ON A KAKEYA-TYPE PROBLEM II
GREGORY A. FREIMAN, YONUTZ V. STANCHESCU

Abstract: Let $A$ be a finite subset of an abelian group $G$. For every element $b_i$ of the sumset $2A = \{b_0, b_1, \ldots, b_{|2A|-1}\}$ we denote by $D_i = \{a - a' : a, a' \in A; a + a' = b_i\}$ and $r_i = |\{(a, a') : a + a' = b_i; a, a' \in A\}|$. After an eventual reodering of $2A$, we may assume that $r_0 \geq r_1 \geq \ldots \geq r_{|2A|-1}$. For every $1 \leq s \leq |2A|$ we define $R_s(A) = |D_0 \cup D_1 \cup \ldots \cup D_{s-1}|$ and $R_s(k) = \max\{R_s(A) : A \subseteq G, |A| = k\}$. Bourgain and Katz and Tao obtained an estimate of $R_s(k)$ assuming $s$ being of order $k$. In this paper we describe the structure of $A$ assuming that $G = \mathbb{Z}^2$, $s = 3$ and $R_3(A)$ is close to its maximal value, i.e. $R_3(A) = 3k - \theta \sqrt{k}$, with $\theta \leq 1.8$.

Keywords: Inverse additive number theory, Kakeya problem.

1. Introduction

Let $A$ be a finite subset of the group $G = \mathbb{Z}$ or $G = \mathbb{Z}^2$. For every element $b_i$ of the sumset $2A = A + A = \{x + x' : x \in A, x' \in A\} = \{b_0, b_1, b_2, \ldots, b_{|2A|-1}\}$ we denote

$$D_i = \{a - a' : a \in A, a' \in A, a + a' = b_i\}, \quad d_i = |D_i|,$$

$$r_i = r_i(A) = |\{(a, a') : a + a' = b_i, a \in A, a' \in A\}|.$$

After an eventual reordering of the set $2A$, we may assume that $r_0 \geq r_1 \geq \ldots \geq r_{|2A|-1}$. We denote

$$c_i = \frac{b_i}{2}, \quad C = \{c_0, c_1, c_2\}, \quad \text{Diff}(A) = D_0 \cup D_1 \cup D_2,$$

$$R_3(A) = |\text{Diff}(A)| = |D_0 \cup D_1 \cup D_2|,$$

$$R_3(k) = \max\{R_3(A) : A \subseteq G, |A| = k\}.$$

In the paper [1], we determined the maximal value of $|\text{Diff}(A)|$ for finite sets $A \subseteq \mathbb{Z}^2$, assuming that $b_0, b_1, b_2$ are non-collinear. We also described the structure

The research of the second named author was supported by The Open University of Israel's Research Fund, Grant No. 100937.

2000 Mathematics Subject Classification: primary 11P70; secondary 11B75.
of planar extremal sets $A^*$, i.e. sets of integer lattice points on the plane $\mathbb{Z}^2$ for which we have

$$R_3(A^*) = R_3(k) = 3k - \sqrt{3}k. \quad (3)$$

More precisely, for every $\alpha \in \mathbb{N}$ we denote by $H_\alpha$ the set of all points $P = (x, y) \in \mathbb{Z}^2$ such that $x$ and $y$ are odd integers and $-2\alpha < x, y, x + y - 1 < 2\alpha$. We proved the following result (see [1], Section 3):

**Theorem 1.** Let $A$ be a finite subset of $\mathbb{Z}^2$, $|A| = k$. Then

$$R_3(A) = |\text{Diff}(A)| \leq 3k - \sqrt{3}k. \quad (4)$$

Moreover, the equality $R_3(A) = 3k - \sqrt{3}k$ holds if and only if $k = 3\alpha^2$ and there is an affine isomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A = \phi(H_\alpha)$.

Note that $H_\alpha$, the canonical form of an extremal set, contains only odd lattice points $(x, y)$ (i.e. both coordinates $x$ and $y$ are odd integers), its convex hull is a hexagon and the set $H_\alpha$ lies on $2\alpha$ lines parallel to the line $y = 0$, on $2\alpha$ lines parallel to the line $x = 0$ and on $2\alpha$ lines parallel to the line $x + y = 1$ (see Figure 1.1). Moreover, $H_\alpha$ satisfies equality (3) with respect to the centers $c_0, c_1, c_2$ given by $e_0 = (0, 0), e_1 = (1, 0), e_2 = (0, 1)$, respectively.

![Figure 1.1: The set $H_\alpha$ and the centers $c_i = e_i$, $i = 0, 1, 2$.](image)

In this paper we continue the study of such finite sets and we will determine the structure of sets of odd lattice points on the plane for which $c_i = e_i$, $i = 0, 1, 2$ and the number of differences $R_3(A)$ is close to its maximal value (3). In order to formulate our main result we will use the following notation. If $u = (u_1, u_2) \in \mathbb{R}^2$, we denote by $u_1$ and $u_2$ its coordinates with respect to the canonical basis $e_1 = (1, 0), e_2 = (0, 1)$ and $e_0 = (0, 0)$ represents the origin point. Let $a = 2\alpha$, $b = 2\beta$ and $c = 2\gamma$ be three natural numbers such that

$$2 \leq c \leq a + b - 2. \quad (5)$$
We denote by $H(a, b, c)$ the set of all points $P = (x, y) \in \mathbb{Z}^2$ which satisfy the following conditions:

$$H(a, b, c) : \begin{cases} -2\alpha + 1 \leq x \leq 2\alpha - 1, & \text{x odd,} \\ -2\beta + 1 \leq y \leq 2\beta - 1, & \text{y odd,} \\ -2\gamma + 1 \leq x + y - 1 \leq 2\gamma - 1. \end{cases} \quad (6)$$

Note that if $a = b = c = 2\alpha$, then $H(a, b, c)$ is the perfect hexagon $H_\alpha$ described in Figure 1.1.

