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ON A KAKEYA-TYPE PROBLEM II

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Abstract: Let A be a finite subset of an abelian group G. For every element b_i of the sumset $2A = \{b_0, b_1, ..., b_{|2A|-1}\}$ we denote by $D_i = \{a - a' : a, a' \in A; a + a' = b_i\}$ and $r_i = |\{(a, a') : a + a' = b_i; a, a' \in A\}|$. After an eventual reordering of $2A$, we may assume that $r_0 \geqslant r_1 \geqslant ... \geqslant r_{|2A|-1}$. For every $1 \leqslant s \leqslant |2A|$ we define $R_s(A) = |D_0 \cup D_1 \cup ... \cup D_{s-1}|$ and $R_s(k) = \max\{R_s(A) : A \subseteq G, |A| = k\}$. Bourgain and Katz and Tao obtained an estimate of $R_s(k)$ assuming s being of order k. In this paper we describe the *structure* of A assuming that $G = \mathbb{Z}^2$, $s = 3$ and $R_3(A)$ is close to its maximal value, i.e. $R_3(A) = 3k - \theta\sqrt{k}$, with $\theta \leqslant 1.8$.

Keywords: Inverse additive number theory, Kakeya problem.

1. Introduction

Let A be a finite subset of the group $G = \mathbb{Z}$ or $G = \mathbb{Z}^2$. For every element b_i of the sumset $2A = A + A = \{x + x' : x \in A, x' \in A\} = \{b_0, b_1, b_2, ..., b_{|2A|-1}\}$ we denote

$$D_i = \{a - a' : a \in A, a' \in A, a + a' = b_i\}, \quad d_i = |D_i|, \tag{1}$$
$$r_i = r_i(A) = |\{(a, a') : a + a' = b_i, a \in A, a' \in A\}|. \tag{2}$$

After an eventual reordering of the set $2A$, we may assume that $r_0 \geqslant r_1 \geqslant ... \geqslant r_{|2A|-1}$. We denote

$$c_i = \frac{b_i}{2}, \quad C = \{c_0, c_1, c_2\}, \quad \mathrm{Diff}(A) = D_0 \cup D_1 \cup D_2,$$
$$R_3(A) = |\mathrm{Diff}(A)| = |D_0 \cup D_1 \cup D_2|,$$
$$R_3(k) = \max\{R_3(A) : A \subseteq G, |A| = k\}.$$

In the paper [1], we determined the maximal value of $|\mathrm{Diff}(A)|$ for finite sets $A \subseteq \mathbb{Z}^2$, assuming that b_0, b_1, b_2 are non-collinear. We also described the structure

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of planar *extremal sets* A^*, i.e. sets of integer lattice points on the plane $\mathbb{Z}^2$ for which we have

$$R_3(A^*) = R_3(k) = 3k - \sqrt{3k}. \tag{3}$$

More precisely, for every $\alpha \in \mathbb{N}$ we denote by H_α the set of all points $P = (x, y) \in \mathbb{Z}^2$ such that x and y are odd integers and $-2\alpha < x, y, x+y-1 < 2\alpha$. We proved the following result (see [1], Section 3):

Theorem 1. *Let A be a finite subset of $\mathbb{Z}^2$, $|A| = k$. Then*

$$R_3(A) = |\mathrm{Diff}(A)| \leqslant 3k - \sqrt{3k}. \tag{4}$$

Moreover, the equality $R_3(A) = 3k - \sqrt{3k}$ holds if and only if $k = 3\alpha^2$ and there is an affine isomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A = \phi(H_\alpha)$.

Note that H_α, the canonical form of an extremal set, contains only *odd lattice points* (x, y) (i.e. both coordinates x and y are odd integers), its convex hull is a hexagon and the set H_α lies on 2α lines parallel to the line $y = 0$, on 2α lines parallel to the line $x = 0$ and on 2α lines parallel to the line $x + y = 1$ (see Figure 1.1). Moreover, H_α satisfies equality (3) with respect to the centers c_0, c_1, c_2 given by $e_0 = (0,0), e_1 = (1,0), e_2 = (0,1)$, respectively.

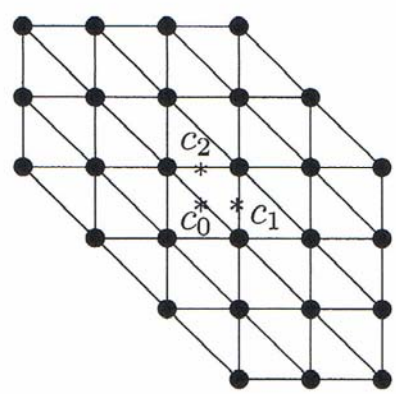

Figure 1.1: The set H_α and the centers $c_i = e_i$, $i = 0, 1, 2$.

In this paper we continue the study of such finite sets and we will determine the structure of sets of odd lattice points on the plane for which $c_i = e_i, i = 0, 1, 2$ and the number of differences $R_3(A)$ is close to its maximal value (3). In order to formulate our main result we will use the following notation. If $u = (u_1, u_2) \in \mathbb{R}^2$, we denote by u_1 and u_2 its coordinates with respect to the canonical basis $e_1 = (1,0), e_2 = (0,1)$ and $e_0 = (0,0)$ represents the origin point. Let $a = 2\alpha$, $b = 2\beta$ and $c = 2\gamma$ be three natural numbers such that

$$2 \leqslant c \leqslant a + b - 2. \tag{5}$$

We denote by $H(a,b,c)$ the set of all points $P=(x,y)\in\mathbb{Z}^2$ which satisfy the following conditions:

$$H(a,b,c):\begin{cases}-2\alpha+1\leqslant x\leqslant 2\alpha-1, & x \text{ odd},\\ -2\beta+1\leqslant y\leqslant 2\beta-1, & y \text{ odd},\\ -2\gamma+1\leqslant x+y-1\leqslant 2\gamma-1. & \end{cases}\tag{6}$$

Note that if $a=b=c=2\alpha$, then $H(a,b,c)$ is the perfect hexagon H_α described in Figure 1.1.

