Functiones et Approximatio
XXXVII.1 (2007), 131–148

ON A KAKEYA-TYPE PROBLEM

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Dedicated with best wishes
to Professor Jean-Marc Deshouillers
on the occasion of his 60th birthday

Abstract: Let A be a finite subset of an abelian group G. For every element b_i of the sumset $2A=\{b_0, b_1, ..., b_{|2A|-1}\}$ we denote by $D_i=\{a - a' : a, a' \in A; a+a'=b_i\}$ and $r_i=|\{(a, a') : a+a'=b_i; a, a' \in A\}|$. After an eventual reordering of $2A$, we may assume that $r_0 \geqslant r_1 \geqslant ... \geqslant r_{|2A|-1}$. For every $1 \leqslant s \leqslant |2A|$ we define $R_s(A) = |D_0 \cup D_1 \cup ... \cup D_{s-1}|$ and $R_s(k) = \max\{R_s(A) : A \subseteq G, |A|=k\}$. Bourgain and Katz and Tao obtained an estimate of $R_s(k)$ assuming s being of order k. In this note we find the *exact value* of $R_s(k)$ in cases $s=1$, $s=2$ and $s=3$. The case $s=3$ appeared to be not simple. The structure of *extremal sets* led us to sets isomorphic to planar sets having a rather unexpected form of a perfect hexagon. The proof suggests the way of dealing with the general case $s \geqslant 4$.
Keywords: inverse additive number theory; Kakeya problem.

1. Introduction

Let A be a finite subset of an abelian group $(G, +)$. Assume

$$A = \{x_0, x_1, x_2, ..., x_{k-1}\}, k = |A|.$$

For every element b_i of the sumset

$$2A = A + A = \{x + x' : x \in A, x' \in A\} = \{b_0, b_1, b_2, ..., b_{|2A|-1}\}$$

we denote

$$r_i = r_i(A) = |\{(a, a') : a + a' = b_i, a \in A, a' \in A\}|, \quad (1)$$
$$D_i = D_i(A) = \{a - a' : a \in A, a' \in A, a + a' = b_i\}, \quad (2)$$
$$d_i = d_i(A) = |D_i(A)|. \quad (3)$$

After an eventual reordering of the set $2A$, we may assume that

$$r_0 \geqslant r_1 \geqslant r_2 \geqslant r_3 \geqslant ... \geqslant r_{|2A|-1}. \quad (4)$$

2000 Mathematics Subject Classification: primary 11P70; secondary 11B75.

For every $1 \leqslant s \leqslant |2A|$ we denote

$$R_s(A) = |D_0 \cup D_1 \cup D_2 \cup ... \cup D_{s-1}|,$$
$$R_s(k) = \max\{R_s(A) : A \subseteq G, |A| = k\}.$$

We shall consider the problem of bounding the quantity $R_s(A)$, for sets A included in $\mathbb{Z}^n$. Without any further assumption on s one can only obtain the trivial bound $R_s(A) \leqslant |A - A| \leqslant k^2 - k + 1$. However, Bourgain [1] proved that under the additional assumption $s \leqslant k$ we have

$$|R_s(A)| \leqslant k^{2-\frac{1}{13}}. \tag{5}$$

This nontrivial upper bound was improved by Katz and Tao [2] who showed that

$$|R_s(A)| \leqslant k^{2-\frac{1}{6}}. \tag{6}$$

These estimates were established with an application in mind: by Bourgain and Katz-Tao methods estimates (5) and (6) imply lower bounds for Minkowski and Hausdorff dimension of Kakeya sets.

We would like to find a *sharp* upper estimate for $R_s(k)$ in terms of k and s and to determine its maximal value. Moreover, in cases when the maximal value is obtained, we would like to describe the structure of *extremal sets* A^* for which we have $R_s(A^*) = R_s(k)$.

In this note we study these questions for the values $s = 1, s = 2$ and $s = 3$. The cases $s = 1$ and $s = 2$ are straightforward and are proved in Section 2. The case $s = 3$ appeared to be not so simple; the structure of extremal sets led us to sets of lattice points in the plane having a rather unexpected form of a perfect hexagon. This is shown in Section 3. Moreover, the proof of this result suggests the way of dealing with the general case. We should mention that the main result we obtained is natural and may be further developed; in Section 4 we discuss briefly some final remarks and open questions. Nevertheless, for the moment, the main result is not connected to the study of dimension of Kakeya sets.

2. The cases $s = 1$ and $s = 2$

Let us begin with some simple remarks, for a finite set $A \subseteq \mathbb{Z}$. We easily see that $d_i = r_i$ for every $0 \leqslant i \leqslant |2A| - 1$. Indeed, using (2) and (3) we get that for two pairs (a_1, a_1') and (a_2, a_2') of $A \times A$ such that $a_1 + a_1' = a_2 + a_2' = b_i$ we have $a_1 - a_1' = a_2 - a_2'$ if and only if the equality $(a_1, a_1') = (a_2, a_2')$ holds.

Using definition (3), we see that d_i is equal to the number of pairs (a, a') such that $a, a' \in A$, and a and a' are symmetric with respect to $c_i = \frac{b_i}{2}$. Moreover, we note that if $a \neq a'$ then the pairs (a, a') and (a', a) give two distinct differences

$$a - a' = a - (b_i - a) = 2a - b_i \quad \text{and} \quad a' - a = -(2a - b_i) \tag{7}$$

and if $a = a'$ we have one pair (a, a) and one difference $d = a - a = 0$.

2.1. Let us look at the case $s=1$. In this situation we have representations

$$b_0 = a + a', \text{ such that } a \in A \text{ and } a' \in A$$

and we count the number $d_0 = |D_0(A)|$ of all distinct differences $a - a'$. Using the above remark, we see that d_0 cannot exceed the number of elements of A, that is $d_0 \leqslant k = |A|$ and we have equality $d_0 = k$ if and only if A is a union of pair of points a, a' symmetric with respect to $c_0 = \frac{b_0}{2}$. We proved the following

Proposition 1. (a) *For every set* $A \subseteq \mathbb{Z}, |A| = k$ *we have*

$$R_1(A) = |D_0(A)| \leqslant k.$$

(b) *A set of integers* $A \subseteq \mathbb{Z}$ *satisfies* $R_1(A) = k$ *if and only if* A *is symmetric with respect to the point* $c_0 = \frac{b_0}{2} = \frac{1}{2}(\min A + \max A)$. ■

2.2. The case $s=2$. We prove the following statement:

Proposition 2. (a) *For every set* $A \subseteq \mathbb{Z}, |A| = k$ *we have*

$$R_2(A) = |D_0(A) \cup D_1(A)| \leqslant 2k - 1.$$

(b) *A set of integers* $A \subseteq \mathbb{Z}$ *satisfies* $R_2(A) = 2k - 1$ *if and only if* A *is an arithmetic progression.*

Proof. (a) Denote $2A = \{b_0, b_1, ..., b_{|2A|-1}\}$ and assume that $r_0 \geqslant r_1 \geqslant ..... \geqslant r_{|2A|-1}$. In view of $r_1 \leqslant r_0 \leqslant k$ we should examine only two cases.

