QUASIFACTORS OF MINIMAL SYSTEMS

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§1. Introduction

In the ring of integers $\mathbb{Z}$ two integers m and n have no common factor if whenever $k|m$ and $k|n$ then $k = \pm 1$. They are disjoint if whenever $m|k$ and $n|k$, then also $mn|k$. Of course in $\mathbb{Z}$ these two notions coincide. In his seminal paper of 1967 [F2], H. Furstenberg introduced the same notions in the context of dynamical systems, both measure preserving transformations and homeomorphisms of compact spaces, and asked whether in these categories as well the two are equivalent. In 1979 D. Rudolph provided the first counter example in the category of measure preserving transformations [R], and in [GW2] a topological counter example consisting of two horocycle flows which have no nontrivial common factor but are nevertheless non-disjoint, is produced using the results of M. Ratner concerning these kind of flows [Ra]. More recently an even more striking example was given by E. Lindenstraus, where two minimal dynamical systems with no nontrivial factor share a common almost 1-1 extension, [L]. The notion of joining was introduced in order to deal with the relationship of two not necessarily disjoint dynamical systems. Beginning with the pioneering works of Furstenberg and Rudolph, the notion of joinings was exploited by many authors; Furstenberg 1977 [F3] and 1981 [F4], Veech 1982 [V2], Ratner 1983 [Ra], del Junco and Rudolph 1987 [JR], Host 1991 [Host], King 1992 [Ki], Glasner, Host and Rudolph 1992 [GHR], Thouvenot 1993 [T], Ryzhikov 1994 [Ry], del Junco, Lemańczyk and Mentzen 1995 [JLM], and Lemańczyk, Parreau and Thouvenot 1999 [LPT], to mention a few.

The abstract theory of minimal $\mathbb{Z}$ topological dynamical systems generalizes easily to actions of a general group of transformations. Accordingly when dealing with topological dynamical systems, unless we say otherwise, the letter T stands for a general (countable discrete) group (rather than a single transformation). Thus a (topological) dynamical system consists of a compact metric space X and a representation of T as a group of homeomorphisms of X. We write (X,T) to denote such a system and we denote by $(x,t) \mapsto tx$ the action of the element of $H(X)$ corresponding to $t \in T$. (This is a slight abuse of notation as the representation $T \to H(X)$ need not be 1-1). We let e be the identity element of T and id will denote the identity homeomorphism of X; again we will often identify the two. When $Y \subset X$ is a closed and T-invariant subset of the system (X,T) we say that the system (Y,T) is a *subsystem* of (X,T). We say that the system (Y,T) is a *factor* of the system (X,T) if there exists a continuous onto map $\pi : X \to Y$ such that

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$\pi(tx) = t\pi(x)$ for every $x \in X$ and $t \in T$. If (X,T) and (Y,T) are two dynamical systems their *product system* is the system $(X \times Y, T)$, where $t(x,y) = (tx, ty)$. The systems (X,T) and (Y,T) are *disjoint* if whenever (X,T) and (Y,T) are factors of a system (Z,T), say $\phi : Z \to X$ and $\psi : Z \to Y$, then the map

$$(\phi, \psi) : Z \to X \times Y$$

is onto; i.e. $(X \times Y, T)$ is also a factor of (Z,T).

A *joining* of (X,T) and (Y,T) is any closed invariant subset (subsystem) W of $X \times Y$ which projects onto both X and Y. It is easy to see that disjointness of (X,T) and (Y,T) is equivalent to the requirement that $X \times Y$ is the only joining of these systems. If a non-trivial system (Z,T) is the common factor of two systems: ϕ :(X,T) $\to$(Z,T) and ψ :(Y,T) $\to$(Z,T), then the relation

$$X \times Y \supset W = \{(x,y) : \phi(x) = \psi(y)\}$$

is a subsystem of the product system $(X \times Y, T)$ and the non-triviality of (Z,T) implies that $W \subsetneqq X \times Y$, so that (X,T) and (Y,T) are not disjoint.

Mostly we will be interested in minimal systems. The dynamical system (X,T) is *minimal* if each orbit, $Tx = \{tx : t \in T\}$ is dense. Zorn's lemma implies the existence of a minimal subsystem in every compact system. For minimal systems (X,T) and (Y,T) any non-empty closed T-invariant subset of $X \times Y$ projects onto both X and Y. Thus for such systems disjointness is equivalent to the condition that the product system $(X \times Y, T)$ be minimal. Usually when dealing with minimal systems, unless we say otherwise all joinings are assumed to be *minimal joinings* i.e. minimal subsets of the product system.

Can one find the information about all the possible joinings of a system (X,T) within the system itself? The key to this question is the notion of a quasifactor.

For a minimal dynamical system (X,T), a *quasifactor* of X is any minimal subsystem of the system $(2^X, T)$ induced by the action of T on the compact metric space of closed subsets of X with its Hausdorff metric.

Up to almost 1-1 extensions (see below) every factor is canonically isomorphic to a quasifactor and to every minimal joining $W \subset X \times Y$ corresponds a quasifactor $\mathcal{X}$ of X, an almost 1-1 extension Y^* of Y and a factor map $Y^* \to \mathcal{X}$. Moreover the quasifactor $\mathcal{X}$ is the trivial one point system iff X and Y are disjoint. Thus if X and Y are not disjoint then a nontrivial quasifactor of X is "almost" a factor of Y. (The quasifactor $\mathcal{X}$ which corresponds to the joining W is the closure of the collection of sets $\{W[y] : y \in Y_0\}$, where for $y \in Y$ the set $W[y]$ is the set $\{x \in X : (x,y) \in W\}$ and the set $Y_0 \subset Y$ is the set of points of continuity of the upper-semi-continuous map $y \mapsto W[y]$ from Y to the space 2^X).

Not every quasifactor arises from a joining and we call those that do "joining quasifactors". Even more special are the "group quasifactors". Our main concern in this work is to analyze these notions and to investigate the following question of J. Auslander. If (X,T) is a minimal system then a proper minimal quasifactor of X can not be disjoint from X. Are there any further restrictions on a quasifactor?

The notion of a joining quasifactor in ergodic theory was investigated in [GW4] and in the present paper I follow some of the ideas developed there.

The examples described in the last section are elaboration on ideas of H. Furstenberg. I thank him and B. Weiss. for many fruitful conversations on these subjects.

