n_\nu\alpha\|}$ converges to some $v_0 \in S^{d-1}$. And, as above, proceeding by induction, we assume likewise that all the corresponding limits $\{v_0, \dots, v_{k-1}\}$ exist.

Next we choose a **sequence** $\{n_i\}$ such that $n_i\alpha \to \beta$, $\frac{\beta - n_i\alpha}{\|\beta - n_i\alpha\|} \to v_0$ etc., We conclude that $\sigma^{n_i} \to p$. Thus every element of $E(\sigma, X)$ is obtained as a limit of a sequence in $\mathbb{Z}$ and is therefore of Baire class 1. □

Remark 7.4. From the proof we see that the elements of $E(\sigma, X) \setminus \mathbb{Z}$ can be parametrized by the set $\mathbb{T}^d \times \mathcal{F}$, where $\mathcal{F}$ is the collection of ordered orthonormal bases for $\mathbb{R}^d$, $p \mapsto (\beta, \{v_0, \dots, v_{d-1}\})$.

For further recent results on tame systems see [73] and [3]. Below we will study the question whether some coding functions are tame.

Definition 7.5.

(1) Let $S \times X \to X$ be an action on a (not necessarily compact) space X, $f : X \to \mathbb{R}$ a bounded (not necessarily continuous) function, $h : S_0 \to S$ a homomorphism of semigroups and $z \in X$. The following function will be called a *coding function*:
$$m(f, z) : S_0 \to \mathbb{R}, \ s \mapsto f(h(s)z).$$

(2) When $S_0 = \mathbb{Z}^k$ and $f(X) = \{0, 1, \dots, d\}$ we say that f is a $(k, d+1)$-*code*. Every such code generates a point transitive subshift of $A^{\mathbb{Z}^k}$, where $A = \{0, 1, \dots, d\}$. In the particular case of the characteristic function $\chi_D : X \to \{0, 1\}$ for a subset $D \subset X$ and $S_0 = \mathbb{Z}$ we get a $(1, 2)$-code, i.e. a binary function $m(D, z) : \mathbb{Z} \to \{0, 1\}$ which generates a $\mathbb{Z}$-subshift of the Bernoulli shift on $\{0, 1\}^{\mathbb{Z}}$.

Question 7.6. When is a coding function tame ?

It follows from results in [33] that a coding bisequence $c : \mathbb{Z} \to \mathbb{R}$ (with $S_0 := \mathbb{Z}$) is tame iff it can be represented as a generalized matrix coefficient of a Rosenthal Banach space representation. That is, iff there exist: a Rosenthal Banach space V, a linear isometry $\sigma \in \mathrm{Iso}(V)$ and two vectors $v \in V$, $\varphi \in V^*$ such that
$$c_n = \langle \sigma^n(v), \varphi \rangle = \varphi(\sigma^n(v)) \qquad \forall n \in \mathbb{Z}.$$

We will see that many coding functions are tame, including some multidimensional analogues of Sturmian sequences. The latter are defined on the groups $\mathbb{Z}^k$ and instead of the characteristic function $f := \chi_D$ (with $D = [0, c)$) one we consider coloring of the space leading to shifts with finite alphabet. Here we give a precise definition which (at least in some partial cases) was examined in several papers. Regarding some dynamical and combinatorial aspects of coding functions (like multidimensional Sturmian sequences) see for example [8, 20, 73], and the survey paper [7].

Definition 7.7. (Multidimensional Sturmian sequences) Consider an arbitrary finite partition
$$\mathbb{T} = \cup_{i=0}^d [c_i, c_{i+1})$$
of $\mathbb{T}$ by the ordered d-tuple of points $c_0 = 0, c_1, \dots, c_d = 1$ and define the natural function
$$f : \mathbb{T} \to A := \{0, \dots, d\}, \quad f(t) = i \text{ iff } t \in [c_i, c_{i+1}).$$

Now for a given k-tuple $(\alpha_1,\dots,\alpha_k)\in\mathbb{T}^k$ and a given point $z\in\mathbb{T}$ consider the corresponding coding function

$$m(f,z):\mathbb{Z}^k\to\{0,\dots,d\}\quad (n_1,\dots,n_k)\mapsto f(z+n_1\alpha_1+\cdots+n_k\alpha_k).$$

We call such a sequence a *multidimensional* (k,d)*-Sturmian like sequence.*

Lemma 7.10 and Remark 7.11 below demonstrate the relevance of Definition 7.5 for coding functions. By Theorem 4.8 a continuous function $f:X\to\mathbb{R}$ on a compact S-system X is tame iff fS does not contain an independent sequence. This fact justifies our terminology in the following definition. For the definition of an independent sequence of functions see Definition 2.8.

Definition 7.8. Let S be a semigroup, X a (not necessarily compact) S-space and $f:X\to\mathbb{R}$ a bounded (not necessarily continuous) function. We say that f is of *tame-type* if the orbit fS of f in $\mathbb{R}^X$ does not contain an independent sequence.

An example of a Baire 1 tame-type function which is not tame, being discontinuous, is the characteristic function χ_D of an arc $D=[a,a+s)\subset\mathbb{T}$ defined on the system $(R_\alpha,\mathbb{T})$, where R_α is an irrational rotation of the circle $\mathbb{T}$. See Theorem 7.16.3.

Lemma 7.9.

1. *Let $q:X_1\to X_2$ be a map between sets and $\{f_n:X_2\to\mathbb{R}\}$ a bounded sequence of functions* (with no continuity assumptions on q and f_n). *If $\{f_n\circ q\}$ is an independent sequence on X_1 then $\{f_n\}$ is an independent sequence on X_2.*
2. *If q is onto then the converse is also true. That is $\{f_n\circ q\}$ is independent if and only if $\{f_n\}$ is independent.*
3. *Let $\{f_n\}$ be a bounded sequence of continuous functions on a topological space X. Let Y be a* dense *subset of X. Then $\{f_n\}$ is an independent sequence on X if and only if the sequence of restrictions $\{f_n|_Y\}$ is an independent sequence on Y.*

Proof. Claims (1) and (2) are straightforward.

(3) Since every f_n is continuous, for every pair of finite disjoint sets $P,M\subset\mathbb{N}$, the set

$$\bigcap_{n\in P} f_n^{-1}(-\infty,a)\cap\bigcap_{n\in M} f_n^{-1}(b,\infty)$$

is non-empty and open. Hence, each of them meets the dense set Y. As $f_n^{-1}(-\infty,a)\cap Y=f_n|_Y^{-1}(-\infty,a)$ and $f_n^{-1}(b,\infty)\cap Y=f_n|_Y^{-1}(b,\infty)$, this implies that $\{f_n|_Y\}$ is an independent sequence on Y.

Conversely if $\{f_n|_Y\}$ is an independent sequence on a subset $Y\subset X$ then by (1) (where q is the embedding $Y\hookrightarrow X$), $\{f_n\}$ is an independent sequence on X. $\square$

Lemma 7.10. *In terms of Definition 7.8 we have:*

1. *Every tame function $f:X\to\mathbb{R}$ is of tame-type.*
2. *If X is compact every continuous tame-type function f is tame.*
3. *Let $f\in\mathrm{RUC}(X)$; then $f\in\mathrm{Tame}(X)$ if and only if f is tame-type. Moreover, there exists an S-compactification $\nu:X\to Y$ where the action $S\times Y\to Y$ is continuous, Y is tame and $f=\tilde{f}\circ\nu$ for some $\tilde{f}\in C(Y)$.*
4. *Let G be a topological group and $f\in\mathrm{RUC}(G)$. Then f is tame if and only if fG does not contain an independent subsequence.*
5. *Let L be a discrete semigroup and $f:L\to\mathbb{R}$ a bounded function. Then $f\in\mathrm{Tame}(L)$ if and only if fL has no independent subsequence.*
6. *Let $h:L\to S$ be a homomorphism of semigroups, $S\times Y\to Y$ be an action (without any continuity assumptions) on a set Y and $f:Y\to\mathbb{R}$ be a bounded function such that fL does not contain an independent subsequence. Then for every point $y\in Y$ the corresponding coding function $m(f,y):L\to\mathbb{R}$ is tame on the discrete semigroup (L,τ_{discr}).*

Proof. For (1), (2) and (3) use Lemma 7.9 and Theorem 4.8.

For (3) consider the cyclic S-compactification $\nu:X\to Y=X_f$ (see Definition 1.10). Since $f\in\mathrm{RUC}(X)$ the action $S\times X_f\to X_f$ is jointly continuous (Remark 1.11). By the basic property of the cyclic compactification there exists a continuous function $\tilde{f}:X_f\to\mathbb{R}$ such that $f=\tilde{f}\circ\nu$.

Since f is of tame-type the family fS has no independent sequence. By Lemma 7.9 we conclude that also $\tilde{f}S$ has no independent sequence. This means, by Theorem 4.8, that $\tilde{f}$ is tame. Hence (by Definition 1.9) so is f. The converse follows from (1) (or, directly from Lemma 7.9).

(4) and (5) follow easily from (3) (with $X = G = L$) taking into account (1) and the fact that on a discrete semigroup L every bounded function $L \to \mathbb{R}$ is in $\mathrm{RUC}(L)$.

(6) By (5) it is enough to show that the coding function $f_0 := m(f, y)$ is of tame-type on L. That is, we have to show that f_0L has no independent subsequence. Define $q : L \to Y, s \mapsto h(s)y$. Then $f_0t = (ft) \circ q$ for every $t \in L$. If f_0t_n is an independent sequence for some sequence $t_n \in L$ then Lemma 7.9.1 implies that the sequence of functions ft_n on Y is independent. This contradicts the assumption that fL has no independent subsequence. $\square$

Remark 7.11. Regarding Lemma 7.10.6 note that the following conditions are equivalent:

(1) The coding function $f_0 = m(f, z) : L \to \mathbb{R}$ is a tame function on the semigroup (L, τ_{discr}).
(2) The cyclic system $X_{f_0} \subset A^L$, induced by the function f_0 with $A := f(h(L)z)$, is a tame L-system (note that in particular for the characteristic function $f = \chi_D$ of $D \subset X$ we get a (cyclic) subshift $X_{f_0} \subset \{0, 1\}^L$ induced by the function $f_0 := m(\chi_D, z)$).
(3) The orbit f_0L contains no independent sequence.