(i) If $r_0 \leqslant k-1$, then $r_0 + r_1 \leqslant 2r_0 \leqslant 2k-2$ and from $d_0 = r_0, d_1 = r_1$ we will get

$$R_2(A) = |D_0(A) \cup D_1(A)| \leqslant d_0 + d_1 = r_0 + r_1 \leqslant 2k - 2.$$

(ii) If $r_0 = k$, then A is symmetric with respect to the point $c_0 = \frac{b_0}{2} = \frac{1}{2}(\min A + \max A)$. But A is a finite set and so $c_1 = \frac{b_1}{2}$ cannot be another center of symmetry. This is equivalent to $r_1 \leqslant k-1$. Consequently if $r_0 = k$, then $r_1 \leqslant k-1$ and (a) follows:

$$R_2(A) = |D_0(A) \cup D_1(A)| \leqslant d_0 + d_1 = r_0 + r_1 \leqslant k + (k-1) = 2k - 1.$$

(b) In order to prove the second assertion of Proposition 2, we first show that for an arithmetic progression $A^* = \{1, 2, 3, ..., k\}$ we have

$$\begin{aligned} &r_0 = k, && r_1 = k-1, \\ &D_0(A^*) \cap D_1(A^*) = \emptyset, && R_2(A^*) = |D_0(A^*) \cup D_1(A^*)| = 2k - 1. \end{aligned} \tag{8}$$

It is obvious that $b_0 = k+1, r_0 = k$ and $b_1 = k, r_1 = k-1$. The sets $D_0(A^*)$ and $D_1(A^*)$ are disjoint. Indeed, (7) implies that every difference $d_0 \in D_0(A^*)$ is of the form $d_0 = 2a - b_0 = 2a - k - 1$, $a \in A^*$, every difference $d_1 \in D_1(A^*)$ is of the form $d_1 = 2a^* - b_1 = 2a^* - k$, $a^* \in A^*$ and it follows that $2a - k - 1 \not\equiv 2a^* - k$

(mod 2), that is $D_0(A^*) \cap D_1(A^*) = \emptyset$. We get that $R_2(A^*) = |D_0(A^*) \cup D_1(A^*)| = d_0 + d_1 = r_0 + r_1 = 2k - 1$ and (8) follows.

It remains to show that if a set of integers $A = \{x_0 < x_1 < ... < x_{k-1}\}$ verifies

$$R_2(A) = |D_0(A) \cup D_1(A)| = 2k - 1, \tag{9}$$

then A is an arithmetic progression of difference $d = |b_1 - b_0|$. Assumption (9) implies that $d_0 = r_0 = k$ and $d_1 = r_1 = k-1$. By Proposition 1, A is a symmetric set with respect to $c_0 = \frac{1}{2}b_0$ and thus

$$b_0 = x_0 + x_{k-1} = x_1 + x_{k-2} = x_2 + x_{k-3} = ... = x_{k-1} + x_0. \tag{10}$$

It is sufficient to examine only the case $b_0 < b_1$. Remark that all $k-1$ solutions of the equation $b_1 = a + a'$, $(a, a') \in A \times A$ are exactly the pairs $(a, b_1 - a)$ with $a \in A \setminus \{x_0\}$. Note that $x_1 < x_2 < ... < x_{k-1}$ implies $b_1 - x_1 > b_1 - x_2 > ... > b_1 - x_{k-1}$ and thus

$$\{x_1, x_2, ..., x_{k-1}\} = \{b_1 - x_1, b_1 - x_2, ..., b_1 - x_{k-1}\}.$$

This implies $x_1 = b_1 - x_{k-1}, x_2 = b_1 - x_{k-2}, ..., x_{k-2} = b_1 - x_2, x_{k-1} = b_1 - x_1$. Using (10) it follows that $x_1 = b_0 - x_{k-2}, x_2 = b_0 - x_{k-3}, ..., x_{k-1} = b_0 - x_0$ and thus

$$x_1 = b_1 - x_{k-1} = b_0 - x_{k-2},$$
$$x_2 = b_1 - x_{k-2} = b_0 - x_{k-3}, ..., x_{k-1} = b_1 - x_1 = b_0 - x_0,$$

which is equivalent to $d = b_1 - b_0 = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = ... = x_{k-1} - x_{k-2}$. Proposition 2 is proved. ■

3. Two-dimensional sets

In this section we shall study the case $s = 3$, i.e. the maximal value of

$$R_3(A) = |D_0(A) \cup D_1(A) \cup D_2(A)|,$$

for a finite set $A \subseteq \mathbb{Z}^2$ such that b_0, b_1, b_2 are *non-collinear points*. We shall obtain a *sharp upper estimate* for $R_3(A)$ depending only on $k = |A|$; moreover we shall describe the structure of planar *extremal* sets.

In order to formulate our main result we need some definitions. If $u = (u_1, u_2) \in \mathbb{R}^2$, we denote by u_1 and u_2 its coordinates with respect to the canonical basis $e_1 = (1, 0), e_2 = (0, 1)$ and $e_0 = (0, 0)$ represents the origin point. Let B and C be finite subsets of $\mathbb{Z}^2$. We say that B is *isomorphic* to C if there is an affine isomorphism $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(B) = C$.

Let $\alpha \in \mathbb{N}$. We denote by $H(\alpha)$ the set of all points $P = (x, y) \in \mathbb{Z}^2$ such that:

(a) $-2\alpha < x < 2\alpha$,
(b) $-2\alpha < y < 2\alpha$,
(c) $-2\alpha < x + y - 1 < 2\alpha$,
(d) x and y are odd integers.

Note that the convex hull of $H(\alpha)$ is a hexagon (see Figure 1) and its points lie on 2α lines parallel to the line $y = 0$, on 2α lines parallel to the line $x = 0$ and on 2α lines parallel to the line $x + y = -1$.

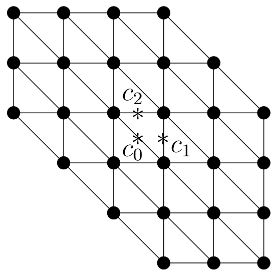

Figure 1. The set $H(\alpha), \alpha = 3$; $c_i = e_i, i = 0, 1, 2$.

We will prove the following

Theorem 1. *Let A be a finite subset of $\mathbb{Z}^2$, $|A| = k$. Then*
(a) $R_3(A) \leqslant 3k - \sqrt{3k}$
and
(b) $R_3(A) = 3k - \sqrt{3k}$ *if and only if $k = 3\alpha^2$ and A is isomorphic to $H(\alpha)$.*

We shall prove assertion (a) of Theorem 1 in Sections 3.1-3.3 below, while the proof of assertion (b) will be postponed to Section 3.4.

