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Glasner, Eli.
Weiss, Benjamin.

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THE TOPOLOGICAL ROHLIN PROPERTY AND TOPOLOGICAL ENTROPY

By ELI GLASNER and BENJAMIN WEISS

Abstract. For a compact metric space X let $G = H(X)$ denote the group of self homeomorphisms with the topology of uniform convergence. The group G acts on itself by conjugation and we say that X satisfies the topological Rohlin property if this action has dense orbits. We show that the Hilbert cube, the Cantor set and, with a slight modification, also even dimensional spheres, satisfy this property. We also show that zero entropy is generic for homeomorphisms of the Cantor set, whereas it is infinite entropy which is generic for homeomorphisms of cubes of dimension $d \geq 2$ and the Hilbert cube.

0. Introduction. Let $(X, \mathcal{X}, \mu, T)$ be an aperiodic probability measure preserving system with μ nonatomic. Given $\epsilon > 0$ and a positive integer n, Rohlin's lemma tells us that there is a measurable subset $A \subset X$ such that $A, TA, T^2A, \ldots, T^{n-1}A$ are disjoint and cover X up to a set of measure less than ϵ. This simple lemma is an essential tool in ergodic theory. It is used in one way or another in most aspects of this theory. One well-known consequence of it is the following.

THEOREM. *For a nonatomic probability space $(X, \mathcal{X}, \mu)$ let G be the Polish group of measure preserving transformations with a measurable inverse, equipped with the weak topology. Then the action of the group G on itself by conjugation is topologically transitive; i.e., there exists a transformation $T \in G$ such that the set $\{STS^{-1}\colon S \in G\}$ is dense in G.*

One can consider a more general situation where a (say countable discrete) group Γ acts by measure preserving transformations on a probability space $(X, \mathcal{X}, \mu)$. Again the space $\mathbb{A} = \mathbb{A}_\Gamma$ of all such Γ-actions can be endowed with the weak topology, making it a Polish space, and the group G of all bi-measure-preserving-transformations of $(X, \mathcal{X}, \mu)$, acts on $\mathbb{A}$ by conjugation. In [GK] the following definition was introduced. Say that the group Γ has the *Rohlin property* if the action of G on $\mathbb{A}_\Gamma$ is topologically transitive. It is observed there that every amenable Γ has the Rohlin property, and the question which groups have the Rohlin property is raised. (See [GK] for more details.)

In the present work we are dealing with an analogous question in the topological context. For a compact metric space X, denote the group of self

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homeomorphisms of X by $G = H(X)$. With the topology of uniform convergence, G is a Polish topological group.

We say that a Polish topological group G has the *topological Rohlin property* (or just the Rohlin property) when it acts transitively on itself by conjugation. We say that the space X has the *Rohlin property* when $G = H(X)$ has the Rohlin property; i.e. $H(X)$ is the closure of a single conjugacy class. Which compact metric spaces possess the Rohlin property? We show that the Hilbert cube and the Cantor set have it. For some connected spaces like spheres the existence of orientation of a homeomorphism, which is clearly preserved under conjugation, means that $H(S^d)$ cannot have the Rohlin property; therefore we say that a sphere satisfies the Rohlin property when the group $H_0(S^d)$—the connected component of the identity in $H(S^d)$—has the Rohlin property. With this definition we show that even dimensional spheres have the Rohlin property. On the other hand it appears that for general compact manifolds of positive finite dimension the answer is rather different. For circle homeomorphisms, Poincaré's rotation number, $\tau\colon H^+(S^1) \to \mathbb{R}/\mathbb{Z},\ h \mapsto \tau(h)$, where $H^+(S^1) = H_0(S^1)$ is the subgroup of index 2 of orientation preserving homeomorphisms, is a continuous conjugation invariant and thus there are at least a continuum of different closed disjoint conjugation invariant subsets.

We refer the reader to the recent paper [AHK], by E. Akin, M. Hurley and J. Kennedy, for a detailed discussion of circle homeomorphisms. Their main result (on the circle) can be briefly formulated by saying that the circle has the local Rohlin property, where a space X has the *local Rohlin property* if $H(X)$ contains an open dense subset which is the union of interior of conjugacy class closures. More precisely they show that for a rational number c, the set $\tau^{-1}(c)$ has a nonempty interior in $H^+(S^1)$ and that in each such set $\tau^{-1}(c)$ there is a—necessarily unique—residual $H^+(S^1)$ conjugacy class. On the other hand for irrational rotation numbers we have the following information. Denote by $\mathcal{T}$ the set of topologically transitive homeomorphisms of S^1, then $\mathcal{T}$ is a G_δ subset of $H^+(S^1)$ on which τ takes irrational values and for an irrational number c the set $\mathcal{T} \cap \tau^{-1}(c)$ *is* the conjugacy class of the "rigid" rotation h_c. It is also easy to see that no odd dimensional sphere has the Rohlin property. For the proof it suffices to note that there are orientation preserving homeomorphisms with an attracting fixed point—hence all small perturbations have a fixed point, while there are orientation preserving homeomorphisms with no fixed points—and any small perturbation will not have one either.

The motivation for the definition of the Rohlin property came from the work [GK]. The Hilbert cube case was done during the special year in ergodic theory at the Institute for Advanced Studies of the Hebrew University in Jerusalem, 1996–97. The question regarding the Cantor set was raised recently by J. King and was answered independently by E. Akin, [AHK].

A related problem is the question: what is the topological entropy of the typical homeomorphism in $H(X)$? The machinery we develop for dealing with the Rohlin property enables us to answer the entropy problem as follows. For the Hilbert cube and spheres $S^d,\ d \geq 2$, the set of homeomorphisms with infinite entropy is residual while for the Cantor set it is the set of zero entropy which is a dense G_δ subset of $H(X)$.

The Hilbert cube is dealt with in Section 1, the Cantor set in Section 2 and in Section 3 we consider finite dimensional cubes and spheres.