For (1) $\Leftrightarrow$ (2) observe that any cyclic space X_{f_0} is a subshift of A^L for any bounded function $f_0 : L \to A$. By the basic minimality property of the cyclic system X_{f_0} we obtain that it is a factor of any tame system (L, Y) which realizes f_0. Hence, f_0 is tame iff (L, X_{f_0}) is tame. Alternatively, use Theorem 4.8. For (1) $\Leftrightarrow$ (3) apply Lemma 7.10.4.

Let $f : X \to Y$ be a function between topological spaces. We denote by $cont(f)$ and $disc(f)$ the points of continuity and discontinuity for f respectively.

Definition 7.12. Let F be a family of functions on X. We say that F is:

(1) *Strongly almost continuous* if for every $x \in X$ we have $x \in cont(f)$ for almost all $f \in F$ (i.e. with the exception of at most a finite set of elements which may depend on x).
(2) *Almost continuous* if for every infinite (countable) subset $F_1 \subset F$ there exists an infinite subset $F_2 \subset F_1$ such that F_2 is strongly almost continuous on X.

Example 7.13.

(1) Let $G \times X \to X$ be a group action, $G_0 \leq G$ a subgroup and $f : X \to \mathbb{R}$ a function such that
$$G_0x \cap disc(f) \text{ and } St(x) \cap G_0 \ \text{ are } \ \text{finite } \forall x \in X,$$
where $St(x) \leq G$ is the stabilizer subgroup of x. Then the family fG_0 is strongly almost continuous (indeed, use the following equality $g^{-1}cont(f) = cont(fg),\ g \in G$).
(2) A coarse sufficient condition for (1) is: $disc(f)$ is finite and $St(x) \cap G_0$ is finite $\forall x \in X$.
(3) As a particular case of (2) we have the following example. For every compact group G and a function $f : G \to \mathbb{R}$ with finitely many discontinuities, fG_0 is strongly almost continuous for every subgroup G_0 of G.

Theorem 7.14. *Let X be a compact metric space and F a bounded family of real valued functions on X such that F is almost continuous. Further assume that:*

(*) *for every sequence $\{f_n\}_{n \in \mathbb{N}}$ in F there exists a subsequence $\{f_{n_m}\}_{m \in \mathbb{N}}$ and a countable subset $C \subset X$ such that $\{f_{n_m}\}_{m \in \mathbb{N}}$ pointwise converges on $X \setminus C$ to a function $\phi : X \setminus C \to \mathbb{R}$ such that $\phi \in \mathcal{B}_1(X \setminus C)$.*

Then there is no independent sequence in F.

Proof. Assuming the contrary let $\{f_n\}$ be an independent sequence in F. Then, by assumption, there exists a countable subset $C \subset X$ and a subsequence $\{f_{n_m}\}$ such that $\{f_{n_m} : X \setminus C \to \mathbb{R}\}$ pointwise converges on $X \setminus C$ to a function $\phi : X \setminus C \to \mathbb{R}$ such that $\phi \in \mathcal{B}_1(X \setminus C)$.

Independence is preserved by subsequences so this subsequence $\{f_{n_m}\}$ remains independent. For simplicity of notation assume that $\{f_n\}$ itself has the properties of $\{f_{n_m}\}$. Moreover we can suppose in addition, by Definition 7.12, that $\{f_n\}$ is strongly almost continuous.

By the definition of independence, for every pair of disjoint finite sets $P, M \subset \mathbb{N}$, there exist $a < b$ such that
$$\bigcap_{n\in P} A_n \cap \bigcap_{n\in M} B_n \neq \emptyset,$$
where $A_n := f_n^{-1}(-\infty, a)$ and $B_n := f_n^{-1}(b, \infty)$. Now define a tree of nested sets as follows:
$$\Omega_1 := X$$
$$\Omega_2 := \Omega_1 \cap A_1 = A_1 \qquad \Omega_3 := \Omega_1 \cap B_1 = B_1$$
$$\Omega_4 := \Omega_2 \cap A_2 \qquad \Omega_5 := \Omega_2 \cap B_2 \qquad \Omega_6 := \Omega_3 \cap A_2 \qquad \Omega_7 := \Omega_3 \cap B_2,$$
and so on. In general,
$$\Omega_{2^{n+1}+2k} := \Omega_{2^n+k} \cap A_{n+1}, \quad \Omega_{2^{n+1}+2k+1} := \Omega_{2^n+k} \cap B_{n+1}$$
for every $0 \leq k < 2^n$ and every $n \in \mathbb{N}$.

We obtain a system $\{\Omega_n\}_{n\in\mathbb{N}}$ which satisfies:
$$\Omega_{2n} \cup \Omega_{2n+1} \subset \Omega_n \ \text{ and } \Omega_{2n} \cap \Omega_{2n+1} = \emptyset \text{ for each } n \in \mathbb{N}.$$

Since $\{(A_n, B_n)\}$ is independent, every Ω_n is nonempty.

For every binary sequence $u = (u_1, u_2, \dots) \in \{0,1\}^{\mathbb{N}}$ we have the corresponding uniquely defined *branch*
$$\alpha_u := \Omega_1 \supset \Omega_{n_1} \supset \Omega_{n_2} \supset \cdots$$
where for each $i \in \mathbb{N}$ with $2^{i-1} \leq n_i < 2^i$ we have
$$n_{i+1} = 2n_i \text{ iff } u_i = 0 \text{ and } n_{i+1} = 2n_i + 1 \text{ iff } u_i = 1.$$

Let us say that $u, v \in \{0,1\}^{\mathbb{N}}$ are *essentially distinct* if they have infinitely many different coordinates. Equivalently, if u and v are in different cosets of the Cantor group $\{0,1\}^{\mathbb{N}}$ with respect to the subgroup H consisting of the binary sequences with finite support. Since H is countable there are uncountably many pairwise essentially distinct elements in the Cantor group. We choose a subset $T \in \{0,1\}^{\mathbb{N}}$ which intersects each coset in exactly one point. Clearly, $|T| = 2^{\omega}$. Now for every branch α_u where $u \in T$ choose one element
$$x_u \in \cap_{i\in\mathbb{N}} cl(\Omega_{n_i}).$$
Here we use the compactness of X which guarantees that $\cap_{i\in\mathbb{N}} cl(\Omega_{n_i}) \neq \emptyset$. We obtain a set $X_T := \{x_u : \ u \in T\} \subset X$ and a function $T \to X_T, \ u \mapsto x_u$.

Claim:

(1) The function $T \to X_T, \ u \mapsto x_u$ is injective. In particular, X_T is uncountable.
(2) $|\phi(x_u) - \phi(x_v)| \geq \varepsilon := b - a$ for every distinct $x_u, x_v \in X_T \setminus C$.

Proof of the Claim: (1) Let $u = (u_i)$ and $v = (v_i)$ are distinct elements in T. Denote by $\alpha_u := \{\Omega_{n_i}\}_{i\in\mathbb{N}}$ and $\alpha_v := \{\Omega_{m_i}\}_{i\in\mathbb{N}}$ the corresponding branches. Then, by the definition of X_T, we have the uniquely defined points $x_u \in \cap_{i\in\mathbb{N}} cl(\Omega_{n_i})$ and $x_v \in \cap_{i\in\mathbb{N}} cl(\Omega_{m_i})$ in X_T.

Since $u, v \in T$ are essentially distinct they have infinitely many different indices.

As $\{f_n\}$ is strongly almost continuous there exists a sufficiently large $t_0 \in \mathbb{N}$ such that the points x_u and x_v are both points of continuity of f_n for every $n \geq t_0$.

Now note that if $u_i \neq v_i$ then the sets Ω_{n_i+1} and Ω_{m_i+1} are contained (respectively) in the pair of disjoint sets $A_k := f_k^{-1}(-\infty, a)$ and $B_k := f_k^{-1}(b, \infty)$. Since u and v are essentially distinct we can assume that i is sufficiently large in order to ensure that $k \geq t_0$. That is, we necessarily have exactly one of the cases:
$$(a) \quad \Omega_{n_i+1} \subset A_k, \quad \Omega_{m_i+1} \subset B_k$$
or
$$(b) \quad \Omega_{n_i+1} \subset B_k, \quad \Omega_{m_i+1} \subset A_k.$$

For simplicity we only check the first case (a). For (a) we have $x_u \in \mathrm{cls}\,(\Omega_{n_i+1}) \subset \mathrm{cls}\,(f_k^{-1}(-\infty, a))$ and $x_v \in \mathrm{cls}\,(\Omega_{n_i+1}) \subset \mathrm{cls}\,(f_k^{-1}(b, \infty))$. Since $\{x_u, x_v\} \subset cont(f_n)$ are continuity points for every $n \geq t_0$ and since $k \geq t_0$ by our choice, we obtain $f_k(x_u) \leq a$ and $f_k(x_v) \geq b$. So, we can conclude that $|f_k(x_u) - f_k(x_v)| \geq \varepsilon := b - a$ for every $k \geq t_0$. In particular, x_u and x_v are distinct. This proves

(1). Furthermore, if our distinct $x_u, x_v \in X_T$ are in addition from $X_T \setminus C$ then $\lim f_k(x_u) = \phi(x_u)$ and $\lim f_k(x_v) = \phi(x_v)$. It follows that $|\phi(x_u) - \phi(x_v)| \geq \varepsilon$ and the condition (2) of our claim is also proved.

Since $X_T \setminus C$ is an uncountable subset of a Polish space X there exists an uncountable subset $Y \subset X_T \setminus C$ such that any point of y is a condensation point in $X_T \setminus C$ (this follows from the proof of Cantor-Bendixson theorem). For every open subset U in X with $U \cap Y \neq \emptyset$ we have $\mathrm{diam}(\phi(U \cap Y)) \geq \varepsilon$. This means that $\phi : X \setminus C \to \mathbb{R}$ is not fragmented. Since C is countable and X is compact metrizable the subset $X \setminus C$ is Polish. On Polish spaces fragmentability and Baire 1 property are the same for real valued functions (Lemma 2.2.2). So, we obtain that $\phi : X \setminus C \to \mathbb{R}$ is not Baire 1. This contradicts the assumption that $\phi \in \mathcal{B}_1(X \setminus C)$. □

Theorem 7.15. *Let X be a compact metric space, and F a bounded family of real valued functions on X such that F is almost continuous. Assume that $\mathrm{cls}_p(F) \subset \mathcal{B}_1(X)$. Then there is no independent sequence in F.*

Proof. Assuming the contrary let F has an independent subsequence $F_1 := \{f_n\}$. Since $\mathrm{cls}_p(F) \subset \mathcal{B}_1(X)$ we have $\mathrm{cls}_p(F_1) \subset \mathcal{B}_1(X)$. By the BFT theorem [10, Theorem 3F] the compactum $\mathrm{cls}_p(F_1)$ is a Fréchet topological space. Every (countably) compact Fréchet space is sequentially compact, [16, Theorem 3.10.31], hence $\mathrm{cls}_p(F_1)$ is sequentially compact. Therefore the sequence $\{f_n\}$ contains a pointwise convergent subsequence, say $f_n \to \phi \in \mathcal{B}_1(X)$. Now apply Theorem 7.14 to get a contradiction, taking into account that the properties almost continuity and independence are both inherited by subsequences. □

Theorem 7.16.