§2. A brief survey of abstract topological dynamics

We begin with a brief survey on the basic definitions and results of abstract topological dynamics and Ellis' algebraic theory of minimal systems (see e.g. [E1], [G1], [A1] and [G4]). A *topological dynamical system* or briefly a system is a pair (X,T), where X is a compact Hausdorff space and T an abstract group which acts on X as a group of homeomorphisms. For a point $x \in X$, we let $\mathcal{O}_T(x) = \{tx : t \in T\}$, and $\bar{\mathcal{O}}_T(x) = \mathrm{cls}\,\{tx : t \in T\}$. These subsets of X are called the *orbit* and *orbit closure* of x respectively. We say that (X,T) is *point transitive* if there exists a point $x \in X$ with a dense orbit. In that case x is called a *transitive point.* If every point is transitive we say that (X,T) is a *minimal system.* If (Y,T) is another system then a continuous onto map $\pi : X \to Y$ satisfying $t \circ \pi = \pi \circ t$ for every $t \in T$ is called a *homomorphism* of dynamical systems. In this case we say that (Y,T) is a *factor* of (X,T) and also that (X,T) is an *extension* of (Y,T). With the system (X,T) we associate the induced action (the *hyper system* associated with (X,T)) on the compact space 2^X of closed subset of X. The system (X,T) can be considered as a *sub-system* (i.e. a closed invariant subset) of the system $(2^X,T)$, by identifying x with $\{x\}$. Recall that if $(X,T) \overset{\pi}{\to} (Y,T)$ is a homomorphism then in general $\pi^{-1} : Y \to 2^X$ is an upper-semi-continuous map and that $\pi : X \to Y$ is open iff $\pi^{-1} : Y \to 2^X$ is continuous. When there is no room for confusion we write X for the system (X,T).

We assume for simplicity that our acting group T is a countable discrete group. βT will denote the Stone-Čech compactification of T. The universal properties of βT make it

(1) a compact semigroup with right continuous multiplication (for a fixed $p \in \beta T$ the map $q \mapsto qp,\ q \in \beta T$ is continuous), and right continuous multiplication by elements of T, considered as elements of βT (for a fixed $t \in T$ the map $q \mapsto tq,\ q \in \beta T$ is continuous).
(2) a dynamical system $(\beta T, T)$ under left multiplication by elements of T.

The system $(\beta T, T)$ is universal point transitive T-system; i.e. for every point transitive system (X,T) and a point $x \in X$ with dense orbit, there exists a homomorphism of systems $(\beta T,T) \to (X,T)$ which sends e, the identity element of T, onto x. For $p \in \beta T$ we let px denote the image of p under this homomorphism. This defines an "action" of the semigroup βT on every dynamical system. When dealing with the hyper system $(2^X,T)$ we write $p \circ A$ for the image of the closed subset $A \subset X$ under $p \in \beta T$, to distinguish it from the (usually non-closed) subset $pA = \{px : x \in A\}$. We always have $pA \subset p \circ A$. The compact semigroup βT has a rich algebraic structure. E.g. there are 2^c minimal left (necessarily closed) ideals in βT all isomorphic as systems and each serving as a universal minimal system. Each such minimal ideal, say M, has a subset J of 2^c idempotents such that $\{Mv : v \in J\}$ is a partition of M into disjoint isomorphic (non-closed) subgroups and M is the union of these groups. The group of dynamical system automorphisms of (M,T), $G = \mathrm{Aut}\,(M,T)$ can be identified with any one of the groups vM as follows: for $\alpha \in vM$ we associate the automorphism $\hat{\alpha} : (M,T) \to (M,T)$ given by right multiplication $\hat{\alpha}(p) = p\alpha,\ p \in M$. The group G plays a central role in the algebraic theory. It carries a natural T_1 compact topology called by Ellis the *τ-topology.* The τ-closure of a subset A of G consists of those $\beta \in G$ for which the set $\mathrm{graph}\,(\beta)$ is a subset of the closure in $M \times M$ of the set $\bigcup\{\mathrm{graph}\,(\alpha) : \alpha \in A\}$.

It is convenient to fix a minimal left ideal M in βT and an idempotent $u \in M$;

we then identify G with uM. In this way we can consider the "action" of G on every system (X,T) via the action of βT on X. With every minimal system (X,T) and a point $x_0 \in X$ we associate a τ-closed subgroup

$$\mathcal{G}(X,x_0) = \{\alpha \in G : \alpha x_0 = x_0\}$$

the *Ellis group* of the pointed system (X,x_0). For a homomorphism with $\pi(x_0) = y_0$ we have

$$\mathcal{G}(X,x_0) \subset \mathcal{G}(Y,y_0).$$

For a τ-closed subgroup F of G the *derived group* F' is given by:

$$F' := \bigcap\{\tau\text{-closure}V : V \ \tau\text{-open neighborhood of } u \text{ in } F\}.$$

F' is a τ-closed normal (in fact characteristic) subgroup of F and it is characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group.

A pair of points $(x,x') \in X \times X$ for a system (X,T) is called *proximal* if there exists a net $t_i \in T$ and a point $z \in X$ such that $\lim t_i x = \lim t_i x' = z$ (iff there exists $p \in \beta t$ with $px = px'$). We denote by P the set of proximal pairs in $X \times X$. A system (X,T) is called *proximal* when $P = X \times X$ and *distal* when $P = \Delta$, the diagonal in $X \times X$. A minimal system (X,T) is called *point distal* if there exists a point $x \in X$ such that if x, x' is a proximal pair then $x = x'$.

More generally an extension $(X,T) \xrightarrow{\pi} (Y,T)$ of minimal systems is called a *proximal extension* if the relation $R_\pi = \{(x,x') : \pi(x) = \pi(x')\}$ satisfies $R_\pi \subset P$ and a *distal extension* when $R_\pi \cap P = \Delta$. One can show that every distal extension is open. π is an *almost 1-1 extension* if there is a point $y \in Y$ with $\pi^{-1}(y) = \{x\}$ a single point of X. It is easy to see that an almost 1-1 extension is proximal. π is called (in the metrizable case)an *isometric extension* if there exists a continuous function $d : R_\pi \to \mathbb{R}^+$ whose restriction, for each $y \in Y$, to $\pi^{-1}(y) \times \pi^{-1}(y)$ is a metric and is $t \times t$-invariant for every $t \in T$.

The algebraic language is particularly suitable for dealing with such notions. For example an extension $(X,T) \xrightarrow{\pi} (Y,T)$ of minimal systems is a proximal extension iff the Ellis groups $\mathcal{G}(X,x_0) = A$ and $\mathcal{G}(Y,y_0) = F$ coincide. It is distal iff for every $y \in Y$, and $x \in \pi^{-1}(y)$, $\pi^{-1}(y) = \mathcal{G}(Y,y)x$; iff for every $y = py_0 \in Y$, p an element of M, $\pi^{-1}(y) = p\pi^{-1}(y_0) = pFx_0$, where $F = \mathcal{G}(Y,y_0)$. In particular (X,T) is distal iff $Gx = X$ for some (hence every) $x \in X$. π is an isometric extension iff it is a distal extension and, denoting $\mathcal{G}(X,x_0) = A$ and $\mathcal{G}(Y,y_0) = F$,

$$F' \subset A$$

in which case the compact group F/F' is the group of the *group extension* associated with isometric extension π (see [G]).