We will prove that if $c_i - b_i^2 - e_i$, for $i = 0, 1, 2$ and if $|\text{Diff}(A)| \geq 3k - 1.8\sqrt{k}$, then $A$ is almost hexagonal, i.e. an essential part of the set $A$ can be approximated by a hexagon similar to the extremal set $H_\alpha$. A precise formulation is given in the following:

**Definition 1.** We say that $A \subseteq \mathbb{Z}^2$ is an almost hexagonal set if there is a subset $A^* \subseteq A$ and a hexagon $H(a, b, c)$ which satisfy the conditions:

1. $|A^*| \geq 0.9|A|$,  
2. $A^*$ is included in $H(a, b, c)$ and $|H(a, b, c)| \leq 1.081|A^*|$,  
3. if $a \preceq b \leq c$, then $a > 0.8\sqrt{|A^*|}$, $b < 1.75a$, $c < 0.75(a + b)$.

Using the above notations, we can state now our main result:

**Theorem 2.** Let $A \subseteq \mathbb{Z}^2$ be a finite subset of odd lattice points on the plane. Assume that $|A| = k$ is sufficiently large and $c_i = e_i$, for $i = 0, 1, 2$. If

$$R_3(A) = |\text{Diff}(A)| = 3k - \theta \sqrt{k}, \quad \theta \leq 1.8, \quad (7)$$

then the set $A$ is almost hexagonal.

We prove Theorem 2 in Sections 2-5. Actually, we will prove a more precise estimate (16). In Section 3 we prove Theorem 2 for connected sets and in Section 5 we complete the proof using properties of disconnected sets obtained in Section 4. In Section 6 we will discuss some directions for further research.

We complete the introduction by recalling some simple remarks from [1]. We will use them whenever necessary without further mention. We easily see that $d_i = r_i$, for every $0 \leq i \leq |2A| - 1$. Indeed, using $(1)$ and $(2)$ we get that for two pairs $(a_1, a_1')$ and $(a_2, a_2')$ of $A \times A$ such that $a_1 + a_1' = a_2 + a_2' = b_i$ we have $a_1 - a_1' = a_2 - a_2'$ if and only if the equality $(a_1, a_1') = (a_2, a_2')$ holds.

Moreover, using (1), we see that $d_i$ is equal to the number of pairs $(a, a')$ such that $a \in A$, $a' \in A$ and $a$ and $a'$ are symmetric with respect to the center $c_i = b_i^2$, i.e.

$$d_i = |D_{c_i}|, \quad \text{where} \quad D_{c_i} = \{(a, a') : a \in A, a' \in A, a + a' = 2c_i\}.$$

We also note that if $a \neq a'$ then the pairs $(a, a')$ and $(a', a)$ give two distinct differences

$$a - a' = a - (b_i - a) = 2a - b_i \quad \text{and} \quad a' - a = -(2a - b_i)$$
and if $a = a'$ we have one pair $(a, a)$ and one difference $d = a - a = 0$. We have 
\[ R_3(A) = |\text{Diff}(A)| = 3k - \theta \sqrt{k} - |D_0(A) \cup D_1(A) \cup D_2(A)| \]
\[ \leq |D_0(A)| + |D_1(A)| + |D_2(A)| \leq d_i + 2k \]
and thus 
\[ d_i \geq R_3(A) - 2k = k - \theta \sqrt{k}, \]
for every $0 \leq i \leq 2$. Let us denote by 
\[ p_i = 2c_i - p \]
the symmetric of $p$ with respect to $c_i$. Denote by $M_i$ the set of points $p \in A$ such that $p_i \notin A$. If $m_i = |M_i|$, then 
\[ d_i = |D_i(A)| = k - m_i \]
and thus 
\[ m_i = k - d_i \leq k - (R_3(A) - 2k) = \theta \sqrt{k}. \quad (8) \]

In other words, Theorem 2 describes the structure of sets of lattice points that are "almost" symmetric with respect to some set $C$ of centers of symmetry. This is a natural question to be studied in geometry and in inverse additive number theory.

2. Normal sets and Covering Hexagons

We will prove first several simple remarks.

**Lemma 1.** Assume that there is a point $p \in A$ such that $p_1 = 2c_1 - p$ and $p_2 = 2c_2 - p$ don't belong to $A$. If 
\[ A' = A \setminus \{p\} \]
is the set obtained from $A$ by removing the point $p$, then 
\[ R_3(A') \geq R_3(A) - 2. \]

**Proof.** Assumptions $p_1 = 2c_1 - p \notin A$ and $p_2 = 2c_2 - p \notin A$ imply that the differences 
\[ d_1 = \pm(p - p_1), \quad d_2 = \pm(p - p_2) \]
do not belong to $D_1(A)$ and $D_2(A)$, respectively. Therefore the removal of $p$ from the set $A$ reduces the cardinality of $\text{Diff}(A)$ by maximum two differences: 
\[ d_0 = \pm(p - p_0). \]
We conclude that 
\[ D_0(A') \geq D_0(A) - 2, D_1(A') = D_1(A), D_2(A') = D_2(A), \]
which implies $R_3(A') = |\text{Diff}(A')| \geq |\text{Diff}(A)| - 2 = R_3(A) - 2. \]
Definition 2. If a point \( p \in A \) satisfies the condition
\[
|\{p_0, p_1, p_2\} \cap A| \leq 1,
\] (9)
i.e. at least two symmetric points of \( p \) with respect to \( \{c_0, c_1, c_2\} \) do not belong to \( A \), then we will say that \( p \) is a removable point of \( A \). If the point \( p \) doesn't satisfy the condition (9), then we will say that \( p \) is an essential point of \( A \).

Assume that \( A \) satisfies inequality (7). In the following Lemma we will estimate the number of removable points of \( A \) and we will show that the subset \( A_0 \) of \( A \) consisting of all essential points of \( A \) has the same property (7).

Lemma 2. Let \( A \) be a finite subset of \( \mathbb{Z}^2 \), \( |A| = k \). Assume that
\[
R_3(A) = |\text{Diff}(A)| = 3k - \theta \sqrt{k}, \quad \theta \leq 1.8.
\] (10)

Let \( A_0 \) be the set of all essential points of \( A \) and let \( A \setminus A_0 \) be the set of removable points of \( A \).

(a) If \( k_0 = |A_0| \), then \( R_3(A_0) \geq 3k_0 - \theta \sqrt{k_0} \).

(b) If \( n = |A \setminus A_0| \), then \( n \leq (\theta - 1.73) \sqrt{k} \leq 0.07 \sqrt{k} \), if \( k \) is sufficiently large.

