SETS WITH SEVERAL CENTERS OF SYMMETRY

GREGORY A. FREIMAN AND YONUTZ V. STANCHESCU

Abstract. Let A be a finite subset of the group $\mathbb{Z}^2$. Let $C = \{c_0, c_1, ..., c_{s-1}\}$ be a finite set of s distinct points in the plane. For every $0 \leq i \leq s-1$, we define $D_i = \{a - a' : a \in A, a' \in A, a + a' = 2c_i\}$ and $R_s(A) = |D_0 \cup D_1 \cup ... \cup D_{s-1}|$. In [1] and [2], we found the maximal value of $R_s(A)$ in cases $s = 1, s = 2$ and $s = 3$ and studied the structure of A assuming that $R_3(A)$ is equal or close to its maximal value. In this paper, we examine the case of $s = 4$ centers of symmetry and we find the *maximal value* of $R_4(A)$. Moreover, in cases when the maximal value is attained, we will describe the *structure of extremal sets*.

1. Introduction

Let A be a finite subset of the group $\mathbb{Z}^2$ of cardinality $|A| = k$. Let $M + N = \{m + n : m \in M, n \in N\}$ be the *algebraic sum* of two finite sets M and N. We call $2A = A + A$ the *sum set* of A and $A - A$ the *difference set* of A. For every $b \in \mathbb{Z}^2$ we define

$$D(b) = \{a - a' : a \in A, a' \in A, a + a' = b\},$$
$$r(b) = |\{(a, a') : a + a' = b, a \in A, a' \in A\}|.$$

We easily see that $|D(b)| = r(b)$. Moreover, $r(b)$ is equal to the number of pairs (a, a') such that $a \in A$, $a' \in A$ and a and a' are symmetric with respect to the center $c = \frac{b}{2}$, i.e.,

$$|D(b)| = |\{(a, a') : a \in A, a' \in A, \frac{a + a'}{2} = c\}|. \tag{1}$$

Let $C = \{c_0, c_1, ..., c_{s-1}\}$ be a finite set of s distinct points in the plane such that $b_i = 2c_i \in \mathbb{Z}^2$, for every $0 \leq i \leq s-1$. We define

$$D_i = D_i(A) = \{a - a' : a \in A, a' \in A, a + a' = b_i\},\ d_i = |D_i|,$$
$$\mathrm{Diff}_s(A) = D_0 \cup D_1 \cup ... \cup D_{s-1},$$
$$R_s(A) = |\mathrm{Diff}_s(A)| = |D_0 \cup D_1 \cup ... \cup D_{s-1}|.$$

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This means that $d_i = |D_i| = r(b_i)$ and thus $R_s(A)$ counts the number of all *distinct differences* $d = a - a' \in A - A$ such that the end points a and a' are symmetric with respect to some set C of centers of symmetry.

In [1] we determined the *maximal value* of $R_3(A)$ for finite sets $A \subseteq \mathbb{Z}^2$, assuming that b_0, b_1, b_2 are non-collinear, and we described the structure of planar *extremal sets* A^*, i.e., sets of integer lattice points in the plane $\mathbb{Z}^2$ for which we have $|A^*| = k$ and $R_3(A^*) = 3k - \sqrt{3k}$. In [2] we studied the *structure* of finite sets $A \subseteq \mathbb{Z}^2$ assuming that $R_3(A)$ is close to its maximal value, i.e., $R_3(A) \geq 3k - \theta\sqrt{k}$, with $\theta \leq 1.8$.

In this paper, we continue the study of finite sets of lattice points in the plane and we will examine the case of four centers of symmetry c_0, c_1, c_2, c_3 defined by

$$b_0 = 2c_0 = (0,0), b_1 = 2c_1 = (1,0), b_2 = 2c_2 = (0,1), b_3 = 2c_3 = (1,1). \tag{2}$$

We will obtain a *sharp upper bound* for $R_4(A) = |D_0 \cup D_1 \cup D_2 \cup D_3|$ and we will determine its *maximal value*

$$R_4(k) = \max\{R_4(A) : A \subseteq \mathbb{Z}^2, |A| = k\}. \tag{3}$$

Moreover, in cases when the maximal value is attained, we will describe the *structure of extremal sets*. The case (2) which we will study in this paper is, of course, a partial one. Nevertheless, we *conjecture* that this case gives the maximal number of differences $\max R_4(A)$ comparing with any other choice of four centers of symmetry.

We should mention that the proof given here, while representing a natural development of [1], is significantly shorter. More importantly, this new method provides clear intuition for the structure of extremal sets, and it seems plausible that our approach can be applied to derive general results for sets of lattice points in $\mathbb{Z}^d$.

In order to describe the canonical form of an *extremal set*, we will use the following notation. If $p = (x, y) \in \mathbb{R}^2$, we denote by x and y its coordinates with respect to the canonical basis $\{e_1 = (1,0), e_2 = (0,1)\}$ and $e_0 = (0,0)$ represents the origin point.

Definition 1. For every integer $t \geq 2$, we denote by E_t the set of all lattice points $p = (x, y) \in \mathbb{Z}^2$ which satisfy the following conditions:

(a) $|x| < t$,
(b) $|y| < t$,
(c) $|x - y| < t$,
(d) $|x + y - \frac{1}{2}| < t$.

The set E_t lies on $2t - 1$ lines parallel to the line $x = 0$, on $2t - 1$ lines parallel to the line $y = 0$, on $2t - 1$ lines parallel to the line $x - y = 0$ and on $2t$ lines parallel to the line $x + y = 0$ (see Figure 1). Note that the set E_t can also be defined using the l_1-norm $||(x, y)||_1 = |x| + |y|$:

$$E_t = \{(x, y) \in \mathbb{Z}^2 : |x - 0.25| + |y - 0.25| < t\}.$$

Thus, E_t is the set of all lattice points that lie inside a two-dimensional open l_1-disk of radius t and center $(0.25, 0.25)$.

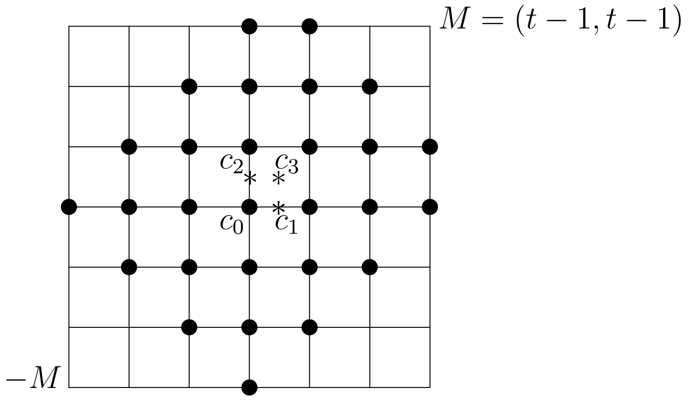


FIGURE 1. The set E_t for $t = 4$ and the centers c_0, c_1, c_2, c_3.

We will prove the following theorem.

Theorem 1. *Let A be a finite subset of $\mathbb{Z}^2$ with $|A| = k$. If k is sufficiently large and if $b_0 = (0,0), b_1 = (1,0), b_2 = (0,1), b_3 = (1,1)$, then*

$$R_4(A) = |\mathrm{Diff}_4(A)| \le 4k - \sqrt{8k+1}. \tag{4}$$

Moreover, the equality

$$R_4(A) = 4k - \sqrt{8k+1} \tag{5}$$

holds if and only if there is $t \in \mathbb{Z}$ such that $k = t(2t-1)$ and A is the extremal set E_t.

This paper is organized as follows. In Sections 2 and 4, we introduce some basic examples: a two-dimensional arithmetic progression S_t, the extremal set E_t and a special octagon P. In Section 3, we state and prove a tight upper bound for $R_4(A)$. Section 5 contains the proof of Theorem 1 for *connected* sets, and in Section 6, we complete the proof by showing that *disconnected* sets are not extremal with respect to (3).

We conclude the introduction with some results obtained in [1]. We will use them in Sections 3 and 6.