3.1. We study first a set $S \subseteq \mathbb{Q}^2$ that lies on two parallel lines. Assume that

$$S = S_+ \cup S_-,$$
$$S_+ = \{P_1, ..., P_{s_+}\},\ P_i = (a_i, 1),\ a_1 < a_2 < ... < a_{s_+},$$
$$S_- = \{Q_1, ..., Q_{s_-}\},\ Q_i = (b_i, -1),\ b_1 < b_2 < ... < b_{s_-}$$

and thus

$$S_+ = S \cap (y = 1), \qquad s_+ = |S_+|, \qquad S_- = S \cap (y = -1), \qquad s_- = |S_-|,$$
$$s = |S| = s_+ + s_-.$$

Denote by $A = \{a_1 < ... < a_{s_+}\}, B = \{b_1 < ... < b_{s_-}\}$ the sets of abscissae of S_+ and S_-, respectively. We study the sets of differences

$$D_0(S) = \{P - P' : P \in S, P' \in S, P + P' = e_0\},\ e_0 = (0, 0),$$
$$D_1(S) = \{P - P' : P \in S, P' \in S, P + P' = e_1\},\ e_1 = (1, 0).$$

If $P_i = e_i - P$ is the symmetric of P with respect to $\frac{1}{2}e_i$, then

$$D_i(S) = \{2P - e_i : P \in S, P_i \in S\}, i = 0, 1.$$

Lemma 1.
(a) $|D_i(S)| \leqslant 2\min\{s_+, s_-\}$, *for* $i = 1, 2$.
(b) $|D_0(S) \cup D_1(S)| \leqslant |D_0(S)| + |D_1(S)| \leqslant 2|S| - 2$.
(c) *If* $|D_0(S) \cup D_1(S)| = 2|S| - 2$, *then the set* S *consists of two parallel arithmetic progressions of common difference* $e_1 - e_0$.

Proof. (a) Let us assume that $s_+ \leqslant s_-$. Each point $P \in S_+$ defines no more than two differences $d = P - P_i$ or $-d = P_i - P$ and so

$$D_i(S) \subseteq \{\pm(P - P') : P \in S_+, P + P' = e_i\}.$$

This implies that $|D_i(S)| \leqslant 2s_+$. Similarly, if $s_- \leqslant s_+$, then $|D_i(S)| \leqslant 2s_-$. Therefore, in both cases we obtain $|D_i(S)| \leqslant 2\min\{s_+, s_-\}$.

(b) Using inequality

$$|D_i(S)| \leqslant 2\min\{s_+, s_-\} = |S| - |s_+ - s_-| \leqslant |S|,$$

we get that if $|s_+ - s_-| \geqslant 1$ then clearly (b) is true. It remains to consider only the case $s_+ = s_-$. Without loss of generality we may assume that $|D_1(S)| \leqslant |D_0(S)|$. We shall distinguish two cases.

Case (i). If there is a point R in S_+ such that $R_0 = e_0 - R$ is not in S_-, then the differences $\pm d = \pm(R - R_0)$ don't belong to $D_0(S)$. Therefore in this case we get $|D_1(S)| \leqslant |D_0(S)| \leqslant 2(s_+ - 1) = |S| - 2$ and so $|D_0(S)| + |D_1(S)| \leqslant 2|S| - 4$.

Case (ii). Assume that for every point $P \in S_+$, its symmetric with respect to $\frac{1}{2}e_0$ belongs to S_-, that is $P_0 = e_0 - P \in S_-$. Using $s_+ = s_-$ we get that S_+ and S_- are symmetric with respect to $\frac{1}{2}e_0$. In this case we have $|D_0(S)| = 2s_+ = 2s_- = |S|$ and in order to prove (b) it remains to show that $|D_1(S)| \leqslant |S| - 2$. The set S_+ lies in the segment determined by the points P_1 and P_{s_+}. The set S_- lies in the segment determined by the points $Q_1 = e_0 - P_{s_+}$ and $Q_{s_-} = e_0 - P_1$. In consequence, $e_1 - P_1$, the symmetric of P_1 with respect to $\frac{1}{2}e_1$ does not belong to S_-. Similarly, $e_1 - Q_1$, the symmetric of Q_1 with respect to $\frac{1}{2}e_1$ does not belong to S_+. Therefore, $|D_1(S)| \leqslant |S| - 2$.

(c) Let $i = 0$ or $i = 1$. Note that D_i is the disjoint union of $D_i = D_i^+ \cup D_i^-$, where

$$\begin{aligned}
D_i^+ &= \{P - P_i : P \in S_+, P_i \in S_-\} = \{\delta = (\delta_1, \delta_2) \in D_i : \delta_2 = +2\},\\
D_i^- &= \{P - P_i : P \in S_-, P_i \in S_+\} = \{\delta = (\delta_1, \delta_2) \in D_i : \delta_2 = -2\},\\
d_i^+ &= |D_i^+| = d_i^- = |D_i^-|.
\end{aligned}$$

Moreover, each difference belonging to D_i^+ can be written as

$$\delta^+ = P - P_i = (x, 1) - (y, -1) = (x - y, 2), \quad \text{with } x \in A, y \in B, x + y = i.$$

Therefore, $x \in A \cap (i - B)$, $d_i^+ = |A \cap (i - B)|$ and

$$d_i = |D_i| = d_i^+ + d_i^- = 2d_i^+ = 2|A \cap (i - B)|.$$

Without loss of generality we may assume that $|D_1(S)| \leqslant |D_0(S)|$. We need to examine only two cases:

Case (i). Assume that $d_0 = |D_0(S)| = |S| > d_1 = |D_1(S)| = |S| - 2$. If $t = |A \cap (-B)|$ then

$$|A| + |B| = |S| = d_0 = 2d_0^+ = 2|A \cap (-B)| = 2t \leqslant 2 \min(|A|, |B|)$$

and thus $|A \cap (-B)| = |A| = |B| = t$. We get that $B = -A$, i.e. S is a symmetric set with respect to $\frac{e_0}{2}$. It follows that $2t-2 = |S|-2 = d_1 = 2d_1^+ = 2|A \cap (1-B)| = 2|A \cap (1+A)|$, and so $|A \cap (1+A)| = t-1 = |A|-1$. We conclude that S consists of two arithmetic progressions S_+ and S_-, with $S_- = -S_+$.

Case (ii). Assume that $|D_0(S)| = |D_1(S)| = |S| - 1$. It $t = |A \cap (-B)|$ then

$$|S|-1 = d_0 = 2d_0^+ = 2|A \cap (-B)| = 2t \text{ and } 2t = |S|-1 = d_1 = 2d_1^+ = 2|A \cap (1-B)|.$$

We get that $|A \cap (-B)| = |A \cap (1-B)| = t, |A| \geqslant t, |B| \geqslant t, |S| = |A|+|B| = 2t+1$. If $|A| = t$, then $|B| = t+1$ and A, B are arithmetic progression such that $B \setminus \{b_{s_-}\} = -A$. If $|A| = t+1$, then $|B| = t$ and A, B are arithmetic progression such that $A \setminus \{a_{s_+}\} = -B$. Lemma 1 is proved. ■

3.2. Let $A \subseteq \mathbb{Q}^2$ be a finite set, $|A| = k$. Using an affine isomorphism that maps the points b_0, b_1, b_2 onto the standard simplex $e_0 = (0,0), e_1 = (1,0), e_2 = (0,1)$, we may assume, without loss in generality that

$$A \subseteq \mathbb{Q}^2, b_0 = e_0 = (0,0), b_1 = e_1 = (1,0), b_2 = e_2 = (0,1).$$

In this section we shall generalize Lemma 1 for sets lying on a lines parallel to $y = 0$ and we will estimate the cardinality of

$$\mathrm{Diff}(A) = D_0 \cup D_1 \cup D_2,$$

where

$$D_i = D_i(A) = \{P - P' : P \in A, P' \in A, P + P' = e_i\}, \quad i = 0,1,2.$$

