The Interface Between Probability Theory and Additive Number Theory (Local Limit Theorems and Structure Theory of Set Addition)

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Abstract

We develop a new method for studying distributions of sums of independent random variables. This method is based on results obtained from Inverse Problems of Additive Number Theory. The notion of an "isomorphism" of random variables with supports in spaces of different dimensions is used, paving the way to new types of local limit theorems.

1 A short review of results in the field of local limit theorems

We list here several of the main local limit theorems [LLT] emphasizing those whose joint features spurred on the new approach in this paper.

Theorem 1 (B. Gnedenko [16]). *Let ξ be a random variable with $\mathbb{P}(\xi \in \mathbb{Z}) = 1$ and $\xi_1, \xi_2, \ldots, \xi_n$ independent random variables distributed as ξ. Let $S_n = \sum_{i=1}^n \xi_i$, $E\xi = a$, and $E(\xi - E\xi)^2 = \sigma^2$. Let the maximal step of ξ equal 1, i.e. if $q, r \in \mathbb{N}$ and $\mathbb{P}(\xi \in q\mathbb{Z} + r) = 1$, then $q = 1$.*

Uniformly on k, we have

$$\sigma\sqrt{n}\mathbb{P}(S_n = k) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k-na)^2}{2\sigma^2 n}\right) \to 0, \qquad \text{if } n \to \infty . \tag{1}$$

A general formulation of the limit problem may be found in M. Loev [18]:

There is a series of sums of independent random variables

$$S_n = \sum_{i=1}^n X_i .$$

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Find conditions for which
$$\mathcal{L}\left(\frac{S_n - ES_n}{\sigma S_n}\right) \to N(0,1) \ , \tag{2}$$
where
$$N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

We can treat (2) in the following way: transform the random variable ξ, centralizing it, $\xi \to \xi - a$, and then use the affine transformation $\mathbb{R}^2$,
$$x' = \frac{x}{\xi\sqrt{n}} \ , \quad y' = \sigma\sqrt{n} y \ ,$$
with corresponding transformation of the random variable.

In a further generalization, the random values
$$\frac{S_n}{b_n} - a_n$$
were studied, and only affine transformations of the random variable S_n were used.

A condition of an arithmetical nature for which Gnedenko's theorem is valid is when the maximal step equals 1.

A natural generalization of Gnedenko's theorem to the case of differently distributed summands is given in [29].

Theorem 2 (Y. Prokhorov). *Let*
$$\xi_1, \xi_2, \ldots, \xi_n, \ldots \tag{3}$$
be a series of independent random variables such that
$$\begin{aligned} &1. \quad \forall_i \mathbb{P}(\xi_i \in \mathbb{Z}) = 1 \ , \\ &2. \quad \forall_i |\xi_i| \le C \ , \end{aligned} \tag{4}$$
where C is an absolute constant.

For LLTs for the sequence (3) *to hold it is necessary and sufficient for the set*
$$A = \left\{ k \in \mathbb{Z} : \sum_{i=1}^{\infty} \mathbb{P}(\xi_i = k) = \infty \right\}$$
to have the maximal step equal to 1.

In Y. Rozanov's paper [30], condition (4) of uniform boundedness is weakened.

In [25] the following is proved:

Theorem 3 (G.A. Freiman, D.A. Moskvin, A.A. Yudin) *Let*

$$\xi_{1n}, \xi_{2n}, \ldots, \xi_{nn}, \quad n = 1, 2, \ldots, \tag{5}$$

be a triangular array of independent random variables distributed as ξ_{1n} *for every given* n, $\mathbb{P}(\xi_{1n} \in \mathbb{Z}) = 1,\ E\xi_{1n} = a_n,\ E(\xi_{1n} - a_n)^2 = \sigma_n^2.$

Suppose that

1. *For* (5) *the central limit theorem is valid*

2. $\sigma_n^2 = O(n^\rho)$, *where*

$$\rho < \frac{\ln(2+c)}{\ln 2} - 1 \ ,$$
$$c < 1,$$

3. *For* $q = 2, 3, \ldots$ *and* ω *positive and sufficiently large*

$$\max_{1 \le r \le q} \mathbb{P}(\xi_{1n} \equiv r (\mathrm{mod}\, q)) < 1 - \omega \max\left(\frac{\rho_n^2}{n\sigma_n^4}, \frac{1}{n^{1-\mu}}\right) \ln n \ ,$$

where

$$\rho_n = E|\xi_{1n} - a_n|^3 \ ,$$
$$\mu = \frac{(1+\rho)\ln 2}{\ln(2+c)} \ .$$

Then the LLT for (5) *holds.*

2 Analysis of results of §1 and directions for future study

The comparison and analysis of conditions formulated in [29, 30, 25, 23] and [24] (we do not give the formulations of theorems from [23] and [24] because they are cumbersome) lead us to the following conclusions.

1. In each theorem there are conditions describing “compactness of a random variable”. In [29] the random variables have to be uniformly bounded. In [30, 25, 23] and [24] there are conditions on the rate of growth of variance, of the third moment, and in [16], only the condition of existence of variance is needed.

2. Conditions characterizing the arithmetical properties of the support of a random variable are always present. For example, values taken by random variable ξ_{1n} have to be well distributed between classes $\bmod q$ for every q and the values

$$\max_r \mathbb{P}(\xi_{1n} \in \{n : n \equiv r (\bmod q)\})$$

must be separated from 1 as in [16] and [29].

Different forms of arithmetic conditions are discussed in more detail in [25] and [26].

The aim of this work is to show how the additive structure of support of a random variable ξ in some cases defines the behavior of

$$\mathbb{P}(S_n = a) \ .$$

Let G be a group where operation is denoted by $+$. For $A \subset G$ and $B \subset G$, we define

$$A + B = \{x : x = a + b, \ a \in A, b \in B\} \ ,$$
$$sA = (s-1)A + A \ .$$

The condition of "small doubling" of the support A of a random variable ξ will play the main role here. Namely, we suppose that

$$|2A| = |A + A| < C|A| \ , \tag{6}$$

where C is some positive constant.

We will show that the condition of small doubling (6) gives a new form of the condition of compactness and will provide a way of obtaining a new type of LLT, valid for distributions for which the usual conditions on random variables from [16, 18, 29, 30, 25, 23, 24] and [26] may not be true (for example, the variance may be missing), and LLT in its usual form may not take place at all.

It appears that, if supp ξ is a set with small doubling, we can build a map of ξ on a random variable ξ' with values in $\mathbb{Z}^s$ for suitable s (see §4). This map, which preserves additive properties of supp ξ may be constructed in such a way that, for supp ξ, conditions of compactness and of arithmetic types will be fulfilled. Applying LLT to S'_n and, knowing the values $\mathbb{P}(S'_n = a')$, we can now find the values $\mathbb{P}(S_n = a)$.

For the history, bibliography and results for the problem of structure of sets with a small doubling the reader is referred to G.A. Freiman [11], [9], Y. Bilu [2], M. Nathanson [27] and T. Gowers [15]. See also the website

www.maths.cam.ac.uk

and [8] and [10].