(1) *Let X be a compact metric S-space, S_0 a subsemigroup of S. Let $f : X \to \mathbb{R}$ be a bounded function such that $\mathrm{cls}_p(fS_0) \subset \mathcal{B}_1(X)$ and with fS_0 almost continuous. Then fS_0 has no independent subsequence.*

(2) *Let X be a compact metric G-space and $G_0 \leq G$ a subgroup of G such that (i) (G_0, X) is tame, (ii) for every $p \in E(G, X)$ and every $x \in X$ the preimage $p^{-1}(x)$ is countable and (iii) $G_0 \cap St(x)$ is finite for every $x \in X$. Suppose further that $f : X \to \mathbb{R}$ is a bounded function with only finitely many points of discontinuity. Then fG_0 has no independent subsequence.*

(3) *In particular, (2) holds in the following useful situation: $X = G$ is a compact metric group, $h : G_0 \to X$ is a homomorphism of groups and $f : X \to \mathbb{R}$ has finitely many discontinuities. Then fG_0 has no independent subsequence.*

(4) *In all the cases above (1), (2), (3), a coding function $m(f, z) : S_0 \to \mathbb{R}$ is a tame function on the discrete semigroup S_0 for every $z \in X$ and every homomorphism $h : S_0 \to S$ (or, $G_0 \to G$). Also the corresponding subshift (S_0, X_f) is tame.*

Proof. (1) Apply Theorem 7.15.

(2) First note that fG_0 is strongly almost continuous (Example 7.13.2). Now assuming the contrary fG_0 has an independent subsequence $\{fg_n\}$. Since (G_0, X) is tame we can assume, with no loss in generality, that the sequence $\{g_n\}$ converges to an element $p \in E(G_0, X)$. Then $fg_n(x)$ converges to $f(px)$ for every $x \in X \setminus C$, where $C := p^{-1}(disc(f))$ is a countable set. Since X is a tame system, $p : X \to X$ is a fragmented function. Then also the restricted function $p_0 : X \setminus C \to X$ is fragmented. Since $f : X \to \mathbb{R}$ is uniformly continuous we obtain that the composition $f \circ p_0 : X \setminus C \to \mathbb{R}$ is fragmented. Since C is countable, $X \setminus C$ is Polish. Therefore, by Lemma 2.2.2, $f \circ p_0 : X \setminus C \to \mathbb{R}$ is Baire 1. This however is in contradiction with Theorem 7.14.

(4) Follows from (1), Lemma 7.10 and Remark 7.11. □

Theorem 7.16.4 directly implies the following:

Example 7.17. For every irrational rotation α of the circle $\mathbb{T}$ and an arc $D := [a, b) \subset \mathbb{T}$ the function

$$\varphi_D := \mathbb{Z} \to \mathbb{R}, \quad n \mapsto \chi_D(n\alpha)$$

is a tame function on the group $\mathbb{Z}$. In particular, for $D := [-\frac{1}{4}, \frac{1}{4})$ we get that $\varphi_D(n) = \mathrm{sgn}\cos(2\pi n\alpha)$ is a tame function on $\mathbb{Z}$.

Theorem 7.18. *The multidimensional Sturmian (k,d)-sequences $\mathbb{Z}^k \to \{0,1,\dots,d\}$ (Definition 7.7) are tame.*

Proof. In terms of Definition 7.7 consider the homomorphism

$$h : \mathbb{Z}^k \to \mathbb{T}, \quad (n_1,\dots,n_k) \mapsto n_1\alpha_1 + \cdots + n_k\alpha_k.$$

The function f induced by a given partition $\mathbb{T} = \cup_{i=0}^{d} [c_i, c_{i+1})$

$$f : \mathbb{T} \to A := \{0,\dots,d\}, \quad f(t) = i \text{ iff } t \in [c_i, c_{i+1}).$$

has only finitely many discontinuities. Now Theorem 7.16 (items (3) and (4)) guarantees that the corresponding (k,d)-coding function $m(f,z) : \mathbb{Z}^k \to A \subset \mathbb{R}$ is tame for every $z \in \mathbb{T}$. □

At least for $(1,d)$-codes, Theorem 7.18, can be derived also from results of Pikula [73], and of Aujogue [3]. See also Remark 8.24.

Lemma 7.19. *Let $\pi : X \to Y$ be a continuous onto S-map of compact metric S-systems. Set*

$$X_0 := \{x \in X : \ |\pi^{-1}(\pi(x))| = 1\}.$$

Then the restriction map $\pi : X_0 \to Y_0$ is a topological homeomorphism of S-subspaces, where $Y_0 := \pi(X_0)$.

Proof. First observe that X_0 and Y_0 are S-invariant and that $\pi : X_0 \to Y_0$ is an onto, continuous, 1-1 map. For every converging sequence $y_n \to y$, where $y_n, y \in Y_0$ the preimage $\pi^{-1}(\{y\} \cup \{y_n\}_{n\in\mathbb{N}})$ is a compact subset of X. On the other hand, $\pi^{-1}(\{y\} \cup \{y_n\}) \subset X_0$ by the definition of X_0. It follows that the restriction of π to $\pi^{-1}(\{y\} \cup \{y_n\})$ is a homeomorphism. In particular, $\pi^{-1}(y_n)$ converges to $\pi^{-1}(y)$. □

Recall that a map $\pi : X \to Y$ as above is said to be an *almost one-to-one extension* if X_0 is a residual subset of X. As a corollary of Theorem 7.14 one can derive the following result which generalizes the above mentioned result of W. Huang from Example 7.1.2.

Theorem 7.20. *Let $\pi : X \to Y$ be a homomorphism of compact metric S-systems such that $X \setminus X_0$ is countable, where*

$$X_0 := \{x \in X : \ |\pi^{-1}(\pi(x))| = 1\}.$$

Assume that (S,Y) is tame and that the set $p^{-1}(y)$ is (at most) countable for every $p \in E(Y)$ and $y \in Y$ (e.g., this latter condition is always satisfied when Y is distal). Then (S,X) is also tame.

Proof. We have to show that every $f \in C(X)$ is tame. Assuming the contrary, suppose fS contains an independent sequence fs_n. Since Y is metrizable and tame, one can assume (by Theorem 1.7) that the sequence s_n converges pointwise to some element p of $E(S,Y)$. Consider the set $Y_0 \cap p^{-1}Y_0$, where $Y_0 = \pi(X_0)$. Since $p^{-1}(y)$ is countable for every $y \in Y \setminus Y_0$ it follows that $Y \setminus (Y_0 \cap p^{-1}Y_0)$ is countable. Therefore, by the definition of X_0 and the countability of $X \setminus X_0$, we see that $X \setminus \pi^{-1}(Y_0 \cap p^{-1}Y_0)$ is also countable. Now observe that the sequence $(fs_n)(x)$ converges for every $x \in \pi^{-1}(Y_0 \cap p^{-1}Y_0)$. In fact, if we denote $y = \pi(x)$ then $s_n y$ converges to py in Y. In fact we have $py \in Y_0$ (by the choice of x) and $s_n y \in Y_0$. By Lemma 7.19, $\pi : X_0 \to Y_0$ is an S-homeomorphism. So we obtain that $s_n x$ converges to $\pi^{-1}(py)$ in X_0. Since $f : X \to \mathbb{R}$ is continuous, $(fs_n)x$ converges to $f(\pi^{-1}(py))$ in $\mathbb{R}$. Every fs_n is a continuous function, hence so is also its restriction to $\pi^{-1}(Y_0 \cap p^{-1}Y_0)$. Therefore the limit function ϕ is Baire 1. Since C:=$X \setminus \pi^{-1}(Y_0 \cap p^{-1}Y_0)$ is countable and fs_n is an independent sequence, Theorem 7.14 provides the sought-after contradiction. □

8. Order preserving systems are tame

In this section all group actions are jointly continuous and representations of systems (and groups) are strongly continuous.

8.1. **Order preserving action on the unit interval.** Recall that for the group $G = H_+[0,1]$ (of orientation preserving self-homeomorphisms) the G-system $X = [0,1]$ with the obvious G-action is tame [35]. One way to see this is to observe that the enveloping semigroup of this dynamical system naturally embeds into the Helly compactum (and hence is a Rosenthal compactum). By Theorem 1.7, (G, X) is tame. By Theorem 4.9 this means that every $p \in E(X)$ is a Baire class 1 map. In fact we can say more. As every monotonic map $[0,1] \to [0,1]$ has at most countably many discontinuities, this holds also for every $p \in E(G, X)$.

We list here some other properties of $H_+[0,1]$.

Theorem 8.1. *Let $G := H_+[0,1]$. Then*

(1) (Pestov [71]) *G is extremely amenable.*
(2) [32] $\mathrm{WAP}(G) = \mathrm{Asp}(G) = \mathrm{SUC}(G) = \{constants\}$ *and every Asplund representation of G is trivial.*
(3) [35] *G is representable on a (separable) Rosenthal space.*
(4) (Uspenskij [84, Example 4.4]) *G is Roelcke precompact.*
(5) $\mathrm{UC}(G) \subset \mathrm{Tame}(G)$*, that is, the Roelcke compactification of G is tame.*
(6) $\mathrm{Tame}(G) \neq \mathrm{UC}(G)$*.*
(7) $\mathrm{Tame}(G) \neq \mathrm{RUC}(G)$*, that is, G admits a transitive dynamical system which is not tame.*
(8) [64] *$H_+[0,1]$ and $H_+(\mathbb{T})$ are minimal groups.*

In properties (5) and (6) we answer two questions of T. Ibarlucia which are related to [46].