Denote by $P_i = e_i - P$ the *symmetric* of P with respect to $\frac{1}{2}e_i$. It follows that each set of differences D_i satisfies

$$D_i = \{P - P_i : P \in A, P_i \in A\} = \{2P - e_i : P \in A, P_i \in A\}, \quad (11)$$
$$d_i = |D_i| = |\{P : P \in A, P_i \in A\}|, \quad i = 0,1,2. \quad (12)$$

For every h, we denote by

$$A_h = A \cap (y = h)$$

the set of points of A that lie on the line $y = h$. The set D_i consists of all differences $d = P - P_i$ such that both P and P_i belong to the set A. Note that if $P \in A_h$, then $P_0 \in A_{-h}$, $P_1 \in A_{-h}$ and $P_2 \in A_{-h+1}$. This remark allows us to split D_i into a *disjoint* union of sets:

$$D_i = \bigcup_h D_i(h),$$

where

$$D_0(h) = D_0(A,h) = \{2P - e_0 : P \in A_h, P_0 \in A_{-h}\}, \tag{13}$$

$$D_1(h) = D_1(A,h) = \{2P - e_1 : P \in A_h, P_1 \in A_{-h}\}, \tag{14}$$

$$D_2(h) = D_2(A,h) = \{2P - e_2 : P \in A_h, P_2 \in A_{-h+1}\}. \tag{15}$$

Let a be the number of lines ℓ' parallel to the line $y = 0$ such that $A \cap \ell' \neq \emptyset$. In a similar way, let b be the number of lines ℓ'' parallel to the line $x = 0$ such that $A \cap \ell'' \neq \emptyset$ and c be the number of lines ℓ^* parallel to the line $x + y = 0$ such that $A \cap \ell^* \neq \emptyset$.

Lemma 2. (a) $|D_0(A) \cup D_1(A)| \leqslant 2k - a$, $|D_0(A) \cup D_2(A)| \leqslant 2k - b$ *and* $|D_1(A) \cup D_2(A)| \leqslant 2k - c$.

(b) $|\mathrm{Diff}(A)| = |D_0(A) \cup D_1(A) \cup D_2(A)| \leqslant 3k - \frac{1}{2}(a + b + c)$.

Proof. (a) Let H be the set of all integers h such that $A_{\pm h} = A_h \cup A_{-h} \neq \emptyset$. We have

$$D_0(A) = \bigcup_{h\in H} D_0(h) = \bigcup_{h\in H,\ h\geqslant 0} D_0(A_{\pm h}),$$

$$D_1(A) = \bigcup_{h\in H} D_1(h) = \bigcup_{h\in H,\ h\geqslant 0} D_1(A_{\pm h}),$$

$$D_0(A) \cup D_1(A) = \bigcup_{h\in H,\ h\geqslant 0} \big(D_0(A_{\pm h}) \cup D_1(A_{\pm h})\big).$$

If $h = 0$ belongs to H, we have $|D_0(A_0) \cup D_1(A_0)| \leqslant 2|A_0| - 1$, in view of Proposition 2(a). For $0 < h \in H$, the set $A_{\pm h}$ is included on two parallel lines. If $|A_h| > 0$ and $|A_{-h}| > 0$, then by Lemma 1(b), we get $|D_0(A_{\pm h}) \cup D_1(A_{\pm h})| \leqslant 2|A_{\pm h}| - 2$. If $|A_h| = 0$ or $|A_{-h}| = 0$, then $A_{\pm h}$ lies on a line and obviously $|D_0(A_{\pm h}) \cup D_1(A_{\pm h})| = 0 \leqslant 2|A_{\pm h}| - 2 < 2|A_{\pm h}| - 1$. We conclude that

$$\begin{aligned} |D_0(A) \cup D_1(A)| &= \sum_{h\in H,\ h\geqslant 0} |D_0(A_{\pm h}) \cup D_1(A_{\pm h})| \leqslant \\ &\leqslant (2|A_0| - 1) + \sum_{h\in H,\ h>0} (2|A_{\pm h}| - 2) \leqslant \\ &\leqslant 2|A| - a = 2k - a. \end{aligned}$$

Note that this inequality was obtained using a partition of A into sets lying on lines parallel to the segment $[e_0, e_1]$. In a similar way, considering lines parallel to the segments $[e_0, e_2]$ and $[e_1, e_2]$, we obtain respectively that

$$|D_0(A) \cup D_2(A)| \leqslant 2k - b \ \text{ and } \ |D_1(A) \cup D_2(A)| \leqslant 2k - c.$$

(b) Denote by

$$d_{ij} = |D_i(A) \cap D_j(A)|, d_{012} = |D_0(A) \cap D_1(A) \cap D_2(A)|.$$

Then $|D_0(A) \cup D_1(A) \cup D_2(A)| = (d_0 + d_1 + d_2) - (d_{01} + d_{02} + d_{12}) + d_{012}$ and thus

$$\begin{aligned}
&2|D_0(A) \cup D_1(A) \cup D_2(A)| = \\
&= (d_0 + d_1 - d_{01}) + (d_0 + d_2 - d_{02}) + (d_1 + d_2 - d_{12}) \\
&\quad - d_{01} - d_{02} - d_{12} + 2d_{012} \\
&= |D_0(A) \cup D_1(A)| + |D_0(A) \cup D_2(A)| + |D_1(A) \cup D_2(A)| \\
&\quad - d_{01} - d_{02} - d_{12} + 2d_{012} \\
&\leqslant 6k - (a + b + c) - d_{ij}.
\end{aligned}$$

We conclude that for every $0 \leqslant i \neq j \leqslant 2$

$$\begin{aligned}
|\mathrm{Diff}(A)| = |D_0(A) \cup D_1(A) \cup D_2(A)| &\leqslant 3k - \frac{a+b+c}{2} - \frac{d_{ij}}{2} \\
&\leqslant 3k - \frac{a+b+c}{2}.
\end{aligned} \tag{16}$$

Lemma 2 is proved. ■

Example 1. Inequality (16) is sharp. The set

$$A^* = \frac{1}{2}H(\alpha) = \{\frac{1}{2}(x, y) : (x, y) \in H(\alpha)\}$$

satisfies $a = b = c = 2\alpha, |D_i(A^*)| = k - \alpha$, for $i = 0, 1, 2$ and $|\mathrm{Diff}(A^*)| = |D_0(A^*)| + |D_1(A^*)| + |D_2(A^*)| = 3k - 3\alpha = 3k - \frac{a+b+c}{2}$; therefore inequality (16) cannot be sharpened by reducing the upper bound for $R_3(A^*)$.

3.3. In this section we obtain the following optimal upper bound

$$|\mathrm{Diff}(A)| = |D_0(A) \cup D_1(A) \cup D_2(A)| \leqslant 3k - \sqrt{3k},$$

valid for every finite set $A \subseteq \mathbb{Q}^2, |A| = k$.

We shall first estimate the cardinality of a finite set $K \subseteq \mathbb{Q}^2$ assuming that K lies on a lines parallel to the line $y = 0$, on b lines parallel to the line $x = 0$ and on c lines parallel to the line $x + y = 0$.

Lemma 3. *Let K be a finite subset of $\mathbb{Q}^2$. If $\max\{a, b, c\} < \frac{a+b+c}{2}$, then*

$$k = |K| \leqslant \frac{1}{3}\frac{(a+b+c)^2}{4} + \frac{\delta}{4},$$

where $\delta = 0$ if $a + b + c$ is even and $\delta = 1$, if $a + b + c$ is odd.