We will prove that if $c_i=\frac{b_i}{2}=e_i$, for $i=0,1,2$ and if $|\mathrm{Diff}(A)|\geqslant 3k-1.8\sqrt{k}$, then A is almost hexagonal, i.e. an essential part of the set A can be approximated by a hexagon similar to the extremal set H_α. A precise formulation is given in the following:

Definition 1. *We say that $A\subseteq\mathbb{Z}^2$ is an almost hexagonal set if there is a subset $A^*\subseteq A$ and a hexagon $H(a,b,c)$ which satisfy the conditions:*

1. $|A^*|\geqslant 0.91|A|$,
2. *A^* is included in $H(a,b,c)$ and $|H(a,b,c)|\leqslant 1.081|A^*|$,*
3. *if $a\leqslant b\leqslant c$, then $a>0.8\sqrt{|A^*|}$, $b<1.75a$, $c<0.75(a+b)$.*

Using the above notations, we can state now our main result:

Theorem 2. *Let $A\subseteq\mathbb{Z}^2$ be a finite subset of odd lattice points on the plane. Assume that $|A|=k$ is sufficiently large and $c_i=e_i$, for $i=0,1,2$. If*

$$R_3(A)=|\mathrm{Diff}(A)|=3k-\theta\sqrt{k},\qquad \theta\leqslant 1.8,\tag{7}$$

then the set A is almost hexagonal.

We prove Theorem 2 in Sections 2-5. Actually, we will prove a more precise estimate (16). In Section 3 we prove Theorem 2 for *connected* sets and in Section 5 we complete the proof using properties of *disconnected* sets obtained in Section 4. In Section 6 we will discuss some directions for further research.

We complete the introduction by recalling some simple remarks from [1]. We will use them whenever necessary without further mention. We easily see that $d_i=r_i$, for every $0\leqslant i\leqslant|2A|-1$. Indeed, using (1) and (2) we get that for two pairs (a_1,a_1') and (a_2,a_2') of $A\times A$ such that $a_1+a_1'=a_2+a_2'=b_i$ we have $a_1-a_1'=a_2-a_2'$ if and only if the equality $(a_1,a_1')=(a_2,a_2')$ holds.

Moreover, using (1), we see that d_i is equal to the number of pairs (a,a') such that $a\in A$, $a'\in A$ and a and a' are symmetric with respect to the center $c_i=\frac{b_i}{2}$, i.e.

$$d_i=|D_{c_i}|,\qquad \text{where}\quad D_{c_i}=\{(a,a'):a\in A,a'\in A,a+a'=2c_i\}.$$

We also note that if $a\neq a'$ then the pairs (a,a') and (a',a) give two distinct differences

$$a-a'=a-(b_i-a)=2a-b_i\qquad\text{and}\qquad a'-a=-(2a-b_i)$$

and if $a = a'$ we have one pair (a, a) and one difference $d = a - a = 0$. We have

$$R_3(A) = |\mathrm{Diff}(A)| = 3k - \theta\sqrt{k} = |D_0(A) \cup D_1(A) \cup D_2(A)| \\ \leqslant |D_0(A)| + |D_1(A)| + |D_2(A)| \leqslant d_i + 2k$$

and thus

$$d_i \geqslant R_3(A) - 2k = k - \theta\sqrt{k},$$

for every $0 \leqslant i \leqslant 2$. Let us denote by

$$p_i = 2c_i - p$$

the symmetric of p with respect to c_i. Denote by M_i the set of points $p \in A$ such that $p_i \notin A$. If $m_i = |M_i|$, then $d_i = |D_i(A)| = k - m_i$ and thus

$$m_i = k - d_i \leqslant k - (R_3(A) - 2k) = \theta\sqrt{k}. \tag{8}$$

In other words, Theorem 2 describes the structure of sets of lattice points that are "almost" symmetric with respect to some set C of centers of symmetry. This is a natural question to be studied in geometry and in inverse additive number theory.

2. Normal sets and Covering Hexagons

We will prove first several simple remarks.

Lemma 1. *Assume that there is a point $p \in A$ such that $p_1 = 2c_1 - p$ and $p_2 = 2c_2 - p$ don't belong to A. If*

$$A' = A \setminus \{p\}$$

is the set obtained from A by removing the point p, then

$$R_3(A') \geqslant R_3(A) - 2.$$

Proof. Assumptions $p_1 = 2c_1 - p \notin A$ and $p_2 = 2c_2 - p \notin A$ imply that the differences

$$d_1 = \pm(p - p_1), \qquad d_2 = \pm(p - p_2)$$

do not belong to $D_1(A)$ and $D_2(A)$, respectively. Therefore the removal of p from the set A reduces the cardinality of $\mathrm{Diff}(A)$ by maximum two differences:

$$d_0 = \pm(p - p_0).$$

We conclude that

$$D_0(A') \geqslant D_0(A) - 2, D_1(A') = D_1(A), D_2(A') = D_2(A),$$

which implies $R_3(A') = |\mathrm{Diff}(A')| \geqslant |\mathrm{Diff}(A)| - 2 = R_3(A) - 2$. ■

Definition 2. *If a point $p \in A$ satisfies the condition*

$$|\{p_0, p_1, p_2\} \cap A| \leqslant 1, \tag{9}$$

i.e. at least two symmetric points of p with respect to $\{c_0, c_1, c_2\}$ do not belong to A, then we will say that p is a removable point of A. If the point p doesn't satisfy the condition (9), *then we will say that p is an essential point of A.*

Assume that A satisfies inequality (7). In the following Lemma we will estimate the number of removable points of A and we will show that the subset A_0 of A consisting of all essential points of A has the same property (7).

Lemma 2. *Let A be a finite subset of $\mathbb{Z}^2$, $|A| = k$. Assume that*

$$R_3(A) = |\mathrm{Diff}(A)| = 3k - \theta\sqrt{k}, \qquad \theta \leqslant 1.8. \tag{10}$$

Let A_0 be the set of all essential points of A and let $A \setminus A_0$ be the set of removable points of A.