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1. The Hilbert cube. Let X be a compact metric space and let $H(X)$ be the group of self homeomorphisms of X equipped with the metric:

$$\rho(S,T) = \sup_{x\in X}\{d(Sx,Tx)\} + \sup_{x\in X}\{d(S^{-1}x,T^{-1}x)\}.$$

With this metric $H(X)$ is a Polish topological group. This metric though, is not in general a right invariant metric. However, every second countable topological group admits a complete right invariant metric (see e.g. [HR], Theorem 8.2), and for convenience we shall fix such an invariant metric D on $H(X)$:

$$D(SR,TR) = D(S,T), \qquad \forall R,S,T \in H(X).$$

We say that X has the *Rohlin property* if the action of the group $H(X)$ on itself by conjugation is topologically transitive; i.e if there exists a homeomorphism $T \in H(X)$ such that the set $\{STS^{-1}\colon\ S \in H(X)\}$ is dense in $H(X)$. From general considerations it then follows that this property holds for a dense G_δ subset of $H(X)$.

THEOREM 1.1. *The Hilbert cube $Q = [-1,1]^{\mathbb{N}} = J^{\mathbb{N}}$ has the Rohlin property.*

Proof. For an element R of $H(Q)$ and a positive ϵ put

$$E(R,\epsilon) = \{T \in H(Q)\colon\ \exists S \in H(Q),\ d(S^{-1}TS,R) < \epsilon\}.$$

Clearly $E(R,\epsilon)$ is an open subset of $H(Q)$. We will show that it is also dense in $H(Q)$. If $\{R_i\}_{i=1}^{\infty}$ is a dense sequence in $H(Q)$ then by Baire's category theorem the G_δ subset

$$E = \bigcap_{i=1}^{\infty}\bigcap_{k=1}^{\infty} E(R_i, 1/k)$$

is dense and in particular nonempty. Clearly every element of E has the required property. Recall that it follows from the Brouwer fixed point theorem that every element of $H(Q)$ has a fixed point (see [M], Corollary 3.5.3).

LEMMA 1.2. *Let $T \in H(Q)$ and $x \in Q$ a fixed point for T: $Tx = x$. Then for every $\epsilon > 0$ there exist $\hat{T} \in H(Q)$ and $\delta > 0$ with*
(1) $D(\hat{T}, T) < \epsilon$.
(2) $\hat{T} \restriction B_\delta(x) = \mathrm{id}$.

Proof. Denote $B = B_\delta(x)$ and let $V = TB$. Choose $\delta > 0$ so small that

$$D(T^{-1} \restriction \partial V, \mathrm{id} \restriction \partial V) < \epsilon.$$

Since ∂B and ∂V are Z-sets, the homeomorphism $T^{-1}\colon\ \partial V \to \partial B$ can be extended to $S \in H(Q)$ with $D(S, \mathrm{id}) < \epsilon$ (see e.g. [M], Theorem 6.4.6). Put $R = ST$, then $D(R, T) = D(S, \mathrm{id}) < \epsilon$ and $ST \restriction \partial B = \mathrm{id}$. Finally let:

$$\hat{T} = \begin{cases} R, & x \notin B \\ \mathrm{id}, & x \in B. \end{cases}$$ □

We set

$$A_+ = [0,1] \times J^{\mathbb{N}}, \qquad A_- = [-1,0] \times J^{\mathbb{N}},$$

and define

$$j\colon\ A_+ \to A_-, \qquad j\colon\ (t,y) \mapsto (-t,y).$$

Then $j \in H(Q)$ is an involution which is the identity map on $A_0 := \partial A_+ = \partial A_-$.

LEMMA 1.3. *If U is an open nonempty set in Q then there exists $\phi \in H(Q)$ with $\phi(A_+) \subset U$.*

Proof. Let $x_1 = (1,1,1,\ldots) \in Q$ and set $V(k,\epsilon) = (1-\epsilon, 1]^k \times J^{\mathbb{N}\setminus\{1,2,\ldots,k\}}$ for $k > 0$ and $0 < \epsilon < 1$. If U is an arbitrary open nonempty subset of Q, choose some $x_0 \in U$ and use the homogeneity of Q ([M], Theorem 6.1.6) to get a homeomorphism $h \in H(Q)$ with $h(x_1) = x_0$. Then, with appropriate k and ϵ, we also get $h(V(k,\epsilon)) \subset U$. Thus it is enough to prove our lemma for $U = V(k,\epsilon)$. Since clearly $V(k,\epsilon)$ is homeomorphic to $V(1,\epsilon) = (1-\epsilon, 1] \times J^{\mathbb{Z}}$, it is enough to prove the lemma for $V(1,\epsilon)$. For this special case however, the lemma is evident. □

To complete our proof we only need to prove the following:

LEMMA 1.4. *For every $T_2 \in H(Q)$ and $\epsilon > 0$ the set $E(T_2,\epsilon)$ is dense in $H(Q)$.*

Proof. Fix $T_2 \in H(Q)$ and let T_1 be an arbitrary element of $H(Q)$. Given $\delta > 0$ we will construct an element $P \in E(T_2,\epsilon)$ with $D(P, T_1) < \delta$.

Starting with T_1 use Lemma 1.2 to get $\hat{T}_1 \in H(Q)$ with

$$D(T_1, \hat{T}_1) < \delta/2 \quad \text{and} \quad \hat{T}_1 \restriction B_1 = \text{id},$$

where $B_1 = B_{\delta_1}(x_1)$ for x_1 a fixed point of T_1 and some $0 < \delta_1 < \delta/4$.

Again use Lemma 1.2 to produce a homeomorphism $\hat{T}_2$ with

$$D(T_2, \hat{T}_2) < \epsilon/2, \quad \text{and} \quad \hat{T}_2 \restriction B_2 = \text{id},$$

where $B_2 = B_{\delta_2}(x_2)$ for a fixed point x_2 of T_2 and some $\delta_2 < \epsilon/4$. Next choose $h \in H(Q)$ with $h(x_2) = x_1$ and take δ_2 small enough to ensure that $h(B_2) \subset B_1$. Use Lemma 1.3 to find a homeomorphic copy of A_+ in $h(B_2)$. Identifying A_+ with its image in $h(B_2)$, we let $j\colon A_+ \to (A_+)^c$ be the corresponding involution.