Proof. We clearly have $k \leqslant \frac{abc}{\max\{a,b,c\}}$. Moreover, let us note that if $\max\{a, b, c\} \geqslant \frac{a+b+c}{2}$, then $k \leqslant \frac{\max^2\{a,b,c\}}{4}$. Indeed, assuming that $a = \max\{a, b, c\}$ we get $a \geqslant b + c$ and thus

$$k \leqslant bc \leqslant \frac{(b+c)^2}{4} \leqslant \frac{a^2}{4} = \frac{\max^2\{a, b, c\}}{4}. \tag{17}$$

In order to prove Lemma 3, we shall apply induction on $\min\{a,b,c\}$.

If $\min\{a,b,c\} = a = 1$, then K lies on a line, $|K| = k, a = 1, b = c = k$, $a+b+c = 1+2k$ is odd and thus $\delta = 1$. In this case Lemma 3 is true because

$$\frac{1}{3}\frac{(a+b+c)^2}{4} + \frac{\delta}{4} = \frac{1}{3}\frac{(2k+1)^2}{4} + \frac{1}{4} = \frac{1}{3}(k^2+k+1) = \frac{1}{3}(k-1)^2 + k \geqslant k.$$

Let us assume that $\min\{a,b,c\} \geqslant 2$ and the assertion holds for smaller values of $\min\{a,b,c\}$. There is no loss in generality if we assume that $a \geqslant b \geqslant c$. Note that $\max\{a,b,c\} < \frac{a+b+c}{2}$ implies

$$b + c > a \geqslant b \geqslant c. \tag{18}$$

Let us choose (see Figure 2)

$$\ell_a : x = u', \ell'_a : x + y = w'; \quad \ell_c : y = v', \ell'_c : x = u; \quad \ell_b : x + y = w, \ell'_b : y = v$$

supporting lines of the convex hull of K such that K is included in the half planes

$$x \geqslant u', x + y \geqslant w', y \geqslant v', x \leqslant u, x + y \leqslant w, y \leqslant v.$$

Denote by

$$K_a = K \setminus (\ell_a \cup \ell'_a), K_b = K \setminus (\ell_b \cup \ell'_b), K_c = K \setminus (\ell_c \cup \ell'_c)$$

and

$$K' = K \setminus (\ell_a \cup \ell'_a \cup \ell_b \cup \ell'_b \cup \ell_c \cup \ell'_c).$$

We have $|K| \leqslant |K_a| + a, |K| \leqslant |K_b| + b, |K| \leqslant |K_c| + c$ and thus

$$|K| \leqslant |K'| + (a + b + c - 3). \tag{19}$$

Clearly, the set K' lies on $a' \leqslant a - 2$ lines parallel to $y = 0$, on $b' \leqslant b - 2$ lines parallel to $x = 0$ and on $c' \leqslant c - 2$ lines parallel to $x + y = 0$. We denote $s' = a' + b' + c'$ and $s = a + b + c$. Put $\delta' = 0$ if $a' + b' + c'$ is even and $\delta' = 1$, if $a' + b' + c'$ is odd.

We shall estimate $|K|$ using the induction hypothesis for the set $K' \subsetneqq K$.

We distinguish several cases.

Case 1. Assume that $\max\{a',b',c'\} < \frac{a'+b'+c'}{2}$. By the induction hypothesis we get $|K'| \leqslant \frac{1}{3}\frac{(a'+b'+c')^2}{4} + \frac{\delta'}{4}$. Using

$$s' \leqslant s - 6$$

we get

$$|K| \leqslant |K'| + (a + b + c - 3) \leqslant \frac{1}{3}\frac{s'^2}{4} + \frac{\delta'}{4} + (s - 3) \tag{20}$$

$$\leqslant \frac{1}{3}\frac{(s-6)^2}{4} + \frac{\delta'}{4} + (s - 3) = \frac{1}{3}\frac{s^2}{4} + \frac{\delta'}{4}. \tag{21}$$

Case 1.1. If $\delta'=0$, then $|K|\leqslant \frac{1}{3}\frac{s^2}{4}\leqslant \frac{1}{3}\frac{(a+b+c)^2}{4}+\frac{\delta}{4}$.
Case 1.2. If $s'=s-6$, then $\delta=\delta'$ and we get $|K|\leqslant \frac{1}{3}\frac{(a+b+c)^2}{4}+\frac{\delta}{4}$.
Case 1.3. If $\delta'=1$ and $s'\leqslant s-7$ then $s\geqslant s'+7\geqslant 10$ and thus

$$\begin{aligned}|K|&\leqslant \frac{1}{3}\frac{s'^2}{4}+\frac{\delta'}{4}+(s-3)\leqslant \frac{1}{3}\frac{(s-7)^2}{4}+\frac{1}{4}+(s-3)=\frac{1}{3}\frac{s^2}{4}-\frac{s}{6}+\frac{4}{3}\\ &<\frac{1}{3}\frac{s^2}{4}+\frac{\delta}{4}=\frac{1}{3}\frac{(a+b+c)^2}{4}+\frac{\delta}{4}.\end{aligned}$$

Case 2. Assume that $\max\{a',b',c'\}\geqslant \frac{a'+b'+c'}{2}$. Using inequality (17) and (19) we get $|K'|\leqslant \left(\frac{\max\{a',b',c'\}}{2}\right)^2\leqslant \left(\frac{\max\{a,b,c\}-2}{2}\right)^2=\frac{(a-2)^2}{4}$ and $|K|\leqslant \frac{(a-2)^2}{4}+(a+b+c-3)$. In order to prove inequality $|K|\leqslant \frac{1}{3}\frac{(a+b+c)^2}{4}+\frac{\delta}{4}$ it is enough to verify that under the hypothesis (18) the number $\Delta=\frac{(a+b+c)^2}{12}+\frac{\delta}{4}-(a+b+c)+3-\frac{(a-2)^2}{4}$ is nonnegative, i.e.

$$\Delta=\frac{(a+b+c-6)^2}{12}+\frac{\delta}{4}-\frac{(a-2)^2}{4}\geqslant 0. \tag{22}$$

Case 2.1. If $a\geqslant 6$ then using $b+c\geqslant a+1$ we get

$$\Delta\geqslant \frac{(2a-5)^2}{12}+\frac{\delta}{4}-\frac{(a-2)^2}{4}\geqslant \frac{(a-4)^2-3}{12}\geqslant \frac{1}{12}>0.$$