(a) If $k_0 = |A_0|$, then $R_3(A_0) \geqslant 3k_0 - \theta\sqrt{k_0}$.
(b) If $n = |A \setminus A_0|$, then $n \leqslant (\theta - 1.73)\sqrt{k} \leqslant 0.07\sqrt{k}$, if k is sufficiently large.

Proof. If $n = |A \setminus A_0| = k - k_0$ denotes the number of removable points of A, then Lemma 1 implies that

$$\begin{aligned} R_3(A_0) &\geqslant R_3(A) - 2n \geqslant 3k - \theta\sqrt{k} - 2n \\ &= 3(k-n) - \theta\sqrt{k-n} + n - \theta(\sqrt{k} - \sqrt{k-n}) \\ &= 3k_0 - \theta\sqrt{k_0} + n\left(1 - \frac{\theta}{\sqrt{k} + \sqrt{k-n}}\right) \\ &\geqslant 3k_0 - \theta\sqrt{k_0}, \end{aligned}$$

in view of $k \geqslant 4 \geqslant \theta^2$. Assertion (a) is proved. We will now estimate the number of removable points of A. We first note that

$$3k - \theta\sqrt{k} \leqslant R_3(A) \leqslant R_3(A_0) + 2n \leqslant 3|A_0| + 2n = 3(k-n) + 2n = 3k - n$$

and thus

$$n = k - k_0 \leqslant \theta\sqrt{k} \leqslant 2\sqrt{k}. \tag{11}$$

This estimate can be improved by using inequality (4) for the set A_0. Indeed, we have

$$R_3(A_0) \leqslant 3|A_0| - \sqrt{3|A_0|} = 3(k-n) - \sqrt{3(k-n)}$$

and inequality

$$3k - \theta\sqrt{k} \leqslant R_3(A) \leqslant R_3(A_0) + 2n \leqslant 3(k-n) - \sqrt{3(k-n)} + 2n$$

clearly implies

$$n \leqslant \theta\sqrt{k} - \sqrt{3(k-n)} \leqslant \theta\sqrt{k} - \sqrt{3}\sqrt{k - 2\sqrt{k}} \leqslant (\theta - 1.73)\sqrt{k} \leqslant 0.07\sqrt{k},$$

if k is sufficiently large. Assertion (b) is proved. ∎

Lemma 2 allows us to study planar sets A consisting only of essential points.

Definition 3. *We say that $A \subseteq \mathbb{Z}^2$ is normal set (with respect to the centers $c_0 = e_0, c_1 = e_1, c_2 = e_2$) if*

(i) *every point of A is an essential point and*
(ii) *every point $p = (x, y) \in A$ has both coordinates x and y odd integers.*

Let us choose six integers $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ such that:

(i) *every point $p = (x, y) \in A$ satisfies the inequalities*

$$H = H(A) : \begin{cases} \alpha_1 \leqslant x \leqslant \alpha_2, & x \text{ odd}, \\ \beta_1 \leqslant y \leqslant \beta_2, & y \text{ odd}, \\ \gamma_1 \leqslant x + y \leqslant \gamma_2. \end{cases}$$

(ii) *on each line $(x = \alpha_1), (x = \alpha_2), (y = \beta_1), (y = \beta_2), (x + y = \gamma_1), (x + y = \gamma_2)$ there is a least one point of A.*

The finite set $H(A) \subseteq (2\mathbb{Z} + 1) \times (2\mathbb{Z} + 1)$ defined by the above two conditions will be called a covering polygon of the set A.

We will prove that if A is *normal set* then the points of A lie on pairs of symmetric lines with respect to three lines defined by

$$l_1 : (x = 0), \qquad l_2 : (y = 0), \qquad l_3 : (x + y = 1). \tag{12}$$

More precisely:

Lemma 3. *Let $A \subseteq \mathbb{Z}^2$ be a finite normal set. Then*

(a) *If $A \cap (x = \alpha) \neq \varnothing$, then $A \cap (x = -\alpha) \neq \varnothing$.*
(b) *If $A \cap (y = \beta) \neq \varnothing$, then $A \cap (y = -\beta) \neq \varnothing$.*
(c) *If $A \cap (x + y - 1 = \gamma) \neq \varnothing$, then $A \cap (x + y - 1 = -\gamma) \neq \varnothing$.*

Proof. In view of (12), the points c_0 and c_2 belong to l_1, c_0 and c_1 belong to l_2 and finally c_1 and c_2 belong to l_2. Therefore there is no loss of generality if we will prove only assertion (a).

To the contrary, assume that $A \cap (x = \alpha) \neq \varnothing$ and $A \cap (x = -\alpha) = \varnothing$. In this case, every point $p \in A \cap (x = \alpha)$ has no symmetric with respect to c_0 and c_2 and therefore p is a removable point of A. This contradicts our assumption that A is normal set. Lemma 3 is proved. ■

Let $A \subseteq \mathbb{Z}^2$ be a normal set. We will now estimate the number of *odd points* belonging to a *covering polygon* $H(A)$. In view of Definition 3 and Lemma 3, the integers $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ that define the covering lines of $H(A)$ satisfy

$$\begin{aligned} &\alpha_1 \text{ and } \alpha_2 \text{ are odd}, && \alpha_2 = -\alpha_1 = 2\alpha - 1, \\ &\beta_1 \text{ and } \beta_2 \text{ are odd}, && \beta_2 = -\beta_1 = 2\beta - 1, \\ &\gamma_1 \text{ and } \gamma_2 \text{ are even}, && \gamma_2 = -\gamma_1 + 2 = 2\gamma. \end{aligned}$$

It follows that $H(A) = H(a,b,c)$, where $a = 2\alpha, b = 2\beta, c = 2\gamma$. Let us denote by

$$\epsilon = \epsilon(a,b,c) = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2}. \tag{13}$$

We have the following estimate

Lemma 4. *The set $H(a,b,c)$ lies on $a = 2\alpha$ lines parallel to $(x = 0)$, on $b = 2\beta$ lines parallel to $(y = 0)$, on $c = 2\gamma$ lines parallel to $(x + y = 1)$ and*

$$|H(a,b,c)| = \begin{cases} c\min\{a,b\}, & \text{if } c \leqslant |a-b| \\ ab - \frac{(a+b-c)^2}{4}, & \text{if } c \geqslant |a-b| + 2. \end{cases} \tag{14}$$