Proof. If \( n = |A \setminus A_0| = k - k_0 \) denotes the number of removable points of \( A \), then Lemma 1 implies that
\[
R_3(A_0) \geq R_3(A) - 2n \geq 3k - \theta \sqrt{k} - 2n
\]
\[
= 3(k - n) - \theta \sqrt{k - n} \leq 3(k - n) - \theta \sqrt{k - n}
\]
\[
= 3k_0 - \theta \sqrt{k_0} + n \left( 1 - \frac{\theta}{\sqrt{k} + \sqrt{k - n}} \right)
\]
\[
\geq 3k_0 - \theta \sqrt{k_0},
\]
in view of \( k \geq 4 \geq \theta^2 \). Assertion (a) is proved. We will now estimate the number of removable points of \( A \). We first note that
\[
3k - \theta \sqrt{k} \leq R_3(A) \leq R_3(A_0) + 2n \leq 3|A_0| + 2n = 3(k - n) + 2n = 3k - n
\]
and thus
\[
n = k - k_0 \leq \theta \sqrt{k} \leq 2\sqrt{k}.
\] (11)
This estimate can be improved by using inequality (4) for the set \( A_0 \). Indeed, we have
\[
R_3(A_0) \leq 3|A_0| - \sqrt{3|A_0|} = 3(k - n) - \sqrt{3(k - n)}
\]
and inequality
\[
3k - \theta \sqrt{k} \leq R_3(A) \leq R_3(A_0) + 2n \leq 3(k - n) - \sqrt{3(k - n)} + 2n
\]
clearly implies
\[
n \leq \theta \sqrt{k} - \sqrt{3(k - n)} \leq \theta \sqrt{k} - \sqrt{3\sqrt{k} - 2\sqrt{k}} \leq (\theta - 1.73) \sqrt{k} \leq 0.07 \sqrt{k},
\]
if \( k \) is sufficiently large. Assertion (b) is proved. \( \blacksquare \)
Lemma 2 allows us to study planar sets \( A \) consisting only of essential points.

**Definition 3.** We say that \( A \subseteq \mathbb{Z}^2 \) is a normal set (with respect to the centers \( c_0 = e_0, c_1 = e_1, c_2 = e_2 \)) if

(i) every point of \( A \) is an essential point and

(ii) every point \( p = (x, y) \in A \) has both coordinates \( x \) and \( y \) odd integers.

Let us choose six integers \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) such that:

(i) every point \( p = (x, y) \in A \) satisfies the inequalities

\[
H = H(A) : \begin{cases}
\alpha_1 \leq x \leq \alpha_2, & x \text{ odd}, \\
\beta_1 \leq y \leq \beta_2, & y \text{ odd}, \\
\gamma_1 \leq x + y \leq \gamma_2.
\end{cases}
\]

(ii) on each line \((x = \alpha_1), (x = \alpha_2), (y = \beta_1), (y = \beta_2), (x + y = \gamma_1), (x + y = \gamma_2)\) there is at least one point of \( A \).

The finite set \( H(A) \subseteq (2\mathbb{Z} + 1) \times (2\mathbb{Z} + 1) \) defined by the above two conditions will be called a covering polygon of the set \( A \).

We will prove that if \( A \) is normal set then the points of \( A \) lie on pairs of symmetric lines with respect to three lines defined by

\[
l_1 : (x = 0), \quad l_2 : (y = 0), \quad l_3 : (x + y = 1).
\]

More precisely:

**Lemma 3.** Let \( A \subseteq \mathbb{Z}^2 \) be a finite normal set. Then

(a) If \( A \cap (x = \alpha) \neq \emptyset \) and \( A \cap (x = -\alpha) \neq \emptyset \) then \( A \cap (x = -\alpha) = \emptyset \).

(b) If \( A \cap (y = \beta) \neq \emptyset \) then \( A \cap (y = -\beta) \neq \emptyset \).

(c) If \( A \cap (x + y - 1 = \gamma) \neq \emptyset \) then \( A \cap (x + y - 1 = -\gamma) \neq \emptyset \).

**Proof.** In view of (12), the points \( c_0 \) and \( c_2 \) belong to \( l_1 \), \( c_0 \) and \( c_1 \) belong to \( l_2 \) and finally \( c_1 \) and \( c_2 \) belong to \( l_2 \). Therefore there is no loss of generality if we will prove only assertion (a).

To the contrary, assume that \( A \cap (x = \alpha) \neq \emptyset \) and \( A \cap (x = -\alpha) = \emptyset \). In this case, every point \( p \in A \cap (x = \alpha) \) has no symmetric with respect to \( c_0 \) and \( c_2 \) and therefore \( p \) is a removable point of \( A \). This contradicts our assumption that \( A \) is normal set. Lemma 3 is proved.

Let \( A \subseteq \mathbb{Z}^2 \) be a normal set. We will now estimate the number of odd points belonging to a covering polygon \( H(A) \). In view of Definition 3 and Lemma 3, the integers \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) that define the covering lines of \( H(A) \) satisfy

\[
\begin{align*}
\alpha_1 \text{ and } \alpha_2 \text{ are odd,} & \quad \alpha_2 = -\alpha_1 = 2\alpha - 1, \\
\beta_1 \text{ and } \beta_2 \text{ are odd,} & \quad \beta_2 = -\beta_1 = 2\beta - 1, \\
\gamma_1 \text{ and } \gamma_2 \text{ are even,} & \quad \gamma_2 = -\gamma_1 + 2 = 2\gamma.
\end{align*}
\]
It follows that $H(A) = H(a, b, c)$, where $a = 2\alpha, b = 2\beta, c = 2\gamma$. Let us denote by

$$
\epsilon = \epsilon(a, b, c) = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2}.
$$

(13)

We have the following estimate

**Lemma 4.** The set $H(a, b, c)$ lies on $a = 2\alpha$ lines parallel to $(x = 0)$, on $b = 2\beta$ lines parallel to $(y = 0)$, on $c = 2\gamma$ lines parallel to $(x + y = 1)$ and

$$
|H(a, b, c)| = \begin{cases}
    c \min\{a, b\}, & \text{if } c \leq |a - b| \\
    ab - \frac{(a+b-c)^2}{4}, & \text{if } c \geq |a - b| + 2.
\end{cases}
$$

(14)

Moreover,

(a) if $c \leq |a - b|$, then $|H(a, b, c)| \leq \frac{1}{4} \frac{(a+b+c)^2}{4}$.