A minimal system (X,T) is called *incontractible* if the union of minimal subsets is dense in every product system $(X \times X \times \cdots \times X, T \times T \times \cdots \times T)$. This is the case iff $p \circ Gx = X$ for some (hence every) $x \in X$. We say that π is a RIC (*relatively incontractible*) extension if for every $y = py_0 \in Y$, p an element of M, $\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0$, where $F = \mathcal{G}(Y,y_0)$. It is not hard to see that every RIC extension is open. Every distal extension is RIC and it follows that every distal extension is open.

We say that a minimal system (X,T) is a *strictly PI system* if there is a (countable) ordinal η and a family of systems $\{(W_\iota, w_\iota)\}_{\iota\le\eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota<\eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1}\to W_\iota$ which is either proximal or isometric, (iii) for a limit ordinal $\nu\le\eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that (X,T) is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X}\to X$.

If in the definition of PI-systems we replace proximal extensions by almost 1-1 extensions we get the notion of *AI-systems*. If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have *I-systems*. In this terminology the structure theorem for distal systems ([F1], 1963) can be stated as follows:

Theorem 2.1. *A minimal system is distal iff it is an I-system.*

And the Veech-Ellis structure theorem for point distal systems ([V1], 1970, [E3], 1973):

Theorem 2.2. *A minimal dynamical system is point distal iff it is an AI-system.*

Finally we have the Structure theorem for minimal systems ([EGS1], 1975 and [M], 1976):

Theorem 2.3. *Given a metric minimal system (X,T), there exists a countable ordinal η and a canonically defined commutative diagram (the canonical PI-Tower)*

$$\begin{tikzcd}
X \arrow[d, "\pi"'] & X_0 \arrow[l, "\tilde\theta_0"'] \arrow[d, "\pi_0"'] \arrow[dr, "\sigma_1"] & & X_1 \arrow[ll, "\tilde\theta_1"'] \arrow[d, "\pi_1"'] & \cdots & X_\nu \arrow[d, "\pi_\nu"'] \arrow[dr, "\sigma_{\nu+1}"] & & X_{\nu+1} \arrow[ll, "\tilde\theta_{\nu+1}"'] \arrow[d, "\pi_{\nu+1}"'] & \cdots & X_\eta = X_\infty \arrow[d, "\pi_\infty"'] \\
pt & Y_0 \arrow[l, "\theta_0"] & Z_1 \arrow[l, "\rho_1"] & Y_1 \arrow[l, "\theta_1"] & \cdots & Y_\nu & Z_{\nu+1} \arrow[l, "\rho_{\nu+1}"] & Y_{\nu+1} \arrow[l, "\theta_{\nu+1}"] & \cdots & Y_\eta = Y_\infty
\end{tikzcd}$$

where for each $\nu\le\eta, \pi_\nu$ is RIC, ρ_ν is isometric, $\theta_\nu, \tilde\theta_\nu$ are proximal and π_∞ is RIC and weakly mixing. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota<\nu$. Thus X_∞ is a proximal extension of X and a RIC weakly mixing extension of the stictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) iff X is a PI-system.

§3. Joining quasifactors in topological dynamics

We are now ready to develop a theory of joining quasifactors analogous to the measure theoretical one developed in [GW4]. As usual the topological analogue of a measure theoretical construction is complicated by the necessity to pass to almost 1-1 or even proximal extensions. Some of the statements (and their proofs) of this section can be found in [G3] (§5). The examples whose existence is claimed in proposition 3.10 will be constructed in §5 below.

If (X,T) is a minimal metrizable dynamical system, then a *quasifactor* of X is any minimal subsystem of the system $(2^X,T)$ induced by the action of T on the compact metric space of closed subsets of X with its Hausdorff metric.

Let (X,T) and (Y,T) be minimal metric systems, $W\subset X\times Y$ a minimal subset of the product system (a minimal joining).

Proposition 3.1.

(1) *The subset* $\mathrm{cls}\,\{W[y] : y \in Y\}$ *of the system* $(2^X, T)$ *contains a unique minimal subset* $\mathcal{X}$*; a quasifactor of* (X,T).

(2) $W[y] \in \mathcal{X}$ *when* y *is in the residual subset* Y_c *of* Y *which consists of the continuity points of the upper-semi-continuous map* $W[\cdot] : y \mapsto W[y]$ *and* $\mathcal{X} = \mathrm{cls}\,\{W[y] : y \in Y_c\} = \{p \circ W[y_0] : p \in M\}$*, for every* $y_0 \in Y_c$.

(3) *For every* $y \in Y_c$ *and every* $x \in W[y] = \xi \in \mathcal{X}$,

$$\xi = \{px : p \in M,\ py = y\}.$$

Proof. By the upper-semi-continuity of the map $y \mapsto W[y]$, it follows that $p \circ W[y] \subset W[py]$ for every $y \in Y$ and $p \in \beta Z$. Now for a continuity point $y_0 \in Y_c$, $p \circ W[y_0] = W[py_0]$ and it follows that for any such y_0, $\mathcal{X} = \{p \circ W[y_0] : p \in M\}$ is the unique minimal subset of $\mathrm{cls}\,\{W[y] : y \in Y\} \subset 2^X$. This proves assertions (1) and (2). For part (3), $W[y] \supset \{px : p \in M,\ py = y\}$ is clear and if we pick any $x \in W[y]$ then $(x, y) \in W$ and, since W is a minimal set, there exists $p \in M$ with $p(x, y) = (x, y)$. $\square$

Fix $y_0 \in Y_c$ and $x_0 \in W[y_0] = \xi_0$ as above, and put

$$Y^* = \mathcal{X} \vee Y = \bar{\mathcal{O}}_T(\xi_0, y_0) \subset \mathcal{X} \times Y.$$

and

$$W^* = X \vee Y^* = \bar{\mathcal{O}}_T(x_0, \xi_0, y_0).$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
 & W^* \arrow[dl] \arrow[d] \arrow[dr] & \\
X \arrow[d] & W \arrow[dl] \arrow[dr] & Y^* \arrow[d] \\
X & & Y
\end{tikzcd}.$$

Proposition 3.2. *In the above diagram*

(1) $W^* = \{(x, \xi, y) : x \in \xi \in \mathcal{X}, (\xi, y) \in Y^*\}$

(2) Y^* *and* W^* *are minimal systems and they are independent of the choice of points* $y_0 \in Y_c$ *and* $x_0 \in W[y_0]$.