Moreover,

(a) *if $c \leqslant |a-b|$, then $|H(a,b,c)| \leqslant \frac{1}{4}\frac{(a+b+c)^2}{4}$.*
(b) *if $c \geqslant |a-b|+2$, then $|H(a,b,c)| \leqslant \frac{1}{3}\Big(\frac{(a+b+c)^2}{4} - \epsilon\Big)$.*

Proof. Every point $P = (x,y) \in H(a,b,c)$ belongs to the rectangle defined by

$$R(A) : |x| \leqslant 2\alpha - 1, \quad |y| \leqslant 2\beta - 1, \quad x \text{ and } y \text{ are odd.}$$

and thus $H(a,b,c)$ lies on $a = 2\alpha$ lines parallel to $(x = 0)$, on $b = 2\beta$ lines parallel to $(y = 0)$. Moreover, if $P = (x,y)$ is a point of $H(a,b,c)$ lying on the supporting line $(x + y = 2\gamma)$, then $x + y \leqslant 2\alpha + 2\beta - 2$ and therefore $H(a,b,c)$ lies on $c = 2\gamma \leqslant 2\alpha + 2\beta - 2 = a + b - 2$ lines parallel to $(x + y = 1)$.

It is enough to examine only the case $a \geqslant b$.

Case 1. If $2 \leqslant 2\gamma \leqslant 2\alpha - 2\beta$, then $2 \leqslant c \leqslant a - b$, the set $H(a,b,c)$ is actually a parallelogram and

$$H(a,b,c) = 2\gamma b = cb = c\min\{a,b\}.$$

Case 2. If $2\gamma = 2\alpha - 2\beta + 2$, then $c = a - b + 2$. The set $H(a,b,c)$ lies on two parallel lines, if $a = b$, or $H(a,b,c)$ is a pentagon, if $a \neq b$. Therefore

$$H(a,b,c) = 2\gamma b - 1 = cb - 1 = (a-b+2)b - 1 = ab - (b-1)^2 = ab - \frac{(a+b-c)^2}{4}.$$

Case 3. If $2\alpha - 2\beta + 4 \leqslant 2\gamma \leqslant 2\alpha + 2\beta - 4$, then $a - b + 4 \leqslant c \leqslant a + b - 4$, the set $H(a,b,c)$ is a hexagon and

$$H(a,b,c) = ab - \sum_{j=1}^{\alpha+\beta-\gamma-1} j - \sum_{j=1}^{\alpha+\beta-\gamma} j = ab - (\alpha+\beta-\gamma)^2 = ab - \frac{(a+b-c)^2}{4}.$$

Case 4. If $2\gamma = 2\alpha + 2\beta - 2$, then $c = a + b - 2$, the set $H(a,b,c)$ satisfies

$$H(a,b,c) = R(A) \setminus \{v\},$$

where v is the vertex $v = (-2\alpha + 1, -2\beta + 1)$. Thus

$$H(a,b,c) = ab - 1 = ab - \frac{(a+b-c)^2}{4}.$$

Equality (14) is proved.
Moreover, in case 1 we have $c \leqslant a - b$, $a \geqslant b + c$ and thus

$$|H(a,b,c)| = cb = \frac{(b+c)^2 - (b-c)^2}{4} \leqslant \frac{1}{4}\Big(\Big(\frac{a+b+c}{2}\Big)^2 - (b-c)^2\Big)$$
$$\leqslant \frac{1}{4}\Big(\frac{a+b+c}{2}\Big)^2.$$

In cases 2, 3 and 4 we have $c \geqslant a - b + 2$ and thus

$$|H(a,b,c)| = ab - \frac{(a+b-c)^2}{4} = \frac{2ab + 2bc + 2ca - a^2 - b^2 - c^2}{4}$$
$$= \frac{(a+b+c)^2}{12} - \frac{\epsilon}{3}.$$

Lemma 4 is proved. ∎

3. Normal connected sets

In this section we prove Corollary 1 which implies Theorem 2 for connected normal sets. We need the following:

Definition 4. *Let $A \subseteq \mathbb{Z}^2$ be a finite normal set and let*

$$x = \pm(2\alpha - 1), \qquad y = \pm(2\beta - 1), \qquad x + y - 1 = \pm(2\gamma - 1)$$

denote the supporting lines of the covering polygon $H(A) = H(a,b,c)$. We say that A is a connected normal set if the following three conditions are true:

(a) *for every odd integer p such that $|p| \leqslant 2\alpha - 1$ we have $A \cap (x = p) \neq \varnothing$.*
(b) *for every odd integer q such that $|q| \leqslant 2\beta - 1$ we have $A \cap (y = q) \neq \varnothing$.*
(c) *for every odd integer r such that $|r| \leqslant 2\gamma - 1$ we have $A \cap (x + y - 1 = r) \neq \varnothing$.*

We will use the following result:

Lemma 5. *Let $A \subseteq \mathbb{Z}^2$ be a connected normal set. If $H(A)$, the covering polygon of A, is equal to $H(a,b,c)$, then*

$$R_3(A) = |\mathrm{Diff}(A)| \leqslant 3k - \frac{a+b+c}{2}. \tag{15}$$

Proof. See assertion (b) of Lemma 2 in [1]. ∎

We can now prove without difficulty the following corollary which describes the structure of a connected normal set A which satisfies $R_3(A) \geqslant 3k - \sqrt{3.241k}$. This condition is less restrictive than inequality (10) and will be used in Section 5.