(b) if $c \geq |a - b| + 2$, then $|H(a, b, c)| \leq \frac{1}{3} \left( \frac{(a+b+c)^2}{4} - \epsilon \right)$.

**Proof.** Every point $P = (x, y) \in H(a, b, c)$ belongs to the rectangle defined by

$$
R(A) : |x| \leq 2\alpha - 1, \quad |y| \leq 2\beta - 1, \quad x \text{ and } y \text{ are odd.}
$$

and thus $H(a, b, c)$ lies on $a = 2\alpha$ lines parallel to $(x = 0)$, on $b = 2\beta$ lines parallel to $(y = 0)$. Moreover, if $P = (x, y)$ is a point of $H(a, b, c)$ lying on the supporting line $(x + y = 2\gamma)$, then $x + y \leq 2\alpha + 2\beta - 2$ and therefore $H(a, b, c)$ lies on $c = 2\gamma \leq 2\alpha + 2\beta - 2 = a + b - 2$ lines parallel to $(x + y = 1)$.

It is enough to examine only the case $a \geq b$.

Case 1. If $2 \leq 2\gamma \leq 2\alpha - 2\beta$, then $2 \leq c \leq a - b$, the set $H(a, b, c)$ is actually a parallelogram and

$$
H(a, b, c) = 2\gamma b = cb = c \min\{a, b\}.
$$

Case 2. If $2\gamma = 2\alpha - 2\beta + 2$, then $c = a - b + 2$. The set $H(a, b, c)$ lies on two parallel lines, if $a = b$, or $H(a, b, c)$ is a pentagon, if $a \neq b$. Therefore

$$
H(a, b, c) = 2\gamma b - 1 - cb - 1 - (a - b + 2)b - 1 = ab - (b - 1)^2 - ab - \frac{(a + b - c)^2}{4}.
$$

Case 3. If $2\alpha - 2\beta + 4 \leq 2\gamma \leq 2\alpha + 2\beta - 4$, then $a - b + 4 \leq c \leq a + b - 4$, the set $H(a, b, c)$ is a hexagon and

$$
H(a, b, c) = ab - \sum_{j=1}^{\alpha + \beta - \gamma - 1} j - \sum_{j=1}^{\alpha + \beta - \gamma} j = ab - (\alpha + \beta - \gamma)^2 = ab - \frac{(a + b - c)^2}{4}.
$$

Case 4. If $2\gamma = 2\alpha + 2\beta - 2$, then $c = a + b - 2$, the set $H(a, b, c)$ satisfies

$$
H(a, b, c) = R(A) \setminus \{v\},
$$
where $v$ is the vertex $v = (-2\alpha + 1, -2\beta + 1)$. Thus

$$H(a, b, c) = ab - 1 = ab - \frac{(a + b - c)^2}{4}.$$  

Equality (14) is proved.

Moreover, in case 1 we have $c \leq a - b$, $a \geq b + c$ and thus

$$|H(a, b, c)| = cb = \frac{(b + c)^2 - (b - c)^2}{4} \leq \frac{1}{4} \left( \left( \frac{a + b + c}{2} \right)^2 - (b - c)^2 \right)$$

$$\leq \frac{1}{4} \left( \frac{a + b + c}{2} \right)^2.$$  

In cases 2, 3 and 4 we have $c \geq a - b + 2$ and thus

$$|H(a, b, c)| = ab - \frac{(a + b - c)^2}{4} = \frac{2ab + 2bc + 2ca - a^2 - b^2 - c^2}{4}$$

$$= \frac{(a + b + c)^2}{12} - \frac{e}{3}.$$  

Lemma 4 is proved.

3. Normal connected sets

In this section we prove Corollary 1 which implies Theorem 2 for connected normal sets. We need the following:

**Definition 4.** Let $A \subseteq \mathbb{Z}^2$ be a finite normal set and let

$$x = \pm(2\alpha - 1), \quad y = \pm(2\beta - 1), \quad x + y - 1 = \pm(2\gamma - 1)$$

denote the supporting lines of the covering polygon $H(A) = H(a, b, c)$. We say that $A$ is a connected normal set if the following three conditions are true:

(a) for every odd integer $p$ such that $|p| \leq 2\alpha - 1$ we have $A \cap (x = p) \neq \emptyset$.
(b) for every odd integer $q$ such that $|q| \leq 2\beta - 1$ we have $A \cap (y = q) \neq \emptyset$.
(c) for every odd integer $r$ such that $|r| \leq 2\gamma - 1$ we have $A \cap (x + y - 1 = r) \neq \emptyset$.

We will use the following result:

**Lemma 5.** Let $A \subseteq \mathbb{Z}^2$ be a connected normal set. If $H(A)$, the covering polygon of $A$, is equal to $H(a, b, c)$, then

$$R_3(A) = |\text{Diff}(A)| \leq 3k - \frac{a + b + c}{2}. \quad (15)$$

**Proof.** See assertion (b) of Lemma 2 in [1].

We can now prove without difficulty the following corollary which describes the structure of a connected normal set $A$ which satisfies $R_3(A) \geq 3k - \sqrt{3.241}k$. This condition is less restrictive than inequality (10) and will be used in Section 5.
Corollary 1. Let \( A \subseteq \mathbb{Z}^2 \) be a connected normal set. Let \( H(A) = H(a, b, c) \) be the covering polygon of \( A \). Denote by

\[
k = |A|, \quad k^* = |H(A)|.
\]

(a) If \( c \leq |a - b| \), then \( R_3(A) \leq 3k - 2\sqrt{k^*} \leq 3k - 2\sqrt{k} \).

(b) If \( c \geq |a - b| + 2 \), then \( R_3(A) \leq 3k - \sqrt{3k^* + \epsilon} \leq 3k - \sqrt{3k} + \epsilon \).

(c) If \( R_3(A) \geq 3k - \sqrt{3.241k} \), then \( |H(A)| < 1.081|A| \). Moreover, if we assume that \( a \leq b \leq c \), then \( a > 0.8\sqrt{k}, b < 1.75a \) and \( c < 0.75(a + b) \).