Proposition 1. *Let A be a finite subset of $\mathbb{Z}^2$.*

(a) *If A lies on the line $(y = 0)$, then $R_2(A) = |D_0(A) \cup D_1(A)| \le 2|A| - 1$.*

(b) *If A lies on two parallel lines $(y = h)$ and $(y = -h)$, then*

$$R_2(A) = |D_0(A) \cup D_1(A)| \le 2|A| - 2.$$

(c) *If A lies on a lines parallel to the line $(x = 0)$, on b lines parallel to the line $(y = 0)$ and on c lines parallel to the line $(x + y = 0)$, then*

$$|A| \le \frac{1}{3}\frac{(a+b+c)^2}{4} + \frac{1}{4}.$$

Proof. Assertion (a) is equivalent to Proposition 2 (a) from [1]. Assertion (b) is true in view of Lemma 1 (b) from [1]. Finally, using the proof of Corollary 1 from [1], we obtain assertion (c). ☐

2. Some Examples

We begin with a simple remark about the sets of differences D_0, D_1, D_2 and D_3. As we mentioned in Section 1, the centers $c_i = \frac{b_i}{2}$ satisfy assumption (2).

Lemma 1. *Let* $A \subseteq \mathbb{Z}^2$ *be a finite set of* k *lattice points in the plane. Assume that* $b_0 = (0,0), b_1 = (1,0), b_2 = (0,1), b_3 = (1,1)$. *We have*

$$D_0(A) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}, D_1(A) \subseteq (2\mathbb{Z}+1) \times 2\mathbb{Z},$$
$$D_2(A) \subseteq 2\mathbb{Z} \times (2\mathbb{Z}+1), D_3(A) \subseteq (2\mathbb{Z}+1) \times (2\mathbb{Z}+1)$$

and thus the sets of differences $D_0(A), D_1(A), D_2(A)$ *and* $D_3(A)$ *are disjoint.*

Proof. For every lattice point $p = (x, y) \in \mathbb{Z}^2$, we denote by

$$p_i = 2c_i - p$$

the symmetric reflection of p with respect to c_i, $0 \leq i \leq 3$. If $d \in D_i(A)$, then there is a point $p = (x, y) \in A$ such that $p_i \in A$ and

$$d = p - p_i = 2p - 2c_i = (2x, 2y) - b_i.$$

Lemma 1 is proved, in view of (2). ☐

We will first examine the case of a two dimensional arithmetic progression S_t, which includes the extremal example E_t (see Figures 2 and 1). We will prove the following result.

Lemma 2. *Let* $t \geq 2$ *be an integer and let* S_t *denote the set of all lattice points* $p = (x, y) \in \mathbb{Z}^2$ *such that* $|x| < t, |y| < t$. *Then* $n = |S_t| = (2t-1)^2$ *and*

$$R_4(S_t) = 4n - 4\sqrt{n} + 1. \tag{6}$$

Proof. We will estimate $|D_i(S_t)|$ using equality (1). Let us now examine Figure 2.

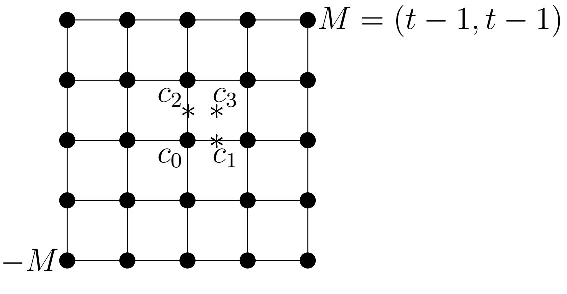


Figure 2. The set S_t for $t = 3$ and the centers c_0, c_1, c_2, c_3.

We have $n = (2t-1)^2$ and $c_0 = \frac{b_0}{2} = \frac{e_0}{2}, c_1 = \frac{b_1}{2} = \frac{e_1}{2}, c_2 = \frac{b_2}{2} = \frac{e_2}{2}, c_3 = \frac{b_3}{2} = \frac{e_1+e_2}{2}$. The set S_t is symmetric with respect to c_0 and thus

$$d_0 = |D_0(S_t)| = n. \tag{7}$$

The set $S_t \setminus (x = -t+1)$ is symmetric with respect to the center c_1 and thus

$$d_1 = |D_1(S_t)| = n - (2t-1). \tag{8}$$

The set $S_t \setminus (y = -t+1)$ is symmetric with respect to the center c_2 and thus

$$d_2 = |D_2(S_t)| = n - (2t-1). \tag{9}$$

Finally, the set $S_t \setminus ((x = -t+1) \cup (y = -t+1))$ is symmetric with respect to the center c_3 and thus

$$d_3 = |D_3(S_t)| = n - 2(2t-1) + 1. \tag{10}$$

Moreover, the sets $D_0(S_t), D_1(S_t), D_2(S_t), D_3(S_t)$ are disjoint by Lemma 1, so we conclude that the total number of differences is

$$\begin{aligned} R_4(S_t) &= |D_0(S_t) \cup D_1(S_t) \cup D_2(S_t) \cup D_3(S_t)| \\ &= d_0 + d_1 + d_2 + d_3 = 4n - 4(2t-1) + 1 = 4n - 4\sqrt{n} + 1. \end{aligned}$$

Lemma 2 is proved. □

In case of $s = 2$ centers of symmetry, the extremal sets are arithmetic progressions of difference $\Delta = 2c_1 - 2c_0$ (see [1], Proposition 2). Surprisingly, the maximal value of $R_4(A)$ is not attained for a two dimensional arithmetic progression S_t. In order to describe the extremal set E_t, let us recall that the canonical form of an extremal set for the case of *three centers* c_0, c_1, c_2 is a hexagon H_α (see [1], Theorem 1). This set (see Figure 3) lies on pairs of symmetric lines with respect to *three lines*

$$l_1 : (x = 0),\ l_2 : (y = 0) \text{ and } l_3 : (x + y = 0.5).$$

Note that c_0 and c_2 belong to l_1, c_0 and c_1 belong to l_2 and c_1 and c_2 belong to l_3. The definition of E_t is similar in the sense that this set also lies on pairs of symmetric lines with respect to *four lines*

$$l_1 : (x = 0),\ l_2 : (y = 0),\ l_3 : (x + y = 0.5) \text{ and } l_4 : (x - y = 0).$$

Note that c_0 and c_3 belong to l_4.

The following result determines the number of differences for E_t and, at the same time, implies that $R_4(k) = \max\{R_4(A) : A \subseteq \mathbb{Z}^2, |A| = k\}$ is at least $4k - \sqrt{8k+1}$:

Lemma 3. *Let $t \geq 2$ be an integer and let E_t denote the set of all lattice points $p = (x, y) \in \mathbb{Z}^2$ such that $|x - y| < t$ and $|x + y - \frac{1}{2}| < t$. Then $k = |E_t| = (2t-1)t$ and*

$$R_4(E_t) = 4k - \sqrt{8k+1}. \tag{11}$$

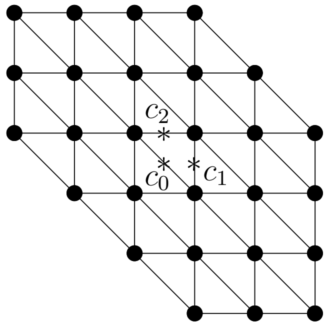


FIGURE 3. The set H_α and the centers $c_i = \frac{e_i}{2}, i = 0, 1, 2$.