Set

$$S = h^{-1}j,$$

and

$$V = S^{-1}\hat{T}_2 S = jh\hat{T}_2 h^{-1} j.$$

For $x \notin A_+$ we have

$$jx \in A_+ \subset h(B_2) \Rightarrow h^{-1}jx \in B_2 \Rightarrow \hat{T}_2 h^{-1}jx = h^{-1}jx$$

and $Vx = jhh^{-1}jx = x$; i.e

$$V \restriction (A_+)^c = \text{id}\,.$$

Put

$$P = \hat{T}_1 V,$$

then on $h(B_2)^c \supset B_1^c$, $P = \hat{T}_1$. Since P leaves B_1 invariant, we still have

$$D(T_1, P) \le D(T_1, \hat{T}_1) + D(\hat{T}_1, P) < \delta.$$

Now

$$\begin{aligned} SPS^{-1} &= S\hat{T}_1 V S^{-1} = h^{-1}j\hat{T}_1 V jh \\ &= h^{-1}j\hat{T}_1 jh\hat{T}_2 h^{-1}jjh = h^{-1}j\hat{T}_1 jh\hat{T}_2. \end{aligned}$$

For $x \notin B_2$, $\hat{T}_2 x \notin B_2$ hence $h\hat{T}_2 x \notin h(B_2)$. Hence

$$\begin{aligned} &h\hat{T}_2 x \notin A_+ \Rightarrow jh\hat{T}_2 x \in A_+ \Rightarrow \hat{T}_1 jh\hat{T}_2 x = jh\hat{T}_2 x \Rightarrow \\ &SPS^{-1}x = h^{-1}j\hat{T}_1 jh\hat{T}_2 x = h^{-1}jjh\hat{T}_2 x = \hat{T}_2 x. \end{aligned}$$

For $x \in B_2$, $\hat{T}_2 x = x$. If $h(x) \in h(B_2)\setminus A_+$ then

$$jh(x) \in A_+ \subset B_1 \Rightarrow \hat{T}_1 jhx = jhx \Rightarrow$$
$$SPS^{-1}x = h^{-1}jjhx = x.$$

On the other hand if $h(x) \in A_+$ then

$$jhx \notin A_+ \Rightarrow j\hat{T}_1 hx \in A_+ \subset h(B_2) \Rightarrow$$
$$SPS^{-1}x = h^{-1}j\hat{T}_1 hx \in h^{-1}h(B_2) = B_2.$$

Since $\operatorname{diam} B_2 < \epsilon/2$ we get

$$D(SPS^{-1}, T_2) \leq D(SPS^{-1}, \hat{T}_2) + D(\hat{T}_2, T_2) < \epsilon. \qquad \square$$

The proof of the Theorem is now complete. $\square$

THEOREM 1.5. *For the Hilbert cube Q, the set $\mathcal{I} \subset H(Q)$ of homeomorphisms having infinite topological entropy contains a dense G_δ subset of $H(Q)$.*

Proof. Our starting point is that in the two dimensional cube $D = [-1, 1]^2$

(1) for every $k \geq 2$ there is a "horse shoe" homeomorphism $\phi \in H(D)$ with topological entropy $h_{\text{top}}(\phi) \geq \log_2 k$, with $\phi \restriction \partial D = \text{id}$ and such that

(2) there exists an open set V around ϕ with $h_{\text{top}}(\psi) \geq \log_2 k$ for every $\psi \in V$.

Given an open ball $B \subset Q$ we can find a homeomorphic copy $\tilde{D}$ of $D \times Q$ inside B, then define a homeomorphism $S \in H(Q)$ such that $S = \text{id}$ outside $\tilde{D}$ and S mimics $\phi \times \text{id}$ on $\tilde{D}$.

Now by Lemma 1.2, starting with an arbitrary element $T \in H(Q)$ and using the ball B on which $\hat{T}$ acts as the identity, we define S as above and let $\tilde{T} = \hat{T}S$. Clearly $D(T, \tilde{T}) < \epsilon$ (with ϵ as in Lemma 1.2) and for $\tilde{T}$, as well as for every homeomorphism in a neighborhood of $\tilde{T}$, the topological entropy is $\geq \log_2 k$.

We now conclude that the set

$$E(k) = \{T \in H(Q) \colon \; h_{\text{top}}(T) \geq \log_2 k\}$$

contains an open dense subset of $H(Q)$. Thus the set $\mathcal{I}$ of homeomorphisms of Q with infinite topological entropy contains the dense G_δ subset $E = \bigcap_{k=2}^{\infty} E(k)$. $\square$

Remark. Again we refer to the work [AHK] for more details on the prevalence of "topological horse-shoes" for manifold homeomorphisms.

2. The Cantor set. In this section we let X be the Cantor set equipped with some compatible metric d with $\operatorname{diam}(X) = 1$. As in the Hilbert cube case we let D be a complete right invariant metric on $H(X)$.

Definition. We say that the homeomorphism $S \in H(X)$ is *simple* if it satisfies the following conditions.

(1) There exist a finite number of nonempty clopen sets U_j and periods $r_j \geq 1, j = 1, 2, \dots k$, such that $S^{r_j} \restriction U_j = \mathrm{id}$ and the collection $\{S^i U_j\colon\ i = 0, 1, \dots r_j - 1;\ j = 1, 2, \dots, k\}$ is pairwise disjoint.

(2) There exists a finite number of clopen subsets $V_s, s = 1, 2, \dots l$, and for each s two periodic points v_s^+ and v_s^- with periods $q_s^{\pm} \geq 1$, such that the two periodic orbits are disjoint. The sets $S^n V_s,\ n \in \mathbb{Z},\ 1 \leq s \leq l$ are pairwise disjoint and the orbits of $V_s^{\pm}$ spiral towards the periodic orbits $\{S^i v_s^{\pm}\}$; i.e. $\lim_{n \to \pm\infty} d(S^n V_s, S^n v_s^{\pm}) = 0$.

(3) The whole space X is represented as:

$$
\begin{aligned}
X = & \bigcup_{j=1}^{k} \bigcup_{i=0}^{r_j - 1} S^i U_j \\
& \cup \bigcup_{s=1}^{l} \bigcup_{n \in \mathbb{Z}} S^n V_s \\
& \cup \bigcup \{S^i v_s^{\pm}\colon\ 0 \leq i \leq q_s^{\pm} - 1,\ s = 1, 2, \dots, l\},
\end{aligned}
$$

a disjoint union.

We observe that the numbers k, $r_j \geq 1, j = 1, 2, \dots, k$, l and $q_s^{\pm}, s = 1, 2, \dots, l$, determine the isomorphism type of a simple homeomorphism: i.e., if S and T are two simple homeomorphisms in $H(X)$ with the same set of periods as above, then there exists a homeomorphism $h \in H(X)$ with $T = hSh^{-1}$. We say that $[r_j,\ j = 1, 2, \dots, k;\ q_s^{\pm}, s = 1, 2, \dots, l]$ is the *set of periods* of T.