Corollary 1. *Let $A \subseteq \mathbb{Z}^2$ be a connected normal set. Let $H(A) = H(a,b,c)$ be the covering polygon of A. Denote by*

$$k = |A|, \qquad k^* = |H(A)|.$$

(a) *If $c \leqslant |a-b|$, then $R_3(A) \leqslant 3k - 2\sqrt{k^*} \leqslant 3k - 2\sqrt{k}$.*
(b) *If $c \geqslant |a-b|+2$, then $R_3(A) \leqslant 3k - \sqrt{3k^* + \epsilon} \leqslant 3k - \sqrt{3k+\epsilon}$.*
(c) *If $R_3(A) \geqslant 3k - \sqrt{3.241k}$, then $|H(A)| < 1.081|A|$. Moreover, if we assume that $a \leqslant b \leqslant c$, then $a > 0.8\sqrt{k}$, $b < 1.75a$ and $c < 0.75(a+b)$.*

Proof. We have $H(A) = H(a,b,c)$, $k \leqslant k^*$ and we may assume without loss of generality that $a \leqslant b$.

Case (a). If $c \leqslant b-a$, then assertion (a) of Lemma 4 implies that

$$\frac{a+b+c}{2} \geqslant 2\sqrt{|H(A)|} = 2\sqrt{k^*} \geqslant 2\sqrt{k}.$$

Using (15), we get $R_3(A) \leqslant 3k - \frac{a+b+c}{2} \leqslant 3k - 2\sqrt{k^*} \leqslant 3k - 2\sqrt{k}$.

Case (b). If $c \geqslant b-a+2$, then assertion (b) of Lemma 4 implies that

$$\frac{a+b+c}{2} \geqslant \sqrt{3k^*+\epsilon} \geqslant \sqrt{3k+\epsilon}.$$

Using (15), we get

$$R_3(A) \leqslant 3k - \frac{a+b+c}{2} \leqslant 3k - \sqrt{3k^*+\epsilon} \leqslant 3k - \sqrt{3k+\epsilon}.$$

We prove now assertion (c). Let us assume that the set A satisfies the inequality

$$R_3(A) \geqslant 3k - \sqrt{3.241k}.$$

Using Corollary 1 (a) and inequalities (5) and (15) we get that

$$2 + |a-b| \leqslant c \leqslant a+b-2$$

and

$$3k - \sqrt{3.241k} \leqslant R_3(A) \leqslant 3k - \frac{a+b+c}{2} \leqslant 3k - \sqrt{3k^*+\epsilon}$$
$$\leqslant 3k - \sqrt{3k+\epsilon} \leqslant 3k - \sqrt{3k}.$$

Therefore $3k^* + \epsilon \leqslant 3.241k$, $\sqrt{3k} \leqslant \frac{a+b+c}{2} \leqslant 3k - R_3$, $\epsilon \leqslant (3k-R_3)^2 - 3k$ and thus

$$|H(A)| < 1.081|A| - \frac{\epsilon}{3}, \tag{16}$$

$$\begin{aligned} 3.464\sqrt{k} \leqslant a+b+c \leqslant 2\sqrt{3.241k}, \\ 2\epsilon = (a-b)^2 + (b-c)^2 + (c-a)^2 \leqslant 0.482k. \end{aligned} \tag{17}$$

We may assume without loss of generality that

$$a \leqslant b \leqslant c.$$

Denote $b = a+u$ and $c = b+v$. Inequality (17) imply that $u^2+v^2+(u+v)^2 \leqslant 0.482k$. Thus $u^2 \leqslant 0.241k$, $v^2 \leqslant 0.241k$, $(u+v)^2 \leqslant 0.322k$. Therefore

$$u \leqslant 0.491\sqrt{k}, \quad v \leqslant 0.491\sqrt{k}, \quad u+v \leqslant 0.568\sqrt{k},$$
$$3.464\sqrt{k} \leqslant a+b+c = 3a+u+(u+v) \leqslant 3a+1.059\sqrt{k},$$
$$a \geqslant \frac{1}{3}2.405\sqrt{k} \geqslant 0.801\sqrt{k}.$$

Moreover, the quotient $\frac{b}{a}$ is less than 1.75 because $2\sqrt{3.241k} \geqslant a+b+c \geqslant a+2b = a(1+2\frac{b}{a})$ implies that

$$\frac{b}{a} \leqslant \frac{1}{2}\Big(\frac{2\sqrt{3.241k}}{a} - 1\Big) \leqslant \frac{1}{2}\Big(\frac{2\sqrt{3.241k}}{0.801\sqrt{k}} - 1\Big) \leqslant 1.748.$$

In order to prove assertion (c), it remains to be shown that $t = \frac{c}{a+b} \leqslant 0.75$. We have

$$2\sqrt{3.241k} \geqslant a+b+c = (1+t)(a+b) \geqslant 2(1+t)\sqrt{ab},$$
$$k \leqslant ab - \Big(\frac{a+b-c}{2}\Big)^2 = ab - \Big(\frac{(1-t)(a+b)}{2}\Big)^2$$

and thus

$$2\sqrt{3.241k} \geqslant 2(1+t)\sqrt{k + \Big(\frac{(1-t)(a+b)}{2}\Big)^2}.$$

Clearly $\sqrt{3.241k} \geqslant (1+t)\sqrt{k}$ and thus $t \leqslant 0.8003$. This last estimate can be slightly improved using the inequalities $a+b \geqslant 2\sqrt{ab} \geqslant 2\sqrt{k}$. Indeed, we obtain

$$2\sqrt{3.241k} \geqslant 2(1+t)\sqrt{k+(1-t)^2k}, \quad 3.241 \geqslant (1+t)^2 + (1-t^2)^2$$

and so $t^4 - t^2 + 2t \leqslant 1.241$. Using $0 \leqslant t \leqslant 1$ we get $t < 0.75$. Corollary 1 is proved. ■

4. Disconnected normal sets

Definition 5. *Let $A \subseteq \mathbb{Z}^2$ be a finite normal set and let*

$$x = 2\alpha - 1, \qquad x = -2\alpha + 1, \qquad y = 2\beta - 1, \qquad y = -2\beta + 1,$$
$$x + y = 2\gamma, \qquad x + y = -2\gamma + 2$$

denote the supporting lines of the covering polygon $H = H(A)$. We say that A is a disconnected normal set if it is normal and at least one of the assertion (a), (b), (c) of Definition 4 is not true.