Proof. We will estimate $|D_i(E_t)|$ using equality (1). Let us now examine Figure 1. We have $k = (2t-1)t$ and $c_0 = \frac{b_0}{2} = \frac{e_0}{2}, c_1 = \frac{b_1}{2} = \frac{e_1}{2}, c_2 = \frac{b_2}{2} = \frac{e_2}{2}, c_3 = \frac{b_3}{2} = \frac{e_1+e_2}{2}$. The set $E_t \setminus (x + y = t)$ is symmetric with respect to the center c_0 and thus

$$d_0 = |D_0(E_t)| = k - (t-1). \tag{12}$$

The set $E_t \setminus (-x + y = t - 1)$ is symmetric with respect to the center c_1 and thus

$$d_1 = |D_1(E_t)| = k - t. \tag{13}$$

The set $E_t \setminus (-x + y = -t + 1)$ is symmetric with respect to the center c_2 and thus

$$d_2 = |D_2(E_t)| = k - t. \tag{14}$$

Finally, the set $E_t \setminus (x + y = -t + 1)$ is symmetric with respect to the center c_3 and thus

$$d_3 = |D_3(E_t)| = k - t. \tag{15}$$

Moreover, the sets $D_0(E_t), D_1(E_t), D_2(E_t), D_3(E_t)$ are disjoint by Lemma 1, so we conclude that the total number of differences is

$$\begin{aligned} R_4(E_t) &= |D_0(E_t) \cup D_1(E_t) \cup D_2(E_t) \cup D_3(E_t)| \\ &= d_0 + d_1 + d_2 + d_3 = 4k - (4t - 1) = 4k - \sqrt{8k+1}. \end{aligned}$$

Lemma 3 is proved. □

3. A SHARP UPPER BOUND

Let $A \subseteq \mathbb{Z}^2$ be a finite set of $k = |A|$ lattice points. In this section, the method of [1] will be used in order to obtain a sharp upper bound for $R_4(A)$.

Let a be the number of lines $\ell_1' : (x = h)$ such that $A \cap (x = \pm h) \neq \emptyset$, let b be the number of lines $\ell_2' : (y = h)$ such that $A \cap (y = \pm h) \neq \emptyset$, let c be the number of lines $\ell_3' : (x + y = h)$ such that $A \cap (x + y - 0.5 = \pm(h - 0.5)) \neq \emptyset$ and finally, let d be the number of lines $\ell_4' : (x - y = h)$ such that $A \cap (x - y = \pm h) \neq \emptyset$.

For example, if $A = E_t$, then $a = b = c - 1 = d = 2t - 1$ and if $A = S_t$, then $2a - 1 = 2b - 1 = c = d = 4t - 3$.

Lemma 4.
(a) *$d_0 + d_1 = |D_0(A) \cup D_1(A)| \leq 2k - b$.*
(b) *$R_4(A) \leq 4k - \max(2a, 2b, c + d)$.*
(c) *$R_4(A) \leq 4k - \frac{1}{2}(a + b + c + d) - \frac{\delta}{2}$, where $\delta = 0$ if $a + b + c + d$ is even and $\delta = 1$ if $a + b + c + d$ is odd.*

Proof. We follow the argument used in the proof of Lemma 2 in [1]. For every integer h, we denote by

$$A_h = A \cap (y = h)$$

the set of points of A that lie on the line $y = h$. For every $0 \leq i \leq 3$, the set $D_i(A)$ consists of all differences $d = p - p_i$ such that both points p and $p_i = 2c_i - p$ belong to the set A. Therefore each difference $d \in D_i$ is of the form

$$d = 2p - 2c_i = 2p - b_i.$$

Note that if $p \in A_h$, then $p_0 \in A_{-h}$, $p_1 \in A_{-h}$ and $p_2 \in A_{-h+1}, p_3 \in A_{-h+1}$. This remark allows us to split each set of differences $D_i, i = 0, 1, 2, 3$, into a *disjoint* union of sets:

$$D_i = \bigcup\nolimits_h D_i(h),$$

where

$$\begin{aligned}
D_0(h) = D_0(A, h) &= \{2p - e_0 : p \in A_h, p_0 \in A_{-h}\},\\
D_1(h) = D_1(A, h) &= \{2p - e_1 : p \in A_h, p_1 \in A_{-h}\},\\
D_2(h) = D_2(A, h) &= \{2p - e_2 : p \in A_h, p_2 \in A_{-h+1}\},\\
D_3(h) = D_3(A, h) &= \{2p - (e_1 + e_2) : p \in A_h, p_3 \in A_{-h+1}\}.
\end{aligned}$$

Let H be the set of all integers h such that $A_{\pm h} = A_h \cup A_{-h} \neq \emptyset$. We have

$$\begin{aligned}
D_0(A) &= \bigcup\nolimits_{h \in H} D_0(h) = \bigcup\nolimits_{h \in H,\ h \geq 0} D_0(A_{\pm h}),\\
D_1(A) &= \bigcup\nolimits_{h \in H} D_1(h) = \bigcup\nolimits_{h \in H,\ h \geq 0} D_1(A_{\pm h}),\\
D_0(A) \cup D_1(A) &= \bigcup\nolimits_{h \in H,\ h \geq 0} \big(D_0(A_{\pm h}) \cup D_1(A_{\pm h})\big).
\end{aligned}$$

If $h = 0$ belongs to H, we have $|D_0(A_0) \cup D_1(A_0)| \leq 2|A_0| - 1$, in view of Proposition 1 (a). For $0 < h \in H$, the set $A_{\pm h}$ is contained by two parallel lines. If $|A_h| > 0$ and $|A_{-h}| > 0$, then $|D_0(A_{\pm h}) \cup D_1(A_{\pm h})| \leq 2|A_{\pm h}| - 2$, by Proposition 1(b). If $|A_h| = 0$ or $|A_{-h}| = 0$, then $A_{\pm h}$ lies on a line and obviously $|D_0(A_{\pm h}) \cup D_1(A_{\pm h})| = 0 \leq$

$2|A_{\pm h}| - 2 < 2|A_{\pm h}| - 1$. We conclude that

$$\begin{aligned} d_0 + d_1 = |D_0(A) \cup D_1(A)| &= \sum_{h \in H,\ h \geq 0} |D_0(A_{\pm h}) \cup D_1(A_{\pm h})| \\ &\leq |D_0(A_0) \cup D_1(A_0)| + \sum_{h \in H,\ h > 0} (2|A_{\pm h}| - 2) \\ &\leq 2|A| - b = 2k - b. \end{aligned}$$

In order to prove (b) we will use Lemma 1 and get

$$R_4(A) = |D_0(A) \cup D_1(A) \cup D_2(A) \cup D_3(A)| = d_0 + d_1 + d_2 + d_3.$$

Note that inequality (a) was obtained using a partition of A into sets lying on lines $\ell_2' : (y = h)$ parallel to the segment $[c_0, c_1]$. In a similar way, considering lines parallel to the segments $[c_2, c_3], [c_0, c_2], [c_1, c_3]$, we obtain respectively that

$$d_2 + d_3 \leq 2k - b,\ d_0 + d_2 \leq 2k - a,\ d_1 + d_3 \leq 2k - a.$$

Moreover, considering lines parallel to the segments $[c_1, c_2], [c_0, c_3]$, we obtain that

$$d_1 + d_2 \leq 2k - c,\ d_0 + d_3 \leq 2k - d.$$

Indeed, these last two inequalities are also valid, because c represents the number of lines $\ell_3' : (x+y = h)$ such that $A \cap (x+y-0.5 = \pm(h-0.5)) \neq \emptyset$ and d is the number of lines $\ell_4' : (x - y = h)$ such that $A \cap (x - y = \pm h) \neq \emptyset$. It follows that

$$R_4(A) \leq 4k - 2a,\ R_4(A) \leq 4k - 2b, R_4(A) \leq 4k - (c + d)$$

and thus

$$R_4(A) \leq 4k - \max(2a, 2b, c + d). \tag{16}$$

Moreover,

$$\begin{aligned} 4R_4(A) &= 4(d_0 + d_1 + d_2 + d_3) \\ &= (d_0 + d_1) + (d_2 + d_3) + (d_0 + d_2) + (d_1 + d_3) + 2(d_1 + d_2) + 2(d_0 + d_3) \\ &\leq (2k - b) + (2k - b) + (2k - a) + (2k - a) + 2(2k - c) + 2(2k - d) \\ &= 16k - 2(a + b + c + d). \end{aligned}$$

and thus

$$R_4(A) = d_0 + d_1 + d_2 + d_3 \leq 4k - \frac{1}{2}(a + b + c + d) - \frac{\delta}{2}, \tag{17}$$

where we put $\delta = 0$ if $a+b+c+d$ is even and $\delta = 1$ if $a+b+c+d$ is odd. Lemma 4 is proved. □