Proof. (3) See [35] (or Theorem 3.6.1 with Lemma 3.5).

(5) (Sketch) Consider the Roelcke G-compactification $G \to R$ of G. That is, the compactification of G induced by the algebra $UC(G) := \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$. One can show that, in the present case, this compactification is a G-factor of the Ellis compactification $G \to E$, where $E = E(G, [0,1])$ is the enveloping semigroup for the action $G \times [0,1] \to [0,1]$, which is tame as we mentioned above. As in [32] one can apply a characterization of Uspenskij, for elements of R; they can be identified with some special relations on $[0,1]$. Namely, those (connected) curves in the square $[0,1] \times [0,1]$ which connect the points $(0,0)$ and $(1,1)$ and never go down. These are not functions in general and may have vertical intervals (as well as, horizontal intervals) in their graphs.

For the enveloping semigroup $E = E(G, [0,1])$ we know [35] that, as a compactum, it is naturally embedded into the Helly compactum of nondecreasing selfmaps of $[0,1]$. Each element p of E has at most countably many discontinuity points, where left and right limits both exist. Our aim is to find a G-factor map $f : E \to R$. Let $p \in E$. At each discontinuity point $x \in [0,1]$ of the function $p : [0,1] \to [0,1]$, add a vertical interval to the graph of p. That is, we "fill" the graph by joining the points (x, y_1) and (x, y_2), where y_1 is the left limit of the function p at x and y_2 is the right limit. Then after this operation, repeated at each discontinuity point, p "becomes" an element of R which we denote by $f(p)$. This defines a natural map $f : E \to R$ which is G-equivariant, onto and continuous (but not 1-1) .

(6) Define $f : G \to [0,1], f(g) = g(\frac{1}{2})$. Then f is tame (since the system $(G, [0,1])$ is tame) and not left uniformly continuous.

(7) We will show that for some $g \in G$ the system $(\mathrm{Exp}\,[0,1], g)$, induced on the space $\mathrm{Exp}\,[0,1]$ of closed subsets of $[0,1]$, contains a subsystem which is isomorphic to a Bernoulli shift.

Define a homeomorphism $g \in G$ as follows. First set $g(0) = 0, g(1) = 1$. Now choose a two sided increasing sequence

$$T = \{\dots, t_{-2}, t_{-1}, t_0, t_1, t_2, \dots\}$$

with $\lim_{i\to-\infty} = 0, \lim_{i\to\infty} = 1$, and let g map each of the closed intervals determined by this sequence affinely onto its right hand neighbor; i.e. $g[t_i, t_{i+1}] = [t_{i+1}, t_{i+2}]$. In particular then $g(t_i) = t_{i+1}$ and we conclude that $g \restriction T \cup \{0,1\}$ defines a dynamical system which is isomorphic to the two point compactification of the shift on the integers, $(\mathbb{Z}_*, \sigma)$, where $\mathbb{Z}_* = \mathbb{Z} \cup \{\pm\infty\}$.

Now it is well known (and easily seen) that the induced action on the space of closed subsets, $(\mathrm{Exp}\,(\mathbb{Z}_*), \sigma)$, contains a copy of the full Bernoulli shift on $\{0,1\}^{\mathbb{Z}}$. Thus the same is true for $(\mathrm{Exp}\,[0,1], g)$.

∎

Regarding Theorem 8.1.2 we note that recently Ben-Yaacov and Tsankov [5] found some other Polish groups G for which $\mathrm{WAP}(G) = \{constants\}$ (and which are therefore also reflexively trivial).

The group $H_+(\mathbb{T})$ is Asplund-trivial. Indeed, it is algebraically simple [24, Theorem 4.3] and contains a copy of $H_+[0,1] = St(z)$ (a stabilizer group of some point $z \in \mathbb{T}$) which is Asplund-trivial [32]. Now, as in [32, Lemma 10.2] use an observation of Pestov, which implies that then any continuous Asplund representation of $H_+(\mathbb{T})$ is trivial.

Theorem 8.2. *The Polish group $G = H_+(\mathbb{T})$ is Roelcke precompact.*

Proof. First a general fact: if a topological group G can be represented as $G = KH$, where K is a compact subset and H a Roelcke-precompact subgroup then G is also Roelcke-precompact. This is easy to verify either directly or by applying [75, Prop. 9.17]. As was mentioned in Theorem 8.1.4, $H_+[0,1]$ is Roelcke precompact. Now, observe that in our case $G = KH$, where $H := St(1) \cong H_+[0,1]$ is the stability group of $1 \in \mathbb{T}$ and $K \cong \mathbb{T}$ is the subgroup of G consisting of the rotations of the circle. Indeed, the coset space G/H is homeomorphic to $\mathbb{T}$ and there exists a natural continuous section $s : \mathbb{T} \to K \subset G$. $\square$

8.2. **Linearly ordered dynamical systems.** A map $f : (X, \leq) \to (Y, \leq)$ between two (partially) ordered sets is said to be *order preserving* or *monotonic* if $x \leq x'$ implies $f(x) \leq f(x')$ for every $x, x' \in X$.

Definition 8.3.

(1) (Nachbin [65]) Let (X, τ) be a topological space and $\leq$ is a partial order on the set X. The triple $(X, \tau, \leq)$ is said to be a *compact ordered space* if X is a compact space and the graph of the relation $\leq$ is closed in $X \times X$.
(2) (See [25, p. 157]) A compact dynamical S-system (X, τ) with a partial order $\leq$ is said to be a *partially ordered dynamical system* if the graph of $\leq$ is closed in $X \times X$ and every translation $\tilde{s} : X \to X$ is an order preserving map.
(3) For every linear order $\leq$ on a set X we have the standard *interval topology* which we denote by $\tau_\leq$. The triple $(X, \tau_\leq, \leq)$ is said to be a *linearly ordered topological space* (LOTS). Sometimes we write just $(X, \leq)$, or even simply X, where no ambiguity can occur.
(4) We say that a compact dynamical S-system (X, τ) is a *linearly ordered* dynamical system if there exists a linear order $\leq$ on X such that $\tau = \tau_\leq$ is the interval topology and every s-translation $X \to X$ is an $\leq$-order preserving map.

Corollary 8.5 below implies that (4) is a particular case of (2).

For every compact G-system X there is a natural partial ordering of inclusion on the hyperspace 2^X. This makes 2^X is a compact partially ordered dynamical G-system. See [25, Section 3] for details and some applications.

Recall that for every linearly ordered set $(X, \leq)$ the rays $(a, \rightarrow)$, $(\leftarrow, b)$ with $a, b \in X$ form a subbase for the standard *interval topology* $\tau_\leq$ on X. It is well known that the interval topology is Hausdorff (and even normal). Moreover it is easy to see that it is *order-Hausdorff* in the following sense.

Lemma 8.4. *Let $(X, \leq)$ be a LOTS. Then for any two distinct points $u_1 < u_2$ in X there exist disjoint $\tau_\leq$-open neighborhoods O_1 and O_2 in X of u_1 and u_2 respectively such that $O_1 < O_2$, meaning that $x < y$ for every $(y, x) \in O_2 \times O_1$. In particular, the graph of $\leq$ is closed in $(X, \tau_\leq) \times (X, \tau_\leq)$.*

Corollary 8.5. *Any* compact *LOTS is a compact ordered space in the sense of Nachbin.*

Theorem 8.6. *Every order preserving map $f : (X, \leq) \to (Y, \leq)$ between compact linearly ordered spaces is fragmented.*

Proof. First note that the question can be reduced to the case of $Y := [0,1]$. Indeed, by Corollary 8.5, $(Y, \tau_\leq)$ is an ordered space in the sense of L. Nachbin. Fundamental results from his book [65, p. 48 and 113] imply that there exists a point separating family of *order preserving* continuous maps $q_i : Y \to [0,1]$, $i \in I$. Clearly the composition of two order preserving maps is order preserving. Now by Lemma 2.2.9 it is enough to show that every map $q_i \circ f$ is fragmented. So we can assume that our order preserving function is of the form $f : X \to Y = [0,1]$. We have to show that f is fragmented. Assuming the contrary, by Lemma 2.2.8, there exists a closed subset $K \subset X$ and $a < b$ in $\mathbb{R}$ such that $K \cap \{f \leq a\}$, $K \cap \{f \geq b\}$ are both dense in K.

Choose arbitrarily two distinct points $k_1 < k_2$ in K. By Lemma 8.4 one can choose disjoint open neighborhoods O_1 and O_2 in X of k_1 and k_2 respectively such that $O_1 < O_2$.

By our assumption we can choose $x \in O_1 \cap K$ such that $b \leq f(x)$. Similarly, there exists $y \in O_2 \cap K$ such that $f(y) \leq a$. Since $a < b$ we obtain $f(x) > f(y)$. On the other hand, $x < y$ (because $O_1 < O_2$), contradicting our assumption that f is order preserving. □

Let $(X, \leq)$ be LOTS. Denote by $M_+ := M_+(X, \leq)$ the set of order preserving real valued maps on $(X, \leq)$ and by $C_+ := C_+(X, \leq)$ the set of order preserving continuous real valued maps.

Theorem 8.7. *Let $(X, \leq)$ be a compact LOTS.*

1. *$(X, \leq)$ is WRN.*
2. *Any bounded subfamily $F \subset C_+(X, \leq)$ is a Rosenthal family for X. In particular, F does not contain an independent sequence.*
3. *Any bounded sequence $\{f_n : X \to \mathbb{R}\}$ of order preserving continuous maps on X contains a pointwise convergent subsequence.*

Proof. (2) Since the natural order is closed in $\mathbb{R}^2$, we have $\operatorname{cls}_p(M_+) = M_+$. By Theorem 8.6 we know that $M_+ \subset \mathcal{F}(X)$ (the set of fragmentable functions). Thus,

$$\operatorname{cls}_p(F) \subset \operatorname{cls}_p(C_+) \subset \operatorname{cls}_p(M_+) = M_+ \subset \mathcal{F}(X).$$

This means that F is a Rosenthal family for X (Definition 2.10). In particular, F does not contain an independent sequence by Theorem 2.11.