Proof. We have \( H(A) = H(a, b, c), k \leq k^* \) and we may assume without loss of generality that \( a \leq b \).

Case (a). If \( c \leq b - a \), then assertion (a) of Lemma 4 implies that

\[
\frac{a + b + c}{2} \geq 2\sqrt{|H(A)|} = 2\sqrt{k^*} \geq 2\sqrt{k}.
\]

Using (15), we get

\[
R_3(A) \leq 3k - \frac{a + b + c}{2} \leq 3k - 2\sqrt{k^*} \leq 3k - 2\sqrt{k}.
\]

Case (b). If \( c \geq b - a + 2 \), then assertion (b) of Lemma 4 implies that

\[
\frac{a + b + c}{2} \geq \sqrt{3k^* + \epsilon} \geq \sqrt{3k} + \epsilon.
\]

Using (15), we get

\[
R_3(A) \leq 3k - \frac{a + b + c}{2} \leq 3k - \sqrt{3k^* + \epsilon} \leq 3k - \sqrt{3k} + \epsilon.
\]

We prove now assertion (c). Let us assume that the set \( A \) satisfies the inequality

\[
R_3(A) \geq 3k - \sqrt{3.241k}.
\]

Using Corollary 1 (a) and inequalities (5) and (15) we get that

\[
2 + |a - b| \leq c \leq a + b - 2
\]

and

\[
3k - \sqrt{3.241k} \leq R_3(A) \leq 3k - \frac{a + b + c}{2} \leq 3k - \sqrt{3k^* + \epsilon} \leq 3k - \sqrt{3k} + \epsilon
\]

Therefore \( 3k^* + \epsilon \leq 3.241k, \sqrt{3k} \leq \frac{a + b + c}{2} \leq 3k - R_3, \epsilon \leq (3k - R_3)^2 - 3k \) and thus

\[
|H(A)| < 1.081|A| - \frac{\epsilon}{3}, \quad (16)
\]

\[
3.464\sqrt{k} \leq a + b + c \leq 2\sqrt{3.241k},
\]

\[
2\epsilon = (a - b)^2 + (b - c)^2 + (c - a)^2 \leq 0.482k. \quad (17)
\]
We may assume without loss of generality that

\[ a \leq b \leq c. \]

Denote \( b = a + u \) and \( c = b + v \). Inequality (17) imply that \( u^2 + v^2 + (u + v)^2 \leq 0.482k \). Thus \( u^2 \leq 0.241k, \ v^2 \leq 0.241k, \ (u + v)^2 \leq 0.322k \). Therefore

\[
\begin{align*}
 u & \leq 0.491\sqrt{k}, \quad v \leq 0.491\sqrt{k}, \quad u + v \leq 0.568\sqrt{k}, \\
3.464\sqrt{k} & \leq a + b + c = 3a + u + (u + v) \leq 3a + 1.059\sqrt{k}, \\
a & \geq \frac{1}{3} \cdot 2.405\sqrt{k} \geq 0.801\sqrt{k}.
\end{align*}
\]

Moreover, the quotient \( \frac{b}{a} \) is less than 1.75 because \( 2\sqrt{3.241k} \geq a + b + c \geq a + 2b = a(1 + 2\frac{b}{a}) \) implies that

\[
\frac{b}{a} \leq \frac{1}{2} \left( \frac{2\sqrt{3.241k}}{a} - 1 \right) \leq \frac{1}{2} \left( \frac{2\sqrt{3.241k}}{0.801\sqrt{k}} - 1 \right) \leq 1.748.
\]

In order to prove assertion (c), it remains to be shown that \( t = \frac{c}{a+b} \leq 0.75 \). We have

\[
2\sqrt{3.241k} \geq a + b + c = (1 + t)(a + b) \geq 2(1 + t)\sqrt{ab},
\]

\[
k \leq ab - \left( \frac{a + b - c}{2} \right)^2 = ab - \left( \frac{(1 - t)(a + b)}{2} \right)^2
\]

and thus

\[
2\sqrt{3.241k} \geq 2(1 + t)\sqrt{k + \left( \frac{(1 - t)(a + b)}{2} \right)^2}.
\]

Clearly \( \sqrt{3.241k} \geq (1 + t)\sqrt{k} \) and thus \( t \leq 0.8003 \). This last estimate can be slightly improved using the inequalities \( a + b \geq 2\sqrt{ab} \geq 2\sqrt{k} \). Indeed, we obtain

\[
2\sqrt{3.241k} \geq 2(1 + t)\sqrt{k + (1 - t)^2k}, \quad 3.241 \geq (1 + t)^2 + (1 - t)^2
\]

and so \( t^4 - t^2 + 2t \leq 1.241 \). Using \( 0 \leq t \leq 1 \) we get \( t < 0.75 \). Corollary 1 is proved.

4. Disconnected normal sets

**Definition 5.** Let \( A \subseteq \mathbb{Z}^2 \) be a finite normal set and let

\[
\begin{align*}
x &= 2\alpha - 1, \quad x = -2\alpha + 1, \quad y = 2\beta - 1, \quad y = -2\beta + 1, \\
x + y &= 2\gamma, \quad x + y = -2\gamma + 2
\end{align*}
\]

denote the supporting lines of the covering polygon \( H = H(A) \). We say that \( A \) is a disconnected normal set if it is normal and at least one of the assertion (a), (b), (c) of Definition 4 is not true.
As we remarked before, this means that the set $A$ is normal and at least one of the following three conditions is true:

(a) there is an odd integer $u$ such that $-2\alpha + 1 \leq u \leq 2\alpha - 1$ and $A \cap (x = \pm u) = \emptyset$.
(b) there is an odd integer $v$ such that $-2\beta + 1 \leq v \leq 2\beta - 1$ and $A \cap (y = \pm v) = \emptyset$.
(c) there is an even integer $w$ such that $-2\gamma + 2 \leq w \leq 2\gamma$ and $A \cap (x + y = \pm w) = \emptyset$.

We will examine now such a set $K \subset \mathbb{Z}^2$ for which only condition (c) is satisfied.