Remark 1. The upper bound $R_4(A) \leq 4k - \frac{1}{2}(a+b+c+d) - \frac{\delta}{2}$ is sharp. Indeed, in view of Lemma 3, the set E_t satisfies $a = b = d = 2t-1, c = 2t$ and we have $\delta = 1$ and

$$R_4(E_t) = d_0 + d_1 + d_2 + d_3 = 4k - \sqrt{8k+1} = 4k - (4t-1) = 4k - \frac{1}{2}(a+b+c+d) - \frac{1}{2}.$$

We conclude that inequality (17) cannot be improved by reducing the upper bound for $R_4(A)$. $\square$

Remark 2. Inequality (17) implies a first non-trivial upper bound for $R_4(A)$ in terms of k:

$$R_4(A) \leq 4k - \frac{1}{2}(a+b) - \frac{1}{2}(c+d) \leq 4k - \sqrt{ab} - \sqrt{cd} \leq 4k - 2\sqrt{k}.$$

In the following sections, we will improve this estimate and we will show that $R_4(A) \leq 4k - \sqrt{8k+1}$, for every finite set of lattice points in the plane. $\square$

4. Special Octagons

In this section, we determine the number of differences $R_4(P) = |\mathrm{Diff}_4(P)|$ for a special octagon $P \subseteq \mathbb{Z}^2$ (see Figure 4).

Let a, b, u and v be four natural numbers such that $a = 2\alpha - 1$, $b = 2\beta - 1$ and

$$u + v + 1 \leq \min\{a, b\} - 1. \tag{18}$$

We denote by $A = (\alpha - 1, \beta - 1), B = (-\alpha + 1, \beta - 1), F = -A$ and $G = -B$ the four vertices of the rectangle $R(a, b)$ defined by $R(a, b) = \{(x, y) \in \mathbb{Z}^2 : |x| < \alpha, |y| < \beta\}$. This finite set lies on $a = 2\alpha - 1$ lines parallel to $(x = 0)$ and on $b = 2\beta - 1$ lines parallel to $(y = 0)$. Let us choose eight points on the edges of $R(a, b)$ as follows:

$$A_1 = A - ve_1, A_2 = A - ve_2, \qquad B_1 = B + ue_1, B_2 = B - ue_2,$$
$$F_1 = F + (v+1)e_1, F_2 = F + (v+1)e_2, \qquad G_1 = G - ue_1, G_2 = G + ue_2$$

and we denote by

$$P = P(a, b, u, v)$$

the set of all lattice points that lie in the convex hull of $\{A_1, B_1, B_2, F_2, F_1, G_1, G_2, A_2\}$. The set $P = P(a, b, u, v)$ is described in Figure 4.

Let $[L, M]$ be the *line segment* $\{(1-t)L + tM : 0 \leq t \leq 1\}$ between two points L and M in the plane. Note that inequality (18) implies that each of the following sets

$$[A, A_1] \cap [B, B_1], \ [B, B_2] \cap [F, F_2], \ [F, F_1] \cap [G_1, G], \ [G, G_2] \cap [A, A_2]$$

contain no more than one point and therefore the eight points $A_1, B_1, B_2, F_2, F_1, G_1, G_2, A_2$ are all vertices of the convex hull $conv(P)$. We conclude that $P = P(a, b, u, v)$

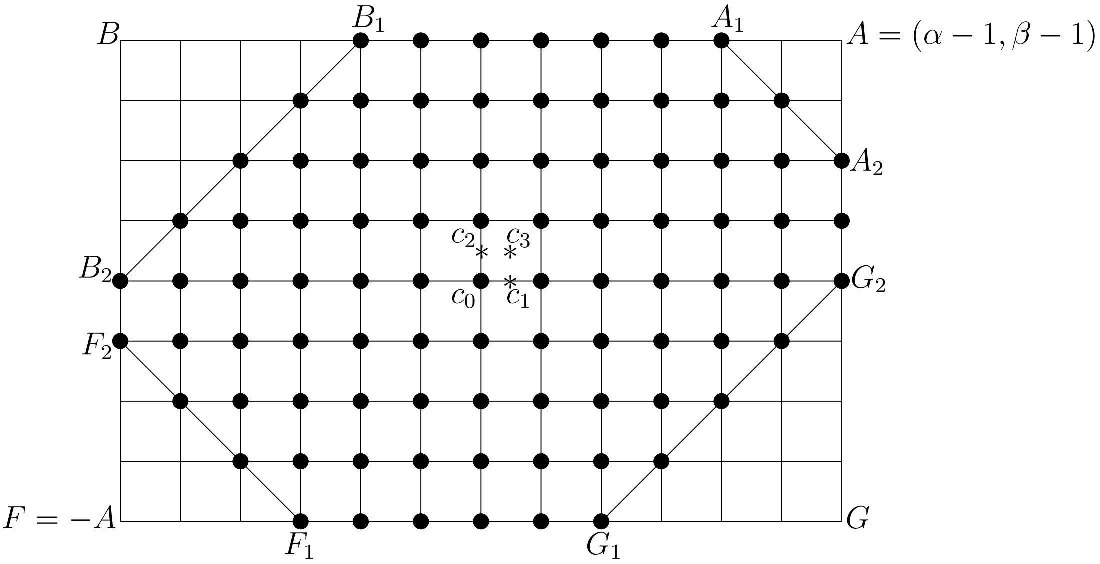
is the set of all points $(x, y) \in \mathbb{Z}^2$ which satisfy the following conditions:

$$P = P(a,b,u,v) : \begin{cases} -\alpha+1 \le x \le \alpha-1, \\ -\beta+1 \le y \le \beta-1, \\ -\gamma+1 \le x+y \le \gamma, \\ -\delta+1 \le x-y \le \delta-1, \end{cases} \tag{19}$$

where γ and δ are given by

$$\gamma = \frac{(a+b-1)-(2v+1)}{2} = \alpha+\beta-2-v, \ \delta = \frac{(a+b-1)-2u+1}{2} = \alpha+\beta-1-u. \tag{20}$$

Note that if $\alpha = \beta = \delta = \gamma$, then $P(a,b,u,v)$ is the extremal set E_α described in Figure 1. We will now find the number of lattice points located inside $P = P(a,b,u,v)$



$A_1 = A - ve_1, \ B_1 = B + ue_1, \ F_1 = F + (v+1)e_1, \ G_1 = G - ue_1$

FIGURE 4. The set $P = P(a,b,u,v), a = 2\alpha-1, b = 2\beta-1$.

and we will determine the number of differences $R_4(P)$.

Lemma 5. *The set* $P = P(a,b,u,v)$ *lies on* $a = 2\alpha - 1$ *lines parallel to* $(x = 0)$, *on* $b = 2\beta - 1$ *lines parallel to* $(y = 0)$, *on* $c = 2\gamma$ *lines parallel to* $(x + y = 0)$ *and on* $d = 2\delta - 1$ *lines parallel to* $(x - y = 0)$. *We have*

(a) $k^* = |P| = ab - u(u+1) - (v+1)^2$.
(b) $R_4(P) = 4k^* - (2a + 2b - 2u - 2v - 3) = 4k^* - (c + d)$.

