Inverse Additive Number Theory.XI. On the detailed structure of sets with small additive property.

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Abstract

It is known that a set of k integers A with small doubling (small $|A+A|$) satisfying the condition $|A+A| = 2k-1+b, 0 \leq b \leq k-3$ is a part of arithmetic progression of $k+b$ terms. It appeared that the structure of A may be described in a much more detailed way. [1]

This paper is the next in the series of papers on Inverse Additive Number Theory which were published in 1955-1964 (see references [84]-[92], [98] in [2]).

Throughout the paper we will work with the set $A \subset \mathbb{Z}$ with cardinality $|A| = k \geq 3$. We assume that

$$A = \{a_0 = 0 < a_1 < ... < a_{k-1}\}$$

and that the greatest common divisor of the numbers from A is one. Let T denote the cardinality of the set $2A = A + A$ of all pairwise sums $a + b$ of numbers from A. Notice that $T \geq 2k - 1$.

In [1],(see also the textbook [3],page 204) we proved the following result.

Theorem 1 *For $0 \leq b < k-2$ and $T = 2k-1+b$ the set A is a subset of the set*

$$L = \{0, 1, 2, ..., k+b-1\}. \tag{1}$$

□

[1]**Keywords:** Inverse additive problems, doubling of the set.

Let us give several examples of such sets for the maximal value of $b = k-3, T = 3k-4$ and $k = 8$.

$$A = \{0,2,4,6,8,10,11,12\}, \tag{2}$$

$$A = \{0,2,4,6,7,8,10,12\}, \tag{3}$$

$$A = \{0,6,7,8,9,10,11,12\}. \tag{4}$$

The fact that a set A with a small doubling (with a small T) may be included in a short interval reflects only a part of a whole picture. Let e denote the maximal $a \in [0, a_{k-1}]$ with $a \notin 2A$,and let c denote the minimal $a \in [0, a_{k-1}]$ with $a + a_{k-1} \notin 2A$. In Lemma 6 we show that one always has $e < c$. We need also the following notion: the set A is called stable if

$$2A \cap [0, a_{k-1}] = A.$$

Examples of stable sets: { 0,6}, {0,2,4,6,},{0,3,4,5 }.

To give an understanding about the content and meaning of results of this paper let us send the reader to the Theorem IV (see page 12) which is the partial case of a Theorem II.

Example. Take the set A, defined in (3). We have $k = 8, T = 3k-4$ so that $a_{k-1} = k-1+b = 12$ and $b = 5$. We have e=5,c=9.

$$A_1 = A \cap [0,6] = \{0,2,4,6\},$$

$$A_2 = 12 - ([8,12] \cap A) = 12 - \{8,10,12\} = \{0,2,4\},$$

$$J = [6,20] \subset 2A,$$

$$I = [6,8] \subset A.$$

We see that the set A may be partitioned into three parts,

$$A = A_1 \bigcup I \bigcup (a_{k-1} - A_2),$$

where A_1 and A_2 are stable, and I is an interval, and the set $2A$ may be partitioned into three parts, $2A = A_1 \bigcup J \bigcup (2a_{k-1} - A_2)$, where J is an interval.

We call the *length* of a segment of integers (or of arithmetic progression) the number of his elements. So, the length of L in (1) is $|L| = |[0, k+b-1]| = k + b$.

We denote by M or $M(A)$ the minimal segment containing A. From Theorem 1 it follows that

$$a_{k-1} = k - 1 + b', \tag{5}$$

where

$$b' \leq b. \tag{6}$$

Thus, the length of $M = [0, a_{k-1}]$ is equal to $k + b'$. We will now evaluate b' from below. From $A \subset [0, k-1+b']$ it follows

$$2A \subseteq 2[0, k-1+b'] = [0, 2k-2+2b'], |2A| \leq |[0, 2k-2+2b']| = 2k-1+2b'.$$

Thus, from $|2A| = T = 2k - 1 + b$ we get

$$b' \geq b/2. \tag{7}$$

An element of M not belonging to A will be called a *hole* in M in respect to A or, simply, a hole in A. So, in (2) we have $A \subset M = [0, 12]$ and the holes are 1,3,5,7,9.