(1) By results of Nachbin [65] the bounded subset $F := C_+((X, \leq), [0,1]) \subset C_+((X, \leq))$ separates points of X. By (2), F is a Rosenthal family for X. Thus we can apply Theorem 3.12 to conclude that X is WRN.

(3) Combine (2) and Theorem 2.11. □

As an application of this theorem we have the following:

Corollary 8.8. *The two arrows space is WRN but not RN.*

Proof. The two arrows space is WRN by Theorem 8.7.1. It is not RN by a result of Namioka [66, Example 5.9]. □

Theorem 8.9. *Let $(X, \leq)$ be a compact linearly ordered dynamical S-system. Then*

1. *the dynamical system (S, X) is representable on a Rosenthal Banach space (i.e., WRN).*
2. *(S, X) is tame.*
3. *Any topological subgroup $G \subset H_+(X)$ is Rosenthal representable.*

Proof. (1) As in the proof of Theorem 8.7 consider the (point separating and eventually fragmented) family $F := C_+((X, \leq), [0,1])$. Since (S, X) is order preserving we obtain that F is S-invariant (i.e., $FS = F$). Now apply Theorem 3.12 and Remark 3.7.

(2) First proof: Apply (1) and Theorem 4.9.

Second (direct) proof: We have to show that every $p \in E(S, X)$ is a fragmented map. Choose a net $\{s_i\}$ in S such that the net $\{j(s_i)\}$ converges to p, where $j : S \to E$ is the Ellis compactification. Since every translation $j(s) = \tilde{s} : X \to X$ is order preserving, and as the order is a closed relation (Corollary 8.5), it follows that p is also order preserving. Now from Theorem 8.6 we can conclude that p is fragmented.

(3) By (1) the system (G, X) is Rosenthal representable. Now by Lemma 3.5 (taking into account Remark 3.7) it follows that G is Rosenthal representable. □

Theorem 8.9.3 suggests the following:

Definition 8.10. Let us say that a topological group G is *pseudolinear* if G is a topological subgroup of $H_+(X, \leq)$ for some linearly ordered compact space.

Thus, every pseudolinear topological group G is Rosenthal representable. For example, $\mathbb{R}$ is pseudolinear as it can be embedded into $H_+([0,1])$, where $[0,1]$ is treated as the two-point compactification of $\mathbb{R}$. Recall that it is unknown yet (see [35, 36]) whether every Polish group is Rosenthal representable. It would be interesting to have concrete examples of Polish pseudolinear groups.

Example 8.11. Let $(X,\leq)$ be a linearly ordered discrete set. The group $\mathrm{A}ut\,(X,\leq)$ of order automorphisms is pseudolinear with respect to the pointwise topology. In fact, the topological group $\mathrm{A}ut\,(X,\leq)$ can be identified with the topological group $H_+(Y,\leq)$ of continuous order automorphisms of the *two-point compactification* $Y := \{-\infty\}\cup X\cup\{\infty\}$.

In particular, the Polish group $\mathrm{A}ut\,(\mathbb{Q},\leq)$ is pseudolinear. As was shown by Pestov [71, 72] this group is extremely amenable. Moreover, the same is true for every $Aut(X,\leq)$ with an ω-transitive linear order $\leq$.

The argument of Example 8.11 shows that any left-ordered discrete group G is pseudolinear. Recall that G is *left-ordered* (LO in short) if there exists a linear order on G which is invariant under left translations (see, for example, [69]). An equivalent condition is that G acts effectively on a linearly ordered set by order preserving translations. Thus, the notion of a pseudolinear group is a topological version of LO. It is easy to see that any LO group is torsion free. Some examples of LO groups are: free groups, braid groups, surface groups and abelian torsion free groups. Any countable LO group can be embedded into $H_+[0,1]$.

8.3. **Orientation preserving actions on the circle.** The following definition is the starting point for one of the approaches to monotone functions on the circle. See, for example, [76].

Definition 8.12. Let $f:\mathbb{T}\to\mathbb{T}$ be a not necessarily continuous selfmap on the circle $\mathbb{T}$. We say that f is *orientation preserving* (notation: $f\in M_+(\mathbb{T},\mathbb{T})$, or $f\in M_+$) if there exists, a not necessarily continuous, map $F:\mathbb{R}\to\mathbb{R}$ which is a monotonic lift of degree 1. More precisely F satisfies the following conditions:

(1) $q\circ F = f\circ q$, where $q:\mathbb{R}\to\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is the quotient map;
(2) $F:\mathbb{R}\to\mathbb{R}$ is order preserving;
(3) $F(x+1)=F(x)+1$ for every $x\in\mathbb{R}$.

In this case we say that F is a *lift* of f.

Remark 8.13. Let k be a fixed integer. Then F is a lift of f iff $F+k$ is. Therefore, among all the possible lifts of f, one may choose F such that $F(0)\in[0,1)$. Clearly, $F(1)=F(0)+1<2$ and $F(x)\leq F(1)<2$ for every $x\in[0,1]$. The restriction $F^*:[0,1]\to[0,2]$ of F to $[0,1]$ uniquely reconstructs F. Indeed, it is easy to see that

$$F(x) := F^*(\{x\}) + n \qquad \forall n\leq x<n+1 \tag{8.1}$$

with $n\in\mathbb{Z}$, where $\{x\}$ is the fractional part of $x\in\mathbb{R}$. Equivalently, $F(x)=F^*(\{x\})+[x]$, where $[x]$ is the integer part. We say that F is a *canonical lift* of f and that F^* is its *kernel*.

Note that for an arbitrary order preserving function $h:[0,1]\to[0,2]$ with $h(1)=h(0)+1$ and $q_2\circ h = f\circ q_1$, the function $F(x) := h^*(\{x\})+[x], x\in\mathbb{R}$ is a lift of f. Observe that h is the kernel F^* of F iff $h(0)<1$. Otherwise, if $h(0)=1$ then $F^*=h-1$.

Lemma 8.14.

(1) *Every orientation preserving* $f:\mathbb{T}\to\mathbb{T}$ *is Baire 1. That is,* $M_+(\mathbb{T},\mathbb{T})\subset\mathcal{B}_1(\mathbb{T},\mathbb{T})$.
(2) M_+ *is pointwise closed in* $\mathbb{T}^{\mathbb{T}}$*;*
(3) M_+ *is a compact right topological submonoid of* $\mathbb{T}^{\mathbb{T}}$ *with respect to the composition.*

Proof. (1) Let F be the canonical lift of f and $F^*:[0,1]\to[0.2]$ be its kernel. Let $q_1:[0,1]\to\mathbb{T}$ and $q_2:[0,2]\to\mathbb{T}$ be the restrictions of $q:\mathbb{R}\to\mathbb{T}$. Then $q_2\circ F = f\circ q_1$. By Lemma 2.2.10 we conclude that f is fragmented (hence, Baire 1, because f is a map between Polish spaces).

(2) Let $p\in\mathrm{cls}\,(M_+)$. Then p is a pointwise limit of some net $f_i\in M_+$. For every f_i consider the canonical lifting F_i and its kernel $F_i^*:[0,1]\to[0,2]$. Passing to subnets if necessary one may assume that F_i^* pointwise converges in $[0,2]^{[0,1]}$ to some $h:[0,1]\to[0,2]$. Then h is order preserving, too.

Moreover, since $0 \leq F_i(0) < 1$ we have $h(0) \in [0,1]$ and $h(1) = h(0)+1 \in [0,2]$. It is easy to show that

$$q_2(h(x)) = \lim q_2(F_i^*(x)) = \lim f_i(q_1(x)) = f(q_1(x)).$$

for every $x \in [0,1]$. Now, as in Remark 8.13, define $F(x) := h(\{x\}) + [x]$. Then F is a lift of p in the sense of Definition 8.12. Note that F is not necessarily the canonical lift of f (though $h(0) \in [0,1]$ but it is possible that $h(0) = 1$).

(3) Clearly, $F := id_{\mathbb{R}}$ is a lift of $f := id_{\mathbb{T}}$. So, $id_{\mathbb{T}} \in M_+$. It is plain to show that if $f_1, f_2 \in M_+$ with lifts F_1, F_2. Then $F_1 \circ F_2$ is a lift for $f_1 \circ f_2$. $\square$

Recall the definition of the natural cyclic ordering on $\mathbb{T}$. Identify $\mathbb{T}$, as a set, with $[0,1)$ and define a ternary relation, a subset $R \subset [0,1)^3$. We say that an ordered triple of pairwise disjoint points $z, y, x \in [0,1)$ has cyclic ordering (and write $[z,y,x] \in R$) if $(x-y)(y-z)(x-z) > 0$. An injective selfmap $f : \mathbb{T} \to \mathbb{T}$ is said to be (cyclic) order preserving if f preserves R, meaning that $[z,y,x] \in R$ implies $[f(z), f(y), f(x)] \in R$.

The following lemma is a version of Lemma 1 in [54, Section 3].

Lemma 8.15. *Every injective order preserving selfmap (e,g., order preserving homeomorphism) $f : \mathbb{T} \to \mathbb{T}$ is orientation preserving in the sense of Definition 8.12.*

Proof. Treating the set $\mathbb{T}$ as $[0,1)$ (so that f is defined as a map $[0,1) \to [0,1)$) consider the partition $[0,1) = I_+(f) \cup I_-(f)$, where

$$I_+(f) := \{x \in [0,1) : f(x) \geq f(0)\}, \quad I_-(f) := \{x \in [0,1) : f(x) < f(0)\}.$$

Define $F^* : [0,1] \to [0,2]$ by

$$F^*(x) = f(x), \text{ if } x \in I_+, \quad F^*(x) = f(x)+1, \text{ if } x \in I_-, \ F^*(1) = f(0)+1.$$

It is easy to see (using the circle ordering) that $I_+(f), I_-(f)$ are intervals, $F^* : [0,1] \to [0,2]$ is order preserving and $q_2 \circ F^* = f \circ q_1$. Then F, defined as in 8.1, is the desired lift of f. $\square$

Let $C_+(\mathbb{T}, \mathbb{T})$ be the topological monoid of all orientation preserving *continuous* selfmaps $\mathbb{T} \to \mathbb{T}$ endowed with the compact open topology. Then, for every submonoid S (in particular, for any subgroup $G \leq H_+(\mathbb{T})$) we have a corresponding (orientation preserving) dynamical system $(S, \mathbb{T})$.

Theorem 8.16.