Proof. Every point $(x, y) \in P(a,b,u,v)$ belongs to the rectangle $R(a,b)$. The set $P = P(a,b,u,v)$ is obtained from $R(a,b)$ by removing the lattice points belonging to four triangles

$$\begin{aligned}
T_A &= \{(x,y) \in R(a,b) : x+y=t, \text{ where } \gamma+1 \le t \le \alpha+\beta-2\},\\
T_B &= \{(x,y) \in R(a,b) : x-y=t, \text{ where } -\alpha-\beta+2 \le t \le -\delta\},\\
T_F &= \{(x,y) \in R(a,b) : x+y=t, \text{ where } -\alpha-\beta+2 \le t \le -\gamma\},\\
T_G &= \{(x,y) \in R(a,b) : x-y=t, \text{ where } \delta \le t \le \alpha+\beta-2\}.
\end{aligned}$$

Thus $P(a,b,u,v)$ lies on exactly $a = 2\alpha - 1$ lines parallel to $(x = 0)$, on $b = 2\beta - 1$ lines parallel to $(y = 0)$, on $c = 2\gamma$ lines parallel to $(x + y = 0)$ and on $d = 2\delta - 1$ lines parallel to $(x - y = 0)$. It is clear that:

$$\begin{aligned}
&n_A = |T_A| = 1+2+...+(v-1)+v, \ n_B = |T_B| = 1+2+...+(u-1)+u,\\
&n_F = |T_F| = 1+2+...+v+(v+1), \ n_G = |T_G| = 1+2+...+(u-1)+u
\end{aligned}$$

and thus

$$k^* = |P| = |R(a,b)| - (n_A+n_B+n_F+n_G) = ab - u(u+1) - (v+1)^2. \tag{21}$$

We will estimate $|D_i(P)|$ using equality (1) and (19). Let us now examine Figure 4. We have

$$c_0 = \frac{b_0}{2} = \frac{e_0}{2}, c_1 = \frac{b_1}{2} = \frac{e_1}{2}, c_2 = \frac{b_2}{2} = \frac{e_2}{2}, c_3 = \frac{b_3}{2} = \frac{e_1+e_2}{2}.$$

The set $P \setminus [A_1, A_2]$ is symmetric with respect to the center c_0 and thus

$$d_0 = |D_0(P)| = k^* - (v+1). \tag{22}$$

The set $P \setminus ([B_1, B_2] \cup [B_2, F_2])$ is symmetric with respect to the center c_1 and thus

$$d_1 = |D_1(P)| = k^* - (b - (v+1)). \tag{23}$$

The set $P \setminus ([F_1, G_1] \cup [G_1, G_2])$ is symmetric with respect to the center c_2 and thus

$$d_2 = |D_2(P)| = k^* - (a - (v+1)). \tag{24}$$

Finally, the set $P \setminus ([B_2, F_2] \cup [F_2, F_1] \cup [F_1, G_1])$ is symmetric with respect to the center c_3 and thus

$$d_3 = |D_3(P)| = k^* - ((b-u) + (a-u) - (v+2)). \tag{25}$$

Moreover, the sets $D_0(P), D_1(P), D_2(P), D_3(P)$ are disjoint by Lemma 1, so we conclude that the total number of differences is

$$\begin{aligned}
R_4(P) &= |D_0(P) \cup D_1(P) \cup D_2(P) \cup D_3(P)|\\
&= d_0 + d_1 + d_2 + d_3 = 4k^* - (2a + 2b - 2u - 2v - 3).
\end{aligned}$$

Note that (20) implies that $c = 2\gamma = (a+b-1)-(2v+1)$, $d = 2\delta-1 = (a+b-1)-2u$ and thus

$$c + d = 2a + 2b - 2u - 2v - 3. \tag{26}$$

We conclude that

$$R_4(P) = 4k^* - (2a + 2b - 2u - 2v - 3) = 4k^* - (c + d). \tag{27}$$

Lemma 5 is proved. □

We will obtain now an upper bound for $R_4(P)$ depending only on $k^* = |P|$:

Lemma 6. *Let us define*

$$\epsilon = (a - b)^2 + (c + d - a - b)^2 + 2(v - u)(v - u + 1) - 1. \tag{28}$$

(a) *ϵ is a non-negative integer and*

$$(c + d)^2 = (8k^* + 1) + 2\epsilon \geq 8k^* + 1. \tag{29}$$

(b) *Every set $P = P(a, b, u, v)$ satisfies*

$$R_4(P) = 4k^* - (c + d) \leq 4k^* - \sqrt{8k^* + 1}. \tag{30}$$

(c) *We have equality $R_4(P) = 4k^* - (c+d) = 4k^* - \sqrt{8k^* + 1}$ if and only if $v = u-1$, $\alpha = \beta = \delta = \gamma = u + 1$ and $P(a, b, u, v)$ is the extremal set E_α.*

Proof. Let us put $w = v - u$. Using (26) and (18) we get that

$$c + d - a - b = (a + b) - (2u + 2v + 3) \geq 1.$$

Thus $(c + d - a - b)^2 \geq 1$. This implies that ϵ is a non-negative integer, because $(v - u)(v - u + 1) = w(w + 1)$, being a product of two consecutive integers, is non-negative. We have

$$\begin{aligned}
8ab + 2\epsilon &= 8ab + 2(a - b)^2 + 2(c + d - a - b)^2 + 4w(w + 1) - 2 \\
&= 8ab + 2(a - b)^2 + 2\Big((a + b) - (2u + 2v + 3)\Big)^2 + 4w(w + 1) - 2 \\
&= (2a + 2b)^2 - 2(2a + 2b)(2u + 2v + 3) + 2(2u + 2v + 3)^2 + 4w(w + 1) - 2 \\
&= ((2a + 2b) - (2u + 2v + 3))^2 + (2u + 2v + 3)^2 + 4w(w + 1) - 2 \\
&= (2a + 2b - 2u - 2v - 3)^2 + (2u + 2v + 3)^2 + 4w(w + 1) - 2 \\
&= (2a + 2b - 2u - 2v - 3)^2 + 8u(u + 1) + 8(v + 1)^2 - 1.
\end{aligned}$$

Therefore (26) gives

$$\begin{aligned}
(c + d)^2 &= (2a + 2b - 2u - 2v - 3)^2 \\
&= 8ab + 2\epsilon - 8u(u + 1) - 8(v + 1)^2 + 1 \\
&= 8\Big(ab - u(u + 1) - (v + 1)^2\Big) + 1 + 2\epsilon \\
&= (8k^* + 1) + 2\epsilon \geq 8k^* + 1.
\end{aligned}$$

Assertion (a) is proved. The upper-bound (30) is an immediate consequence of (29) and (27). Moreover, we have equality $R_4(P) = 4k^* - (c+d) = 4k^* - \sqrt{8k^* + 1}$ if and only if $\epsilon = 0$, which means

(i) $a = b, c + d = a + b + 1, v = u$

or

(ii) $a = b, c + d = a + b + 1, v = u - 1.$

Case (i) is impossible because $(c+d) - (a+b+1) = (a+b) - (2u+2v+3) - 1 = 2a - 4v - 4 = 4\alpha - 4v - 6 \neq 0$.

In case (ii), equality (26) and $c+d = a+b+1$ imply that $2a+2b-2u-2v-3 = a+b+1$ and so

$$2a = a + b = 2u + 2v + 4 = 4u + 2.$$

We apply (20) and get

$$\begin{aligned} a &= b = 2\alpha - 1 = 2\beta - 1 = 2u + 1, \\ c &= 2\gamma = (a + b - 1) - (2v + 1) = 2a - 2u = 2u + 2, \\ d &= 2\delta - 1 = (a + b - 1) - 2u = (2a - 1) - 2u = 2u + 1. \end{aligned}$$

In conclusion, $\alpha = \beta = \gamma = \delta = u + 1$. The proof of Lemma 6 is complete. □

5. Connected Sets and Covering Octagons

We can now prove Theorem 1 for *connected* sets. We need the following definition.

Definition 2. Let $A \subseteq \mathbb{Z}^2$ be a finite set of $k = |A|$ lattice points. Let us choose the parameters a, b, u, v such that:

(i) $A \subseteq P(a, b, u, v)$,
(ii) a and b are minimal,
(iii) u and v are maximal.