In §§3 and 7, we formulate results for random variables with finite support. Let us mention that this condition does not limit generality. If supp ξ is infinite we can always approximate ξ by a random variable ξ' with finite support.

We conclude this section with a discussion of the methods of the proofs of LLT, usually the method of characteristic functions. Let

$$f_\xi(t) = \sum_{k\in\mathbb{Z}} P(\xi = k)e^{2\pi ikt}$$

be a characteristic function of the random variable ξ. To obtain the LLT, we have only to estimate the asymptotics of $f_\xi^n(t)$ in some neighborhood of $t = 0$, and estimates $|f_\xi(t)|^n$ from above for points t which are not in this neighborhood. In [16], [29, 30], [23, 24, 26] these estimates were obtained explicitly with the use of the usual characteristics of a random variable ξ (variance, moments). Only in [22], [1] were the conditions and methods proposed, which enabled the study of the behavior of characteristic functions $f_\xi(t)$ in a more subtle way; and in [22], the behavior of the resolvent of a random walk was discussed. Estimates of precision for asymptotics of LLT may be found in [26],[14],[33],[32]. In [14] it was shown that the precision of LLTs depends on the structure of the set of those t for which $|f_\xi(t)|$ is close to 1. In [5] and [6] it was shown how the distribution of values of a positively defined function (the characteristic function is, evidently, such a function) is defined by the additive arithmetic structure of its support, supp ξ.

3 An example

The discussion of §2 will now be illustrated by a simple example.

Let us consider a triangular array of independent random variables

$$\xi_{1n}, \xi_{2n}, \ldots, \xi_{nn} \ , \tag{7}$$

where

$$K_n = \operatorname{supp} \xi_{in} = \{0, 1, 2, 2n+2, 2n+3, 4n+4\} \ .$$

and $\forall_i, 1 \le i \le n$ and $\forall a$, $a \in K_n$ we have

$$\mathbb{P}(\xi_{in} = a \in K_n) = \frac{1}{6} \ .$$

Let us show that the LLT in the usual form (1) does not hold. Denoting $\xi_{1n} = \xi$, we have,

$$E\xi = \frac{4}{3}n + O(1)$$

$$E(\xi - E\xi)^2 = b^2 \sim cn^2 \ .$$

If for $S_n = \sum_{i=1}^{n} \xi_{in}$ LLT is valid, then

$$\sqrt{cn^3}\mathbb{P}(S_n = k) = \frac{1}{\sqrt{2\pi}} \exp\Big(- \frac{(k - \frac{4}{3}n^2)^2}{2cn^3}\Big) + o(1)$$

and for all k such that

$$\left| k - \frac{4}{3}n^2 \right| < 2n$$

or for

$$\frac{4}{3}n^2 - 2n \leq k \leq \frac{4}{3}n^2 + 2n \tag{8}$$

we will obtain

$$\sqrt{cn^3}\mathbb{P}(S_n = k) = \frac{1}{\sqrt{2\pi}} + o(1) \ . \tag{9}$$

Let us study now the numbers x from S_n. They have a form

$$x = x_1 + x_2 + \cdots + x_n \tag{10}$$

where $\forall_j x_j \in \{0, 1, 2, 2n+2, 2n+3, 4n+4\}$ and therefore

$$x_j = c_j + (2n+2)d_j \tag{11}$$

and

$$x_j \equiv c_j \pmod{2n+2} \ , \tag{12}$$

where c_j may be equal to only one of three values $0, 1$ or 2.

We see that in the case

$$x \equiv c \pmod{2n+2}$$

where c is a residue from the system of nonnegative minimal residues, we have

$$0 \leq c \leq 2n$$

from (10) and (12), and $2n + 1 + (2n+2)t \notin S_n$ for any integer t. Therefore,

$$\mathbb{P}(S_n = 2n + 1 + (2n+2)t) = 0 \ .$$

Inequality (8) defines an interval containing more than $2n + 2$ points. So in this interval we can find a number $k^* \equiv 2n + 1 (\mathrm{mod}\, 2n+2)$ for which $\mathbb{P}(S_n = k^*) = 0$ and (9) gives us

$$0 = \frac{1}{\sqrt{2\pi}} + o(1) \ ,$$

a contradiction.

We will now show that the LLT in the sense of the discussion in §2 is nevertheless valid. For this, we relate the random variable ξ_{1n} with random variable η with values in $\mathbb{Z}^2$.

Define η in the following way:

$$K' = \operatorname{supp} \eta = \{(0,0),(1,0),(2,0),(0,1),(1,1),(0,2)\}$$

and $\forall a' \in K'$

$$\mathbb{P}(\eta = a') = \frac{1}{6}\ .$$

The map $\varphi : K_n \to K'$;

$$\varphi(0) = (0,0), \varphi(1) = (1,0), \varphi(2) = (2,0), \varphi(2n+2) = (0,1)\ ,$$

$$\varphi(2n+3) = (1,1), \varphi(4n+4) = (0,2)$$

(see Fig. 1) is a bijection which, because of (11), may be written in the following way

$$\varphi(c + (2n+2)d) = (c,d)\ .$$

For a sequence of independent random variables

$$\eta_1, \eta_2, \ldots, \eta_n, \ldots \tag{13}$$

where, $\forall_i \eta_i$ is distributed as η, the multidimensional LLT is valid (see [21] and [20]).

These papers showed that the necessary and sufficient condition for validity of LLT for the sequence (13) is the following: let a be any fixed element of $\operatorname{supp}\xi$, then $\operatorname{supp}\xi - a$ generates $\mathbb{Z}^2$. In our example this fact is evident.

Therefore, we will have (see Fig. 2)

$$\begin{aligned} &\mathbb{P}(S'_n = \eta_1 + \cdots + \eta_n = h \in \mathbb{Z}^2) = \\ &= \frac{1}{2\pi n\sqrt{\Delta}}\left(\exp\left(-\frac{1}{2}Q\left(\frac{h-na}{\sqrt{n}}\right)\right) + o(1)\right)\ , \end{aligned} \tag{14}$$

where $a = E_\eta$, Q is the quadratic form with a matrix inverse to a covariance matrix of a random variable η, Δ is the determinant of this covariation matrix.