We see from examples (2)-(4) that the holes in A form a set which is a union of arithmetic progressions which tend to be nearer to the ends of M.

These properties of the structure of the set A will be called the detailed structure of A and their description in an exact way is the aim of this paper.

To describe the set A we will obtain first some properties of the set $2A$ which are, of course, the properties of A himself.

From $A \subset M = [0, a_{k-1}]$ and (5) we see that the number of holes in A is equal to b'. Denote

$$B = A \cup (a_{k-1} + A). \tag{8}$$

We have $B \subset M_1$, where $M_1 = [0, 2a_{k-1}]$. From $|B| = 2k - 1$ and

$$|M_1| = 2a_{k-1} + 1 = 2(k - 1 + b') + 1 = 2k - 1 + 2b', \tag{9}$$

it follows that the number of holes in B is equal to $2b'$. We can partition the set of holes in B into pairs $(a, a + a_{k-1})$, where a is a hole in A and $a + a_{k-1}$ is a hole in $a_{k-1} + A$.

Lemma 1 *For each pair $(a, a + a_{k-1})$ of holes in B we have or*

$$a \in 2A, \tag{10}$$

or

$$a + a_{k-1} \in 2A. \tag{11}$$

Proof Let us look at A as a set of residues moda_{k-1}. Our modulo, a_{k-1}, contains $k + b' \leq k + b \leq 2k - 3$ residues, the sets A (moda_{k-1}) and $a - A$ (moda_{k-1}) contain $k - 1$ residues each, because the numbers 0 and a_{k-1} are congruent modulo a_{k-1}. Thus, the sets of residues A and $a - A$ have a non-zero intersection, and therefore

$$a \in 2A \ (\mathrm{mod} a_{k-1}). \tag{12}$$

But in the set of integers the residue a is represented or by a or by $a + a_{k-1}$. If both these numbers does not belong to A then this contradicts (12). Therefore we have or (10) or (11).

We will need the following auxiliary

Lemma 2 *The set $C = a_{k-1} - A$ is isomorphic of order two to the set A.*

Proof For definition of isomorphism of subsets look, for example, Nathanson [3], page 233.

The map $\phi : A \rightarrow C$ we will define as follows: for each $a \in A$ take $e = \phi(a) \in C$ as $e = a_{k-1} - a$. The induced map $\phi^{(2)} : 2A \rightarrow 2C$ is well defined by $\phi^{(2)}(a_1 + a_2) = \phi(a_1) + (a_2)$. This map is, evidently, one-to-one map and ϕ is an isomorphism, which means that for $x \in 2A$ and $x = a_1 + a_2$, where $a_1 \in A$ and $a_2 \in A$ we have $y = \phi^{(2)}(x) = a_{k-1} - a_1 + a_{k-1} - a_2 = 2a_{k-1} - x$. $\square$

For the pair $(a, a + a_{k-1})$ of Lemma 1 one of numbers of the pair belongs to $2A$. And the second one?

Definition. If in the pair $(a, a + a_{k-1})$ both numbers of the pair belong to $2A$, i.e. (10) and (11) are valid, we call the pair *unstable* . The pair is called *stable* if one of the numbers of the pair does not belong to the $2A$,and this number will be called *stable hole.* If we have

$$a \notin 2A \tag{13}$$

the pair will be called left; if

$$a + a_{k-1} \notin 2A \tag{14}$$

the pair will be called right.

The number and location of pairs of different types depends to a large extent, as we will see, on the structure of both $2A$ and A.

Lemma 3 *The set of b' pairs of holes in B consists of $2b' - b$ stable pairs and $b - b'$ unstable pairs.*

Proof. The number of holes in B is equal to $2b'$. To get all $2k-1+b$ numbers of $2A$ we have to add to $2k - 1$ numbers of B, b numbers more, which are holes in B, so the number of stable points is equal to $2b' - b$, and the same is the number of stable pairs (one stable point in a stable pair). The whole number of pairs of holes in B is equal to b'. The number of unstable pairs is equal to

$$b' - (2b' - b) = b - b'.$$

$\square$

Denote $K = \Big\{(a, a + a_{k-1})\Big\}$ the set of all left stable pairs. The points a in it are stable and we denote e the maximal point of them. In the same way denote $N = \Big\{(a, a + a_{k-1})\Big\}$ the set of all right stable pairs. We denote c the point a in the minimal such pair.