PROPOSITION 2.1. *Let (X, T) be a dynamical system and let $\delta > 0$ be given, then there exist a positive integer N, a finite number of points $u_1, u_2, \dots, u_k \in X$, clopen neighborhoods $B(u_j)$, and numbers $r_j \geq 2$ such that for every $1 \leq j \leq k$:*

(i) *$B(u_j), TB(u_j), \dots, T^{r_j - 1} B(u_j)$ are disjoint.*

(ii) $\operatorname{diam} T^i B(u_j) < \delta, \quad 0 \leq i \leq r_j - 1.$

(iii) $\operatorname{diam} (T^{r_j} B(u_j) \cup B(u_j)) < \delta.$

and

(iv) *for every $x \in X$ there exist j_1 and j_2 such that $T^m x \in B(u_{j_1})$ and $T^{-l} x \in T^{r_{j_2}} B(u_{j_2})$ for some $0 \leq m, l \leq N$.*

Proof. Let x be an arbitrary point in X. By compactness it follows that there exist positive integers $m \geq 0$, and $r \geq 2$ such that, with $u = T^m x$, we have $d(u, T^r u) < \delta/2$. We can now choose a clopen neighborhood $B(u)$ of the point u such that, with $r = r(u)$, the conditions (i)–(iii) are satisfied. Clearly

$$X = \bigcup \{T^{-m(x)} B(u)\colon\ u = u(x), x \in X\}.$$

By compactness we get a finite subcover of the form

$$X = \bigcup \{T^{-m_j} B(u_j)\colon\ j = 1, 2, \ldots, k\}.$$

Repeating the argument with T^{-1}, and adding the finite cover we obtained this way to the previous one, we get our claim (with $N = \max\{m_j\}$). □

THEOREM 2.2. *The set of simple homeomorphisms is dense in $H(X)$.*

Proof. Let $T \in H(X)$ and $\epsilon > 0$ be given; we will construct a simple $S \in H(X)$ with $D(T, S) < \epsilon$. Choose a $\delta > 0$ with $2\delta < \epsilon$, such that

$$\forall x, y \in X,\ d(x, y) < \delta \Rightarrow d(T^j x, T^j y) < \epsilon,\ |j| \le 1.$$

(1) Applying Proposition 2.1 to the dynamical system (X, T) we get a positive integer N, a finite number of points $u_1, u_2, \ldots, u_k \in X$, clopen neighborhoods $B_j = B(u_j)$, and numbers $r_j \ge 2$ satisfying the conditions (i)–(iv) of the proposition. We wish to make these "periodic orbits" $\{T^i B_j\colon\ 0 \le i < r_j\}$ disjoint while preserving the property (iv). To this end, suppose that we have already modified $B_j,\ j = 1, \ldots, k$ to clopen $B_j^p \subset B_j$ for $p < k$ so that the sets $C_j^p = \bigcup_{i=0}^{r_j-1} T^i B_j^p$ are disjoint for $j = 1, \ldots, p$, and so that (iv) still holds for the modified sets. We take up B_{p+1} and study for each $1 \le j \le p$ the intersections between $C_{p+1} = \bigcup_{i=0}^{r_{p+1}-1} T^i B_{p+1}$ and C_j^p. Focus on $A = C_j^p \cap (B_{p+1} \cup T^{r_{p+1}-1} B_{p+1})$ and remove a clopen set from B_j^p to define B_j^{p+1} so that $C_j^{p+1} = \bigcup_{i=0}^{r_j-1} T^i B_j^{p+1}$ is now disjoint from C_{p+1}.

It is easily seen that property (iv) remains valid with this modification. Note that $B_j^{p+1} = B_j$ for $j \ge p+1$. When we have finished we will have formed $B'_j = B_j^k \subset B_j$ so that all the sets $\{T^i B'_j\colon\ i = 0, \ldots, r_j - 1,\ j = 1, \ldots, k\}$ are disjoint and property (iv) still holds. It may be that some B'_j are empty; we assume in the sequel that they are not; clearly a relabelling will achieve this. To simplify the notation we write $B'_j = B_j$. Notice that for every i and j

$$\forall x, y \in T^i B_j,\ d(T^j x, T^j y) < \epsilon,\ |j| \le 1.$$

Now set

$$Z = \bigcup \{T^i B_j\colon\ i = 0, 1, \ldots r_j - 1,\ j = 1, 2, \ldots, k\}.$$

If $Z = X$, we define $U_j = B_j,\ j = 1, 2, \ldots k$ and set $S = T$ on $\bigcup \{T^i U_j\colon\ i = 0, 1, \ldots r_j - 2,\ j = 1, 2, \ldots, k\}$. Finally on $T^{r_j - 1} U_j$ we let $S = T^{-(r_j-1)},\ j = 1, 2, \ldots, k$, so that $S^{r_j} \restriction U_j = \mathrm{id}$. It is now easy to check that S is simple and that $D(S, T) < \epsilon$.

(2) If the clopen set $Z = \bigcup\{T^i B_j\colon\ i = 0, 1, \ldots r_j - 1,\ j = 1, 2, \ldots, k\}$ is not all of X, then $Y = X \setminus Z$ is a nonempty clopen set. For $y \in Y$ there are $i < 0 < j$ such that $\{T^i y, T^{i+1} y, \ldots, T^j y\} \subset Y$ but $T^{i-1} y, T^{j+1} y \notin Y$. Choose a clopen neighborhood $V \subset Y$ of y such that the sets $T^n V,\ i - 1 \le n \le j + 1$ are pairwise disjoint, $T^n V,\ i \le n \le j$ are contained in Y while $T^{i-1} V, T^{j+1} V \subset Z$, and each of them is of diameter $< \delta$. Passing to a finite subcover and disjointifying we obtain a finite collection $y_s \in V_s,\ s = 1, 2, \ldots, l$, such that

$$\{T^n V_s\colon\ i_s \le n \le j_s,\ s = 1, 2, \ldots, l\}$$

is a cover of Y by pairwise disjoint clopen sets.