What kind of progress have we made by introducing the series (13)? First, instead of the triangular array (7) we obtained a simpler case of a series of equally distributed random variables. Of course, this is not the most general situation but it may occur often. Second, these random variables appeared to be “good”, i.e. the compactness condition and arithmetic

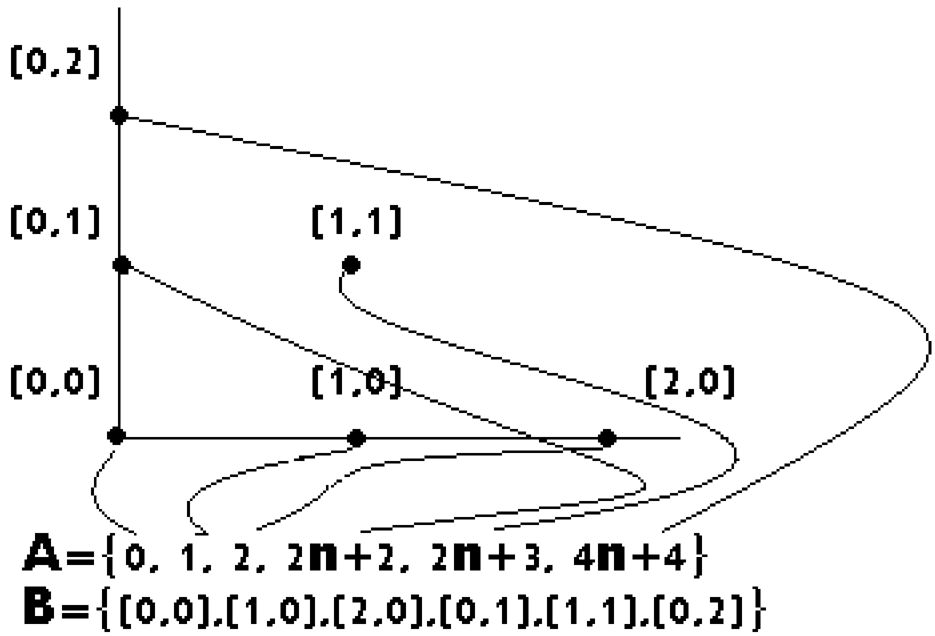

Figure 1

conditions were fulfilled automatically, which gave us the possibility of obtaining the value of $\mathbb{P}(S'_n)$ in (14). And last, we will now show how formula (14) will enable us to find the distribution for the array ξ.

Let us build a bijection

$$f : nK_n \to nK' \ .$$

For the element $x \in nK_n$, we have

$$x = x_1 + x_2 + \cdots + x_n, \tag{15}$$

where

$$\forall_i x_i \in K_n \ ,$$

and (15) is one of the possible representations of the element x.

In view of (11), x_i may be presented as

$$x_j = h_{1j} + h_{2j}(2n+2) \ .$$

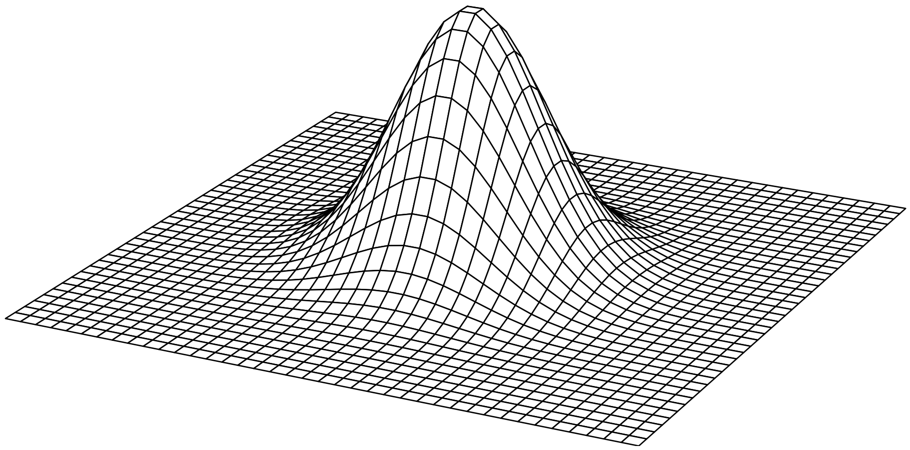

Figure 2

Define

$$
\begin{aligned}
f(x) &= \varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n) = \\
&= \sum_{j=1}^{n} \varphi(h_{1j} + h_{2j}(2n+2)) \\
&= \sum_{j=1}^{n} (h_{1j}, h_{2j}) = (h_1, h_2) \ .
\end{aligned}
$$

We have still to prove that the map f is defined correctly. Let some other representation of x be

$$x = y_1 + y_2 + \cdots + y_n \ ,$$

where

$$\forall_i \ y_i \in K_n \ .$$

Then

$$y_j = h'_{1j} + h'_{2j}(2n+2) \ ,$$

$$x = \sum_{i=1}^{n} h'_{1j} + (2n+2)\sum_{j=1}^{n} h'_{2j} = h'_1 + (2n+2)h'_2$$

and

$$f(x) = (h'_1, h'_2) \ .$$

We have

$$x = h_1 + (2n+2)h_2 = h'_1 + (2n+2)h'_2$$

From $h_1 = \sum_{j=1}^{n} h_{1j}$ and $0 \le h_{ij} \le 2$ we get

$$0 \le h_1 \le 2n$$

and, in the same way, we have

$$0 \le h'_1 \le 2n$$

It follows that

$$h_1 \equiv h'_1 \pmod{2n+2} \ ,$$
$$h_1 = h'_1$$

and

$$h_2 = h'_2 \ .$$

Let us show that f is a surjection. Take some element from nK':

$$(h_1, h_2) \in K'$$

Then (15) gives

$$(h_1, h_2) = \sum_{j=1}^{n} (h_{1j}, h_{2j}) \ ,$$

where

$$\forall_j (h_{1j}, h_{2j}) \in K'$$

We see that for

$$x = x_1 + x_2 + \cdots + x_n \ ,$$

where

$$x_j = h_{1j} + h_{2j}(2n+2)$$

we have

$$x = \sum_{j=1}^{n} h_{1j} + (2n+2) \sum_{j=1}^{n} h_{2j} = h_1 + (2n+2)h_2$$

and

$$f(x) = (h_1, h_2) \ .$$

We have shown that $f(x)$ is a surjection.

Let us show that f is an injection. Suppose that

$$f(x) = f(y) = (h_1, h_2) \ . \tag{16}$$

We have

$$x = h_1 + (2n+2)h_2, \quad y = h_1' + (2n+2)h_2$$

and therefore

$$f(x) = (h_1, h_2), \quad f(y) = (h_1', h_2')$$

(16) gives

$$h_1 = h_1', \quad h_2 = h_2'$$

and

$$x = y \ .$$

f is an injection.