(3) *The extension* $W^* \to Y^*$ *is an open map.*

(4) *The extensions* $Y^* \to Y$ *and* $W^* \to W$ *are almost 1-1 extensions. They are both isomorphisms iff the map* $y \mapsto W[y]$*, from* Y *to* 2^X*, is a continuous map.*

Proof. Given $x \in W[y_0] = \xi_0$, from part (3) of the previous proposition we can choose $p \in M$ with $p(x_0, y_0) = (x, y_0)$. The fact that y_0 is a continuity point implies: $p(x_0, \xi_0, y_0) = (x, \xi_0, y_0) \in W^*$. In particular for $x = x_0$, $p(x_0, \xi_0, y_0) = (x_0, \xi_0, y_0)$ so that (x_0, ξ_0, y_0) is an almost periodic point. This shows that W^* and hence also Y^* are minimal. If we let $W' = \{(x, \xi, y) : x \in \xi \in \mathcal{X}, (\xi, y) \in Y^*\}$, then clearly $W^* \subset W'$. On the other hand if (x, ξ, y) is in W' then there exists $p \in M$ with $p(\xi_0, y_0) = (\xi, y)$; i.e. $py_0 = y$ and $p \circ W[y_0] = \xi$. Since we have already shown that

$W[y_0] \times \{(\xi_0, x_0)\} \subset W^*$, it follows that $(x, \xi, y) \in p \circ (W[y_0] \times \{(\xi_0, x_0)\}) \subset W^*$. The independence of $\mathcal{X}, Y^*$ and W^* on the choice of generating points is now clear.

Denoting by ϕ the projection map from W^* to Y^*, we now see that $\phi^{-1}((\xi, y)) = \xi \times \{(\xi, y)\}$ (where in the last expression ξ appears first as a subset of X then as a point of $\mathcal{X}$). This clearly implies that $\phi^{-1} : Y^* \to 2^{W^*}$ is a continuous set-valued map, which is equivalent to the fact that ϕ is an open map.

Finally the assertions of part (4) are easy consequences of the continuity of $W[\cdot]$ at the points of Y_c. $\square$

Thus starting from a minimal joining W of two minimal metric systems X and Y—by replacing Y by an almost 1-1 extension Y^*—we can always obtain a "nicer" minimal joining W^* of X and Y^*, where now in the diagram

$$\begin{tikzcd} & W^* \arrow[dl] \arrow[dr] & \\ X & & Y^* \end{tikzcd},$$

the map $y^* \mapsto W^*[y^*]$ is a continuous homomorphism of Y^* onto $\mathcal{X} \subset 2^X$, and the map $W^* \to Y^*$ is open.

Now put

$$X^* = X \vee \mathcal{X} = \{(x, \xi) : x \in \xi\} \subset X \times \mathcal{X},$$

and

$$W^{**} = X^* \vee Y^* = X^* \underset{\mathcal{X}}{\times} Y^* = \{((x, \xi), (\xi, y)) : x \in \xi \in \mathcal{X}, (\xi, y) \in Y^*\}.$$

Then we have the following commutative diagram:

$$\begin{tikzcd} & & W^{**} \arrow[dl] \arrow[dr] & & \\ & X^* \arrow[dl] \arrow[dr] & & Y^* \arrow[dl] \arrow[dr] & \\ X & & \mathcal{X} & & Y \end{tikzcd},$$

and the proof of next proposition follows easily from that of proposition 3.2.

Proposition 3.3. *In the above diagram*

1. *X^* is a minimal system.*
2. *The system W^{**} is canonically isomorphic to the system W^* and in particular W^{**} is a minimal system; i.e. the systems X^* and Y^* are relatively disjoint over their common factor $\mathcal{X}$.*
3. *The extensions $X^* \to \mathcal{X}$ and $W^{**} \to Y^*$ are open maps.*

Proposition 3.4. *Let (X, T) be a minimal metric system, $\mathcal{X}$ a minimal quasifactor of X. The following conditions on $\mathcal{X}$ are equivalent:*

1. *There exists a minimal system (Y, T) and a minimal set $W \subset X \times Y$ (i.e. a minimal joining) such that $\mathcal{X}$ is the unique minimal subset of the closed invariant set $\mathrm{cls}\,\{W[y] : y \in Y\}$ in 2^X.*

(2) *There exists a minimal system* (Y,T) *, a point* $y_0 \in Y$ *and a point* $\xi_0 \in \mathcal{X}$ *such that for every* $x_0 \in \xi_0$

$$\xi_0 = \{px_0 : p \in M,\ py_0 = y_0\}.$$

(3) *There exists a point* $\xi_0 \in \mathcal{X}$ *such that for every* $x_0 \in \xi_0$

$$\xi_0 = \{px_0 : p \in M,\ p \circ \xi_0 = \xi_0\}.$$

(4) *The set*

$$X^* = \{(x,\xi) : x \in \xi \in \mathcal{X}\},$$

is a minimal subset of $X \times \mathcal{X}$*.*

Definition. A quasifactor satisfying the equivalent conditions (1)–(4) of proposition 3.4 will be called a *joining quasifactor*, or jqf for short.

Proposition 3.5. *Let* (X,T) *be a minimal metric system and* $\mathcal{X}$ *any minimal quasifactor of* X*. Then there exists a minimal joining quasifactor* $\mathcal{Z}$ *of* X *with the following properties.*

(1) $\mathcal{Z} \succ \mathcal{X}$ *(i.e. for every* $\zeta \in \mathcal{Z}$ *there exists* $\xi \in \mathcal{X}$ *with* $\zeta \subset \xi$*) and* $\mathcal{Z} = \mathcal{X}$ *iff* $\mathcal{X}$ *is jqf.*

(2) *There exist* $\zeta_0 \in \mathcal{Z}, \xi_0 \in \mathcal{X}$ *such that* $\zeta_0 \subset \xi_0$*,* $\mathcal{X} \vee \mathcal{Z} = \bar{\mathcal{O}}_T(\xi_0,\zeta_0) := \mathcal{X}^*$ *is a minimal system and the extension* $\mathcal{X}^* \to \mathcal{X}$ *is an almost 1-1 extension. Thus an almost 1-1 extension of* $\mathcal{X}$ *has* $\mathcal{Z}$ *as a factor. Since in particular an almost 1-1 extension is a proximal extension, denoting* $B = \mathcal{G}(\mathcal{X},\xi_0)$ *and* $C = \mathcal{G}(\mathcal{Z},\zeta_0)$*, we have*

$$\mathcal{G}(\mathcal{X} \vee \mathcal{Z}, (\xi_0,\zeta_0)) = B \cap C = B,$$

and therefore $B \subset C$*.*

Let $\mathcal{X}$ be a jqf of the minimal system (X,T) . Choose $x_0 \in \xi_0 \in \mathcal{X}$ with $ux_0 = x_0$ and $u \circ \xi_0 = \xi_0$. As usual we let $A = \mathcal{G}(X,x_0)$ and let $B = \mathcal{G}(\mathcal{X},\xi_0)$. Then for $\beta \in B$ we have $\beta x_0 \in \beta\xi_0 \subset u \circ \xi_0 = \xi_0$. Conversely, if $\gamma x_0 \in \xi_0$ then the jqf property of $\mathcal{X}$ implies that there exists $p \in M$ with $p \circ \xi_0 = \xi_0$ and $\gamma x_0 = px_0$. If we write $\delta = up$ then clearly $\delta \circ \xi_0 = \xi_0$ i.e. $\delta \in B$, and $\delta x_0 = \gamma x_0$, hence $\delta^{-1}\gamma \in A$ and $\gamma \in BA$. We have shown that for a jqf $u\xi_0 = Bx_0$. Warning: this of course does not mean that $A \subset B$.