(3) Next consider the point $T^{j_1+1} y_1 \in Z$. There exists a unique $1 \le j \le k$ with $T^{j_1+1} y_1 \in B = B_j$. We write B as a disjoint union $B = \{v^+\} \cup \bigcup_{t=0}^{\infty} C_t$ with C_t clopen and $\lim_{t \to +\infty} \operatorname{diam}(C_t \cup \{v^+\}) \to 0$. Define S on $T^{j_1} V_1$ as any homeomorphism from this set onto C_1. Now let $S = T$ on $T^m C_1$ for $m = 0, 1, \ldots, r_j - 2$ and let $S \restriction T^{r_j - 1} C_1$ be an arbitrary homeomorphism from that set onto C_2. Next let $S = T$ on $T^m C_2$ for $m = 0, 1, \ldots, r_j - 2$ and let $S \restriction T^{r_j - 1} C_2$ be an arbitrary homeomorphism from that set onto C_3, etc.. Thus the orbit of V_1 under positive powers of S spirals down to the target periodic orbit $v^+, T v^+ = S v^+, \ldots, T^{r_j - 1} v^+ = S^{r_j - 1} v^+, v^+ = S^{r_j} v^+$. Notice that we have not yet defined S on the "spare" set $C_0 \subset B = B_j$.

We repeat the same construction with the point $T^{i_1} y_1$ which falls into one of the sets $T^{r_i - 1} B_i$, only going backwards with T^{-1} instead of T. If i happens to be equal to j, whether $T^{i_1 - 1} y_1$ falls in $T^{r_i - 1} C_0$ or not, we proceed by presenting the spare set $T^{r_i - 1} C_0$ as a disjoint union $T^{r_i - 1} C_0 = \{v^-\} \cup \bigcup_{t=0}^{\infty} C_{0,t}$, defining S^{-1} on $T^{i_1} V_1$ as any homeomorphism from that set onto $C_{0,1}$ and then spiraling down, with iterations of S^{-1}, to the target periodic orbit $v^-, T v^- = S v^-, \ldots, T^{r_j - 1} v^- = S^{r_j - 1} v^-, v^- = S^{r_j} v^-$. Again S is not yet defined on the spare set $C_{0,0} \subset B_j$.

Inductively S is defined on the rest of the space and we end up with a decomposition:

$$\begin{aligned} X = &\bigcup_{j=1}^{k} \bigcup_{i=0}^{r_j - 1} S^i U_j \\ &\cup \bigcup_{s=1}^{l} \bigcup_{n \in \mathbb{Z}} S^n V_s \\ &\cup \bigcup \{S^i v_s^{\pm}\colon\ 0 \le i < r_{j(v_s^{\pm})},\ s = 1, 2, \ldots, l\}, \end{aligned}$$

where, for $1 \le j \le k$, the S-periodic nonempty clopen set $U_j \subset B_j$ is the spare set left in B_j at the end of our (finite) inductive procedure. Our construction of S is now complete and it is easy to see that S is simple and that $D(S, T) < \epsilon$. $\square$

Remark 2.3. Observe that the proof of Theorem 2.1 shows in fact that the set of simple homeomorphisms with periods $[r_j,\ j = 1, 2, \ldots, k;\ r_{j(s^{\pm})}, s = 1, 2, \ldots, l]$ (i.e., the periods of the cycles associated with the spiraling orbits, $q_s^{\pm}$, are chosen from among the set of periods of the periodic clopen sets $\{r_j\colon\ 1 \le j \le k\}$) is also dense in $H(X)$. From now on we will only deal with simple homeomorphisms of this type. We will refer to the sets $U_j,\ j = 1, 2, \ldots, k$ as the *periodic sets*; to the sets $V_s,\ s = 1, 2, \ldots, l$ as the *spiraling sets*; to the set $\{r_j\colon\ j = 1, 2, \ldots, k\}$, as the set of *basic periods*, and to the periods of the corresponding periodic points $v_s^{\pm}$, $\{r_{j(s^{\pm})}\colon\ s = 1, 2, \ldots, l\}$ as the *target periods*.

LEMMA 2.4. *For every compact metric space Z the set of homeomorphisms of Z having zero topological entropy,* $\mathcal{Z} = \{T \in H(Z)\colon\ h_{\mathrm{top}}(T) = 0\}$*, is a* G_δ *subset of* $H(Z)$.

Proof. Let $\mathcal{U}$ be a finite open cover of Z, $\epsilon > 0$, and $n \ge 1$ a positive integer. Then, clearly the set

$$F(\mathcal{U}, \epsilon, n) = \left\{T \in H(Z)\colon\ \frac{1}{n} \log\left(N(\mathcal{U}_{-n}^{n}, T)\right) < \epsilon\right\}$$

is an open subset of $H(Z)$. (As usual, $\mathcal{U}_{-n}^{n} = \bigvee_{|j| \le n} T^j\mathcal{U}$ is the joint refinement of the covers $T^j\mathcal{U}$, and $N(\mathcal{V})$ denotes the least cardinality of a subcover of the cover $\mathcal{V}$.) Now chose a sequence $\mathcal{U}_j$ of finite open covers with $\operatorname{diam}(\mathcal{U}_j) \to 0$ and observe that

$$\mathcal{Z} = \bigcap_{k=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcup_{n=1}^{\infty} F(\mathcal{U}_j, 1/k, n),$$

is a G_δ set. □

COROLLARY 2.5. *For the Cantor set X, the set* $\mathcal{Z} \subset H(X)$ *of homeomorphisms having zero topological entropy is a dense* G_δ *subset of* $H(X)$.

Proof. Clearly every simple homeomorphism has zero topological entropy. Now Theorem 2.2 and Lemma 2.4 complete the proof of the corollary. □

THEOREM 2.6. *The Cantor set has the Rohlin property.*

Proof. As in the case of the Hilbert cube, it is enough to show that for an arbitrary element R of $H(X)$ and a positive ϵ, the open set

$$E(R, \epsilon) = \{T \in H(X)\colon\ \exists S \in H(X),\ d(S^{-1}TS, R) < \epsilon\}$$

is dense in $H(X)$. Moreover, by Theorem 2.2 and Remark 2.4, it is enough to show that for any T_1 and T_2, simple homeomorphism and $\epsilon > 0$, there exist simple $S_1, S_2 \in H(X)$ such that S_1 and S_2 have the same set of periods, $[r_j,\ j = 1, 2, \ldots, k;\ r_{j(s^{\pm})}, s = 1, 2, \ldots, l]$, and such that $D(T_i, S_i) < \epsilon,\ i = 1, 2$.