**Example 1.** Let $t \in \mathbb{Z}$ be a positive integer. Let us define

$$K(t) = H_t \pm (2t, 2t).$$

![Figure 4.1: The set $K(t)$ for $t = 3$. $K(t)$ is included in $(2\mathbb{Z} + 1) \times (2\mathbb{Z} + 1)$.](image)

The set $K(t)$ is described in Figure 4.1 and is defined by the following conditions: a point $(x, y)$ belongs to $K(t)$ if and only if:

(i) $1 \leq x, y \leq 4t - 1, 2t + 2 \leq x + y \leq 6t$ and $x$ and $y$ are both odd integers.

or

(ii) $-4t + 1 \leq x, y \leq -1, -6t + 2 \leq x + y \leq -2t$ and $x$ and $y$ are both odd integers.
Lemma 6. The set $K = K(t)$ satisfies $k = |K| = 6t^2$ and

$$R_3(K) = 3k - \frac{a + b + c}{2} = 3k - 6t = 3k - \sqrt{6k}. \quad (18)$$

Proof. The set $K(t)$ consists of two disjoint translates of $H_t$ and thus

$$k = |K(t)| = 2|H_t| = 6t^2.$$

Using the properties of the set $H_\alpha$ it follows that $K(t)$ lies on $a = 4t$ lines parallel to $e_2$, $b = 4t$ lines parallel to $e_1$ and $c = 4t$ lines parallel to $e_1 - e_2$. Each line $(x - x_0), x_0$ odd, $-4t + 1 \leq x_0 \leq 4t - 1$ intersects the set $K$. Each line $(y = y_0)$, $y_0$ odd, $-4t + 1 \leq y_0 \leq 4t - 1$ intersects the set $K$. Nevertheless, the lines $(x + y = s)$, $s$ even, $-2t + 2 \leq s \leq 2t$ does not intersect $K$. It follows that only condition (c) of Definition 4 is satisfied. Moreover, the three centers of symmetry of $K$ are $c_i = e_i$, for $i = 0, 1, 2$, $K$ is a normal set and we clearly have:

$$d_0 = |D_0(K)| = |\{ p \in K : p_0 = 2c_0 - p \in K \}|$$
$$= k - |K \cap ((x + t = 6t) \cup (x + y = -2t))|,$$

$$d_1 = |D_1(K)| = |\{ p \in K : p_1 = 2c_1 - p \in K \}|$$
$$= k - |K \cap ((x = 1) \cup (x = -4t + 1))|,$$

$$d_2 = |D_2(K)| = |\{ p \in K : p_2 = 2c_2 - p \in K \}|$$
$$= k - |K \cap ((y = 1) \cup (y = -4t + 1))|.$$

We conclude that $K$ is a disconnected normal set and

$$R_3(K) = d_0 + d_1 + d_2 = (k - 2t) + (k - 2t) + (k - 2t) = 3k - 6t = 3k - \sqrt{6k}. \quad \blacksquare$$

We will now examine in detail a normal disconnected set satisfying case (a). Cases (b) and (c) are similar. The following result generalizes inequality (18):

Lemma 7. Assume that the set $A$ is a normal disconnected set satisfying condition (a). Let us choose $u \geq 1$ minimal such that $u$ is odd and

$$A \cap (x = \pm u) = \emptyset.$$

Define $A_1 = A \cap (-u < x < u)$, $A_2 = A \setminus A_1$, $k_1 = |A_1|$, $k_2 = k - k_1$. Then

$$R_3(A) = R_3(A_1) + R_3(A_2) \leq 3k - \sqrt{3k_1} - \sqrt{6(k_2 - n_0 - 0.5)}, \quad (19)$$

where $n_0$ is the number of points $p \in A_2$ such that $p_0 = 2c_0 - p \notin A_2$.

Proof. We will first show that the subset $A_2$ satisfies an inequality similar to (18). More precisely, we have

$$R_3(A_2) \leq 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}. \quad (20)$$
The set $A_2$ is a disjoint union of

$$A_+ = A \cap (x > u)$$

and

$$A_- = A \cap (x < -u).$$

Denote by $\pi_1(x, y) = x$ the projection parallel to line $(x = 0)$, by $\pi_2(x, y) = y$ the projection parallel to line $(y = 0)$ and by $\pi_3(x, y) = x + y$ the projection parallel to line $(x + y = 0)$. We claim that there is an integral vector $w \in \mathbb{N}^2$ such that the sets

$$B_+ = A_+ + w \quad \text{and} \quad B_- = A_- - w$$

satisfy the following assertions:

(i) $B_+$ and $B_-$ are disjoint,
(ii) the projections $\pi_i(B_+)$ and $\pi_i(B_-)$ are disjoint, for $i = 1, 2, 3$,
(iii) the set $B = B_+ \cup B_-$ satisfies $R_3(A_2) \leq R_3(B)$.

If both coordinates of $w$ are large enough, then assertions (i) and (ii) are clearly true. Let us explain now (iii). Each difference $d = (d_1, d_2) \in \text{Diff}(A)$ can be written as $d = p - p'$, where $p + p' = 2c_i = 2e_i$ and $p, p' \in A$. Therefore, we have either

$$p \in A_+, \quad p' \in A_-, \quad d_1 \geq 2(u + 2) \geq 6$$

or

$$p \in A_-, \quad p' \in A_+, \quad d_1 \leq -2(u + 2) \leq -6.$$ 

This remark allows us to define a one to one map $\varphi$ from

$$\text{Diff}(A_2) = D_0(A_2) \cup D_1(A_2) \cup D_2(A_2)$$

to

$$\text{Diff}(B) = D_0(B) \cup D_1(B) \cup D_2(B).$$

More precisely, if $p_i = 2e_i - p$ denotes the symmetric of $p$ with respect to $e_i$, then $\varphi$ is given by

$$\varphi(d) = \begin{cases} 
  d + 2w, & \text{if } d = p - p_i, \ p \in A_+, \ p_i \in A_-; \\
  d - 2w, & \text{if } d = p - p_i, \ p \in A_-, \ p_i \in A_+.
\end{cases}$$

The image $\varphi(d) \in \text{Diff}(B)$; indeed, if $d = p - p_i, p \in A_+, p_i \in A_-$, then

$$d + 2w = p - p_i + 2w = (p + w) - (p_i - w),$$

$$p + w \in B_+ \subseteq B, \quad p_i - w \in B_- \subseteq B,$$

$$(p + w) + (p_i - w) = p + p_i = 2c_i = 2e_i.$$
and if \( d = p - p_i, \ p \in A_-, \ p_i \in A_+ \), then
\[
d - 2w = p - p_i - 2w = (p - w) - (p_i + w),
\]
\[
 p - w \in B_- \subseteq B, \quad p_i + w \in B_+ \subseteq B,
\]
\[
 (p - w) + (p_i + w) = p + p_i = 2c_i = 2c_i.
\]

Moreover, we may choose the vector \( w \) such that \( d' + 2w \neq d'' - 2w \), for every \( d', d'', d'' \in \text{Diff}(A_2) \). This implies that \( \varphi \) is one to one and assertion (iii) follows.