Definition. A jqf $\mathcal{X}$ with Ellis group $B = \mathcal{G}(\mathcal{X},\xi_0)$ is called a *group quasifactor*, gqf for short, if $\xi_0 = u \circ Bx_0 = u \circ u\xi_0$.

Proposition 3.6. *Let* (X,T) *be a minimal metric system,* $\mathcal{X}$ *a minimal quasifactor of* X *with* $A = \mathcal{G}(X,x_0)$ *and* $B = \mathcal{G}(\mathcal{X},\xi_0)$ *(*$ux_0 = x_0 \in u \circ \xi_0 = \xi_0$*). Then the following conditions are equivalent.*

(1) $\mathcal{X}$ *is a group quasifactor.*

(2) *For every* $\xi \in \mathcal{X}$ *and* $v = v^2 \in M$ *with* $v \circ \xi = \xi$*,* $v \circ v\xi = \xi$*; in other words for every* $p \in M$*,* $p \circ \xi_0 = p \circ u\xi_0$*.*

(3) *The extension* $X^* \to \mathcal{X}$ *is a RIC-extension.*

Proposition 3.7. *Let (X,T) be a minimal metric system with $A=\mathcal{G}(X,x_0)$, $\mathcal{X}$ a group quasifactor of X with $B=\mathcal{G}(\mathcal{X},\xi_0)$ $(x_0\in\xi_0)$. Then $A^B\subset B$ and B is a maximal τ-closed subgroup of BA. In particular if AB is a group then $A\subset B$, ([EGS2], see also [A2]).*

Proposition 3.8. *Let (X,T) be a minimal metric system with $A=\mathcal{G}(X,x_0)$, $\mathcal{X}$ a joining quasifactor of X with $B=\mathcal{G}(\mathcal{X},\xi_0)$ $(x_0\in\xi_0)$. Put $\zeta_0=u\circ u\xi_0=u\circ Bx_0$ and let $\mathcal{Z}=\{p\circ\zeta_0,\ p\in M\}$. Then the quasifactor $\mathcal{Z}$ of X has the following properties.*

(1) *$\mathcal{Z}$ is a gqf.*
(2) *$\mathcal{Z}\succ\mathcal{X}$ and $\mathcal{Z}=\mathcal{X}$ iff $\mathcal{X}$ is a gqf.*
(3)
$$B\subset\mathcal{G}(\mathcal{Z},\zeta_0)\subset AB$$
and $\mathcal{G}(\mathcal{Z},\zeta_0)$ is the largest τ-closed subgroup satisfying these inclusions, [EGS2], [A2]. In particular
$$\mathcal{G}(\mathcal{Z},\zeta_0)\supset A^B:=\bigcap_{\beta\in B}\beta A\beta^{-1}.$$
(4) *Put $\tilde{\mathcal{X}}=\mathcal{X}\vee\mathcal{Z}=\bar{\mathcal{O}}_T(\xi_0,\zeta_0)$ and $\tilde{X}=X^*\vee\mathcal{Z}=\bar{\mathcal{O}}_T(x_0,\xi_0,\zeta_0)$ then the diagram*
$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X^* \arrow[dr] & & \tilde{\mathcal{X}} \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$
is the RIC shadow diagram of the map $X^*\to\mathcal{X}$.

Proposition 3.9. *Notations as in the previous proposition. If (X,T) is a point distal system then every jqf of X is a gqf and in particular $A^B\subset B$ for every jqf $\mathcal{X}$ of X.*

Proof. Use lemma 5.2 in [G3] to conclude that $u\xi_0$ is dense in ξ_0 for some $\xi_0\in\mathcal{X}$, whence $u\circ u\xi_0=\xi_0$. $\square$

Proposition 3.10. *Notations as in proposition 3.7,*

(1) *There exists a minimal proximal system (X,T) —thus with $A=\mathcal{G}(X,x_0)=G$—that admits a non-proximal jqf $\mathcal{X}$, hence with $B=\mathcal{G}(\mathcal{X},\xi_0)\not\supset G=A^B=A^G$.*
(2) *There exists a minimal $\mathbb{Z}$-action $(X,\mathbb{Z})$ with normal Ellis group $A=\mathcal{G}(X,x_0)\lhd G$ and a jqf $\mathcal{X}$ of X for which $B=\mathcal{G}(\mathcal{X},\xi_0)\not\supset A=A^B=A^G$.*

§4. How far can a quasifactor of a system be from the system?

If (X,T) is a minimal system then a proper minimal quasifactor of X can not be disjoint from X. Are there any further restrictions on a quasifactor? Of course this question is meaningful in both the measure and the topological categories. Here are some cases where results restricting the variety and size of quasifactors can be proven. We mainly discuss the topological category and in parentheses comment on the measure case.