LEMMA 2.7. *Given a simple homeomorphism R in $H(X)$ with periods $[r_j,\ j = 1, 2, \dots k;\ r_{j(s^\pm)}, s = 1, 2, \dots, l]$, and an $\epsilon > 0$,*

(1) *There exists, for every $k' \geq k$ and $l' \geq l$, a simple homeomorphism $S \in H(X)$ with periods $[r_j,\ j = 1, 2, \dots, k';\ r_{j(s^\pm)}, s = 1, 2, \dots, l']$, and such that $D(R, S) < \epsilon$.*

(2) *There exists, for every set of positive integers $a_j,\ j = 1, 2, \dots k$, a simple homeomorphism $S \in H(X)$ with periods*

$$[a_j r_j \geq 1, j = 1, 2, \dots, k;\ a_{j(s^\pm)} r_{j(s^\pm)}, s = 1, 2, \dots, l],$$

and such that $D(R, S) < \epsilon$.

Proof. (1) It is enough to show this for $k' = k+1$ and $l' = l+1$. In order to get $k+1$ we need not change R at all; all we have to do is split one of the periodic sets U_j into two clopen nonempty disjoint sets, $U_j = U_{j'} \cup U_{j''}$. Of course we then have $r_j = r_{j'} = r_{j''}$.

In order to get one more spiraling set, again we split one of the spiraling sets into two clopen nonempty disjoint sets $V_s = V_{s'} \cup V_{s''}$. However, now we pick a positive integer m such that $\operatorname{diam}(R^{m+i} V_s \cup \{R^{m+i} v_s^+\}) < \epsilon$ for all $i \geq 0$. On the complement in X of

$$R^{m+r_j-1} V_{s'} \cup R^{m-1} V_{s''} \cup \bigcup \{R^{m+i} V_{s''}\colon\ i = 0, 1, 2, \dots, r_j - 1\},$$

where $r_j = r_{j(s^+)}$ is the target period corresponding to v_s^+, we set $S = R$. We define S on $R^{m+r_j-1} V_{s'}$ as any homeomorphism of this set onto $R^{m+r_j} V_s$. Next we represent each of the sets $\{R^{m+i} V_{s''}\colon\ i = 0, 1, 2, \dots, r_j - 1\}$ as a disjoint union

$$R^{m+i} V_{s''} = \{v^+_{s'',i}\} \cup \bigcup_{t=0}^{\infty} C_{t,i},$$

define S on the new periodic orbit as $S^i v^+_{s'',0} = v^+_{s'',i}$, $S^i v^+_{s'',r_j-1} = v^+_{s'',0}$, and let S spiral down to this new target periodic orbit along the $C_{t,i}$, starting with $C_{0,0}$. The definition of S is completed by mapping $R^{m-1} V_{s''}$ homeomorphically onto $C_{0,0}$. Finally a similar construction is applied to the negative orbit of V_s under R^{-1}.

(2) We first deal with the periodic sets of R. It is clearly enough to consider a single periodic set U of R with period r. Now define $S = R$ on the complement of the periodic R-orbit of U. We will define S on this orbit in such a way that it will become a periodic set for S with period ar and so that $D(R, S) < \epsilon$. One way to do this is as follows: Represent U as a disjoint union of say d clopen sets $D_j,\ j = 1, 2, \dots, d$, each of diameter less than ϵ. Next represent each D_j as a

disjoint union of a nonempty clopen sets $D_{j,i},\ i = 1, 2, \ldots, a$, and let S map these sets periodically:

$$
\begin{aligned}
&D_{j,1} \to RD_{j,1} \to R^2 D_{j,1} \to \cdots \to R^{r-1} D_{j,1} \to \\
&D_{j,2} \to RD_{j,2} \to R^2 D_{j,2} \to \cdots \to R^{r-1} D_{j,2} \to \\
&\ldots \\
&D_{j,a} \to RD_{j,a} \to R^2 D_{j,a} \to \cdots \to R^{r-1} D_{j,a} \to D_{j,1}.
\end{aligned}
$$

Finally we have to deal with the case of changing the target period r of R on the spiraling set V to a target period ar of S. We do this by a combination of the methods described above. Our first step is to find, using part (1) of the lemma, R' $\epsilon/2$ close to R with a (positively) spiraling sets $V_1, V_2, \ldots, V_a$ and target periodic orbits $\{R'^i v_j^+ \colon\ 0 \le i \le r-1,\ 1 \le j \le a\}$. Moreover we can assume that all of these target orbits are $\epsilon/2$ close; i.e.

$$\max\{d(R'^i v_s^+, R'^i v_t^+) \colon\ 0 \le i \le r-1,\ 1 \le s, t \le a\} < \epsilon/2.$$

Next change R' to S with $D(R', S) < \epsilon/2$ in such a way that the S orbit of the R'-spiraling set V_1 spirals towards the ar S-periodic orbit

$$
\begin{aligned}
&v_1^+ \to S v_1^+ \to S^2 v_1^+ \to \cdots \to S^{r-1} v_1^+ \to \\
&S^r v_1^+ = v_2^+ \to S^{r+1} v_1^+ = R' v_2^+ \to \cdots \to S^{2r-1} v_1^+ = R'^{r-1} v_2^+ \to \\
&\ldots \\
&S^{(a-1)r} v_1^+ = v_a^+ \to S^{(a-1)r+1} v_1^+ = R' v_a^+ \to \cdots \to S^{ar-1} v_1^+ = R'^{r-1} v_a^+ \to v_1^+.
\end{aligned}
$$

A similar construction for the negative orbits completes the proof of the lemma. □

We can now complete the proof of Theorem 2.6. We are given T_1 and T_2 simple homeomorphism with sets of periods, $[r_j^{(i)},\ j = 1, 2, \ldots, k_i;\ r_{j(s\pm)}^{(i)}, s = 1, 2, \ldots, l_i],\ i = 1, 2$ respectively, and $\epsilon > 0$. We want to show that there exist simple $S_1, S_2 \in H(X)$ such that S_1 and S_2 have the same set of periods, $[r_j,\ j = 1, 2, \ldots, k;\ r_{j(s\pm)}, s = 1, 2, \ldots, l]$, and such that $D(T_i, S_i) < \epsilon,\ i = 1, 2$. Now by Lemma 2.7 we can find such S_i with a common set of periods $[r_j,\ j = 1, 2, \ldots, k;\ r_{j(s\pm)}, s = 1, 2, \ldots, l]$, where $k = \max(k_1, k_2)$, $l = \max(l_1, l_2)$, and $r_j = r_j^{(1)} \cdot r_j^{(2)},\ j = 1, 2, \ldots, k$. □

3. On finite dimensional cubes and spheres. Let G be the group of homeomorphisms of the cube I^d which fix each point of the boundary ∂I^d. The following lemma is the finite-dimensional analogue of Lemma 1.1 on the Hilbert cube.