Assume that the set \( B_+ \) lies on exactly \( a_1 \) lines parallel to the line \((x = 0)\), on \( b_1 \) lines parallel to the line \((y = 0)\) and on \( c_1 \) lines parallel to the line \((x + y = 0)\). In other words:
\[
a_1 = |\pi_1(B_+)|, \quad b_1 = |\pi_2(B_+)|, \quad c_1 = |\pi_3(B_+)|.
\]
The set \( B_- \) determines the parameters \( a_2, b_2 \) and \( c_2 \) in a similar way, i.e.
\[
a_2 = |\pi_1(B_-)|, \quad b_2 = |\pi_2(B_-)|, \quad c_2 = |\pi_3(B_-)|.
\]
Therefore, property (ii) implies that the set \( B \) lies on exactly \( a_1 + a_2 \) lines parallel to the line \((x = 0)\), on \( b_1 + b_2 \) lines parallel to the line \((y = 0)\) and on \( c_1 + c_2 \) lines parallel to the line \((x + y = 0)\). Using Lemma 2.b. and Corollary 1 from [1] we get
\[
R_3(B) \leq 3|B| - \frac{(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2)}{2}
- 3|B_+| - \frac{a_1 + b_1 + c_1}{2} + 3|B_-| - \frac{a_2 + b_2 + c_2}{2}
\leq 3|B_+| - \sqrt{3(|B_+| - 0.25)} + 3|B_-| - \sqrt{3(|B_-| - 0.25)}.
\]
Let us estimate the cardinalities of the sets \( B_+ \) and \( B_- \) using the fact that \( A, A_2 \) and \( B \) are all "almost symmetric" with respect to \( c_0 \). Let us recall that \( n_0 \) denotes the number of points \( p \in A_2 \) such that \( p_0 = 2c_0 - p \notin A_2 \); therefore we get
\[
n_0 = |\{p : p \in B, p_0 \notin B\}| \leq |B| = |A_2| = k_2
\]
and
\[
|B_+| = |A_+| \geq \frac{|B| - n_0}{2}, \quad |B_-| = |A_-| \geq \frac{|B| - n_0}{2};
\]
inequality (20) follows from:
\[
R_3(A_2) \leq R_3(B) \leq 3|B| - \sqrt{3(|B_+| - 0.25)} - \sqrt{3(|B_-| - 0.25)}
\leq 3|B| - 2\sqrt{3\left(\frac{|B| - n_0}{2} - 0.25\right)} = 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}.
\]
We will show that inequality (19) is true. The set $A$ is a disjoint union of $A_1$ and $A_2$. Using Corollary 1 from [1] we get $R_3(A_1) \leq 3k_1 - \sqrt{3k_1}$. For every $i = 0, 1, 2$ the sets $D_i(A_1)$ and $D_i(A_2)$ are disjoint and thus

$$R_3(A) = R_3(A_1) + R_3(A_2) \leq 3k_1 - \sqrt{3k_1} + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}$$

$$= 3k - \sqrt{3k_1} - \sqrt{6(k_2 - n_0 - 0.5)}.$$

Lemma 7 is proved.

5. The general case and proof of Theorem 2

Assume that $A$ is a finite set that satisfies the hypothesis of Theorem 2. Let $A_0$ be the set of all essential points of $A$. Using inequality (11) or in view of Lemma 2 we have

$$k_0 = |A_0|, \quad 0 \leq k - k_0 \leq 2\sqrt{k}, \quad R_3(A_0) \geq 3k_0 - \theta \sqrt{k_0}. \quad (21)$$

$A_0$ is a finite normal set. If $A_0$ is connected we apply Corollary 1 and Theorem 2 is proved. Assume that $A_0$ is disconnected. In what follows, we will apply three times Lemma 7 in order to obtain a large normal connected proper subset $A_5 \subset A_0$. Let us choose $u \geq 1$ minimal such that $u$ is odd and

$$A_0 \cap (x = \pm u) = \emptyset.$$

Define $A_1 = A_0 \cap (-u < x < u), A_2 = A_0 \setminus A_1, k_1 = |A_1|, k_2 = k_0 - k_1$. The sets $A_1$ and $A_2$ form a partition of $A_0$ and in view of Lemma 7 we have

$$R_3(A_0) = R_3(A_1) + R_3(A_2) \leq R_3(A_1) + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}, \quad (22)$$

where $n_0$ is the number of points $p \in A_2$ such that $p = 2c_0, -p \notin A_2$.

Let us choose $v \geq 1$ minimal such that $v$ is odd and

$$A_1 \cap (y = \pm v) = \emptyset.$$

Define $A_3 = A_1 \cap (-v < y < v), A_4 = A_1 \setminus A_3, k_3 = |A_3|, k_4 = k_1 - k_3$. The sets $A_3$ and $A_4$ form a partition of $A_1$ and using a similar argument as in the proof of Lemma 7, we get

$$R_3(A_1) = R_3(A_3) + R_3(A_4) \leq R_3(A_3) + 3k_4 - \sqrt{6(k_4 - n_1 - 0.5)}, \quad (23)$$

where $n_1$ is the number of points $p \in A_4$ such that $p = 2c_0 - p \notin A_4$.