Now we prove that

$$\mathbb{P}(S_n = x) = \mathbb{P}(S_n' = (h_1, h_2)) \ ,$$

where

$$f(x) = (h_1, h_2) \ .$$

Take one of the possible representations of x:

$$x = x_1 + x_2 + \cdots + x_n \ , \tag{17}$$

where

$$x_j \in K_n$$

and

$$x_j = h_{1j} + (2n+2)h_{2j} \ .$$

The event $\xi_j = x_j$ has the probability of $1/6$ and the probability of event x under condition (15) is $1/6^n$. If Q is the number of representations (17), then

$$\mathbb{P}(S_n = x) = \frac{Q}{6^n}$$

The value x_j is defined by the value of (h_{1j}, h_{2j}). Then:

$$f(x) = (h_1, h_2) = (\Sigma(h_{1j}, h_{2j}))$$

$$\mathbb{P}(\eta_j = (h_{1j}, h_{2j})) = \frac{1}{6}$$

and $\mathbb{P}(S = (h_1, h_2)) = \mathbb{P}(S = \sum_{j=1}^{n}(h_{1j}, h_{2j})) = 1/6^n$ if (h_{1j}, h_{2j}) is given. The number of all n possible pairs (h_{1j}, h_{2j}) is equal to Q and

$$\mathbb{P}(S = (h_1, h_2)) = \frac{Q}{6^n} ,$$

which means

$$P(S_n = x) = \mathbb{P}(S = (h_1, h_2)) \ . \tag{18}$$

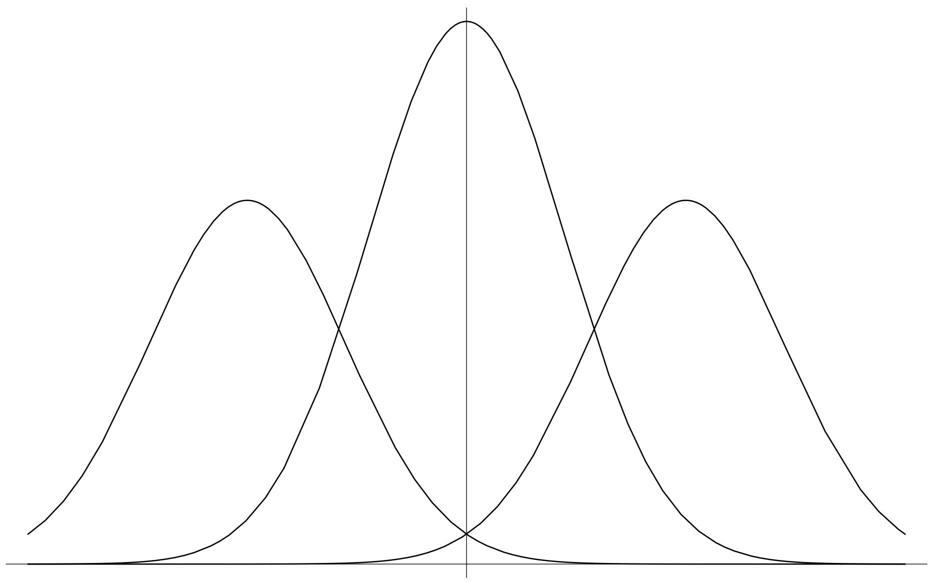

Figure 3

As a result (see Fig. 3), we obtain formula (18) for finding the value of $P(S_n = a)$ through the value of $P(S'_n = h)$. For this we don't need to know the variance of the random variable ξ_{1n} (of the order n^2). It is sufficient to know that the mean and variance of the random variable of η_i are uniformly bounded, they have all the moments and values of $\mathbb{P}(S'_u = h)$ that could be found with the required precision.

If the distribution of a random variable S_n were presented graphically on $\mathbb{Z}$, it would be difficult to see any regularity. At the same time, the distribution of S'_n on $\mathbb{Z}^2$, even for rather small ($n \approx 10$) values of n, is very near to a normal distribution.

We have discussed a rather simple example: the support $K_n = \operatorname{supp} \xi_{1n}$ has a small cardinality, the distribution of ξ_{1n} is uniform.

To obtain results for wider classes of random variables we need to introduce notions from Structure Theory of Set Addition (see the review [11], and Nathanson's book [27]).

4 Isomorphism of random variables

Definition 1 Let $A \subset G_1$ and $B \subset G_2$ and in each of sets G_1 and G_2 an algebraic operation is defined. Sets A and B will be called isomorphic if:

1. There exists a bijection $\varphi : A \to B$,

2. Bijection φ induces the bijection

$$\chi : 2A \to 2B$$

in the following sense: if $b \in 2A$, and $b = a_1 + a_2$, $a_1, a_2 \in A$, then

$$\chi(b) = \varphi(a_1) + \varphi(a_2) \ .$$

We can formulate the second point in another form:

$$\forall x_1, x_2, x_3, x_4 \in A$$
$$x_1 + x_2 = x_3 + x_4 \underset{\chi^{-1}}{\overset{\chi}{\rightleftarrows}} \varphi(x_1) + \varphi(x_2) = \varphi(x_3) + \varphi(x_4) \ .$$

Example 1 Let $A = \{0,1,3\}$, $B = \{0,1,5\}$. Then A is isomorphic to B. We will denote this by $A \sim B$.

Example 2 $A = \{0,1,3\}$ and $B = \{(0,0),(1,0),(0,1)\}$ are isomorphic. $\varphi(0) = (0,0)$, $\varphi(1) = (1,0)$, $\varphi(3) = (0,1)$.

Example 3 Here we will give an example of two sets which are not isomorphic: $A = \{0,1,2\}$, $B = \{(0,0),(1,0),(0,1)\}$. If these two sets would be isomorphic then the sets 2A and 2B must have the same cardinality but this is not the case.

In Definition 1 we describe the notion of isomorphism for the case when only two sets are added. This is a partial case ($n = 2$) of the following definition.

Definition 2 Subsets $A \subset G$ and $B \subset G_2$ will be called isomorphic of the n-th order, if

1. There exists a bijection $\varphi : A \to B$.

2. There exists the map $\chi : nA \to nB$ which is induced by φ and which is a bijection.

Example 4 The sets $A = \{0,1,n+1\}$ and $B = \{0,1,n+2\}$ give an example of sets which are isomorphic of the n-th order but are not isomorphic of the $n+1$-th order.

Now let us introduce a definition for the isomorphism of n-th order for random variables. Let ξ_1, and ξ_2 be random variables with values in $\mathbb{Z}^{s_1}$ and $\mathbb{Z}^{s_2}$, respectively,

$$K_j = \operatorname{supp} \xi_j = \{a \in \mathbb{Z}^{S_j},\ \mathbb{P}(\xi_j = a) \neq 0\} \ , \quad j = 1, 2 \ .$$

Definition 3 Random variables ξ_1 and ξ_2 we will call s-isomorphic if the sets K_1 and K_2 are s-isomorphic, and for bijection $\varphi : K_1 \to K_2$ we have for $\forall a \in K_1$

$$\mathbb{P}(\xi_1 = a) = \mathbb{P}(\xi_2 = \varphi(a)) \ .$$

Example 5 Let ξ_1 be a random variable support of which $K_1 = \{-h, -1, 0, 1, h\}$ with uniform probability distribution, and for a random variable ξ_2, $\operatorname{supp} \xi_2 = K_2 = \{(0,0),$ $(0,1)$, $(1,0)$, $(-1,0)$, $(0,-1)\}$ also with probabilities $1/5$ in each point.

The bijection

$$\varphi : 0 \to (0,1),\ 1 \to (1,0),\ -1 \to (-1,0),\ h \to (0,1),\ -h \to (0,-1)$$

is an $(h-1)$ - isomorphism.

At the same time K_1 and K_2 are not h-isomorphic, and therefore ξ_1 and ξ_2 are not isomorphic either.