LEMMA 3.1. *Let T: $U \to V$ be a homeomorphism between two open neighborhoods U and V of a point $x_0 \in U \subset \mathbb{R}^d$ with $Tx_0 = x_0$. Then for every $\epsilon > 0$, there exists a homeomorphism S: $U \to V$ and a $\delta > 0$ such that $D(S, T) < \epsilon$ and $Sx = x$, for every $x \in B_\delta(x_0)$. Moreover, the homeomorphism S coincides with T outside a ball of radius ϵ. Thus this assertion holds on any finite-dimensional manifold M instead of R^d.*

Proof. In this proof we let, for any $\eta > 0$, $B_\eta = B_\eta(x_0)$, and $C_\eta = \partial B_\eta(x_0)$ the sphere of radius η around x_0. Choose $0 < \eta < \epsilon$ such that $TB_\eta \subset B_{\epsilon/2}$ and let $\delta > 0$ satisfy: $B_\delta \subset B_{\eta/2} \cap TB_\eta$. The annulus theorem (see e.g. [E]) provides a homeomorphism f: $A_{\delta,\eta} \to \tilde{C}$, where $A_{\delta,\eta}$ is the (d-dimensional) annulus with inner diameter δ and outer diameter η and $\tilde{A}$ the topological annulus bounded by C_δ and TC_η. Of course we can assume that $f \restriction C_\eta = T \restriction C_\eta$ (let h: $A_{\delta,\eta} \to \tilde{A}$ be an arbitrary homeomorphism, then let ϕ: $C_\eta \to C_\eta$, $\phi := h^{-1} \circ T \restriction C_\eta$; now extend ϕ to a homeomorphism Φ: $A_{\delta,\eta} \to A_{\delta,\eta}$, e.g. by scaling ϕ down to C_t for $\delta \le t \le \eta$, and put $f = h \circ \Phi$).

The following notation will ease the reading of the rest of the proof. For a map Φ defined on an annulus let Φ_0 and Φ_1 denote the restriction of Φ to the inner and outer bounding spheres respectively. We also write $\Phi \sim (\Phi_0, \Phi_1)$ to indicate this relation. Using this notation we have $f \sim (f_0, T \restriction C_\eta)$. Note that the latter relation and our assumption that T is orientation preserving imply that the map f_0: $C_\delta \to C_\delta$ is orientation preserving and therefore that there exists an isotopy g: $A_{\delta,\eta} \to A_{\delta,\eta}$ such that $g \sim (f_0^{-1}, \mathrm{id})$. In order to see this we refer to [BG], Remark 1, page 13. In this paper the authors show that the statement (i) "the annulus conjecture is true in dimensions less than or equal to n," is equivalent to the statement (ii) "$SH(S^k) = H^+(S^k)$ for $k \le n$". Here $H^+(S^k)$ is the subgroup of orientation preserving homeomorphisms and $SH(S^k)$ the subgroup of stable homeomorphisms (i.e. those homeomorphisms that can be written as a finite product of homeomorphisms having a nonempty open set of fixed points), the latter (as is well known and also easy to see) is arcwise connected. Since the annulus conjecture is now proven in all dimensions (including 4), this shows that $H^+(S^k)$ is arcwise connected for all k.

Define F: $A_{\delta,\eta} \to \tilde{A}$ by $F = f \circ g$, then $F \sim (\mathrm{id}, T_1)$ and clearly now the map S: $\mathbb{R}^d \to \mathbb{R}^d$ defined by

$$S = \begin{cases} \mathrm{id} & \text{on } B_\delta \\ F & \text{on } A_{\delta,\eta} \\ T & \text{on } B_\eta^c, \end{cases}$$

is the required homeomorphism S. □

LEMMA 3.2. *For every $T \in G$ and $\epsilon > 0$ there exists $\hat{T} \in G$ with $D(T, \hat{T}) < \epsilon$ and such that $\hat{T}$ fixes every point in an ϵ neighborhood of ∂I^d.*

Proof. Define $\tilde{T}$ on $I^d_\epsilon = [-\epsilon, 1+\epsilon]^d$ as T on I^d and id on the rest of I^d_ϵ. Now scale everything affinely down to I^d to define $\hat{T}$. □

THEOREM 3.3. *The group G of homeomorphisms of the cube I^d which fix each point of the boundary ∂I^d has the Rohlin property.*

Proof. Let $\mathcal{U}_i$, $i = 1, 2$ be two nonempty open sets in G. It suffices to find $S \in G$ such that $S\mathcal{U}_1 S^{-1} \cap \mathcal{U}_2 \neq \emptyset$. It is clear now from Lemma 3.2 that we can find an $\epsilon > 0$ and homeomorphisms $T_i \in \mathcal{U}_i$ with the following properties: (i) each T_i acts as the identity on an ϵ neighborhood V of ∂I^d, and (ii) the ϵ ball in G around T_i is contained in $\mathcal{U}_i$, $i = 1, 2$.

Now choose two ϵ-balls B_1 and B_2 whose closures are mutually disjoint and also disjoint from ∂I^d. Let ϕ_i be homeomorphisms in G such that (1) $\phi_i(V^c) \subset B_i$, $i = 1, 2$, (2) $\phi_1^{-1}(B_2)$ is an ϵ-ball contained in V, and (3) $\phi_2^{-1}(B_1)$ is an ϵ-ball contained in V.

Set $R_i = \phi_i T_i \phi_i^{-1}$, $i = 1, 2$ (so that R_i mimics T_i inside B_i and acts as the identity on B_i^c). Set $R = R_1 R_2$. It is now easy to check that (a) $D(\phi_1^{-1} R \phi_1, T_1) < \epsilon$ and (b) $D(\phi_2^{-1} R \phi_2, T_2) < \epsilon$. Thus $T_i' := \phi_i^{-1} R \phi_i \in \mathcal{U}_i \, i = 1, 2$, and for $S = \phi_2^{-1}\phi_1$ we get: $S T_1' S^{-1} = T_2'$. □

THEOREM 3.4. *Every even dimensional sphere S^{2d} has the Rohlin property.*

Proof. Again it suffices to show that for every two nonempty open sets $\mathcal{U}_i$, $i = 1, 2$ in $H = H_0(S^{2d})$, there is an $S \in H$ such that $S\mathcal{U}_1 S^{-1} \cap \mathcal{U}_2 \neq \emptyset$. Now every element of H has a fixed point in S^{2d} and Lemma 3.2 implies that there exist $\epsilon > 0$ and elements $T_i \in \mathcal{U}_i$, $i = 1, 2$ with the properties: (i) there exists an ϵ ball B_i in S^{2d} on which T_i acts as the identity, and (ii) the ϵ ball in H around T_i is contained in $\mathcal{U}_i$.