The finite set $P(a, b, u, v) \subseteq \mathbb{Z}^2$ defined by the above three conditions will be called *a covering polygon* of the set A and we will denote it by $P(A)$. Let

$$x = \pm(\alpha - 1), y = \pm(\beta - 1), x + y = \gamma, x + y = -\gamma + 1, x - y = \pm(\delta - 1)$$

denote the supporting lines of the covering polygon $P = P(A)$. We say that A is a *connected* set if the following conditions are true:

(a) $A \cap (x = \pm t) \neq \emptyset$, for every integer $-\alpha + 1 \leq t \leq \alpha - 1$,
(b) $A \cap (y = \pm t) \neq \emptyset$, for every integer $-\beta + 1 \leq t \leq \beta - 1$,
(c) $A \cap (x + y - \frac{1}{2} = \pm(t - \frac{1}{2})) \neq \emptyset$, for every integer $-\gamma + 1 \leq t \leq \gamma$,
(d) $A \cap (x - y = \pm t) \neq \emptyset$, for every integer $-\delta + 1 \leq t \leq \delta - 1$.

Lemma 7. *Let A be a finite subset of $\mathbb{Z}^2$ with $|A| = k$. If A is connected, then*

$$R_4(A) \leq 4k - \sqrt{8k + 1}.$$

Moreover, the equality $R_4(A) = 4k - \sqrt{8k+1}$ holds if and only if there is an integer α such that $k = \alpha(2\alpha - 1)$ and A is the extremal set E_α.

Proof. Let $A^* = P(a,b,u,v)$ be the covering polygon of A and denote $k^* = |A^*|$. We clearly have

$$k \leq k^*.$$

Define c and d as in Lemma 5. The set A^* lies on exactly a lines parallel to the line $l_1 : (x = 0)$, on b lines parallel to the line $l_2 : (y = 0)$, on c lines parallel to the line $l_3 : (x + y - \frac{1}{2} = 0)$ and on d lines parallel to the line $l_4 : (x - y = 0)$. The set A is a connected set and we may use inequality (b) of Lemma 4 with the same parameters a, b, c, d. Combining with (29), we get that

$$R_4(A) \leq 4k - (c+d) = 4k - \sqrt{(8k^* + 1) + 2\epsilon} \leq 4k - \sqrt{8k^* + 1} \leq 4k - \sqrt{8k + 1}. \quad (31)$$

We conclude that inequality (4) holds for every connected set of lattice points.

Let us assume now that the connected set A satisfies equality $R_4(A) = 4k - \sqrt{8k + 1}$. Using (31), we get that

$$R_4(A) = 4k - (c+d) = 4k - \sqrt{(8k^* + 1) + 2\epsilon} = 4k - \sqrt{8k^* + 1} = 4k - \sqrt{8k + 1}. \quad (32)$$

and thus

$$k = k^*, \epsilon = 0, A = A^*, R_4(A) = R_4(A^*) = 4k - (c + d) = 4k^* - (c + d).$$

We can use now assertion (c) of Lemma 6 and conclude that

$$A = A^* = P(a, b, u, v) = E_\alpha.$$

Lemma 7 is proved. ☐

6. Disconnected sets

In this section, we describe the set of differences $\mathrm{Diff}_4(A)$ for disconnected sets. We will show that such sets satisfy inequalities (35) and (40) below and Theorem 1 will be an easy consequence of Lemmas 7 and 10.

Definition 3. Let $A \subseteq \mathbb{Z}^2$ be a finite set. Let

$$x = \pm(\alpha - 1), y = \pm(\beta - 1), x + y = \gamma, x + y = -\gamma + 1, x - y = \pm(\delta - 1)$$

denote the supporting lines of the covering polygon $P = P(A)$. We say that A is a *disconnected* set if at least one of the following conditions is true:

(I) There is an integer t such that $-\alpha + 1 \leq t \leq \alpha - 1$ and $A \cap (x = \pm t) = \emptyset$.
(II) There is an integer t such that $-\beta + 1 \leq t \leq \beta - 1$ and $A \cap (y = \pm t) = \emptyset$.
(III) There is an integer t such that $-\gamma + 1 \leq t \leq \gamma$ and $A \cap (x + y - \frac{1}{2} = \pm(t - \frac{1}{2})) = \emptyset$.
(IV) There is an integer t such that $-\delta + 1 \leq t \leq \delta - 1$ and $A \cap (x - y = \pm t) = \emptyset$.

We will now present a detailed study of disconnected sets satisfying case (I). Cases (II), (III) and (IV) are similar. Let $(m < x < n)$ be the region of the plane equal to $\{(x, y) : (x, y) \in \mathbb{R}^2, m < x < n\}$.

Lemma 8. *Let* $A \subseteq \mathbb{Z}^2$ *be a finite disconnected set satisfying condition (I). Let us choose* $u \geq 0$ *minimal such that*

$$A \cap (x = \pm u) = \emptyset. \tag{33}$$

Define $k = |A|$, $A_1 = A \cap (-u < x < u), A_2 = A \setminus A_1$, $k_1 = |A_1|, k_2 = |A_2| = k - k_1$. *Let* n_0 *be the number of points* $p \in A_2$ *such that* $-p \notin A_2$. *We have* $R_4(A) = R_4(A_1) + R_4(A_2)$ *and*

(a) $R_4(A_2) \leq 4k_2 - n_0$,
(b) *if* $n_0 < k_2$, *then* $R_4(A_2) \leq 4k_2 - \frac{8}{\sqrt{6}}\sqrt{k_2 - n_0 - 0.5}$.

Proof. The set A is disconnected and thus $k_2 = |A_2| \geq 1$. The sets $D_i(A_1)$ and $D_i(A_2)$ are disjoint, for $i = 0, 1, 2, 3$ and thus

$$R_4(A) = R_4(A_1) + R_4(A_2). \tag{34}$$

Indeed, if $u = 0$, then $A_1 = \emptyset$ and equality (34) is obvious. Assume that $u \geq 1$. Using (33), we get that every difference $d = (d_1, d_2) \in \mathrm{Diff}_4(A_1)$ satisfies $|d_1| \leq 2(u-1)$ and every difference $d = (d_1, d_2) \in \mathrm{Diff}_4(A_2)$ satisfies $|d_1| \geq 2(u+1)$. Therefore, $\mathrm{Diff}_4(A_1)$ and $\mathrm{Diff}_4(A_2)$ are disjoint and (34) follows.

Using equality (1), we get that $|D_0(A_2)| = k_2 - n_0$ and therefore $R_4(A_2) = |D_0(A_2) \cup D_1(A_2) \cup D_2(A_2) \cup D_3(A_2)| \leq 3k_2 + |D_0(A_2)| = 3k_2 + k_2 - n_0 = 4k_2 - n_0$. It remains to show that if $n_0 < k_2$, then the subset A_2 satisfies the inequality

$$R_4(A_2) \leq 4k_2 - \frac{8}{\sqrt{6}}\sqrt{k_2 - n_0 - 0.5}. \tag{35}$$

The set A_2 is a disjoint union of

$$A_+ = A \cap (x > u)$$

and

$$A_- = A \cap (x < -u).$$

Let $k_2^+ = |A_+|$ and $k_2^- = |A_-|$. If $k_2^+ = 0$ or $k_2^- = 0$, then $R_4(A_2) = 0$. Therefore, there is no loss of generality in assuming that $k_2^+ \geq 1$ and $k_2^- \geq 1$.

Denote by $\pi_1(x, y) = x$ the projection parallel to line $(x = 0)$, by $\pi_2(x, y) = y$ the projection parallel to line $(y = 0)$, by $\pi_3(x, y) = x + y$ the projection parallel to line $(x + y = 0)$ and by $\pi_4(x, y) = x - y$ the projection parallel to line $(x - y = 0)$.

We *claim* that there is an integral vector $w \in \mathbb{N}^2$ such that the sets

$$B_+ = A_+ + w \text{ and } B_- = A_- - w$$

satisfy the following assertions:

(i) B_+ and B_- are disjoint,
(ii) the projections $\pi_i(B_+)$ and $\pi_i(B_-)$ are disjoint, for $i = 1, 2, 3, 4$,
(iii) the set $B = B_+ \cup B_-$ satisfies $R_4(A_2) \leq R_4(B)$.