(1) *For every submonoid S of $C_+(\mathbb{T}, \mathbb{T})$ the dynamical system $(S, \mathbb{T})$ is tame. In particular, this is true for any subgroup $S := G$ of $H_+(\mathbb{T})$.*
(2) *$H_+(\mathbb{T})$ is Rosenthal representable as a Polish topological group.*

Proof. Part (2) follows from (1), Theorem 3.6 and Lemma 3.5. For (1) we have to show that every $p \in E(\mathbb{T})$ is a Baire 1 class function $\mathbb{T} \to \mathbb{T}$. By our assumption, $S \subset C_+(\mathbb{T}, \mathbb{T}) \subset M_+(\mathbb{T}, \mathbb{T})$ and $M_+(\mathbb{T}, \mathbb{T})$ is pointwise closed (Lemma 8.14.2). So, we obtain $\mathrm{cls}\,(S) = E(S, \mathbb{T}) \subset M_+(\mathbb{T}, \mathbb{T})$. By Lemma 8.14.1 we have $M_+(\mathbb{T}, \mathbb{T}) \subset \mathcal{B}_1(\mathbb{T}, \mathbb{T})$. Therefore, $p \in E(S, \mathbb{T}) \subset \mathcal{B}_1(\mathbb{T}, \mathbb{T})$. $\square$

The Ellis compactification $j : G \to E(G, \mathbb{T})$ of the group $G = H_+(\mathbb{T})$ is a topological embedding. In fact, observe that the compact open topology on $j(G) \subset C_+(\mathbb{T}, \mathbb{T})$ coincides with the pointwise topology. This observation implies, by [36, Remark 4.14] that $\mathrm{Tame}(G)$ separates points and closed subsets.

Although G is representable on a (separable) Rosenthal Banach space, we have $\mathrm{Asp}(G) = \{constants\}$ and therefore any Asplund representation of this group is trivial (this situation is similar to the case of the group $H_+[0,1]$, [32]). Indeed, we have $\mathrm{SUC}(G) = \{constants\}$ by [32, Corollary 11.6] for $G = H_+(\mathbb{T})$, and we recall that for every topological group $\mathrm{Asp}(G) \subset \mathrm{SUC}(G)$.

8.3.1. *Functions of bounded variation.*

Definition 8.17. Let $(X, \leq)$ be a linearly ordered set. We say that a bounded function $f : X \to \mathbb{R}$ has variation not greater than r if

$$\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \leq r$$

for every choice of $x_0 \le x_1 \le \cdots \le x_n$ in X. The least upper bound of all such possible sums is the *variation* of f. Notation: $\Upsilon(f)$. If $\Upsilon(f) \le r$ then we write $f \in BV_r(X)$. If $f(X) \subset [c,d]$ for some reals $c \le d$ then we write also $f \in BV_r(X,[c,d])$.

Denote by $M_+(X,[c,d])$ the set of all increasing functions $X \to [c,d]$.

Theorem 8.18. [63] *For every linearly ordered set X the set of functions $BV_r(X,[c,d])$ does not contain independent subsequences. In particular, this is true also for $M_+(X,[c,d])$.*

This result together with Lemma 7.10 of the present work leads to the following application. See also Corollary 8.22.

Theorem 8.19. *Let X be a linearly ordered set, $f : X \to \mathbb{R}$ be a (not necessarily continuous) function in $BV_r(X)$ and $S \subset M_+(X,X)$ be a semigroup of order preserving (not necessarily continuous) selfmaps. Then for every point $z \in X$ the coding function*

$$m(f,z) : S \to \mathbb{R}, \ \ s \mapsto f(s(z))$$

is tame on the discrete copy of S.

Proof. The orbit fS of f is a bounded family of functions with bounded total variation. This orbit does not contain an independent sequence by Theorem 8.18. Hence by Lemma 7.10.6 the function $m(f,z)$ is tame. □

Definition 8.20. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. We say that it has total variation $\le r$ and write $f \in BV_r(\mathbb{T})$ if the induced function $f \circ q_1 : [0,1] \to \mathbb{R}$ belongs to $BV_r[0,1]$.

Lemma 8.21. *For every $\phi \in M_+(\mathbb{T},\mathbb{T})$ and every $f \in BV_r(\mathbb{T})$ we have $f \circ \phi \in BV_{2r}(\mathbb{T})$.*

Proof. Let Φ be the canonical lift of ϕ and $\Phi^* : [0,1] \to [0,2]$ be its kernel. Then we have to show that $f \circ \phi \circ q_1 : [0,1] \to \mathbb{R}$ belongs to $BV_{2r}[0,1]$. Since, $\phi \circ q_1 = q_2 \circ \Phi^*$, it is equivalent to showing that $f \circ q_2 \circ \Phi^* \in BV_{2r}[0,1]$. Since Φ^* is monotone it suffices to show that $f \circ q_2 \in BV_{2r}[0,2]$. Now it is enough to see that the restrictions of $f \circ q_2$ to the subintervals $[0,1]$ and $[1,2]$ are in BV_r. The first is clear by the definition of $f \in BV_r[0,1]$. The rest is easy using the fact that q_2 has period 1. □

Corollary 8.22. *Any bounded family of functions $A := \{f_i : \mathbb{T} \to \mathbb{R}\}_{i \in I}$ with finite total variation does not contain independent subsequences.*

Proof. Assuming the contrary let $f_n \in BV_r(\mathbb{T})$ be an independent sequence of functions. Then since $q_1 : [0,1] \to \mathbb{T}$ is onto we obtain by Lemma 7.9.2 that the sequence $f_n \circ q_1$ of functions on $[0,1]$ is independent, too. This contradicts Theorem 8.18 because $f_n \circ q_1 : [0,1] \to \mathbb{R}$ is a bounded family of bounded total variation.

□

Theorem 8.23. *Let $f : \mathbb{T} \to \mathbb{R}$ be a (not necessarily continuous) function in $BV_r(\mathbb{T})$.*

(1) *Let $S \subset M_+(\mathbb{T},\mathbb{T})$ be a semigroup of orientation preserving (not necessarily continuous) selfmaps. Then for every point $z \in \mathbb{T}$ the coding function*

$$m(f,z) : S \to \mathbb{R}, \ \ s \mapsto f(s(z))$$

is tame on the discrete copy of S.

(2) *Let $\sigma : \mathbb{T} \to \mathbb{T}$ be an orientation preserving selfmap. Then the coding function*

$$m(f,z) : \mathbb{N} \cup \{0\} \to \mathbb{R}, \ \ n \mapsto f(\sigma^n(z))$$

is tame. If σ is a bijection then one can replace $\mathbb{N} \cup \{0\}$ by $\mathbb{Z}$.

Proof. (1) By Lemma 8.21 we have $fS \subset BV_{2r}(\mathbb{T})$. Clearly, fS is bounded. Then by Corollary 8.22 the orbit fS is free of independent sequences. Now Lemma 7.10.6 finishes the proof.

(2) This is a particular case of (1). □

Remark 8.24. (Coloring function on the circle) Theorem 8.23 gives an additional way of proving Theorem 7.18, which asserts that the multidimensional Sturmian sequences of Example 7.7 are tame. Indeed, consider a finite partition of $\mathbb{T} = \cup_{i=0}^{d} I_i$, where each I_i is an arc on $\mathbb{T}$ (open, closed or containing one of the boundary points). Then any coloring of this partition, that is, any function $f : \mathbb{T} \to \mathbb{R}$ which is constant on each I_i is of finite variation.

Remark 8.25. Several results of this subsection (the case of $\mathbb{T}$) can be generalized to general cyclically ordered sets and orientation preserving actions. We intend to deal with this issue in some future publication.

9. Intrinsically tame groups

Recall that for every topological group G there exists a unique universal minimal G-system $M(G)$. Usually $M(G)$ is nonmetrizable. For example, this is the case for every locally compact noncompact G. On the other hand, many interesting massive Polish groups are extremely amenable that is, having trivial $M(G)$. See for example [71, 72, 84]. The first example of a nontrivial yet computable small $M(G)$ was found by Pestov. In [71] he shows that for $G := H_+(\mathbb{T})$ the universal minimal system $M(G)$ can be identified with the natural action of G on the circle $\mathbb{T}$. Glasner and Weiss [41, 42] gave an explicite description of $M(G)$ for the symmetric group S_∞ and for $H(C)$ (the Polish group of homeomorphisms of the Cantor set C). Using model theory Kechris, Pestov and Todorčević gave in [48] many new examples of computations of $M(G)$ for various subgroups G of S_∞.

Definition 9.1. We say that a topological group G is *intrinsically tame* if the universal minimal G-system $M(G)$ is tame. Equivalently, if every continuous action of G on a compact space X admits a G-subsystem $Y \subset X$ which is tame.

By Lemma 4.7.1 every G-system X admits a largest tame G-factor. Every topological group G has a universal minimal tame system $M_t(G)$ (which is the largest tame G-factor of $M(G)$). So G is intrinsically tame iff the natural projection $M(G) \to M_t(G)$ is an isomorphism.

The G-space $M_t(G)$ can also be described as a minimal left ideal in the universal space G^{Tame}. Recall that the latter is isomorphic to its own enveloping semigroup and thus has a structure of a compact right topological semigroup. Moreover, any two minimal left ideals there are isomorphic as dynamical systems.

In [31] we defined, for a topological group G and a dynamical property P, the notion of P-fpp (P fixed point property). Namely G has the P-fpp if every G-system which has the property P admits a G fixed point. Clearly this is the same as demanding that every minimal G- system with the property P be trivial. Thus for $P = Tame$ a group G has the tame-fpp iff $M_t(G)$ is trivial.

We will need the following theorem which extends a result in [29].

Theorem 9.2. *Let (G, X) be a metrizable minimal tame dynamical system and suppose it admits an invariant probability measure. Then (G, X) is point distal. If moreover, with respect to μ the system (G, μ, X) is weakly mixing then it is a trivial one point system.*

Proof. With notations as in [29] we observe that for any minimal idempotent $v \in E(G, X)$ the set C_v of continuity points of v restricted to the set $\overline{vX}$, is a dense G_δ subset of $\overline{vX}$ and moreover $C_v \subset vX$ ([29, Lemma 4.2.(ii)]). Also, by [29, Proposition 4.3] we have $\mu(vX) = 1$, and it follows that $\overline{vX} = X$. The proof of the claim that (G, X) is point distal is now finished as in [29, Proposition 4.4].