Now $S^{2d} \backslash B_i$ is homeomorphic to I^{2d} and fixing homeomorphisms, say ϕ_i: $I^{2d} \to S^{2d} \backslash B_i$, we obtain 1-1 correspondences between $H(I^{2d})$ and the subgroups H_i of H which fix every point of B_i, $i = 1, 2$. Denote by $\tilde{T}_i$ the element of $H(I^{2d})$ corresponding to T_i and use Lemma 3.1 to obtain elements $\hat{T}_i \in H(I^{2d})$ with the properties (i),(ii) of the proof of Theorem 3.3. Next follow this proof to construct $T_1', T_2', \hat{S} \in H(I^{2d})$, with $D(\hat{T}_i', \hat{T}_i) < \epsilon$, $i = 1, 2$, and $\hat{S}^{-1} \hat{T}_1' \hat{S} = \hat{T}_2$. Finally pull $\hat{S}$ back to H to obtain the required S. □

THEOREM 3.5. *For the cube I^d ($d \geq 2$), the set $\mathcal{I} \subset H(I^d)$ of homeomorphisms having infinite topological entropy contains a dense G_δ subset of $H(I^d)$.*

Proof. Use Brouwer's fixed point theorem and Lemma 3.1 and then follow the proof of Theorem 1.5. (Every $T \in H(I^d)$ can be extended to a homeomorphism $\hat{T} \in H(\mathbb{R}^d)$. For example we can replace I^d by the closed unit ball and then define $\hat{T}$ on the sphere of radius $r > 1$ to mimic the action of T on S^{d-1}. Now if the fixed point of $T \in H(I^d)$ is on the boundary we can extend it to $\hat{T}$ and then use Lemma 3.1.) □

Our next goal is to show that infinite entropy is generic in the groups $H(S^d)$ ($d \geq 2$). Since homeomorphisms of odd dimensional spheres need not have fixed points we need a variation on Lemma 3.1 dealing with periodic points rather than fixed points. Call a homeomorphism S *robustly periodic* if there is a nonempty open set U and an integer $q \geq 1$ such that $(S \restriction U)^q = \mathrm{id}$. The next Rohlin type lemma is our main tool:

LEMMA 3.6. *For $d \geq 2$, the subset of robustly periodic homeomorphisms is dense in $H(S^d)$.*

Proof. Let T, an element of $H(S^d)$, and $\epsilon > 0$ be given; we will show that an ϵ neighborhood of T contains a robustly periodic homeomorphism. If T has a fixed point, Lemma 3.1 applies; otherwise since by compactness, the set of nonwandering points for T is nonempty, we can find a point x_0 and a positive integer $q \geq 2$ such that the points $x_0, Tx_0, \ldots T^{q-1}x_0$ are distinct and $|T^q x_0 - x_0| < \epsilon/2$. Choose $0 < \eta < \epsilon/4$ with the property that for every $0 \leq j \leq q-1$, $T^j B_\eta(x_0) \subset B_{\epsilon/2}(T^j x_0)$ and the sets $T^j B_\eta(x_0)$ are mutually disjoint.

Put $\tilde{C} = T^q C_{\eta/2}(x_0) \subset B_{\epsilon/2}(x_0)$, where again we let $C_\delta = C_\delta(x_0)$ be the sphere of radius δ centered at x_0. Let R be a homeomorphism such that $R \restriction B_{\epsilon/2}(x_0)^c \equiv \mathrm{id}$ and $R\tilde{C} \subset B_{\eta/4}(x_0)$. Set $S_0 = RT$, then

$$D(S_0, T) < \epsilon/2, \quad \text{and} \tag{1}$$

$$S_0^q B_{\eta/2}(x_0) \subset B_{\eta/4}. \tag{2}$$

Let $\theta > 0$ satisfy $B_\theta(x_0) \subset \operatorname{int} S_0^q B_{\eta/2}(x_0)$ and define ϕ from $S_0^{q-1} C_\theta$ onto C_θ by

$$\phi = S_0^{-(q-1)}\colon\ S_0^{q-1} C_\theta \to C_\theta.$$

Let ϕ denote as well the map from $S_0^{q-1} C_{\eta/2}$ onto $S_0^q C_{\eta/2}(x_0)$,

$$\phi = S_0\colon\ S_0^{q-1} C_{\eta/2} \to S_0^q C_{\eta/2}(x_0).$$

Use the Annulus Theorem to extend the map ϕ to a homeomorphism, Φ: $A_1 \to A_2$, where A_1 is the annulus bounded by $S_0^{q-1} C_{\eta/2}$ and $S_0^{q-1} C_\theta$, and A_2 is the annulus bounded by $S_0^q C_{\eta/2}(x_0)$ and C_θ.

Next let S be defined as follows:

(i) $S = S_0$ on $S_0^{q-1} B_{\eta/2}(x_0)^c$,

(ii) $S = \Phi$ on A_1,

(iii) $S = S_0^{-(q-1)}$ maps $S_0^{q-1} B_\theta$ onto B_θ.

Since S is robustly periodic and $D(S, T) < \epsilon$, our proof is complete. □

THEOREM 3.7. *For the sphere S^d ($d \geq 2$), the set $\mathcal{I} \subset H(S^d)$ of homeomorphisms having infinite topological entropy, contains a dense G_δ subset of $H(S^d)$.*

Proof. Again follow the proof of Theorem 1.5 by constructing in an ϵ neighborhood of a robustly periodic S an open subset every element of which has entropy $\geq \log_2 k$. □

DEPARTMENT OF MATHEMATICS, TEL AVIV UNIVERSITY, TEL AVIV, ISRAEL
E-mail: glasner@math.tau.ac.il

INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL
E-mail: weiss@math.huji.ac.il

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