Case 2.2. If $a=5$, then $b+c\geqslant a+1=6$. We distinguish two subcases.
If $b+c\geqslant 7$, then $\Delta=\frac{(b+c-1)^2}{12}+\frac{\delta}{4}-\frac{9}{4}\geqslant 3+\frac{\delta}{4}-\frac{9}{4}\geqslant\frac{3}{4}>0$.
If $b+c=6$, then $\delta=1$ and thus $\Delta=\frac{(b+c-1)^2}{12}+\frac{\delta}{4}-\frac{9}{4}=\frac{25}{12}+\frac{1}{4}-\frac{9}{4}=\frac{1}{12}>0$.
Case 2.3. If $a=4$, then $b+c\geqslant a+1=5$. We distinguish two subcases.
If $b+c\geqslant 6$, then $\Delta=\frac{(b+c-2)^2}{12}+\frac{\delta}{4}-1\geqslant\frac{16}{12}-1>0$.
If $b+c=5$, then $\delta=1$ and thus $\Delta=\frac{(b+c-2)^2}{12}+\frac{\delta}{4}-1=\frac{9}{12}+\frac{1}{4}-1=0$.
Case 2.4. If $a=3$, then $b+c\geqslant a+1=4$. We distinguish two subcases.
If $b+c\geqslant 5$, then $\Delta=\frac{(b+c-3)^2}{12}+\frac{\delta}{4}-\frac{1}{4}\geqslant\frac{4}{12}-\frac{1}{4}>0$.
If $b+c=4$, then $\delta=1$ and thus $\Delta=\frac{(b+c-3)^2}{12}+\frac{\delta}{4}-\frac{1}{4}=\frac{1}{12}>0$.
Case 2.5. If $a=2$, then $\min\{a,b,c\}\geqslant 2$ implies $b=c=2$ and thus $\Delta=\frac{\delta}{4}=0$. ■

An easy consequence of Lemma 3 is

Lemma 4. *Let $K\subseteq\mathbb{Q}^2$ be a finite set such that $\max\{a,b,c\}=m$. We have:*
(a) *If $m\geqslant\frac{a+b+c}{2}$, then $|K|\leqslant\frac{m^2}{4}$.*
(b) *If $m<\frac{a+b+c}{2}$, then $|K|\leqslant\frac{1}{3}\left(\frac{a+b+c}{2}\right)^2+\frac{\delta}{4}\leqslant\frac{3m^2}{4}+\frac{\delta}{4}$, where $\delta=0$ if $a+b+c$ is even and $\delta=1$, if $a+b+c$ is odd.*

Proof. (a) Assertion (a) is equivalent to inequality (17).

(b) Assume that $c \leqslant b \leqslant a = \max\{a,b,c\} < \frac{a+b+c}{2}$. In view of Lemma 3 we obtain

$$k = |K| \leqslant \frac{1}{3}\Big(\frac{a+b+c}{2}\Big)^2 + \frac{\delta}{4} \leqslant \frac{1}{3}\frac{(3m)^2}{4} + \frac{\delta}{4} = \frac{3m^2}{4} + \frac{\delta}{4}. \tag{23}$$

Lemma 4 is proved. ■

Example 2. Lemmas 3 and 4 cannot be sharpened by reducing the upper bound for $|K|$. Indeed, the set $K = H(\alpha)$ satisfies $a = b = c = 2\alpha$ and

$$k = |K| = 3\alpha^2 = \frac{1}{3}\frac{(a+b+c)^2}{4} = \frac{3\max^2(a,b,c)}{4}.$$

Moreover, we may note at this point that Example 1 implies the following equality

$$R_3(A^*) = |\mathrm{Diff}(A^*)| = 3k - \sqrt{3k}, \tag{24}$$

valid for the set $A^* = \frac{1}{2}H(\alpha)$.

We shall complete this section by proving inequality (a) of Theorem 1; Lemma 3 implies the following

Corollary 1. *Let A be a finite subset of $\mathbb{Q}^2$, $|A| = k$. Then*

$$R_3(A) = |\mathrm{Diff}(A)| = |D_0(A) \cup D_1(A) \cup D_2(A)| \leqslant 3k - \sqrt{3k}. \tag{25}$$

Proof. In order to prove (25) we shall examine two cases separately.

(i) If $\max\{a,b,c\} < \frac{a+b+c}{2}$, then Lemma 3 implies $k = |A| \leqslant \frac{1}{3}\left(\frac{a+b+c}{2}\right)^2 + \frac{\delta}{4}$. This is equivalent to $\frac{1}{2}(a+b+c) \geqslant \sqrt{3k - \frac{3}{4}\delta}$; using Lemma 2(b) we get

$$|\mathrm{Diff}(A)| = |D_0(A) \cup D_1(A) \cup D_2(A)| \leqslant 3k - \frac{a+b+c}{2} - \frac{\delta}{2} \tag{26}$$

$$\leqslant 3k - \sqrt{3(k - \frac{\delta}{4})} - \frac{\delta}{2} \leqslant 3k - \sqrt{3k}. \tag{27}$$

(ii) Assume $\max\{a,b,c\} \geqslant \frac{a+b+c}{2}$ and, without loss of generality, $a = \max\{a,b,c\}$. We get $b + c \leqslant a$ and thus $k = |A| \leqslant bc = \frac{(b+c)^2-(b-c)^2}{4} \leqslant \frac{(\frac{a+b+c}{2})^2-(b-c)^2}{4}$. It follows that

$$|\mathrm{Diff}(A)| \leqslant 3k - \frac{a+b+c}{2} \leqslant 3k - \sqrt{4k + (b-c)^2} \leqslant 3k - \sqrt{4k} < 3k - \sqrt{3k}. \tag{28}$$

Corollary 1 is proved. ■

3.4. In this section we shall prove assertion (b) of Theorem 1. In view of equality (24), it remains to show that if A is an extremal set, i.e.

$$R_3(A) = |\mathrm{Diff}(A)| = 3k - \sqrt{3k},\ k = |A|, \tag{29}$$

then A is isomorphic to $H(\alpha)$, for $\alpha = \sqrt{\frac{k}{3}}$. Assume that the set A lies on a lines parallel to $y = 0$, on b lines parallel to $x = 0$ and on c lines parallel to $x + y = 0$.

Assertion 1. (i) *There is a natural number α such that $k = 3\alpha^2$.*
(ii) $R_3(A) = |\mathrm{Diff}(A)| = 3k - \frac{a+b+c}{2} = 3k - \sqrt{3k}$.
(iii) $a + b + c = 6\alpha$ *and* $k = \frac{(a+b+c)^2}{12}$.

Proof. In view of (29) and (28) we may assume that there is $\alpha \in \mathbb{N}$ such that

$$k = 3\alpha^2 \quad \text{and} \quad \max\{a, b, c\} < \frac{a+b+c}{2}. \tag{30}$$

From (26), (27) and (29) we get that:

$$R_3(A) = |\mathrm{Diff}(A)| = 3k - \frac{a+b+c}{2} - \frac{\delta}{2} = 3k - \sqrt{3k - \frac{3}{4}\delta} - \frac{\delta}{2} = 3k - \sqrt{3k}. \tag{31}$$

If $\delta = 1$ then $3k - \sqrt{3k - \frac{3}{4}\delta} - \frac{\delta}{2} < 3k - \sqrt{3k}$, which contradicts (31). Thus $\delta = 0$, which means that $a + b + c = 2\sqrt{3k} = 6\alpha$. We get

$$R_3(A) = |\mathrm{Diff}(A)| = 3k - \frac{a+b+c}{2} = 3k - \sqrt{3k}, \quad k = \frac{(a+b+c)^2}{12} \tag{32}$$

and so Assertion 1 is proved. ■

Assertion 2. (i) $|D_0(A) \cup D_1(A)| = |D_0(A)| + |D_1(A)| = 2k - a$.
(ii) $|D_0(A) \cup D_2(A)| = |D_0(A)| + |D_2(A)| = 2k - b$.
(iii) $|D_1(A) \cup D_2(A)| = |D_1(A)| + |D_2(A)| = 2k - c$.
(iv) *Let $0 \leqslant i \neq j \leqslant 2$. For every line ℓ parallel to the segment $[e_i, e_j]$ the set $A \cap \ell$ is an arithmetic progression of difference $e_i - e_j$.*