The random variable ξ_1 can be presented in a "good" way if we study its s-fold convolutions only up to $S \leq h-1$.

Now we shall discuss the notion of dimension of a set K and of random variable ξ.

Definition 4 Let φ be an isomorphism $A \to \mathbb{Z}^m$ such that there exists no proper subgroup $G \subset \mathbb{Z}^m$ or $G + x$, such that $\varphi(A) \subset G + x$. Maximal m with such property will be called the "dimension" of the set A for s-isomorphism φ which we will denote by $\dim_s A$.

Definition 5 The dimension of the support ξ will be called the dimension of ξ, denoted by $\dim_s \xi$.

Some properties of $\dim_s \xi$ are as follows:

1. Dimension $\dim_s \xi$ is invariant under action on ξ of the group of non-singular linear transformation g, i.e.

$$\dim_s \xi = \dim(g\xi + b) \ ,$$

   where g is a nonsingular linear transformation, where b is in the domain of values of ξ.

2. $s_1 \geq s_2 \Rightarrow \dim_{s_1} A \leq \dim_{s_2} A$.

3. Let $A \subset B$; is it true that

$$\dim_s A \leq \dim_s B?$$

The answer is negative. An example:

$$A = \{1, 2, 2^2, 2^3, 2^4, 2^5\}$$

$$B = \{0, 1, 2, 3, \dots\}$$

We have $A \subset B$, $\dim_2 B = 1$, $\dim_2 A = 5$.

4. Is it true, that for $A_1 \subset \mathbb{Z}^m$ and $A_2 \subset \mathbb{Z}^m$ which are s-isomorphic, $\exists$ a nonsingular affine map $g : \mathbb{Z}^m \to \mathbb{Z}^m$, such that $g(A_1) = g(A_2)$?

The answer is negative. The sets $A = \{(0,0), (2,0), (0,1), (1,1)\}$ and $B = \{(0,0), (3,0), (4,0), (0,1)\}$ are 2-isomorphic but not affine equivalent.

The problem: Find or estimate

$$\dim_s(\{1, 2^k, 3^k, \dots, n^k\})$$

as a function of s, k and n.

An important step in the study of the s-isomorphism of discrete random variables can be found in [4].

5 A new type of local limit theorem

The final result of Gnedenko's theorem is formula (1). Formula (14) is an additional example, when for the series of random variables (13), $\eta_1, \eta_2, \dots, \eta_n$, the formula for $P(S'_n)$ was obtained, and it may be used to compute this probability. Other forms of the formula for $P(S_n)$ are possible. They may depend on the structure of the random variables that are being added, on their dimension, and on the precision needed. In all these cases it may be stated that a LLT takes place in an "explicit" form. Now the LLT for series of random variables can be formulated for cases, when in general, the explicit form of the LLT is not applicable.

Theorem 4 *Let us suppose that the triangular array ξ*

$$\begin{array}{llll} \xi_{11} & & & \\ \xi_{21} & \xi_{22} & & \\ & \cdots & & \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{array}$$

is isomorphic to the triangular array η

$$\begin{array}{llll} \eta_{11} & & & \\ \eta_{21} & \eta_{22} & & \\ & \dots & & \\ \eta_{n1} & \eta_{n2} & \dots & \eta_{nn} \end{array}$$

and the LLT for η *takes place in an explicit form. Then for* ξ *we obtain LLT induced by LLT for* η*. The conditions, ensuring the description of distributions for the scheme* ξ *are the conditions ensuring that the LLT for the scheme* η *is true. These conditions may already have the usual character (existence of variance, conditions on the order of its growth, or the order of growth of a determinant of covariation matrix of r.v.* η *and so on).*

The problem of finding the series η for which good LLT exist (if at all) in a family of isomorphic series may not be simple.

In the example in §3, the very structure of $\operatorname{supp}\xi$ helped to find the map of scheme ξ on a plane $\mathbb{Z}^2$.

Look at two more examples.

Let ξ be a scheme of series where ξ_{n1} have a homogeneous distribution on a set

$$\{0, 1, 2, 2n+1, 2n+2, 2n+3, 4n+2, 4n+3\} .$$

Let ρ be a scheme of series, where ρ_{n+1} is homogeneously distributed on a set $\{0, 2n, 3n+1, 4n, 6n+1, 6n+2, 7n+1, 8n+8\}$. To find a two-dimensional scheme η isomorphic to ξ is not a difficult task, as $\operatorname{supp}\xi_{1n}$ is very similar to $\operatorname{supp}\xi_{1n}$ in the scheme of §3.

The situation is quite different for the scheme ρ.

The schemes ρ and ξ are isomorphic (an exercise for the reader). However, beginning from scheme ρ it may not be simple to find an isomorphic scheme on a plane.

We have shown how to use the theorem in only a few cases.

Naturally, we can ask how wide the domain of applications of this theorem is. How large is the number of classes of the isomorphic schemes ξ for which an explicit form of LLT is applicable?

In §7 we will try to develop the first approach to this problem. To insist that two different schemes be isomorphic up to the n-th order is a rather strong condition to impose.

In §6 we develop the notion of isomorphism of subsets, (s, δ, ϵ)-isomorphism, with the help of which we can strongly lessen the order of the isomorphism in question.

6 (s,δ,ϵ)–isomorphism of random variables

Definition Let P and Q be two probabilistic measures with finite supports A and B respectively. Let $s \in \mathbb{Z}_+$, δ and ϵ be two nonnegative real numbers, and measures P and Q we will call (s,δ,ϵ)-isomorphisms if

1. There exists a bijection
$$\varphi : A \to B \ ;$$
2. For two subsets
$$A^{(s)} \subseteq sA, \quad B^{(s)} \subseteq sB \ ,$$
there exists a bijection
$$\varphi^{(s)} : A^{(s)} \to B^{(s)} \ ;$$
3. (i)
$$P^{*s}(A^{(s)}) \geq 1-\delta \ ,$$
$$Q^{*s}(B^{(s)}) \geq 1-\delta \ ;$$
   (ii) For $x_1,\ldots,x_s,y_1,\ldots,y_s \in \mathcal{A}$ we have
$$x_1+x_2+\cdots+x_s = y_1+y_2+\cdots+y_s \Longleftrightarrow \varphi(x_1)+\cdots+\varphi(x_s) = \varphi(y_1)+\cdots+\varphi(y_s)$$
   (iii) $\forall x \in A^{(s)} |P^{*s}(x) - Q^{*s}(\varphi^{(s)}(x))| \leq \epsilon.$

This notion has already been used in [4]. We will now return to the Example 5 in §4 and show how the results of this section can be generalized.