Let us choose $w \geq 1$ minimal such that $w$ is odd and

$$A_3 \cap (x + y - 1 = \pm w) = \emptyset.$$

Define $A_5 = A_3 \cap (-w < x + y - 1 < w), A_6 = A_3 \setminus A_5, k_5 = |A_5|, k_6 = k_3 - k_5$. The sets $A_5$ and $A_6$ form a partition of $A_3$ and using a similar argument as in the proof of Lemma 7, we get

$$R_3(A_3) = R_3(A_5) + R_3(A_6) \leq R_3(A_5) + 3k_6 - \sqrt{6(k_6 - n_2 - 0.5)}, \quad (24)$$
where \( n_2 \) is the number of points \( p \in A_0 \) such that \( p_0 = 2c_0 - p \notin A_6 \). In view of (22), (23), (24) and using \( k_0 = k_5 + k_2 + k_4 + k_6 \) and \( R_3(A_5) \leq 3k_5 - \sqrt{3k_5} \) we get:

\[
R_3(A_0) \leq R_3(A_1) + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\
\leq R_3(A_3) + 3k_4 - \sqrt{6(k_4 - n_1 - 0.5)} + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\
\leq R_3(A_5) + 3k_6 - \sqrt{6(k_6 - n_2 - 0.5)} + 3k_4 - \sqrt{6(k_4 - n_1 - 0.5)} \\
+ 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\
\leq R_3(A_5) + 3(k_0 - k_5) - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\
\leq 3k_0 - \sqrt{3k_5} - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\
\leq 3k_0 - \sqrt{3k_5 + 6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9}.
\]

Inequality (21) gives a lower bound for \( R_3(A_0) \) and implies that

\[
3k_5 + 6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9 \\
= 3k_0 + 3(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9 \leq \theta^2 k_0 \leq 3.24k_0.
\]

Thus

\[
k_0 - k_5 \leq 0.08k_0 + 2(n_0 + n_1 + n_2) + 3 \\
\leq 0.08k_0 + 6m_0 + 3 \leq 0.08k_0 + 10.8\sqrt{k_0} + 3, \\
k_5 \geq 0.92k_0 - 10.8\sqrt{k_0} - 3.
\]

We applied here (8) and the obvious inequality \( n_i \leq m_0, i = 0, 1, 2 \).

We claim that the set \( A_5 \) satisfies an inequality similar to (7), namely

\[
R_3(A_5) \geq 3k_5 - \sqrt{3.241k_5}.
\]

Indeed, assume to the contrary that \( R_3(A_5) < 3k_5 - \sqrt{3.241k_5} \). Using (25) we get

\[
R_3(A_0) \leq R_3(A_5) + 3(k_0 - k_5) - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\
< 3k_5 - \sqrt{3.241k_5} + 3(k_0 - k_5) - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\
\leq 3k_0 - \sqrt{3.241k_5 + 6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\
\leq 3k_0 - \sqrt{3.241k_0 - 6(n_0 + n_1 + n_2) - 9} \\
\leq 3k_0 - \sqrt{3.241k_0 - 10.8\sqrt{k_0} - 9},
\]

which contradicts inequality (21), if \( k = |A| \) is sufficiently large.

Choose a proper subset \( A_5 \subset A_0 \) such that (26) and (27) are true and \( k_5 = |A_5| \) is minimal. The choice of \( u, v, w \) and the minimality of \( k_5 \) imply that \( A_5 \) is normal and connected. Let

\[
H(A_5) : \begin{cases} 
-2\alpha + 1 \leq x \leq 2\alpha - 1, & x \text{ odd,} \\
-2\beta + 1 \leq y \leq 2\beta - 1, & y \text{ odd,} \\
-2\gamma + 2 \leq x + y \leq 2\gamma
\end{cases}
\]

(28)
be the covering polygon of \( A_5 \). Then \( H(A_5) \) lies on \( a = 2\alpha \) lines parallel to \( (x = 0) \), on \( b = 2\beta \) lines parallel to \( (y = 0) \), on \( c = 2\gamma \) lines parallel to \( (x + y = 1) \) and \( 2 \leq c \leq a + b - 2 \). We will use now inequality (27) and assertion (c) of Corollary 1. We may assume without loss of generality that \( a \leq b \leq c \). We get that

\[
|H(A_5)| < 1.081|A_5|, \quad a > 0.8\sqrt{k_5}, \quad b < 1.75a \quad \text{and} \quad c < 0.75(a + b).
\]

Define \( A^* = A_5 \) and \( H(a, b, c) = H(A_5) \). Using (21) and (26), we conclude that

\[
k - k_5 = (k - k_0) + (k_0 - k_5) \leq 2\sqrt{k} + 0.08k_0 + 10.8\sqrt{k_0} + 3
\]

\[
\leq 0.08k + 12.8\sqrt{k} + 3
\]

and thus \( |A^*| = |A_5| = k_5 \geq 0.92k - 12.8\sqrt{k} - 3 \). Theorem 2 is proved, if \( k \) is sufficiently large. ■

6. Remarks

We use now the notations of Section 1 for finite sets of integers. It is a natural question whether it is possible to describe the structure of sets of integers \( A \subseteq \mathbb{Z} \) such that \( R_3(A) \geq 3k - 1.8\sqrt{k} \).

We propose the following:

**Conjecture.** Let \( A \subseteq \mathbb{Z} \) be a finite set of integers. Assume that \( |A| = k \) and

\[
R_3(A) = |\text{Diff}(A)| \geq 3k - 1.8\sqrt{k}.
\]

Then there is a two dimensional set of odd lattice points on the plane \( \tilde{A} \subseteq \mathbb{Z}^2 \) with the following properties:

(a) \( |\tilde{A}| = |A| = k \),

(b) \( 3k - 1.8\sqrt{k} \leq R_3(A) \leq R_3(\tilde{A}) \leq 3k - \sqrt{3k} \),

(c) the canonical projection \( \pi: \tilde{A} \to \mathbb{Z}, \pi(x, y) = x \) has the image \( \pi(\tilde{A}) = A \).

Inequality (29) for integers is similar to condition (7) for sets of lattice points in the plane and in a subsequent paper we will show that it is possible to apply Theorem 2 in order to study the structure of such sets of integers.

References


Address: Gregory A. Freiman: School of Mathematical sciences, Tel Aviv University, Tel Aviv 69978, Israel; Yonutz V. Stanchescu: The Open University of Israel, Raanana 43107, Israel and Afeka Academic College, Tel Aviv 69107, Israel.

E-mail: ionut@openu.ac.il, yonit@afeka.ac.il

Received: 22 January 2009; revised: 20 April 2009