We will show that mutual location of K and N is not accidental.

In the next two lemmas we begin to explain why the holes in A under conditions of Theorem 1 are concentrated in the neighbourhoods of 0 and a_{k-1} which are endpoints of $M = [0, a_{k-1}]$.

Lemma 4 *When a is a stable point i.e. $a \notin 2A$, the number of holes of A which belong to the segment $[0, a]$ is greater or equal than $[a/2 + 1]$.*

Proof. If a is even, let us look on pairs of numbers

$$(0, a), (1, a - 1), .., (a/2, a/2).$$

The number of these pairs is equal to $a/2+1$. Each pair $(i, a-i)$, $0 \leq i \leq a/2$, contains at least one hole of A (if $i \in A$ and $a - i \in A$ we would have $a = (i + (a - i)) \in 2A$, which would be in contradiction with (13).

If a is odd we look on the pairs

$$(0, a), (1, a - 1), ..., ((a - 1)/2, (a + 1)/2).$$

We finish now the proof for a odd in the same way as we did it for a even. $\square$

Lemma 5 *The number of holes in A which belong to a segment $I = [a, a_{k-1}]$ when $a + a_{k-1}$ is a right stable point is greater or equal than $[(a_{k-1} - a)/2 + 1]$.*

Proof. Let us use Lemma 2 and formulate the condition (14) in terms of isomorphic set C. We have $x \in 2A$ if and only if $\phi^{(2)}(x) = y = 2a_{k-1} - x \in 2C$. Thus the condition (14) is equivalent to

$$2a_{k-1} - (a_{k-1} + a) = a_{k-1} - a = \phi(a) = m \notin 2C.$$

Using Lemma 4 to the set C we see that number of holes in C, which belong to a segment $[0, m]$ is $\geq [m/2 + 1]$, The interval $[0, c]$ under transformation $\phi^{-1}(x) = a_{k-1} - x$ becomes I (ignoring the direction) and the last inequality proves Lemma 5 because of $m = a_{k-1} - a$. □

We are now ready to prove that the numbers of the set of left stable points are less than the numbers of the set of right stable points; the set of numbers between these two sets in A contains only holes which are unstable, and this ensures the existence of a long interval in $2A$.

Lemma 6 *We have*

$$e < c.$$

Proof. We know that e is stable in a left stable pair and so $e \notin 2A$ and from the fact that c is stable in a right stable pair $(c, c + a_{k-1})$ we get $(c + a_{k-1}) \notin 2A$. If $e = c$, then the pair $(e, e + a_{k-1})$ would have both elements not in $2A$, in contradiction to Lemma 1.

Suppose now, contrary to the conclusion of Lemma 6, that $e > c$.

The number of holes in A is equal to b'. We will estimate this number from below, using estimates of the values e and c .

Let

$$K_1 = \{a : (a, a + a_{k-1}) \in K\},$$

$$N_1 = \{a : (a, a + c_{k-1}) \in N\}.$$

Now we build a finite sequence of pairs of numbers

$$(c_1, e_1), (c_2, e_2), ..., (c_i, e_i), ..., (c_{i_0}, e_{i_0}) \qquad (15)$$

in the following manner.

Define $c_1 = c, e_1 = e$. Suppose that the pair (c_i, e_i) is already built,where c_i is a stable point from a right stable pair and e_i is a stable point from a left stable pair. There exist the following possibilities:

(i) There exist left stable pair $(a, a + a_{k-1}) \in K$ such that $c_i < a < e_i$

(ii) The case (i) is not valid but there exist right stable pair $(a, a+a_{k-1}) \in N$ such that $c_i < a < e_i$,

(iii) the cases (i) and (ii) are not valid.