Examine the triangular array
$$\xi_{1s}, \xi_{2s}, \ldots, \xi_{ss}, \quad s = 1,2,\ldots$$
where
$$\mathcal{A} = \operatorname{supp} \xi_{1s} = \{-h(s),-1,0,1,h(s)\}, \quad \forall h(s) \in \mathbb{N}$$
and p_0, p_1, p_2 are positive numbers, for which
$$p_2 = P(\xi_{1s} = h(s)) = P(\xi_{1s} = -h(s)) \ ,$$
$$p_1 = P(\xi_{1s} = 1) = P(\xi_{1s} = -1) \ ,$$
$$p_0 = 1 - p_1 - p_2 \ ,$$

Let η be a two-dimensional random variable and

$$B = \operatorname{supp}\eta = \{(-1,0),(0,-1),(0,0),(1,0),(0,1)\}\ ,$$
$$p_1 = P(\eta = (1,0)) = P(\eta = (-1,0))\ ,$$
$$p_2 = P(\eta = (0,1)) = P(\eta = (0,-1))\ .$$

We will study the following cases:

1.
$$s \le h(s) - 1\ . \tag{19}$$

Define the map $\varphi^{(1)} : B \to A$: $\varphi^{(1)}((0,-1)) = -h(s)$, $\varphi^{(1)}((-1,0)) = -1$, $\varphi^{(1)}((0,0)) = 0$, $\varphi^{(1)}((1,0)) = 1$, $\varphi^{(1)}((0,1)) = h(s)$.

In view of (19) the map $\varphi^{(1)}$ induces a one-to-one map

$$\varphi^{(s)} : sB \to sA\ .$$

The map $\varphi^{(s)}$ is induced also on sB by a map φ of $\mathbb{Z}^2$ with a basis $e_1 = (1,0)$ and $e_2 = (0,1)$ on a line $\mathbb{Z} \subset \mathbb{Z}^2$ ($\mathbb{Z} = \{ke_1, k \in \mathbb{Z}$) which is a projection parallel to a vector $(h(s),-1)$.

For the probabilistic distributions we have

$$\forall x \in B^{(s)} \quad P^{*s}(\varphi^s(x)) = Q^{*s}\ .$$

In this first case, distributions P and Q are $(S,0,0)$-isomorphic.

2.
$$h(s) \le s < \alpha h^2(s)\ ,$$

where α is a sufficiently small positive number.

In this case, the map $\varphi^{(s)} : sB \to sA$ will not already be a bijection and we can illustrate here the usefulness of the notion of (s,ϵ,δ)-isomorphism to its full extent.

We have to define the sets $B^{(s)}$ and $\mathcal{A}^{(s)}$ from condition 2 of Definition 2.

The full preimage G_m of the number $m \in \mathbb{Z}$ under the map φ is

$$G_m = \{(m,0) + \ell(h(s),-1), \ell \in \mathbb{Z}\} \tag{20}$$

For each one of these m which will be in $\mathcal{A}^{(s)}$ we will choose only one point in G_m, in this way building a one-to-one map $\varphi^{(s)}(B^{(s)}) = \mathcal{A}^{(s)}$. This choice will be realized in the following way.

For the random variable η the LLT is valid and

$$P(\eta_1 + \cdots + \eta_s = (m_1,m_2)) = P(H_s = (m_1,m_2)) =$$

$$= \frac{1}{8\pi p_1 p_2 s} \exp\left(-\frac{1}{2}\left(\frac{m_1^2}{2p_1 s} + \frac{m_2^2}{2p_2 s}\right)\right) + R \ ,$$

where the error R is small and we will disregard it.

In view of

$$P(\xi_{1s} + \cdots + \xi_{ss} = m) \doteq P(M_s = m) = P(H_s \in G_m)$$

we get

$$P(M_s = m) = \sum_{(m_1,m_2)\in G_m} P(H_s = (m_1, m_2)) =$$
$$= \frac{1}{8\pi p_1 p_2 s} \sum_{(m_1,m_2)\in G_m} P(H_s = (m_1, m_2)) \ .$$

Now let us find the error if we omit all summands in the last sum for which

$$\frac{m_1^2}{2p_1 s} + \frac{m_2^2}{2p_2 s} > \lambda \ . \tag{21}$$

Easy computation shows that

$$\begin{aligned} &\frac{1}{8\pi p_1 p_2 s} \sum_{\frac{m_2^2}{2p_1 s} + \frac{m_2^2}{2p_2 s} > \lambda} \exp\left(-\frac{1}{2}\left(\frac{m_1^2}{2p_1 s} + \frac{m_2^2}{2p_2 s}\right)\right) \\ &\leq \frac{1}{\sqrt{p_1 p_2}} e^{-\frac{\lambda}{2}} \ . \end{aligned} \tag{22}$$

Now we choose λ in such a way that the set G_m and the set of points for which

$$\frac{m_1^2}{2p_1} + \frac{m_2^2}{2p_2} \leq \lambda \tag{23}$$

intersect at no more than one point.

This will be so if we choose

$$\lambda = \frac{1}{\alpha p_1 p_2} \tag{24}$$

We can now show that distributions P and Q are (s, δ, ϵ)-isomorphic for

$$\delta = \frac{1}{\sqrt{p_1 p_2}} e^{-\frac{1}{\alpha p_1 p_2}} \tag{25}$$

and

$$\epsilon = \delta \ . \tag{26}$$

The set $B^{(s)}$ we define as follows:

$$B^{(s)} = \left\{(m_1, m_2) : \frac{m_1^2}{2p_1 s} + \frac{m_2^2}{2p_2 s} \leq \lambda\right\} \tag{27}$$

The line

$$(m_1, m_2) + \ell(h(s), -1)$$

is intersecting the line $s(1,0)$ in the point

$$m = m_1 + m_2 h(s) \ . \tag{28}$$

We have $\varphi^{(s)}((m_1, m_2)) = m$ where m is defined by (28). For all points on the line G_m except the point in $B^{(s)}$, see (27), condition (23) is valid and the map $\varphi^{(s)} B^{(s)} = \mathcal{A}^{(s)}$ is now completely defined.

From (24) and estimate (22) we get the values δ and ϵ in (25) and (26).

3. $$\alpha h^2(s) \leq s \ .$$

In this case the classical LLT is valid, i.e.

$$P(M_s = m) = \frac{1}{\sigma\sqrt{2\pi s}} \exp\left(-\frac{m^2}{2\sigma^2 s}\right) \quad (1 + 0(1)) \tag{29}$$

where

$$\sigma^2 = 2h^2(s)p_2 + p_1 \ .$$

We would like to stress that LLT for H_s gives a more exact estimate than (29). In this connection it is important to mention the following: the distribution ξ_{1s} depends on three parameters p_2, p_2 and h, at the time that the distribution of H_s (the main term of it) depends only on

$$E\xi_{11}^2 = 2p_2 h^2(s) + 2p_1 \ .$$

We see that, from the information on the limit distribution of M_s, we cannot obtain estimates for all parameters of ξ_{11}, but from distribution H_s it is possible.

7 Density characteristics of probability distributions

The most important and most common characteristic of a random variable ξ is the variance

$$E(\xi - E\xi)^2 = \delta^2 \ .$$

Knowledge of σ^2, with the help of Chebishev's inequality enables us to find a segment on which a substantial part of the probability mass is concentrated. We will say that variance is a "density condition".