If both coordinates of w are distinct and large enough, then assertions (i) and (ii) are clearly true. Let us now explain (iii). Each difference $d=(d_1,d_2)\in \mathrm{Diff}_4(A_2)$ can be written as $d=p-p'$, where $p+p'=2c_i$ and $p,p'\in A_2$. Therefore, we have either

$$p\in A_+, p'\in A_-, d_1\geq 2(u+1)\geq 2 \tag{36}$$

or

$$p'\in A_+, p\in A_-, d_1\leq -2(u+1)\leq -2. \tag{37}$$

This remark allows us to define a one to one map φ from

$$\mathrm{Diff}_4(A_2)=D_0(A_2)\cup D_1(A_2)\cup D_2(A_2)\cup D_3(A_2)$$

to

$$\mathrm{Diff}_4(B)=D_0(B)\cup D_1(B)\cup D_2(B)\cup D_3(B).$$

More precisely, if $p_i=2c_i-p$ denotes the symmetric reflection of p with respect to c_i, then φ is given by

$$\varphi(d)=\begin{cases} d+2w, & \text{if } d=p-p_i,\ p\in A_+,\ p_i\in A_-,\\ d-2w, & \text{if } d=p-p_i,\ p\in A_-,\ p_i\in A_+.\end{cases}$$

The image $\varphi(d)\in \mathrm{Diff}(B)$; indeed, if $d=p-p_i,\ p\in A_+,\ p_i\in A_-$, then

$$\begin{aligned} &d+2w=p-p_i+2w=(p+w)-(p_i-w),\\ &p+w\in B_+\subseteq B, p_i-w\in B_-\subseteq B,\\ &(p+w)+(p_i-w)=p+p_i=2c_i; \end{aligned}$$

if $d=p-p_i,\ p\in A_-,\ p_i\in A_+$, then

$$\begin{aligned} &d-2w=p-p_i-2w=(p-w)-(p_i+w),\\ &p-w\in B_-\subseteq B, p_i+w\in B_+\subseteq B,\\ &(p-w)+(p_i+w)=p+p_i=2c_i. \end{aligned}$$

In order to show that φ is one to one, it is enough to examine only differences $d',d''\in \mathrm{Diff}_4(A_2)$ of the form

$$\begin{aligned} &d'=(d'_1,d'_2)=p'-p'_i,\ \text{where } p'\in A_+,\ p'_i\in A_-,\\ &d''=(d''_1,d''_2)=p''-p''_i,\ \text{where } p''\in A_-,\ p''_i\in A_+. \end{aligned}$$

In view of (36) and (37), we have $d'_1\geq 2$ and $d''_1\leq -2$ and thus $d'+2w\neq d''-2w$, because of $w\in \mathbb{N}^2$. This implies that φ is one to one and assertion (iii) follows.

Assume that the set B_+ lies on exactly a_1 lines parallel to the line $(x=0)$, on b_1 lines parallel to the line $(y=0)$, on c_1 lines parallel to the line $(x+y=0)$ and on d_1 lines parallel to the line $(x-y=0)$. In other words:

$$a_1=|\pi_1(B_+)|, b_1=|\pi_2(B_+)|, c_1=|\pi_3(B_+)|, d_1=|\pi_4(B_+)|.$$

The set B_- determines the parameters a_2, b_2, c_2 and d_2 in a similar way, i.e.,

$$a_2 = |\pi_1(B_-)|, b_2 = |\pi_2(B_-)|, c_2 = |\pi_3(B_-)|, d_2 = |\pi_4(B_-)|.$$

Therefore, property (ii) implies that the set B lies on exactly $a_1 + a_2$ lines parallel to the line $(x = 0)$, on $b_1 + b_2$ lines parallel to the line $(y = 0)$, on $c_1 + c_2$ lines parallel to the line $(x + y = 0)$ and on $d_1 + d_2$ lines parallel to the line $(x - y = 0)$. Using inequality (17) and (i), we obtain

$$\begin{aligned} R_4(A_2) \le R_4(B) &\le 4|B| - \frac{(a_1+a_2)+(b_1+b_2)+(c_1+c_2)+(d_1+d_2)}{2} \\ &= 4|A_2| - (\frac{a_1+b_1+c_1+d_1}{2} + \frac{a_2+b_2+c_2+d_2}{2}). \end{aligned}$$

Note that ${k_2}^+ = |B_+|$ and ${k_2}^- = |B_-|$. We clearly have

$$\begin{aligned} \frac{a_1+b_1+c_1+d_1}{2} &\ge \sqrt{a_1b_1} + \sqrt{c_1d_1} \ge \sqrt{|B_+|} + \sqrt{|B_+|} = 2\sqrt{{k_2}^+}, \\ \frac{a_2+b_2+c_2+d_2}{2} &\ge \sqrt{a_2b_2} + \sqrt{c_2d_2} \ge \sqrt{|B_-|} + \sqrt{|B_-|} = 2\sqrt{{k_2}^-}. \end{aligned}$$

In order to prove inequality (35), we will improve these two estimates, using assertion (c) of Proposition 1. Indeed, this result implies that

$$\frac{a_1+b_1+c_1}{2} \ge \sqrt{3{k_2}^+ - \frac{3}{4}}$$

and thus

$$\frac{a_1+b_1+d_1}{2} \ge \sqrt{3{k_2}^+ - \frac{3}{4}}, \ \frac{a_1+c_1+d_1}{2} \ge \sqrt{3{k_2}^+ - \frac{3}{4}}, \ \frac{b_1+c_1+d_1}{2} \ge \sqrt{3{k_2}^+ - \frac{3}{4}}.$$

These four estimates give $\frac{3}{2}(a_1+b_1+c_1+d_1) \ge 4\sqrt{3{k_2}^+ - \frac{3}{4}}$ and thus

$$\frac{a_1+b_1+c_1+d_1}{2} \ge \frac{4}{\sqrt{3}}\sqrt{{k_2}^+ - \frac{1}{4}}.$$

A similar inequality is valid for the sum $a_2 + b_2 + c_2 + d_2$ and therefore

$$\begin{aligned} R_4(A_2) &\le 4|A_2| - (\frac{a_1+b_1+c_1+d_1}{2} + \frac{a_2+b_2+c_2+d_2}{2}) \\ &\le 4|A_2| - \frac{4}{\sqrt{3}}\sqrt{{k_2}^+ - \frac{1}{4}} - \frac{4}{\sqrt{3}}\sqrt{{k_2}^- - \frac{1}{4}}. \end{aligned}$$

Let us estimate the cardinalities of the sets B_+ and B_-. Let us recall that n_0 denotes the number of points $p \in A_2$ such that $p_0 = 2c_0 - p \notin A_2$; therefore, the number of points $p \in A_2$ with $-p \in A_2$ is equal to $k_2 - n_0$, and so

$${k_2}^+ = |B_+| = |A_+| \ge \frac{k_2 - n_0}{2}, {k_2}^- = |B_-| = |A_-| \ge \frac{k_2 - n_0}{2}.$$

Using $n_0 < k_2$, we get

$$R_4(A_2) \le 4k_2 - \frac{4}{\sqrt{3}}\sqrt{{k_2}^+ - \frac{1}{4}} - \frac{4}{\sqrt{3}}\sqrt{{k_2}^- - \frac{1}{4}} \le 4k_2 - \frac{8}{\sqrt{3}}\sqrt{\frac{k_2 - n_0}{2} - \frac{1}{4}}$$
$$= 4k_2 - \frac{8}{\sqrt{6}}\sqrt{k_2 - n_0 - 0.5}.$$

and Lemma 8 is proved. ☐

The following result is an easy consequence of Lemma 8.

Lemma 9. *Let $A \subseteq \mathbb{Z}^2$ be a finite set of $k = |A|$ lattice points in the plane. If A is disconnected, then we can split it into two disjoint subsets A_1 and B_1 such that:*

(a) $|A_1| \ge 0, |B_1| \ge 1$ *and* $R_4(A) = R_4(A_1) + R_4(B_1)$,
(b) $R_4(B_1) \le 4|B_1| - 2.9\sqrt{|B_1|}$ *or* $R_4(B_1) < 3.99|B_1|$.