Finally, as the measure preserving system (G, μ, X) is weakly mixing it follows that it is also topologically weakly mixing. By the Veech-Ellis structure theorem for point distal systems [86, 14], if (G, X) is nontrivial it admits a nontrivial equicontinuous factor, say (G, Y). However (G, Y), being a factor of (G, X), is at the same time also topologically weakly mixing which is a contradiction. □

Remark 9.3. It seems that this observation, namely that the existence of an invariant measure can replace the assumption that G is abelian in proving point distally, can be pushed to a proof of the full statement of Proposition 5.1 in [29] (modulo some obvious modifications) under the assumption that X supports an invariant measure.

Theorem 9.4.

(1) *Every extremely amenable group is intrinsically tame.*
(2) *The Polish group $H_+(\mathbb{T})$ of orientation preserving homeomorphisms of the circle is intrinsically tame.*
(3) *The Polish groups* $\mathrm{Aut}\,(\mathbf{S}(2))$ *and* $\mathrm{Aut}\,(\mathbf{S}(3))$*, of automorphisms of the circular directed graphs* $S(2)$ *and* $S(3)$*, are intrinsically tame.*
(4) *A discrete group which is intrinsically tame is finite.* [2]
(5) *For an abelian infinite countable discrete group G, its universal minimal tame system $M_t(G)$ is a highly proximal extension of its Bohr compactification G^{AP} (see e.g.* [29]*).*
(6) *The Polish group $H(C)$, of homeomorphisms of the Cantor set, is not intrinsically tame.*
(7) *The Polish group $G = S_\infty$, of permutations of the natural numbers, is not intrinsically tame. In fact $M_t(G)$ is trivial; i.e. G has the tame-fpp.*

Proof. Claim (1) is trivial and claim (2) follows from Pestov's theorem [71] which identifies, for $G = H_+(\mathbb{T})$, the universal minimal dynamical system $(G, M(G))$ with the tautological action $(G, \mathbb{T})$, and from Theorem 8.16 which asserts that this action is tame.

The universal minimal G-systems for the groups $\mathrm{A}ut\,(\mathbf{S}(2))$ and $\mathrm{A}ut\,(\mathbf{S}(3))$ are computed in [85]. In both cases it is easy to check that every element of the enveloping semigroup $E(M(G))$ is an order preserving map. As there are only $2^{\aleph_0}$ order preserving maps, it follows that the cardinality of $E(M(G))$ is $2^{\aleph_0}$, whence, in both cases, the dynamical system $(G, M(G))$ is tame.

In order to prove Claim (4) we assume, to the contrary, that G is infinite and apply a result of B. Weiss [87], to obtain a minimal model, say (G, X, μ), of the Bernoulli probability measure preserving system $(G, \{0,1\}^G, (\frac{1}{2}(\delta_0 + \delta_1))^G)$. Now (G, X, μ) is metrizable, minimal and tame, and it carries a G-invariant probability measure with respect to which the system is weakly mixing. Applying Theorem 9.2 we conclude that X is trivial. This contradiction finishes the proof.

In [44], [51] and [29] it is shown that a metric minimal tame G-system is an almost one-to-one extension of an equicontinuous system. (Note that not every minimal almost one-to-one extension of a minimal equicontinuous G system is tame, such systems e.g. can have positive topological entropy.) Of course every minimal equicontinuous G-system is tame. Now tameness is preserved under sub-products, and because our group G is countable, it follows that $M_t(G)$ is a minimal sub-product of all the minimal tame metrizable systems. In turn this implies that $M_t(G)$ is a (non-metrizable) highly proximal extension of the Bohr compactification G^{AP} of G.

To see that $G = H(C)$ is not intrinsically tame it suffices to show that the tautological action (G, C), which is a factor of $M(G)$, is not tame. To that end note that the shift transformation σ on $X = \{0,1\}^{\mathbb{Z}}$ is a homeomorphism of the Cantor set. Now the enveloping semigroup $E(\sigma, X)$ of the cascade (σ, X), a subset of $E(G, X)$, is homeomorphic to $\beta\mathbb{N}$.

To see that $G = S_\infty$ is not intrinsically tame we recall first that, by [39], the universal minimal dynamical system for this group can be identified with the natural action of G on the compact metric space $X = LO(\mathbb{N})$ of linear orders on $\mathbb{N}$. Also, it follows from the analysis of this dynamical system that for any minimal idempotent $u \in E(G, X)$ the image of u contains exactly two points, say $uX = \{x_1, x_2\}$. A final fact that we will need concerning the system (G, X) is that it carries a G-invariant probability measure μ of full support [39]. Now to finish the proof, suppose that (G, X) is tame. Then there is a **sequence** $g_n \in G$ such that $g_n \to u$ in $E(G, X)$. If $f \in C(X)$ is any continuos real valued function, then we have, for each $x \in X$,

$$\lim_{n\to\infty} f(g_n x) = f(ux) \in \{f(x_1), f(x_2)\}.$$

But then, choosing a function $f \in C(X)$ which vanishes at the points x_1 and x_2 and with $\int f\,d\mu = 1$, we get, by Lebesgue's theorem,

$$1 = \int f\,d\mu = \lim_{n\to\infty} \int f(g_n x)\,d\mu = \int f(ux)\,d\mu = 0.$$

---

[2] Modulo an extension of Weiss' theorem, which does not yet exist, a similar idea will work for any locally compact group. The more general statement would be: A locally compact group which is intrinsically tame is compact.

Finally, the property of supporting an invariant measure, as well as the fact that the cardinality of the range of minimal idempotents is ≤ 2, are inherited by factors and the same argument shows that $M(G)$ admits no nontrivial tame factor. Thus $M_t(G)$ is trivial. $\square$

Remark 9.5. A theorem of Huang, Kerr-Li and Glasner ([44], [51], [29]) asserts that: for G abelian any metrizable minimal tame action is an almost 1-1 extension of an equicontinuous system. The fact that the minimal action of $H_+(\mathbb{T})$ on $\mathbb{T}$ is tame shows that some restrictive assumption on the group G is really necessary.

It would be interesting to find other examples of intrinsically tame Polish groups.

The (nonamenable) group $G = H_+(\mathbb{T})$ has one more remarkable property. Besides $M(G)$, one can also effectively compute the affine analogue of $M(G)$. Namely, the *universal irreducible affine system* of G (we denote it by $IA(G)$) which was defined and studied in [25, 26]. It is uniquely determined up to affine isomorphisms. The corresponding affine compactification $G \to IA(G)$ is equivalent to the affine compactification $G \to P(M_{sp}(G))$, where, $M_{sp}(G)$ is the *universal strongly proximal minimal system* of G and $P(M_{sp}(G))$ is the space of probability measures on the compact space $M_{sp}(G)$.

Definition 9.6. We say that G is *convexly intrinsically tame* (or *conv-int-tame* for short) if the G-system $IA(G)$ is tame.

Note that this condition holds iff every compact affine dynamical system (G, Q) admits an affine tame G-subsystem, iff every compact affine dynamical system (G, Q) admits a tame G-subsystem. The latter assertion follows from the fact that $P(X)$ is tame whenever X is [33, Theorem 6.11], and by the affine universality of $P(X)$. In particular, it follows that any intrinsically-tame group is convexly intrinsically tame. Of course any amenable topological group (i.e. a group with trivial $IA(G)$) has this property. Thus we have the following diagram which emphasizes the analogy between the two pairs of properties:

$$\begin{array}{ccc} \text{extreme amenability} & \Longrightarrow & \text{intrinsically tame} \\ \Downarrow & & \Downarrow \\ \text{amenability} & \Longrightarrow & \text{convexly intrinsically tame} \end{array}$$

Remark 9.7. Given a class P of compact G-systems which is stable under subdirect products, one can define the notions of intrinsically P-group and convexly intrinsically P group in a manner analogous to the one we adopted for $P = Tame$. We then note that in this terminology a group is convexly intrinsically HNS (and , hence, also conv-int-WAP) iff it is amenable. This follows easily from the fact that the algebra Asp(G) is left amenable, [34]. Also, at least for discrete groups, if G is intrinsically HNS then it is finite. In fact, for any group, an HNS minimal system is equicontinuous, so that for a group G which is intrinsically HNS the universal minimal system $M(G)$ coincides with its Bohr compactification G^{AP}. Now for a discrete group, it is not hard to show that an infinite minimal equicontinuous system admits a nontrivial almost one to one (hence proximal) minimal extension. Thus $M(G)$ must be finite. However, by a theorem of Ellis [13], for discrete groups the group G acts freely on $M(G)$, so that G must be finite as claimed. Probably similar arguments will show that a locally compact intrinsically HNS group is necessarily compact. This "collapsing effect" together with the special role of tameness in the dynamical BFT dichotomy 1.7 suggest that the notions of intrinsic tameness and con-int-tameness are natural analogues of extreme amenability and amenability, respectively.

The Polish group S_∞ is amenable (hence convexly intrinsically tame) but not intrinsically tame.

The group $H(C)$ is not convexly intrinsically tame. In fact, its natural action on the Cantor set C is minimal and strongly proximal, but this action is not tame; it contains as a subaction a copy of the full shift $(\mathbb{Z}, C) = (\sigma, \{0,1\}^{\mathbb{Z}})$. The group $H([0,1]^{\mathbb{N}})$ is a universal Polish group (see Uspenskij [83]). It also is not convexly intrinsically tame. This can be established by observing that the action of this group on the Hilbert cube is minimal, strongly proximal and not tame. The strong proximality of this action can be easily checked. The action is not tame because it is a *universal action* (see [58]) for all Polish groups on compact metrizable spaces.

The (universal) minimal G-system $\mathbb{T}$ for $G = H_+(\mathbb{T})$ is strongly proximal. Hence, $IA(G)$ in this case is easily computable and it is exactly $P(\mathbb{T})$ which, as a G-system, is tame (by Theorem 9.4 and a remark above). So, $H_+(\mathbb{T})$ is a (convexly) intrinsically tame nonamenable topological group.