As we remarked before, this means that the set A is normal and at least one of the following three conditions is true:

(a) there is an odd integer u such that $-2\alpha+1 \leqslant u \leqslant 2\alpha-1$ and $A \cap (x = \pm u) = \varnothing$.
(b) there is an odd integer v such that $-2\beta+1 \leqslant v \leqslant 2\beta-1$ and $A \cap (y = \pm v) = \varnothing$.
(c) there is an even integer w such that $-2\gamma+2 \leqslant w \leqslant 2\gamma$ and $A \cap (x+y = \pm w) = \varnothing$.

We will examine now such a set $K \subset \mathbb{Z}^2$ for which only condition (c) is satisfied.

Example 1. Let $t \in \mathbb{Z}$ be a positive integer. Let us define

$$K(t) = H_t \pm (2t, 2t).$$

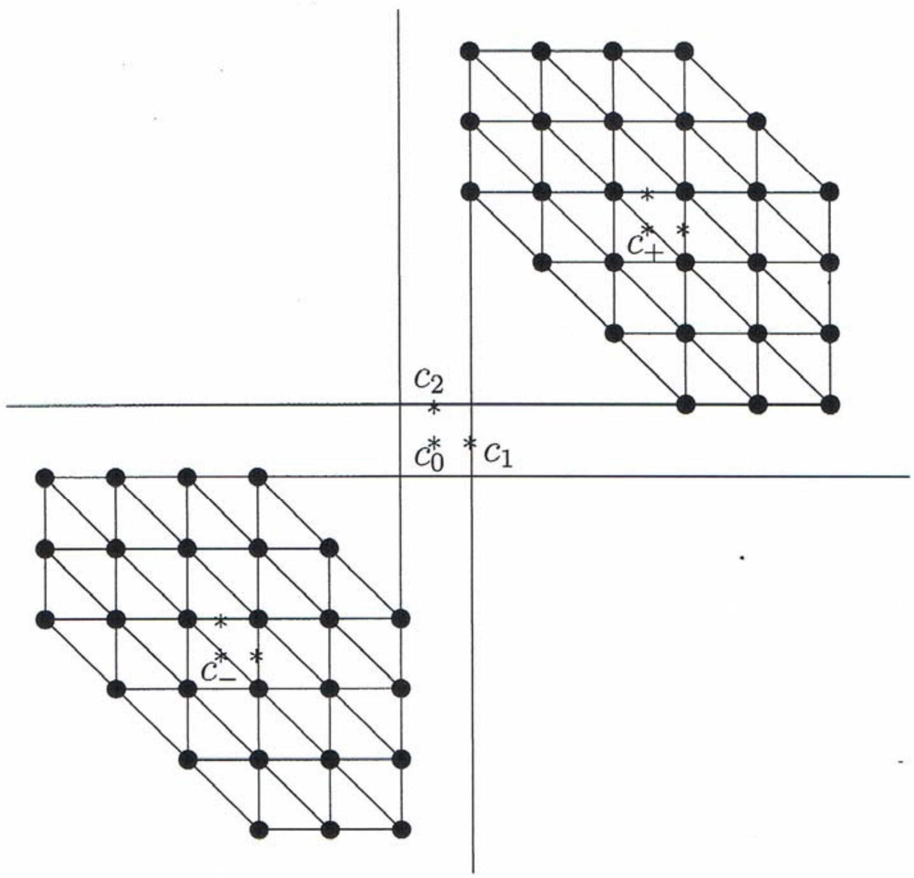

Figure 4.1: The set $K(t)$ for $t = 3$. $K(t)$ is included in $(2\mathbb{Z}+1) \times (2\mathbb{Z}+1)$.

The set $K(t)$ is described in Figure 4.1 and is defined by the following conditions: a point (x, y) belongs to $K(t)$ if and only if:

(i) $1 \leqslant x, y \leqslant 4t-1, 2t+2 \leqslant x+y \leqslant 6t$ and x and y are both odd integers.

or

(ii) $-4t+1 \leqslant x, y \leqslant -1, -6t+2 \leqslant x+y \leqslant -2t$ and x and y are both odd integers.

Lemma 6. *The set $K = K(t)$ satisfies $k = |K| = 6t^2$ and*

$$R_3(K) = 3k - \frac{a+b+c}{2} = 3k - 6t = 3k - \sqrt{6k}. \qquad (18)$$

Proof. The set $K(t)$ consists of two disjoint translates of H_t and thus

$$k = |K(t)| = 2|H_t| = 6t^2.$$

Using the properties of the set H_α it follows that $K(t)$ lies on $a = 4t$ lines parallel to e_2, $b = 4t$ lines parallel to e_1 and $c = 4t$ lines parallel to $e_1 - e_2$. Each line $(x = x_0), x_0$ odd, $-4t+1 \leqslant x_0 \leqslant 4t-1$ intersects the set K. Each line $(y = y_0)$, y_0 odd, $-4t+1 \leqslant y_0 \leqslant 4t-1$ intersects the set K. Nevertheless, the lines $(x + y = s)$, s even, $-2t+2 \leqslant s \leqslant 2t$ does not intersect K. It follows that *only* condition (c) of Definition 4 is satisfied. Moreover, the three centers of symmetry of K are $c_i = e_i$, for $i = 0, 1, 2$, K is a normal set and we clearly have:

$$\begin{aligned} d_0 = |D_0(K)| &= |\{p \in K : p_0 = 2c_0 - p \in K\}| \\ &= k - |K \cap ((x + t = 6t) \cup (x + y = -2t))|, \\ d_1 = |D_1(K)| &= |\{p \in K : p_1 = 2c_1 - p \in K\}| \\ &= k - |K \cap ((x = 1) \cup (x = -4t + 1))|, \\ d_2 = |D_2(K)| &= |\{p \in K : p_2 = 2c_2 - p \in K\}| \\ &= k - |K \cap ((y = 1) \cup (y = -4t + 1))|. \end{aligned}$$

We conclude that K is a disconnected normal set and

$$R_3(K) = d_0 + d_1 + d_2 = (k - 2t) + (k - 2t) + (k - 2t) = 3k - 6t = 3k - \sqrt{6k}. \blacksquare$$

We will now examine in detail a normal disconnected set satisfying case (a). Cases (b) and (c) are similar. The following result generalizes inequality (18):

Lemma 7. *Assume that the set A is a normal disconnected set satisfying condition (a). Let us choose $u \geqslant 1$ minimal such that u is odd and*

$$A \cap (x = \pm u) = \emptyset.$$

Define $A_1 = A \cap (-u < x < u)$, $A_2 = A \setminus A_1$, $k_1 = |A_1|$, $k_2 = k - k_1$. Then

$$R_3(A) = R_3(A_1) + R_3(A_2) \leqslant 3k - \sqrt{3k_1} - \sqrt{6(k_2 - n_0 - 0.5)}, \qquad (19)$$

where n_0 is the number of points $p \in A_2$ such that $p_0 = 2c_0 - p \notin A_2$.