Proof. There is no loss of generality in assuming that the disconnected set A satisfies condition (I): there is an integer t such that $-\alpha+1 \le t \le \alpha-1$ and $A \cap (x = \pm t) = \emptyset$. Choose $u \ge 0$ minimal such that $A \cap (x = \pm u) = \emptyset$ and define $A_1 = A \cap (-u < x < u)$ and $B_1 = A \setminus A_1$. Let n_0 be the number of points $p \in B_1$ such that $p_0 = 2c_0 - p \notin B_1$. Using Lemma 8, we get that $k_1 = |A_1| \ge 0, l_1 = |B_1| \ge 1, 0 \le n_0 \le l_1, R_4(A) = R_4(A_1) + R_4(B_1)$ and

$$R_4(B_1) \le 4l_1 - n_0, \text{ if } 0 \le n_0 \le l_1, \tag{38}$$

and

$$R_4(B_1) \le 4l_1 - \frac{8}{\sqrt{6}}\sqrt{l_1 - n_0 - 0.5}, \text{ if } n_0 < l_1. \tag{39}$$

We complete the proof by showing that inequalities (38) and (39) imply assertion (b) of Lemma 9. We may assume that $l_1 \ge 3$. Indeed, if $l_1 = 1$, then $R_4(B_1) \le 1 < 3.99l_1$, and if $l_1 = 2$, then $R_4(B_1) \le 3 < 3.99l_1$. We distinguish several cases.

Case 1. Assume that $6.77l_1 \ge 32n_0 + 16$. It follows that $n_0 < l_1, l_1 \ge 1$, and inequality (39) implies that

$$R_4(B_1) \le 4l_1 - \frac{8}{\sqrt{6}}\sqrt{l_1 - n_0 - 0.5} \le 4l_1 - 2.9\sqrt{l_1}.$$

Case 2. Assume that $6.77l_1 < 32n_0 + 16$. Using $l_1 \ge 3$, we get $n_0 \ge 1$, and inequality (38) implies that

$$R_4(B_1) \le 4l_1 - n_0 < 4l_1 - \frac{6.77l_1 - 16}{32} < 4l_1 - \frac{6.72l_1 - 16}{32} = 3.79l_1 + 0.5 < 3.99l_1.$$

Lemma 9 is proved. ☐

Theorem 1 follows from Lemma 7 and Lemma 10 below.

Lemma 10. *Let $A \subseteq \mathbb{Z}^2$ be a finite disconnected set of $k = |A|$ lattice points in the plane. If k is sufficiently large, then*

$$R_4(A) < 4k - \sqrt{8k+1}. \tag{40}$$

Proof. Note that Lemma 9 does not imply directly inequality (40), because the set A_1 is not necessarily connected. Therefore, we apply Lemma 9 several times. Let $A_0 = A$. There is an integer $n \geq 1$ and a finite sequence of subsets of A

$$\{A_1, B_1, A_2, B_2, ..., A_n, B_n\},$$

such that, for every $1 \leq j \leq n$, we have

$$\begin{aligned}
&A_{j-1} = A_j \cup B_j,\ A_j \cap B_j = \emptyset \text{ and } B_j \neq \emptyset,\\
&R_4(A_{j-1}) = R_4(A_j) + R_4(B_j), \qquad (41)\\
&R_4(B_j) \leq 4|B_j| - 2.9\sqrt{|B_j|} \text{ or } R_4(B_j) < 3.99|B_j|,\\
&A_n \text{ is connected or } A_n = \emptyset.
\end{aligned}$$

Denote $k_j = |A_j|, l_j = |B_j|$ and define a partition of $\{1, 2, ..., n\}$ as follows:

$$\begin{aligned}
I_1 &= \{j : 1 \leq j \leq n, R_4(B_j) \leq 4l_j - 2.9\sqrt{l_j}\},\ u = \sum_{j \in I_1} l_j,\\
I_2 &= \{j : 1 \leq j \leq n, R_4(B_j) < 3.99l_j\},\ v = \sum_{j \in I_2} l_j.
\end{aligned}$$

It follows that $k_j + l_j = k_{j-1},\ l_j \geq 1$ for $1 \leq j \leq n$, and we have

$$k = |A| = k_n + l_1 + l_2 + ... + l_n,\ k = k_n + u + v,\ u + v \geq 1.$$

Using equality (41) several times, we get

$$\begin{aligned}
R_4(A) &= R_4(A_1) + R_4(B_1) = \big(R_4(A_2) + R_4(B_2)\big) + R_4(B_1) = ...\\
&= R_4(A_n) + R_4(B_n) + R_4(B_{n-1}) + ... + R_4(B_2) + R_4(B_1)\\
&= R_4(A_n) + \sum_{j \in I_1} R_4(B_j) + \sum_{j \in I_2} R_4(B_j)\\
&\leq R_4(A_n) + \sum_{j \in I_1}(4l_j - 2.9\sqrt{l_j}) + \sum_{j \in I_2}(4l_j - 0.01l_j)\\
&= R_4(A_n) + \sum_{1 \leq j \leq n} 4l_j - 2.9\sum_{j \in I_1}\sqrt{l_j} - 0.01\sum_{j \in I_2} l_j\\
&\leq R_4(A_n) + \sum_{1 \leq j \leq n} 4l_j - 2.9\sqrt{\sum_{j \in I_1} l_j} - 0.01\sum_{j \in I_2} l_j\\
&= R_4(A_n) + 4(u + v) - 2.9\sqrt{u} - 0.01v. \qquad (42)
\end{aligned}$$

We distinguish two cases.

Case 1. The set A_n is connected. Using (42), $k = k_n + u + v$ and Lemma 7, we get

$$\begin{aligned} R_4(A) &\leq 4k_n - \sqrt{8k_n + 1} + 4(u + v) - 2.9\sqrt{u} - 0.01v \\ &\leq 4k - \sqrt{8k_n + 8.41u + 1} - 0.01v. \end{aligned}$$

If $v = 0$, then $k = k_n + u, u \geq 1$ and thus

$$R_4(A) \leq 4k - \sqrt{8k_n + 8.41u + 1} - 0.01v = 4k - \sqrt{8k_n + 8.41u + 1} < 4k - \sqrt{8k + 1}.$$

If $v \geq 1$, then $k = k_n + u + v$ and thus

$$R_4(A) \leq 4k - \sqrt{8k_n + 8.41u + 1} - 0.01v \leq 4k - \sqrt{8k_n + 8u + 1} - 0.01v < 4k - \sqrt{8k + 1}.$$

Note that the last inequality is true in view of

$$\sqrt{8k + 1} - \sqrt{8k_n + 8u + 1} = \frac{8v}{\sqrt{8k + 1} + \sqrt{8k_n + 8u + 1}} < \frac{8v}{\sqrt{8k + 1}} < 0.01v$$

and if we assume $k \geq 80000$.

Case 2. The set A_n is empty. Using (42) and $k = u + v$, we get

$$R_4(A) \leq 4(u + v) - 2.9\sqrt{u} - 0.01v = 4k - 2.9\sqrt{u} - 0.01v.$$

If $v = 0$, then $k = u, u \geq 1$ and thus $k \geq 3$ implies

$$R_4(A) \leq 4k - \sqrt{8.41u} < 4k - \sqrt{8k + 1}.$$

If $v \geq 1$, then $k = u + v$ and thus

$$R_4(A) \leq 4k - \sqrt{8.41u} - 0.01v \leq 4k - \sqrt{8u} - 0.01v < 4k - \sqrt{8k + 1}.$$

Note that the last inequality is true in view of

$$\sqrt{8k + 1} - \sqrt{8u} = \frac{8v + 1}{\sqrt{8k + 1} + \sqrt{8u}} \leq \frac{9v}{\sqrt{8k + 1}} < 0.01v$$

and if we assume $k \geq 101250$. Lemma 10 is proved. ☐

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Gregory A. Freiman
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.
E-mail address: `grisha@post.tau.ac.il`

Yonutz V. Stanchescu
The Open University of Israel, Raanana 43107, Israel
and
Afeka Academic College, Tel Aviv 69107, Israel.
E-mail address: `ionut@openu.ac.il and yonis@afeka.ac.il`