In the case (i) put $c_{i+1} = c_i, e_{i+1} = a$; if (ii) is true put $c_{i+1} = a, e_{i+1} = e_i$; if (iii)is true put $i = i_0$, and the sequence is built. Let us mention that the sequence (15) was built in such a way that we have

$$c_i < e_i, i = 1, 2, ..., i_0$$

and

$$[c_1, e_1] \supset [c_2, e_2] \supset ...[c_{i_0}, e_{i_0}].$$

Denote x the number of holes in A which belong to the interval (c_{i_0}, e_{i_0}). All these holes because of the manner we had build them are unstable and we have because of Lemma 4 an estimate

$$x \leq b - b'. \tag{16}$$

For the holes in A which are in the segment $[c_{i_0}, e_{i_0}]$ we have an estimate from above by the number of elements of this segment, $e_{i_0} - c_{i_0} + 1$. The number of these holes is equal to $x + 2$; thus we get the inequality

$$x + 2 \leq e_{i_0} - c_{i_0} + 1. \tag{17}$$

The holes in A which are in the segment $[c_{i_0}, e_{i_0}]$ are perhaps counted twice when we estimate the number of holes in A belonging to $[1, a_{i_0}]$ with the help of Lemma 4 and when we estimate the number of holes in A belonging to $[c_{i_0}, a_{k-1}]$ with the help of Lemma 5. Putting all what was said together we obtain the inequality

$$b' \geq (e_{i_0} + 1)/2 + (a_{k-1} - c_{i_0} + 1)/2 - x - 2.$$

In view of (5) we get

$$b' \geq (k+b')/2 + (e_{i_0} - c_{i_0})/2 - 3/2 - x$$

and therefore

$$b' \geq k + e_{i_0} - c_{i_0} - 3 - 2x.$$

Using (17) we get

$$b' \geq k - 2 - x + e_{i_0} - c_{i_0} - 1 - x \geq k - 2 - x.$$

Because of (16) we have

$$0 \geq k - b + (b - b') - x - 2 \geq k - b - 2 \geq k - (k-3) - 2 = 1,$$

contradiction. $\square$

We will use now Lemmas 4, 5 and 6 and will estimate the length of the interval which belong to $2A$.

Theorem 2 *We have*

$$J = [e+1, c + a_{k-1} - 1] \subset 2A, \tag{18}$$

and for cardinality of J we have an estimate

$$|J| \geq 2k - 1 + 2d, \tag{19}$$

where d is the number of unstable pairs $(a, a + a_{k-1})$ for which

$$e < a < c.$$

Proof. We will prove first that (18) is valid. We represent the set J as a union of the four following subsets:

$$J = J_1 \cup J_2 \cup J_3 \cup J_4,$$

where

$$J_1 = B \cap J,$$

$$J_2 = \bigcup \{a, a + a_{k-1}\},$$

where $(a, a + a_{k-1})$-is an unstable pair and so $a \in 2A, a + a_{k-1} \in 2A$,

$$J_3 = (\bigcup \{a, a + a_{k-1}\}) \cap J,$$

where $(a, a + a_{k-1})$ are left stable pairs,

$$J_4 = (\cup \{a, a + a_{k-1}\}) \cap J,$$

where $(a, a + a_{k-1})$ are right stable pairs.

We have $J_1 \subset 2A$, because $B \subset 2A$. Definition of an unstable pair shows that $J_2 \subset 2A$. If $(a, a+a_{k-1})$ is a left stable pair, then $a \leq e$, $a \notin J, a+a_{k-1} \in 2A \cap J$.

If $(a, a + a_{k-1})$ is in $2A$ and is a right stable pair, then $a \in 2A \cap J$.

We will prove now the estimate (19).

From (18) and (5) we get

$$|J| = c + a_{k-1} - 1 - e = k - 2 + b' + c - e. \tag{20}$$

We will estimate now $c - e$ from below. For this aim we will estimate the number of holes in A, which are less than e or bigger than c and denote it P. For the number P_1 of holes which are less than e we have, according to Lemma 4

$$P_1 \geq (e + 1)/2 \tag{21}$$

and for the number P_2 of holes which are greater than c we have, according to Lemma 5

$$P_2 \geq (a_{k-1} - c + 1)/2. \tag{22}$$

The sets P_1 and P_2 have an empty intersection, in view of Lemma 6 and thus, in view of (21) and (22) we have

$$P \geq P_1 + P_2 \geq (e + 1)/2 + (k - 1 + b' - c + 1)/2. \tag{23}$$

We will get an estimate of P from above taking the number of all pairs b' minus the number d of those a, which are unstable holes between e and c.