Proof. We will first show that the subset A_2 satisfies an inequality similar to (18). More precisely, we have

$$R_3(A_2) \leqslant 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}. \qquad (20)$$

The set A_2 is a disjoint union of

$$A_+ = A \cap (x > u)$$

and

$$A_- = A \cap (x < -u).$$

Denote by $\pi_1(x, y) = x$ the projection parallel to line $(x = 0)$, by $\pi_2(x, y) = y$ the projection parallel to line $(y = 0)$ and by $\pi_3(x, y) = x + y$ the projection parallel to line $(x + y = 0)$. We *claim* that there is an integral vector $w \in \mathbb{N}^2$ such that the sets

$$B_+ = A_+ + w \qquad \text{and} \qquad B_- = A_- - w$$

satisfy the following assertions:

(i) B_+ and B_- are disjoint,
(ii) the projections $\pi_i(B_+)$ and $\pi_i(B_-)$ are disjoint, for $i = 1, 2, 3$,
(iii) the set $B = B_+ \cup B_-$ satisfies $R_3(A_2) \leqslant R_3(B)$.

If both coordinates of w are large enough, then assertions (i) and (ii) are clearly true. Let us explain now (iii). Each difference $d = (d_1, d_2) \in \mathrm{Diff}(A)$ can be written as $d = p - p'$, where $p + p' = 2c_i = 2e_i$ and $p, p' \in A$. Therefore, we have either

$$p \in A_+, \qquad p' \in A_-, \qquad d_1 \geqslant 2(u + 2) \geqslant 6$$

or

$$p \in A_-, \qquad p' \in A_+, \qquad d_1 \leqslant -2(u + 2) \leqslant -6.$$

This remark allows us to define a one to one map φ from

$$\mathrm{Diff}(A_2) = D_0(A_2) \cup D_1(A_2) \cup D_2(A_2)$$

to

$$\mathrm{Diff}(B) = D_0(B) \cup D_1(B) \cup D_2(B).$$

More precisely, if $p_i = 2e_i - p$ denotes the symmetric of p with respect to e_i, then φ is given by

$$\varphi(d) = \begin{cases} d + 2w, & \text{if } d = p - p_i, \ p \in A_+, \ p_i \in A_-, \\ d - 2w, & \text{if } d = p - p_i, \ p \in A_-, \ p_i \in A_+. \end{cases}$$

The image $\varphi(d) \in \mathrm{Diff}(B)$; indeed, if $d = p - p_i$, $p \in A_+$, $p_i \in A_-$, then

$$d + 2w = p - p_i + 2w = (p + w) - (p_i - w),$$

$$p + w \in B_+ \subseteq B, \quad p_i - w \in B_- \subseteq B,$$

$$(p + w) + (p_i - w) = p + p_i = 2c_i = 2e_i$$

and if $d = p - p_i$, $p \in A_-$, $p_i \in A_+$, then

$$d - 2w = p - p_i - 2w = (p - w) - (p_i + w),$$

$$p - w \in B_- \subseteq B, \quad p_i + w \in B_+ \subseteq B,$$

$$(p - w) + (p_i + w) = p + p_i = 2c_i = 2e_i.$$

Moreover, we may choose the vector w such that $d' + 2w \neq d'' - 2w$, for every $d' \neq d'', d', d'' \in \mathrm{Diff}(A_2)$. This implies that φ is one to one and assertion (iii) follows.

Assume that the set B_+ lies on exactly a_1 lines parallel to the line $(x = 0)$, on b_1 lines parallel to the line $(y = 0)$ and on c_1 lines parallel to the line $(x + y = 0)$. In other words:

$$a_1 = |\pi_1(B_+)|, \qquad b_1 = |\pi_2(B_+)|, \qquad c_1 = |\pi_3(B_+)|.$$

The set B_- determiries the parameters a_2, b_2 and c_2 in a similar way, i.e.

$$a_2 = |\pi_1(B_-)|, \qquad b_2 = |\pi_2(B_-)|, \qquad c_2 = |\pi_3(B_-)|.$$

Therefore, property (ii) implies that the set B lies on exactly $a_1 + a_2$ lines parallel to the line $(x = 0)$, on $b_1 + b_2$ lines parallel to the line $(y = 0)$ and on $c_1 + c_2$ lines parallel to the line $(x + y = 0)$. Using Lemma 2.b. and Corollary 1 from [1] we get

$$\begin{aligned} R_3(B) &\leqslant 3|B| - \frac{(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2)}{2} \\ &= 3|B_+| - \frac{a_1 + b_1 + c_1}{2} + 3|B_-| - \frac{a_2 + b_2 + c_2}{2} \\ &\leqslant 3|B_+| - \sqrt{3(|B_+| - 0.25)} + 3|B_-| - \sqrt{3(|B_-| - 0.25)}. \end{aligned}$$