Proof. Using inequality (16) and equality (ii) of Assertion 1 we get $d_{ij} = |D_i(A) \cap D_j(A)| = 0$ and thus $|D_i(A) \cup D_j(A)| = |D_i(A)| + |D_j(A)|$, for every $0 \leqslant i \neq j \leqslant 2$. Moreover, from the proof of Lemma 2(b) we get that an extremal set A satisfies

$$3k - \frac{a+b+c}{2} = R_3(A) = |\mathrm{Diff}(A)| \leqslant \frac{1}{2} \sum_{0 \leqslant i \neq j \leqslant 2} (|D_i(A) \cup D_j(A)|) \tag{33}$$

$$\leqslant \frac{1}{2}((2k - a) + (2k - b) + (2k - c)) = 3k - \frac{a+b+c}{2}. \tag{34}$$

We conclude that for an extremal set A equalities (i), (ii) and (iii) are true. In order to check part (iv), we may assume without loss of generality that $i = 0, j = 1$ and we will show that if $|D_0(A) \cup D_1(A)| = 2k - a$, then for every line ℓ parallel to the segment $[e_0, e_1]$ the set $A \cap \ell$ is an arithmetic progression of difference $e_1 - e_0$.

Let us recall that $A_h = A \cap (y = h)$ and $A_{\pm h} = A_h \cup A_{-h}$. From the proof of Lemma 2 we get that (33) implies:

(i) $|D_0(A_0) \cup D_1(A_0)| = 2|A_0| - 1$, if $A_0 \neq \emptyset$

and

(ii) $|D_0(A_{\pm h}) \cup D_1(A_{\pm h})| = 2|A_{\pm h}| - 2$, for every integer $h > 0$ such that $A_{\pm h} \neq \emptyset$. In view of Proposition 2 and assertion (c) of Lemma 1 we get that for every line ℓ parallel to the segment $[e_0, e_1]$ the set $A \cap \ell$ is an arithmetic progression of difference $e_1 - e_0$. ■

For the next two results we shall use the notations of Lemma 3. Let K be a finite nonempty subset of $\mathbb{Q}^2$. Assume that K lies on $a \geqslant 1$ lines parallel to $y = 0$, on $b \geqslant 1$ lines parallel to $x = 0$ and on $c \geqslant 1$ lines parallel to $x + y = 0$. If $P = (x, y) = K \setminus K'$, we say that P is a *boundary point* of K and write $P \in Bd(K)$ (see Figure 2).

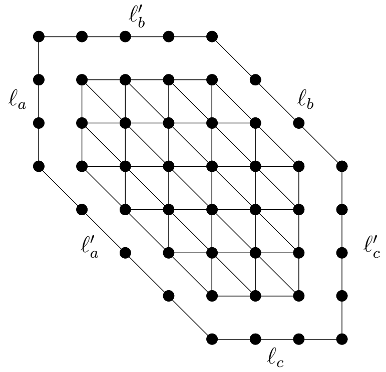

Figure 2. The set $H(3)$ and the boundary $Bd(H(4)) = H(4) \setminus H(3)$.

Assertion 3. *If* $k = |K| = \frac{(a+b+c)^2}{12} = 3\alpha^2$, *then*

(i) $a \geqslant 2, b \geqslant 2, c \geqslant 2$ *and* $\max\{a, b, c\} < \frac{a+b+c}{2}$.

(ii) *the set of all boundary points of* K *satisfies* $|Bd(K)| = a + b + c - 3$.

(iii) *the set* $K' = K \setminus Bd(K)$ *lies on* $a' = a - 2$ *lines parallel to* $y = 0$, *on* $b' = b - 2$ *lines parallel to* $x = 0$ *and on* $c' = c - 2$ *lines parallel to* $x + y = 0$. *Moreover, if* $K' \neq \emptyset$, *then* $\max\{a', b', c'\} < \frac{a'+b'+c'}{2}$ *and if* $K' = \emptyset$, *then* $a = b = c = 2, k = 3$.

(iv) $|K'| = \frac{(a'+b'+c')^2}{12} = 3(\alpha - 1)^2$.

Proof. If $\min\{a, b, c\} = 1$, then $a + b + c$ is odd and this contradicts $k = |K| = \frac{(a+b+c)^2}{12}$. Therefore, in what follows we will assume that

$$\max\{a, b, c\} = a \geqslant b \geqslant c \geqslant 2.$$

We prove first that the set K satisfies

$$\max\{a, b, c\} < \frac{a + b + c}{2}. \tag{35}$$

To the contrary, if $a = \max\{a, b, c\} \geqslant \frac{a+b+c}{2}$, then $bc \geqslant k = \frac{(a+b+c)^2}{12} \geqslant \frac{(b+c)^2}{3}$ implies $b^2 - bc + c^2 \leqslant 0$, which is impossible for a nonempty set K.

We clearly have

$$a' \leqslant a-2, b' \leqslant b-2, c' \leqslant c-2, a'+b'+c' \leqslant a+b+c-6. \tag{36}$$

Case 1. Assume that $\max\{a',b',c'\} \geqslant \frac{a'+b'+c'}{2}$. We shall prove that

$$a'=b'=c'=0, K'=\emptyset, a=b=c=2, k=|K|=3. \tag{37}$$

Using (36) we get $\max\{a',b',c'\} \leqslant \max\{a,b,c\}-2 = a-2$ and Lemma 4(a) implies $k'=|K'| \leqslant \frac{(a-2)^2}{4}$. From inequality (19) we get that

$$\frac{(a+b+c)^2}{12} = k \leqslant k' + (a+b+c-3) \leqslant \frac{(a-2)^2}{4} + (a+b+c-3)$$

and so

$$(a+b+c-6)^2 \leqslant 3(a-2)^2.$$

Note that $a=\max\{a,b,c\} < \frac{a+b+c}{2}$ and $a+b+c$ is even imply that $b+c \geqslant a+2$ and we obtain that $4(a-2)^2 \leqslant (a+b+c-6)^2 \leqslant 3(a-2)^2, a=\max\{a,b,c\}=2$; we conclude that $a=b=c=2, k=3, a'=b'=c'=0$, which proves (37).