Another example of a Polish group which is nonamenable yet convexly intrinsically tame (that is, with tame $IA(G)$) is any semisimple Lie group G with finite center and no compact factors. Indeed, by Furstenberg's result [22] the universal minimal strongly proximal system $M_{sp}(G)$ is the homogeneous space $X = G/P$, where P is a minimal parabolic subgroup (see [26]). Results of Ellis [15] and Akin [1] (Example 7.1.1) show that the enveloping semigroup $E(G, X)$ in this case is a Rosenthal compactum, whence the system (G, X) is tame by the dynamical BFT dichotomy (Theorem 1.7).

Question 9.8. For which (Polish) groups G the following universal constructions lead to tame G-systems:

(a) The Roelcke compactification $R(G)$ (of a Roelcke precompact group);
(b) the universal minimal system $M(G)$;
(c) the universal irreducible affine system $IA(G)$?

A related question is to compute the largest tame factor for these (and some additional) compactifications of G.

10. Appendix

The proof of the following result was communicated to us by Stevo Todorčević.

Theorem 10.1. *$\beta\mathbb{N}$ is not WRN.*

For the proof we will need several definitions and lemmas.

Definition 10.2. A family of disjoint pairs of subsets $\{(F_i, G_i) : i \in I\}$ of a set S is *independent* if for every two disjoint finite sets $K, L \subset I$

$$\bigcap_{i \in K} F_i \ \cap \ \bigcap_{i \in L} G_i \neq \emptyset.$$

Definition 10.3. A sequence $\{F_n : n \in \mathbb{N}\}$ of subsets of a set S is *convergent* if for every $s \in S$ there is n_0 such that, either $s \in F_n$ for every $n \geq n_0$, or $s \notin F_n$ for every $n \geq n_0$.

Lemma 10.4. *There exists a family $\{(F_r, G_r) : r \in \mathbb{R}\}$ of pairs of disjoint closed sets of $\beta\mathbb{N}$ which is independent.*

Proof. As is well known the Cantor cube $\Omega = \{0,1\}^{\mathbb{R}}$ contains a dense countable sequence, say $\{\omega_n\}_{n\in\mathbb{N}}$. Let $\rho : \beta\mathbb{N} \to \Omega$ be the unique continuous extension to $\beta\mathbb{N}$ of the map $\mathbb{N} \to \Omega$, $n \mapsto \omega_n$. Then ρ is a continuous surjection and for $r \in \mathbb{R}$ we set

$$F_r = \{x \in \beta\mathbb{N} : \rho(x)(r) = 0\}, \qquad G_r = \{x \in \beta\mathbb{N} : \rho(x)(r) = 1\}.$$

□

The next lemma is a crucial tool. Its proof is very similar to the proof of a theorem of Rosenthal [77] (see also [18] and [57, page 100]).

Lemma 10.5. *Let $\{(A_i, B_i) : i \in \omega\}$ be an independent family of disjoint pairs of subsets of a set S. Suppose there is a positive integer $k \geq 1$, and k families of disjoint pairs $\{(A_{ij}, B_{ij}) : i \in \omega\}$, $1 \leq j \leq k$, such that:*

$$A_i \times B_i \ \subset \ \bigcup_{j=1}^{k} A_{ij} \times B_{ij}.$$

Then there is an infinite $M \subset \omega$ and $j_0 \in \{1, 2, \ldots, k\}$ such that the family

$$\{(A_{ij_0}, B_{ij_0}) : i \in M\}$$

is an independent family.

Proof. We let $[\omega]^\omega$ denote the collection of infinite subsets of ω. More generally, if $M \in [\omega]^\omega$ then $[M]^\omega$ denotes the collection of infinite subsets of M. The space $[\omega]^\omega$ carries a natural topology when we identify it with the subset of the Cantor space $\{0,1\}^\omega$ consisting of sequences with infinitely many 1's.

For $1 \le j \le k$ let

$$\mathcal{X}_j = \{M = \{m_1 < m_2 < \cdots\} \in [\omega]^\omega : \ \forall n < \omega, \\ \bigcap_{l=1}^{n} A_{m_{2l},j} \ \cap \ \bigcap_{l=1}^{n} B_{m_{2l+1},j} \neq \emptyset\}$$

Each $\mathcal{X}_j$ is a closed subset of $[\omega]^\omega$ and the Galvin-Prikry theorem [23] implies that either:

(i) there is some $1 \le j \le k$ and $M \in [\omega]^\omega$ such that $[M]^\omega \subset \mathcal{X}_j$, or

(ii) there is an $M \in [\omega]^\omega$ such that for every $1 \le j \le k$, $[M]^\omega \cap \mathcal{X}_j = \emptyset$.

In the first case, where $[M]^\omega \subset \mathcal{X}_j$, let $M = \{m_n : n < \omega\}$ and set $N = \{m_{2n} : n < \omega\}$. Then $\{(A_{n,j}, B_{n,j}) : n \in N\}$ is independent. In fact, given a positive integer $u \ge 1$ and two disjoint finite sets $K, L \subset N$, such that $K \cup L = \{m_2, m_4, \ldots, m_{2u}\}$, construct a sequence $N_1 = \{n_h\}_{h\in\mathbb{N}} \subset N$ which contains the integers $\{m_2, m_4, \ldots, m_{2u}\}$, scattered among $\{n_1, n_2, \ldots, n_{2u}\}$ in such a way that for $m_{2p} \in K, m_{2p} = n_{2h}$ for some $1 \le h \le u$, and for $m_{2p} \in L, m_{2p} = n_{2h+1}$ for some $1 \le h \le u$. Since $N_1 \in \mathcal{X}_j$ we now have

$$\bigcap_{m_{2p}\in K} A_{m_{2p},j} \ \cap \ \bigcap_{m_{2p}\in L} B_{m_{2p},j} \ \supset \ \bigcap_{h=1}^{u} A_{n_{2h},j} \ \cap \ \bigcap_{h=1}^{u} B_{n_{2h+1},j} \neq \emptyset.$$

Our proof will be complete when we show next that in our situation the case (ii) can not occur. In fact, we show that if $[M]^\omega \cap \mathcal{X}_j = \emptyset$, then the sequence $\{A_{ij} : i \in M\}$ converges. For otherwise we can find a point $s \in S$ and an infinite subsequence $N = \{n_1 < n_2 < \cdots\} \subset M$ such that $s \in A_{n_{2l},j}$ and $s \notin A_{n_{2l+1},j}$ for every $l \ge 1$. But then, $N \in [M]^\omega \cap \mathcal{X}_j$, contradicting our assumption.

Thus under the assumption that (ii) holds, for every $1 \le j \le k$, the sequence $\{A_{ij} : i \in M\}$ converges. This however clearly contradicts the independence of the family $\{(A_i, B_i) : i \in \omega\}$, and our proof is complete. $\square$

Lemma 10.6. (Rosenthal [77, Proposition 4]) *Let S be a set and $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^S$ a uniformly bounded sequence of functions on S. Suppose there are real numbers $p < q$ such that the pairs of sets*

$$F_n = \{s \in S : f_n(s) \le p\} \quad \text{and} \quad G_n = \{s \in S : f_n(s) \ge q\}$$

form an independent family. Then the sequence $\{f_n\}_{n\in\mathbb{N}}$ is an ℓ_1-sequence in the Banach space $\ell_\infty(S)$.

Proof of Theorem 10.1. Suppose to the contrary that V is a Rosenthal Banach space and that B_{V^*}, equipped with its weak* topology, contains a copy Φ of $\beta\mathbb{N}$. By Lemma 10.4 there exists a family $\{(F_i, G_i) : i \in \mathbb{R}\}$ of pairs of disjoint closed subsets of Φ which is independent. By the nature of the weak* topology, for each i there exist a finite set $\{v_{ij} : 1 \le j \le k_i\} \subset B_V$, the unit ball of V, and a finite set of pairs $\{q_{ij} < q'_{ij} : 1 \le j \le k_i\} \subset \mathbb{Q}$, such that the sets

$$Q_i = \bigcap_j \{\phi \in V^* : \phi(v_{ij}) \le q_{ij}\}, \qquad Q'_i = \bigcap_j \{\phi \in V^* : \phi(v_{ij}) \ge q'_{ij}\},$$

separate the pair (F_i, G_i); i.e. $F_i \subset Q_i$ and $G_i \subset Q'_i$. Now, as $|\mathbb{R}| = \mathbf{c}$ and there are only countably many possible choices for the values k_i we conclude, by the pigeon holes principle, that there exists a finite positive integer $k \ge 1$ and an uncountable subset $D \subset \mathbb{R}$ with $k_i = k$ for every $i \in D$. Next we chose an arbitrary infinite countable subset $C \subset D$ and we now have a countable subfamily $\{(F_i, G_i) : i \in C\}$ (for a countable subset $C \subset \mathbb{R}$) such that $k_i = k$ for every $i \in C$. Clearly the family $\{(Q_i \cap \Phi, Q'_i \cap \Phi) : i \in C\}$ is a *countable* independent family.

Applying Lemma 10.5 to the sets

$$A_i = Q_i \cap \Phi, \qquad B_i = Q'_i \cap \Phi,$$
$$A_{ij} = \{\phi \in \Phi : \phi(v_{ij}) \le q_{ij}\}, \qquad B_{i,j} = \{\phi \in \Phi : \phi(v_{ij}) \ge q'_{ij}\},$$

with $i \in C$ and $1 \le j \le k$, we conclude that for some infinite $M \subset C$ and some $j_0 \in \{1, 2, \ldots, k\}$ the family

$$\{(A_{ij_0}, B_{ij_0}) : i \in M\}$$

is an independent family.

Applying Lemma 10.6 to the sequence of functions $f_i = v_{ij_0} \restriction \Phi : \Phi \to \mathbb{R}$, we conclude that this sequence is an ℓ_1-sequence in the Banach space $C(\Phi)$. However, as the restriction map $v \mapsto f_v := v \restriction \Phi$, $V \to C(\Phi)$ satisfies

$$\|v\| = \sup_{\phi \in B_{V^*}} |\phi(v)| \ge \|f_v\| = \sup_{\phi \in \Phi} |\phi(v)|,$$

it follows that $\{v_{ij_0} : i \in M\}$ is also an ℓ_1-sequence in V. This contradicts our assumption that V is a Rosenthal space and the proof is complete. $\square$

Remark 10.7. The proof of Theorem 10.1 actually shows that the cube $Q(\omega_1) = [0,1]^{\omega_1}$, as well as any compactum K which maps continuously onto $Q(\omega_1)$, is not a WRN compactum.

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