The main idea of this paper is to propose the use of density conditions which may be applied even in cases where variance does not exist.

Let K be a finite set, $K \subset \mathbb{Z}$, $|K| = card K$.

We say that the set K has a "small doubling" if

$$|2K| < c|K| \ ,$$

where $c \in \mathbb{R}^+$.

To have some understanding of the behavior of cardinality of $2K$ defined by the structure of K, we allow $K \subset \mathbb{R}^n$. Instead of cardinality of K we will use its measure.

Let, first, K be an interval I in $\mathbb{R}^1$. Then

$$\mu(2I) = 2\mu(I) \ .$$

If $A \subset \mathbb{R}^n$ then

$$\mu(2A) \geq 2^n \mu(A)$$

and equality occurs if A is a convex set (Brunn–Minkowski theorem).

We see that in $\mathbb{R}^n$ the sets with small doubling are the convex sets. A remarkable fact is that in the case of small doubling for $K \subset \mathbb{Z}$, the structure of K is such that it may be described as a dense subset of a convex set.

Theorem 5 *Let $K \subset \mathbb{Z}$, $|K| < +\infty$ and*

$$|2K| < 2|K| - 1 + b$$

where

$$0 \leq b \leq |K| - 3 \ .$$

Then K is a part of arithmetic progression of length $|K| + b$, i.e. $K \subset \mathcal{L}$, where

$$\mathcal{L} = \{a, a+d, a+2d, \ldots, a+(k+b-1)d\}$$

where $a, d \subset \mathbb{Z}$, $d > 0$.

Corollary *Take $b = |K| - 3$. We obtain that if*

$$|2K| \leq 3|K| - 4 \ ,$$

then K is part of an arithmetic progression of length $2|K| - 3$.

For the proof, see [8, page 11]. Further results on the structure of sets with small doubling can be found in [11],[19],[17], [31].

Theorem 6 *Let ξ be a scheme*

$$\begin{array}{llll}\xi_{11} & & & \\ \xi_{21} & \xi_{22} & & \\ & \cdots & & \\ \xi_{n1} & \xi_{n2} & \ldots & \xi_{nn}\end{array}$$

Let $K_n = \operatorname{supp}\xi_{n1}$, $p_{a,n} = \mathbb{P}(\xi_{n_1} = a \in K_n)$ and there exists a constant $p > 0$ such that

$$\forall_n \forall_n \exists K_n \; p_{a,n} \geq p \; .$$

Let

$$|2K_n| \leq 3|K| - 4 \; .$$

Then for the scheme ξ, LLT hold where for the isomorphic scheme η the support η_{n1} is an arithmetic progression with cardinality $2|K| - 3$ and a maximal step equal to one.

Using the conditions of the theorem in [25], we can verify that the scheme η is isomorphic to the scheme ξ, thus, proving our theorem.

Let us stress that the theorem is, in fact, only a partial case of Gnedenko's theorem and is given only to illustrate the method which gives the real results in the Main Theorem to follow.

Definition Let the set

$$\mathcal{D} = \{(x_1, x_2, \ldots, x_s) \mid x_1 \in \mathbb{Z}, 0 \leq x_i < h_i, h_i \in \mathbb{Z}, h_i \geq 1, 1 \leq i \leq s\}$$

be called a d-dimensional parallelepiped. The number of integer points in $\mathcal{D}$ is $\prod_{i=1}^{s} h_i = |\mathcal{D}|$.

Theorem 7 (G.A. Freiman, 1964). *For every finite subset $K \subset \mathbb{Z}$ for which*

$$|2K| < C|K| \; , \tag{30}$$

where the constant C does not depend on $|K|$, there exists $c = c(C)$, $d \leq [C - 1]$ and a d-dimensional parallelepiped $\mathcal{D}$,

$$|\mathcal{D}| < c|K|$$

such that

$$K \subset \varphi(\mathcal{D})$$

where φ is an isomorphism of the second order.

The proofs of this theorem can be found in [2],[8],[27],[9],[3].

Let us discuss what this theorem contributes to the better understanding of the structure of r.v. ξ with finite $\operatorname{supp}\xi$. From the formulation of the theorem it follows that dimension of r.v. is connected with a doubling coefficient of its support. Degree of "compactness" is given by a volume of the parallelepiped $\mathcal{D}$, for which we have an estimate

$$|\mathcal{D}| < c(C)|\operatorname{supp}\xi| \ .$$

The volume $\mathcal{D}$ shows to what degree we can "compress" $\operatorname{supp}\xi$ under 2-isomorphism. What is very important here is the connection of the volume of $\mathcal{D}$ with the cardinality of $\operatorname{supp}\xi$ which is defined by a constant $c(C)$. The importance of the study of $c(C)$ was stressed by Gowers in [15]. The best estimate for the moment is

$$\log\frac{|D|}{|\operatorname{supp}\xi|} < \mathcal{A}\cdot C^2(\log C)^3$$

where A is an absolute positive constant. This result was obtained by M.-C. Chang in her deep work [3].

In the book [8] the examples of sets K were given which show that

$$c(C) > c^{AC} \ ,$$

where A is an absolute constant.

Let us point out that the theorem ensures the existence of an isomorphism of second order and the necessity of prolonging it to higher orders creates additional difficulties. Irregularities of probabilistic distribution, if they exist, must also be taken into account. We now formulate the central result of this paper.

Main Theorem *Consider the scheme* ξ

$$\begin{array}{llll} \xi_{11} & & & \\ \xi_{21} & \xi_{22} & & \\ \xi_{n1} & \xi_{n2} & \ldots & \xi_{nn} \end{array}$$

with the following properties:

1. $\forall_j \mathbb{P}(\xi_{j1}\in\mathbb{Z}) = 1$

2. $\forall_j \mathbb{P}(\xi_{j1} = a \in \operatorname{supp}\xi_{j1}) \geq p > 0$, *$p$-an absolute constant.*
3. $$\forall_j |2\operatorname{supp}\xi_{j1}| < C|\operatorname{supp}\xi_{j1}| \tag{31}$$

4. $\forall_j$ *2-isomorphism for* $\operatorname{supp}\xi_{j1}$ *derived from the theorem for the scheme* ξ *is a* j*-isomorphism.*

Then for the scheme ξ*, LLT holds.*

Proof. Condition (31) of this theorem is the condition (30) of Theorem 7. We obtain a 2-isomorphism $\varphi : \mathcal{D}_n \subset \mathbb{Z}$ such that

$$\varphi^{-1}(K_n) \subset \mathcal{D}_n \ .$$

Let η_{n1} be a r.v., the image of r.v. ξ_{n1} under the 2-isomorphism map φ^{-1},

$$\varphi^{-1}(\xi_{n1}) = \eta_{n1} \ .$$

We have obtained the scheme η isomorphic to a scheme ξ. We will now show that for the scheme η the conditions of multidimensional LLT from [7] are fulfilled.