(1) A quasifactor of a minimal equicontinuous system is isomorphic to a factor of the system, when the acting group T is abelian. In particular it is itself equicontinuous. (This is also true for discrete spectrum).
(2) A minimal quasifactor of a minimal distal system X is a factor of the enveloping group of X. In particular it is itself distal, [K]. (The latter statement applies also to measure distal systems, see [GW3]).
(3) By proposition 3.9 above, every minimal joining quasifactor $\mathcal{X}$ of a point distal minimal system X is a group quasifactor. In particular $A^B \subset B$, where A and B are the Ellis groups of X and $\mathcal{X}$ respectively.
(4) A quasifactor of a uniformly rigid system is uniformly rigid. (The same holds for rigidity in the ergodic theoretical sense).
(5) Take X to be Chacon's system. By [J] there exists a subset X_0, whose complement is the union of two orbits (hence X_0^c is a countable set) such that for every $x = (x_1, x_2, ..., x_k)$ with $\{x_i\} \subset X_0$ and belonging to k different orbits, the orbit closure of x in X^k is X^k. It is now clear that for every closed uncountable subset A of X, its orbit closure in 2^X contains the fixed point $\{X\}$. Thus the only non-trivial minimal quasifactors are those whose elements are finite or countable subsets of X, and every such quasifactor is isomorphic to X. (The quasifactors of minimal-self-joinings systems — or more generally of simple systems — are treated in [G2], see also [JR] and [GHR]).
(6) For almost simple (AS) systems it is shown in [G3] that every minimal jqf is, up to almost 1-1 extension, a quasifactor corresponding to a group factor.
(7) As was shown above (proposition 3.7), for every minimal system X (say with Ellis group A), for any gqf $\mathcal{X}$, the Ellis group $B = \mathcal{G}(\mathcal{X})$ must contain the τ-closed group $A^B = \bigcap\{\beta A \beta^{-1} : \beta \in B\}$. In particular when A is a normal subgroup of G, it follows that $B \supset A$. Thus in that case every gqf of X is proximally equivalent to a factor of X.
(8) A quasifactor of a metrizable system is metrizable (separable in the measure category).
(9) Call a τ-closed group A *metrizable* if there exists a metric minimal pointed system (X, x_0) with $\mathcal{G}(X, x_0) = A$. Otherwise we say that A is *nonmetrizable*. Thus A is nonmetrizable if whenever $A = \mathcal{G}(X, x_0)$ then the system X is necessarily nonmetrizable. Here are few examples of nonmetrizable groups:

a. $A = \{u\}$ is nonmetrizable.

Proof: If $\mathcal{G}(X) = \{u\}$ for a minimal T-system then the extension $M \to X$ is a proximal extension. Now we have $M \to K$, where K is the Bohr compactification of T. Since K is distal we also have $X \to K$, hence for groups T with nonmetrizable K, also X is nonmetrizable.

b. One can construct uncountably many Chacon-like minimal systems $\{X_\iota : \iota \in I\}$ such that any finite collection of them is disjoint. Let $\{A_\iota : \iota \in I\}$ be the corresponding collection of Ellis groups and let $X = \prod X_\iota$. Then $\mathcal{G}(X) = A = \bigcap A_\iota$, is nonmetrizable.

Proof: If $\mathcal{G}(Y) = A$ for some minimal system Y, then for each ι, $A_\iota \supset A$, implies that Y extends X_ι; whence Y extends X which is nonmetrizable.

c. In [EG] (proposition 4.2 and lemma 4.3) it is shown that the Ellis groups A_i of the maximal weakly mixing minimal systems, are all nonmetrizable groups.

Claim: If A is a nonmetrizable τ-closed group and X is a metrizable minimal system then A can not appear as Ellis group of a minimal quasifactor of X.

Proof: A quasifactor of a metrizable system is metrizable.

On the other hand the following results show that a quasifactor can sometimes be surprisingly "remote" from the system itself.

(1) In [G2] and [JR] there is a construction of a (measure) weakly mixing group extension of a simple system (hence with zero entropy), which admits an ergodic joining quasifactor having -1 as an eigenvalue. The same construction yields a minimal topologically weakly mixing system with a minimal, joining quasifactor having the two point system as a factor.
(2) As we have seen (proposition 2.3), each ergodic system of positive entropy admits every ergodic system of positive entropy as a quasifactor.
(3) For the (non-minimal) system $X = \mathbb{Z} \cup \{\infty\}$, the one point compactification of $\mathbb{Z}$ with the shift, the system 2^X is isomorphic to the Bernoulli system $\{0,1\}^{\mathbb{Z}}$.
(4) In [GW3] a minimal metric system of zero topological entropy is constructed possessing a minimal joining quasifactor of positive topological entropy. (This is in contrast to the situation for ergodic measure preserving systems where a quasifactor of a zero entropy system has zero entropy as well, [GW3]).
(5) The systems of proposition 3.10 above provide examples of a minimal nonproximal quasifactor of a metric minimal proximal system. And for $\mathbb{Z}$-actions, an example of a minimal metric X with normal Ellis group $A \lhd G$ and a jqf $\mathcal{X}$ of X, with Ellis group B that does not contain A.

Problems. Does there exist a minimal metric $\mathbb{Z}$-system (X, T) with any of the following properties?

(1) Every minimal metrizable system Y which is not disjoint from X appears as a quasifactor of X.
(2) For every minimal metrizable system Y which is not disjoint from X there is a quasifactor Y^* of X which is almost 1-1 equivalent to Y (i.e Y and Y^* have a common almost 1-1 extension).
(3) For every minimal metrizable system Y which is not disjoint from X there is a quasifactor Y^* of X which is proximally equivalent to Y. In other words, denoting $A = \mathcal{G}(X, x_0)$ we want for every metrizable τ-closed subgroup $B \subset G$ with $AB \neq G$, a quasifactor Y of X with $\mathcal{G}(Y) = B$.

One can ask the same questions about joining quasifactors or group quasifactors instead of quasifactors.

§5. The examples of proposition 3.10

Our first example will present a minimal action (X, G) of a discrete countable group G with the following properties

(1) (X, G) is non-proximal and moreover does not admit a nontrivial proximal

factor.

(2) (X, G) does not admit a nontrivial incontractible factor.

(3) (X, G) admits a nontrivial minimal proximal quasifactor $\mathcal{X}$.

(4) The minimal proximal system $\mathcal{X}$ admits the nonproximal system X as a quasifactor.

We let $X = \{0, 1, 2\}^{\mathbb{Z}}$ and σ will denote the shift on X. The symmetric group S_3 will act on X by permuting values of the zero coordinate of a point $x \in X$. For every pair of integers n and m we let $\tau = \tau_{n,m}$ be the homeomorphism of X defined by $(\tau x)_n = x_m$, $(\tau x)_m = x_n$ and $(\tau x)_p = x_p$ for $p \notin \{n, m\}$. We let $G = \langle \sigma, \tau_{n,m}, S_3 : n, m \in \mathbb{Z}\rangle$ be the subgroup of homeomorphisms of X generated by σ, S_3 and the various $\tau_{n,m}$. Notice that the group generated by the homeomorphisms $\tau_{n,m}$ is the group of homeomorphisms given by permutations of $\mathbb{Z}$ with finite support. We call elements of this subgroup τ-permutations.

Clearly every homeomorphism $g \in G$ has the property

$$(*) \qquad \forall x, y \in X \ [x_n \neq y_n \ \forall n \in \mathbb{Z} \implies (gx)_n \neq (gy)_n \ \forall n \in \mathbb{Z}].$$

It can be shown that in fact G is the group of all homeomorphisms of X satisfying this property.

To facilitate our description we introduce the following terminology. Call a pair of points $x, y \in X$ an *edge* if $x_n \neq y_n$ for every $n \in \mathbb{Z}$. If $x_n \neq y_n$ for infinitely many $n \in \mathbb{Z}$ we say that x and y are *opposed*.