Thus,

$$b' - d \geq (e + k + b' - c + 1)/2$$

and

$$c - e \geq k + b' + 1 - 2(b' - d) = k + 2d + 1 - b'. \tag{24}$$

Because of (24) we get in (20)

$$|J| \geq k - 2 + b' + k + 2d + 1 - b' = 2k + 2d - 1.$$

□

We will explain now how under some conditions it is possible to get a set $A'' \supset A$ (we will name the set A'' a completion of A) which helps to understand and describe the properties of A.

Take the set $A' = \{p\}$ of numbers which are holes in A where p belong to pairs $(p, p + a_{k-1})$ which are unstable and for which

$$e < p < c. \tag{25}$$

Define a set $A'' = A \bigcup A'$.

Theorem 3 *The set A'' has the following properties:*

(a)

$$2A'' = 2A, \tag{26}$$

(b) The interval $J' = [e + 1, c - 1]$ is included in A''

(c) We have

$$|A + A| = 2k - 1 + b' = |A'' + A''| = 2|A''| - 1 + b' - 2d. \tag{27}$$

Proof. We will prove first the equality (26). Let $p \in A'$; for the number $p+x$, if $x \in A''$, we have $p+x \in 2A''$. But in view of $x \in [0, a_{k-1}]$ we have

$$p + 0 \leq p + x \leq p + a_{k-1}$$

and using (25) we get

$$e + 1 \leq p + x \leq a_{k-1} + c - 1$$

and we see from (18) that $p + x \in J \in 2A$.

To prove (b) we have to take $x \in J'$ and $x \notin A$. If $(x, x + a_{k-1})$ is a left stable, we have $x \notin 2A, x + a_{k-1} \in 2A$ and therefore $x \leq e$ and $x \notin J'$. Thus, $(x, x+a_{k-1})$ is unstable and $x \in A'$. If $(x, x+a_{k-1})$ is a right stable, we have $x \in 2A, x + a_{k-1} \notin 2A$ and $x \notin A$, i.e. $x \geq c$ and $x \notin J'$. So, $(x, x + a_{k-1})$ is unstable and $x \in A''$. Now we will check assertion (c). The cardinality of A' is equal to d. We have $|A''| = k + d$. □

Let us give an example illustrating Theorem 3.

Example. Let us assume that $8|k$.

$$A = \left\{0, 1, ..., \frac{3k}{8}, \frac{5k}{8}, \frac{5k}{8} + 1, ..., k-1, k\right\}.$$

The completion of A is the set $A' = \{0, 1, 2, ..., k\}$ and we have

$$2A = 2A'', J = 2A'' = [0, 2k].$$

□

The most interesting results are those when $a_{k-1} = k - 1 + b$ i.e. when the interval containing A has the maximal length for a given T.

Theorem 4 *Let for A be $T = 2k - 1 + b$ where $0 \leq b < k - 2$ and the length of the interval $M(A)$ be maximal, i.e. $|M| = k + b$. Then*

(a) $A_1 = A \cap [0, e+1]$ is stable, i.e. $A \cap [0, e] = 2A \cap [0, e]$,

(b) $A_2 = a_{k-1} - ([c - 1, a_{k-1}] \cap A)$ is stable,

(c) $J = [e+1, c+a_{k-1}-1] \subset 2A$,

(d) $I = [e+1, c-1] \subset A$.

Proof. This Theorem is a corollary of Theorem 2. We have $b' = b$. Thus the set of unstable pairs is empty,every point of the interval $[e+1, c+a_{k-1}-1]$ belongs to $2A$.

Example.

$A = \{0, 2, 4, 6, 7, 8, 9, 10, 14\}$.

We have here l=5, c=11 ,for (a) the set A_1 =$\{$0,2,4,5$\}$, which is stable, for (b) A_2 =$\{$0,3$\}$,which is stable, J=[6,24], I=$\{$6,7,8,9,10$\}$.

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