Conditions of variance and the third moment are fulfilled in an obvious way in view of the fact then $|K_n|$ are uniformly bounded which follows from condition 2 of the Main Theorem.

Now, with respect to the arithmetic condition, suppose first that for every sublattice $G \subset \mathbb{Z}^s$ and every $x \in \mathbb{Z}^s$ we have

$$\mathbb{P}(\eta_{n1} \in G + x) \neq 1 \ .$$

This means that there exists $y \in \operatorname{supp}\eta_{n1}$ and $y \notin G + x$, and then

$$\max_x \mathbb{P}(\eta_{n1} \in G + x) \leq 1 - \mathbb{P}(\eta_{n1} = y) \leq 1 - p \ .$$

If the arithmetic condition is not fulfilled, i.e. $\mathbb{P}(\eta_{n1} \in G + x) = 1$ then, with the help of a suitable affine transformation, $G + x$ may be transformed to a lattice with the value of the volume of a fundamental parallelepiped equal to 1. Repeating this reasoning we arrive at a r.v. for which the arithmetic condition is fulfilled.

The Main Theorem enables us to obtain new LLT for a wide class of schemes and therefore enables us to compute values of probabilities of n-fold convolutions, the larger the number n the more exact.

In conclusion, we express our gratitude to Professor Jean-Marc Deshouillers for his invaluable help throughout the many years of our collaboration.

Appendix

1. An explicit formula for the example in §3 is as follows:

$$\mathbb{P}(\xi_1 + \cdots + \xi_n = m_1 + (2n+2)m_2) = \frac{3}{\pi \cdot \sqrt{35} \cdot n} \cdot \exp\Bigg(-\frac{1}{2n}\Big[(m_1 - 2/3n)^2 - \frac{1}{3}(m_1 - 2/3n)(m_2 - 2/3n) + (m_2 - 2/3n)^2\Big]\Bigg) \cdot (1 + o(1)) \ .$$

2. The formulation of LLT from [25].

Let $\xi_{n1}, \xi_{n2}, \dots, \xi_{nn}$ be a scheme of independent equally distributed random variables, taking integer values and having finite variance. Denote: $E(\xi_{n1} - E\xi_{n1})^2 = \sigma_n^2$; $E\xi_{n1} = a_n$.

Theorem *Let $E|\xi_{n1}|^3 < \infty$ and $\rho_n = E|\xi_{n1} - a_n|^3$. Suppose that*

I for $n \to \infty$, $\frac{\rho u^2}{u\sigma_n^u} \ln n \to \infty$

II $\sigma_n^2 = O(n^\rho)$, where $\rho < \frac{\ln(2+c)}{\ln 2} - 1$ and $c < 1$,

III for $q = 2, 3, \dots$ and sufficiently large $\omega > 0$

$$\max_{1 \le r \le q} P(\xi_1 \equiv r(\mathrm{mod} q)) \le 1 - \omega \max\left(\frac{\rho_n^2}{n\sigma_n^4}, \frac{1}{n^{1-\mu}}\right) \ln n \ ,$$

where $\mu = \frac{(1+\rho)\ln 2}{\ln(2+c)}$. Then for series $\xi_{n1}, \xi_{n2}, \dots, \xi_{nn}$ the LLT takes place.

3. Formulation of LLT from [7].

Theorem. *Let $\{\vec{\xi}_\ell^{(n)}\}_{\ell=1}^n$ be the scheme of independent vectors $\vec{\xi}_e^{(n)} = (\xi_{\ell 1}^{(n)}, \dots, \xi_{\ell d}^{(n)})$, $\ell = 1, 2, \dots, n$; $n = 1, 2, \dots$*

We take the vector of mathematical expectations of vectors of series to be zero, and variances of components $\sigma_{n_i}^2$, $i = 1, 2, \dots, d, \dots$ finite, but dependent on the number of series n. $\|\rho_{ij}^{(n)}\|$ are matrix covariations of quadratic form connected with $\|\rho_{c_j}^{(n)}\|$ we suppose to be positively defined

$$f(\overline{x}) = (2\pi)^{-d/2} (\det \|\rho_{ij}\|)^{-1/2} \cdot \exp\left\{-1/2 \sum_{i,j=1}^d \alpha_{ij} x_i x_j\right\} \ , \ \text{where } \|\alpha_{ij}^{(n)}\| = \|\rho_{ij}^{(n)}\|^{-1} \ .$$

Standard notation: $\mathbb{P}\left\{\overline{\xi}_1^{(n)} = \overline{m}\right\} = p(\overline{m})$, $\overline{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, $\overline{s}_u = \sum_{k=1}^u \overline{\xi}_k^{(n)} = (S_{n1}, \dots, S_{nd})$; $P\{\overline{s}_n = \overline{z}\} = P_n(\overline{z})$, $\overline{z} = (z_1, \dots, z_d) \in \mathbb{Z}^d$; $B_{ni}^2 = DS_{ni} = n\sigma_{ni}^2$, $i = 1, 2, \dots, d$; $\overline{\xi}_n = (S_{n1}/B_{n1}, \dots, S_{nd}/B_{nd})$, $\overline{z}/B_n = (z_1/B_{n1}, \dots, zd/B_{nd})$.

We suppose that

$$\beta_{ni} = E|\xi_{1i}^{(n)}|^3 < +\infty, \ i = 1, \dots, d \ ,$$

$\forall a \in \mathbb{Z}$, $q \ge 2$ $\forall \overline{a} = (a_1, \dots, a_d)$, $a_i \in \mathbb{Z}$, $i = 1, \dots, d$, $(a_1, \dots, a_d, q) = 1$; $\max_{0 \le r \le a-1} \mathbb{P}\{(\overline{a}, \overline{\xi}^{(n)}) \equiv r(mod\ q)\} \le 1 - \alpha_n$,

$$\alpha_n = K \max_{1 \le i \le d} \ \max\left\{\frac{\beta_{ni}^2}{\sigma_{ni}^4, \Delta_n^2 \cdot n} \ , \ \frac{\sigma_{ni}}{\sqrt{n}}\right\} \ln n \ ,$$

where K is some constant, $\Delta_n = \det \|\rho_{ij}\|$. Let $\beta_{n_i} = 0(\sigma_{ni}^3 \Delta_n^2 \sqrt{n})$. Then there exists the constant $c(d)$ and $k \geq c(d)$ such that the LLT is valid, i.e.

$$B_{n1}B_{n2}\cdots B_{nd}P_n(\overline{z}) - f\left(\frac{\overline{z}}{B_n}\right) \to 0 \ .$$

4. Inequality of S.N. Bernstein, (Probability Theory, 4th Edition, Gostechizdat, 1946, Ch. IV).

In the case of uniformly bounded summands

$$S_n = \sum_{j=1}^{n} \xi_j \ ,$$

then,

$$1 - F_n(xB_n) < \exp(-x^2/4) \quad (0 < x < B_n/H) \ ,$$
$$1 - F_n(xB_n) < \exp(-xB_n/4H) \quad (x > B_n/H) \ .$$

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