Claim 1. *(X, G) is minimal.*

Proof. This is clear already when one considers the action of the subgroup $H = \langle \sigma, S_3 \rangle$. $\square$

Claim 2. *Every edge is an edge of a unique triangle.*

Proof. Clearly an edge $\{x, y\}$ determines a unique point z such that $\{x, y, z\}$ is a triangle; i.e. $\{x, y\}$, $\{x, z\}$ and $\{z, y\}$ are edges. $\square$

Let $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$ denote the points of X whose coordinates are constantly 0, 1 and 2 respectively.

Claim 3. *Given four points $\{x^0, x^1, x^2, x^3\}$ in X, there exists a sequence $g_n \in G$ with $\lim g_n x^j \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ for $j = 0, 1, 2, 3$.*

Proof. By minimality we can assume that $x^0 = \mathbf{0}$. If there exists j with x^j opposed to $\mathbf{0}$ (i.e. x^j has infinitely many coordinates different from zero) we can apply a sequence of τ-permutations, elements of S_3 and various powers of σ to the pair $\{\mathbf{0}, x^j\}$ to get in the limit the pair $\{\mathbf{0}, \mathbf{1}\}$. Otherwise $0 = x^1_n = x^2_n = x^3_n$ for all but finitely many n, and then $\lim_{k\to\infty} \sigma^k x^j = \mathbf{0}$ for $j = 0, 1, 2, 3$. In either case passing to a further subsequence we now have:

$$\{\lim_{n\to\infty} g_n x^j\} \subset \{\mathbf{0}, \mathbf{1}, y^2, y^3\}$$

for some $y^2, y^3 \in X$. If either y^2 or y^3 has infinitely many coordinates with the value 2, we can similarly pass to a limit which is a subset of $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, z^3\}$ for some $z^3 \in X$. Otherwise we can pass to a subset of $\{\mathbf{0}, \mathbf{1}\}$. Finally from $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, z^3\}$ we can get into $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. $\square$

Recall that a subset A of the dynamical system (X, G) is called an *almost periodic set* if for every finite subset $\{x_1, \dots, x_n\} \subset A$, the point $(x_1, \dots, x_n)$ is an almost periodic point of the n-fold product system X^n; i.e the orbit closure of $(x_1, \dots, x_n)$ in X^n is minimal.

Claim 4. *The maximal almost periodic subsets of X are exactly the triangles. In particular (X, G) is not proximal.*

Proof. Clear from the previous claim and the property $(*)$ of homeomorphisms in G. $\square$

Claim 5. *The collection of triangles $\mathcal{X}$ is a minimal proximal quasifactor of (X, G).*

Proof. Clearly the set $\mathcal{X}$ of triangles in X is a closed invariant subset of 2^X. From the previous claims it is clear that the triangle $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ is in the orbit closure of any other triangle. On the other hand it is also easy to see that conversely, every triangle is in the orbit closure of $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. Thus $\mathcal{X}$ is a minimal quasifactor. Finally if $\{x^0, x^1, x^2\}$ is an arbitrary triangle then the set $\{x^0, x^1, x^2, \mathbf{0}, \mathbf{1}, \mathbf{2}\}$ can be squeezed into $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$, whence $\mathcal{X}$ is a proximal system. $\square$

Claim 6. *The system (X, G) admits no nontrivial proximal factors.*

Proof. Suppose $\pi : X \to Y$ is a homomorphism with (Y, G) proximal. First note that for any triangle $\{x^0, x^1, x^2\}$ we have $\pi(x^0) = \pi(x^1) = \pi(x^2)$. Next let x, x' be distinct points in X. There is a point $z \in X$ such that $\{x, z\}$ is an edge and z is free of say, zeroes. Likewise there is a point $z' \in X$ such that $\{x', z'\}$ is an edge and z' is free of ones. Then the chain: $\{x, z\}$, $\{z, \mathbf{0}\}$, $\{\mathbf{0}, \mathbf{1}\}$, $\{\mathbf{1}, z'\}$, $\{z', x'\}$ shows that $\pi(x) = \pi(x')$, so that Y is a one point system. $\square$

Recall that a minimal system (X, G) is called *incontractible* if for every $\epsilon > 0$ there exists an ϵ-dense almost periodic subset of X. An equivalent condition is that (X, G) admits no nontrivial minimal proximal quasifactor.

Claim 7. *The system (X, G) admits no nontrivial incontractible factors.*

Proof. Suppose $\pi : X \to Y$ is a homomorphism with (Y, G) incontractible. Then $\pi(\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}) = Y$. Since clearly $P[\mathbf{0}]$, the proximal cell of $\mathbf{0}$, is dense in X, it follows that $\{\pi(\mathbf{0})\} = Y$. $\square$

Claim 8. *Let $\mathcal{X}$ be the quasifactor of triangles. Then X is isomorphic to a quasifactor of $\mathcal{X}$.*

Proof. Set for $x \in X$:

$$A_x = \{\xi \in \mathcal{X} : x \in \xi\}.$$

Clearly $\{A_x : x \in X\}$ is a quasifactor of $\mathcal{X}$ which is isomorphic to X via the map $x \mapsto A_x$, with inverse $A_x \mapsto \bigcap\{\xi : \xi \in A_x\} = \{x\}$. $\square$

Our next example is of a similar nature, but with a Lie group as the acting group. Let G be the closed subgroup of the Lie group $GL(4, \mathbb{R})$ consisting of all 4×4 matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

with $A, B \in GL(2,\mathbb{R})$. We let G act on the subspace Y of the projective space $\mathbb{P}^3$ consisting of the disjoint union of the two one dimensional projective spaces $\mathbb{P}^1$, which are naturally embedded in $\mathbb{P}^3$, the quotient space of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Call these two copies Y_1 and Y_2 respectively. There is a natural projection from (Y, G) onto the two points G-system $(\{Y_1, Y_2\}, G)$ and it is easy to see that the system (Y, G) is a minimal proximal extension of this two points system. It is also easy to see that the maximal almost periodic sets of (Y, G) are the sets of the form $\{y_1, y_2\}$ with $y_i \in Y_i, i = 1, 2$. It is now easy to establish the remaining assertion of the following:

Claim 9. *The G-system (Y, G) is a minimal non-proximal system, admitting the isometric factor which is the "flip" on two points. The system (Y, G) admits no nontrivial proximal system as a factor.*