Case 2. Assume that $\max\{a',b',c'\} < \frac{a'+b'+c'}{2}$. Using (19) and Lemma 4(b) for the set K' we obtain

$$\frac{(a+b+c)^2}{12} = k \leqslant k' + (a+b+c-3) \leqslant \frac{(a'+b'+c')^2}{12} + \frac{\delta'}{4} + (a+b+c-3).$$

Note that if $a'+b'+c' \leqslant a+b+c-7$, then $k \leqslant \frac{(a+b+c-7)^2}{12} + \frac{1}{4} + (a+b+c-3) < \frac{(a+b+c)^2}{12}$, which contradicts the hypothesis $k = \frac{(a+b+c)^2}{12}$. Therefore, inequalities (36) imply

$$\begin{aligned} &a'+b'+c' = a+b+c-6,\ a'=a-2,\ b'=b-2,\ c'=c-2,\\ &a'+b'+c' \equiv a+b+c \equiv 0 (\bmod 6),\ \delta'=\delta=0 \end{aligned}$$

and we conclude that

$$\frac{(a+b+c)^2}{12} = k \leqslant k'+(a+b+c-3) \leqslant \frac{(a'+b'+c')^2}{12}+(a+b+c-3) = \frac{(a+b+c)^2}{12}$$

and so

$$k-k' = |Bd(K)| = a+b+c-3,\ k' = \frac{(a'+b'+c')^2}{12}.$$

Assertion 3 is proved. ■

Using the same notations we have

Assertion 4. *If* $k = |K| = \frac{(a+b+c)^2}{12} = 3\alpha^2$, *then* $a = b = c = 2\alpha$.

Proof. We shall use induction on $\min = \min\{a, b, c\} \geqslant 2$. Note that $\min\{a, b, c\} = 1$ is impossible, because of assertion 3(i).

If $\min = 2$, then we may assume that $2 = c \leqslant b \leqslant a$ and we get

$$k = |K| = \frac{(a+b+c)^2}{12} \leqslant 2b. \tag{38}$$

These inequalities have only one solution, namely $a = b = c = 2$. Indeed, if $a \geqslant 4$, then $24b \geqslant (a+b+c)^2 \geqslant (b+6)^2$ implies $a \geqslant b = 6, c = 2$ and this contradicts (38). If $a = 3$, then $(a, b, c) = (3, 3, 2)$ or $(a, b, c) = (3, 2, 2)$ and this contradicts $a + b + c \equiv 0 (\text{mod } 6)$. We get that $a = b = c = 2$.

Let $\min \geqslant 3$ and assume that Assertion 4 holds for smaller values of min. Using Assertion 3 (iii) and (iv) we get that

$$k' = |K'| = \frac{(a'+b'+c')^2}{12} = 3(\alpha-1)^2;$$

moreover, the set $K' = K \setminus Bd(K)$ lies on $a' = a - 2 \geqslant 1$ lines parallel to $y = 0$, on $b' = b - 2 \geqslant 1$ lines parallel to $x = 0$ and on $c' = c - 2 \geqslant 1$ lines parallel to $x + y = 0$. Note that $\min\{a', b', c'\} = 1$ is impossible, in view of assertion 3(i). By the induction hypothesis for the set K' we obtain $a' = b' = c' = 2(\alpha - 1)$ and thus $a = b = c = 2\alpha$. Assertion 4 is proved. ∎

We shall use the following definition. We say that a planar $S \subseteq \mathbb{Q}^2$ is an $\alpha-$*regular set* with respect to the basis $B = \{e_0, e_1, e_2\}$ if:

(a) $|S| = 3\alpha^2$,

(b) S lies on $a = 2\alpha$ lines parallel to the segment $[e_0, e_1]$, on $b = 2\alpha$ lines parallel to the segment $[e_0, e_2]$ and on $c = 2\alpha$ lines parallel to the segment $[e_1, e_2]$.

(c) For every line ℓ parallel to the segment $[e_i, e_j]$, $0 \leqslant i \neq j \leqslant 2$, the set $S \cap \ell$ is an arithmetic progression of difference $e_i - e_j$.

Note that assertions 1, 2, 3 and 4 imply that every extremal set A is an $\alpha-$*regular set*. Moreover, if f_0 is an arbitrary point on the plane and f_1 and f_2 are defined by

$$\left(f_i = f_0 + \frac{e_i}{2}, i = 1, 2\right) \text{ or } \left(f_i = f_0 - \frac{e_i}{2}, i = 1, 2\right),$$

then the affine isomorphism of the plane T that maps e_i to f_i, $i = 1, 2$ defines an $\alpha-$*regular set* denoted by

$$H(\alpha, f_0, f_1, f_2) = T\big(H(\alpha)\big).$$

For example, (see Figure 3) if $f_0 = (\frac{1}{2}, \frac{1}{2}), f_1 = (0, \frac{1}{2}), f_2 = (\frac{1}{2}, 0)$, this set contains all the lattice points $P = (x, y) \in \mathbb{Z}^2$ such that

$$-\alpha < x, y, x + y \leqslant \alpha.$$

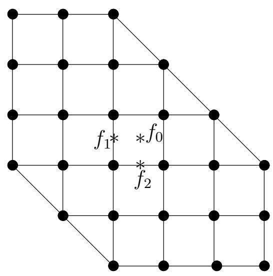

Figure 3. The set $H(\alpha, f_0, f_1, f_2), \alpha = 3$.

In order to conclude the proof of assertion (b) of Theorem 1 it is enough to check

Assertion 5. *For every $\alpha-$ regular set S there are three non-collinear points f_0, f_1, f_2 such that $S = H(\alpha, f_0, f_1, f_2)$.*

Proof. We shall use induction on α.

For $\alpha = 1$ the set S contains three points and is affine isomorphic to $H(1) = \{(-1,1),(1,-1),(1,1)\}$.

Let $\alpha \geqslant 2$ and assume that Assertion 5 holds for smaller values for α. Let S be an $\alpha-$ *regular set*. Clearly $S = 3\alpha^2 = \frac{1}{12}(a+b+c)^2$. We note first that

$$S' = S \setminus Bd(S)$$

is an $(\alpha-1)-$ regular set. Indeed, Assertion 3 implies that: $|Bd(S)| = a+b+c-3 = 6\alpha - 3$, $|S'| = 3(\alpha-1)^2$ and the set S' lies on $a' = 2\alpha - 2$ lines parallel to $y = 0$, on $b' = 2\alpha - 2$ lines parallel to $x = 0$ and on $c' = 2\alpha - 2$ lines parallel to $x+y = 0$. Moreover, the third property of an $\alpha-$ *regular set* is still valid for the set S'. By the induction hypothesis, we get that the set S' is equal to $S' = H(\alpha-1, f_0, f_1, f_2)$. The set S is obtained from the set S' by adding the $6\alpha - 3$ points of $Bd(S)$. This can be done in an unique way, in view that S and S' satisfy both condition (c). We conclude that $S = H(\alpha, f_0, f_1, f_2)$. ■

4. Some final remarks

This section contains some concluding remarks and open problems.

4.1. Theorem 1 describes the structure of extremal sets A such that $k = |A| = 3\alpha^2$. We conjecture that an extremal set A with $3(\alpha-1)^2 < k < 3\alpha^2$ is *included* in a set isomorphic to $H(\alpha)$.

4.2. It would be interesting to generalize Theorem 1 and Propositions 1 and 2 to the general case $s \geqslant 4$. More precisely, we should obtain sharp upper estimates for $R_s(A) = |D_0 \cup D_1 \cup D_2 \cup ... \cup D_{s-1}|$ and to describe the structure of extremal sets A^*.

4.3. We conjecture that an isomorphic projection $P : \mathbb{Z}^2 \to \mathbb{Z}$ maps the set $H(\alpha) \subseteq \mathbb{Z}^2$ into a set of *integers* $A^* = P(H(\alpha))$ that is the extremal set for the problem

$$R_3(k) = \max\{R_3(A) : A \subseteq \mathbb{Z}, |A| = k\}.$$

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Received: 6 December 2006