Let $\mathcal{Y}$ be the quasifactor of (Y, G) defined by:

$$\mathcal{Y} = \{\{y_1, y_2\} : y_i \in Y_i, i = 1, 2\}.$$

Again it is easy to check that the system $(\mathcal{Y}, G)$ is a minimal and proximal G-system. Next consider the quasifactor $\hat{Y}$ of the system $(\mathcal{Y}, G)$ which consists of all closed subsets of $\mathcal{Y}$ of the form:

$$F_y = \{\{y_1, y_2\} \in \mathcal{Y} : y \in \{y_1, y_2\}\}, \qquad y \in Y.$$

We clearly have an isomorphism of G-systems between Y and $\hat{Y}$ given by $y \mapsto F_y$ (the inverse of this map is given by $F_y \mapsto \bigcap F_y = \{y\}$). We have thus established the following:

Claim 10. *The proximal minimal G-system $(\mathcal{Y}, G)$ admits a non-proximal minimal quasifactor $(\hat{Y}, G)$, isomorphic to the original system (Y, G).*

Next we take up the setup of [GW1] and consider a minimal infinite system (Z, σ) with Z zero dimensional and σ a homeomorphism of Z. Thus we let $X = Z \times Y$ and consider the subset $\mathcal{S}_G(X)$ of the Polish group $\mathcal{H}(X)$, of self homeomorphisms of X, given by:

$$\mathcal{S}_G(X) = \{h \circ (\sigma \times \mathrm{id}) \circ h^{-1} : h \in \mathcal{H}(X) \ \& \ \forall z \in Z, \ \ h(\{z\} \times Y) = \{z\} \times Y\}.$$

(For such h we write h_z for the restriction of h to the set $\{z\} \times Y$). theorems 1 and 3 of [GW1] assert that under certain assumptions on the action of G on Y, there exists a residual subset $\mathcal{R}$ of cls $\mathcal{S}_G(X)$, such that for each member $T \in \mathcal{R}$, the system (X, T) is minimal and the extension $\pi : (X, T) \to (Z, \sigma)$ is a proximal extension. The requirements on the G-action on Y are as follows:

(1) (Y, G) is minimal
(2) For every pair of points $y_1, y_2 \in Y$ there exist neighborhoods U and V of y_1 and y_2 respectively, such that for every $\epsilon > 0$, there exits $g \in G$ with $\mathrm{diam}\,(g(U \cup V)) < \epsilon$.
(3) G is pathwise connected.

We would like to apply these results to the G-actions (Y, G) and $(\mathcal{Y}, G)$ above. Since these actions are minimal and the action $(\mathcal{Y}, G)$ is proximal in the strong sense stated in (2), we have the requirements (1) and (2) satisfied. However the requirement (3) is not satisfied, as the topological group G has *two* path components. In order to overcome this difficulty, let us go back to the proof of, say theorem 1 in [GW1]. The point where one uses the connectivity of G is in lemma 2.1, where the map $t \mapsto h_t$, from the finite set $\{0, 1/n, 2/n \dots, n-1/n\}$ into G, is extended to a continuous map of $I = [0,1]$ into G.

Now instead of dealing with one interval I we work here with two disjoint closed intervals, say $I = [0,1]$ and $J = [2,3],$. The set $\{h_0, h_{1/n}, \dots, h_{1-1/n}\}$ is then replaced by two sets, $\{h_0, h_{1/n}, \dots, h_{1-1/n}\}$ and $\{h_2, h_{2+1/n}, \dots, h_{3-1/n}\}$, where the first is contained in the identity path component of G and the second in the other component. The map $t \mapsto h_t$ can now be extended to a continuous map from $I \cup J$ into G. The proof proceeds as before and we only have to notice that, since Z is zero dimensional, the map $\tilde{\theta} : \bigcup_{i=0}^{m-1} \sigma^i K \to I \cup J$ can be extended continuously to a map $\tilde{\theta} : Z \to I \cup J$. lemma 4.1 in [GW1], needed for the proof of theorem 3, is treated similarly.

We first apply our modified theorem 1 of [GW1], to obtain a residual subset $\mathcal{R}$ of cls $\mathcal{S}_G(X)$, such that for each member $T \in \mathcal{R}$, the $\mathbb{Z}$-system (X, T) is minimal. Next consider for any $T \in$ cls $\mathcal{S}_G$ the corresponding quasifactor $(\mathcal{X}, T)$ of the system (X, T), consisting of all subsets of X of the form

$$\mathcal{X} = \{\{z\} \times \{y_1, y_2\} : z \in Z,\ y_i \in Y_i,\ i = 1, 2\} \subset Z \times \mathcal{Y}.$$

Applying the modified theorems 1 and 3 of [GW1], we obtain a residual subset of $\mathcal{R}$ with the property that for every T in this subset, $(\mathcal{X}, T)$ is minimal and the extension $(\mathcal{X}, T) \to (Z, \sigma)$ is a proximal extension. Since an extension of minimal systems preserves Ellis groups iff it is a proximal extension, it follows that $A = \mathcal{G}(\mathcal{X}, \xi_0) = \mathcal{G}(Z, z_0)$. On the other hand, since (Y, G) is not proximal, it follows that the extension $(X, T) \to (Z, \sigma)$ is not a proximal extension, whence $B = \mathcal{G}(X, x_0) \underset{\neq}{\subset} A = \mathcal{G}(Z, z_0)$.

Finally we form the quasifactor $\hat{X}$ of the system $(\mathcal{X}, T)$ which consists of all closed subsets of $\mathcal{X}$ of the form:

$$F_{(z,y)} = \{\{z\} \times \{y_1, y_2\} \in \mathcal{X} : y \in \{y_1, y_2\}\}, \qquad (z, y) \in X.$$

There is an isomorphism of G-systems between X and $\hat{X}$ given by $(z, y) \mapsto F_{(z,y)}$ (the inverse of this map is given by $F_{(z,y)} \mapsto \bigcap F_{(z,y)} = \{(z, y)\}$), and we have thus proved the following:

Claim 11. *There exists a residual subset $\mathcal{R}$ of* cls $\mathcal{S}_G(X)$*, such that for each member $T \in \mathcal{R}$, the $\mathbb{Z}$-systems (X, T) and $(\mathcal{X}, T)$ are minimal, and the extension $(\mathcal{X}, T) \to (Z, \sigma)$ is a proximal extension while the extension $(X, T) \to (Z, \sigma)$ is not. Each of these two systems is a quasifactor of the other. In particular, for the quasifactor (X, T) of $(\mathcal{X}, G)$, denoting $A = \mathcal{G}(Z, z_0) = \mathcal{G}(\mathcal{X}, \xi_0)$ and $B = \mathcal{G}(X, x_0)$, we have $B \underset{\neq}{\subset} A$, hence $B \not\supset A^B = A$. If we choose Z to be a normal system (i.e. a minimal system whose Ellis group is normal; every regular system is normal, see e.g. [GMS]), say the dyadic adding machine, we have $B \not\supset A^G = A$.*

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