Let us estimate the cardinalities of the sets B_+ and B_- using the fact that A, A_2 and B are all "almost symmetric" with respect to c_0. Let us recall that n_0 denotes the number of points $p \in A_2$ such that $p_0 = 2c_0 - p \notin A_2$; therefore we get

$$n_0 = |\{p : p \in B, p_0 \notin B\}| \leqslant |B| = |A_2| = k_2$$

and

$$|B_+| = |A_+| \geqslant \frac{|B| - n_0}{2}, \qquad |B_-| = |A_-| \geqslant \frac{|B| - n_0}{2};$$

inequality (20) follows from:

$$\begin{aligned} R_3(A_2) \leqslant R_3(B) &\leqslant 3|B| - \sqrt{3(|B_+| - 0.25)} - \sqrt{3(|B_-| - 0.25)} \\ &\leqslant 3|B| - 2\sqrt{3(\frac{|B| - n_0}{2} - 0.25)} = 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}. \end{aligned}$$

We will show that inequality (19) is true. The set A is a disjoint union of A_1 and A_2. Using Corollary 1 from [1] we get $R_3(A_1) \leqslant 3k_1 - \sqrt{3k_1}$. For every $i = 0, 1, 2$ the sets $D_i(A_1)$ and $D_i(A_2)$ are disjoint and thus

$$\begin{aligned} R_3(A) &= R_3(A_1) + R_3(A_2) \leqslant 3k_1 - \sqrt{3k_1} + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\ &= 3k - \sqrt{3k_1} - \sqrt{6(k_2 - n_0 - 0.5)}. \end{aligned}$$

Lemma 7 is proved. ■

5. The general case and proof of Theorem 2

Assume that A is a finite set that satisfies the hypothesis of Theorem 2. Let A_0 be the set of all essential points of A. Using inequality (11) or in view of Lemma 2 we have

$$k_0 = |A_0|, \qquad 0 \leqslant k - k_0 \leqslant 2\sqrt{k}, \qquad R_3(A_0) \geqslant 3k_0 - \theta\sqrt{k_0}. \tag{21}$$

A_0 is a finite normal set. If A_0 is connected we apply Corollary 1 and Theorem 2 is proved. Assume that A_0 is disconnected. In what follows, we will apply three times Lemma 7 in order to obtain a large normal connected proper subset $A_5 \subset A_0$. Let us choose $u \geqslant 1$ minimal such that u is odd and

$$A_0 \cap (x = \pm u) = \emptyset.$$

Define $A_1 = A_0 \cap (-u < x < u), A_2 = A_0 \setminus A_1, k_1 = |A_1|, k_2 = k_0 - k_1$. The sets A_1 and A_2 form a partition of A_0 and in view of Lemma 7 we have

$$R_3(A_0) = R_3(A_1) + R_3(A_2) \leqslant R_3(A_1) + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)}, \tag{22}$$

where n_0 is the number of points $p \in A_2$ such that $p_0 = 2c_0 - p \notin A_2$.

Let us choose $v \geqslant 1$ minimal such that v is odd and

$$A_1 \cap (y = \pm v) = \emptyset.$$

Define $A_3 = A_1 \cap (-v < y < v), A_4 = A_1 \setminus A_3, k_3 = |A_3|, k_4 = k_1 - k_3$. The sets A_3 and A_4 form a partition of A_1 and using a similar argument as in the proof of Lemma 7, we get

$$R_3(A_1) = R_3(A_3) + R_3(A_4) \leqslant R_3(A_3) + 3k_4 - \sqrt{6(k_4 - n_1 - 0.5)}, \tag{23}$$

where n_1 is the number of points $p \in A_4$ such that $p_0 = 2c_0 - p \notin A_4$.

Let us choose $w \geqslant 1$ minimal such that w is odd and

$$A_3 \cap (x + y - 1 = \pm w) = \emptyset.$$

Define $A_5 = A_3 \cap (-w < x + y - 1 < w), A_6 = A_3 \setminus A_5, k_5 = |A_5|, k_6 = k_3 - k_5$. The sets A_5 and A_6 form a partition of A_3 and using a similar argument as in the proof of Lemma 7, we get

$$R_3(A_3) = R_3(A_5) + R_3(A_6) \leqslant R_3(A_5) + 3k_6 - \sqrt{6(k_6 - n_2 - 0.5)}, \tag{24}$$

where n_2 is the number of points $p \in A_6$ such that $p_0 = 2c_0 - p \notin A_6$. In view of (22), (23), (24) and using $k_0 = k_5 + k_2 + k_4 + k_6$ and $R_3(A_5) \leqslant 3k_5 - \sqrt{3k_5}$ we get:

$$\begin{aligned} R_3(A_0) &\leqslant R_3(A_1) + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\ &\leqslant R_3(A_3) + 3k_4 - \sqrt{6(k_4 - n_1 - 0.5)} + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\ &\leqslant R_3(A_5) + 3k_6 - \sqrt{6(k_6 - n_2 - 0.5)} + 3k_4 - \sqrt{6(k_4 - n_1 - 0.5)} \\ &\quad + 3k_2 - \sqrt{6(k_2 - n_0 - 0.5)} \\ &\leqslant R_3(A_5) + 3(k_0 - k_5) - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \qquad (25) \\ &\leqslant 3k_0 - \sqrt{3k_5} - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\ &\leqslant 3k_0 - \sqrt{3k_5 + 6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9}. \end{aligned}$$

Inequality (21) gives a lower bound for $R_3(A_0)$ and implies that

$$\begin{aligned} &3k_5 + 6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9 \\ &\qquad = 3k_0 + 3(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9 \leqslant \theta^2 k_0 \leqslant 3.24k_0. \end{aligned}$$

Thus

$$\begin{aligned} k_0 - k_5 &\leqslant 0.08k_0 + 2(n_0 + n_1 + n_2) + 3 \\ &\leqslant 0.08k_0 + 6m_0 + 3 \leqslant 0.08k_0 + 10.8\sqrt{k_0} + 3, \\ k_5 &\geqslant 0.92k_0 - 10.8\sqrt{k_0} - 3. \qquad (26) \end{aligned}$$

We applied here (8) and the obvious inequality $n_i \leqslant m_0$, $i = 0, 1, 2$.

We claim that the set A_5 satisfies an inequality similar to (7), namely

$$R_3(A_5) \geqslant 3k_5 - \sqrt{3.241k_5}. \qquad (27)$$

Indeed, assume to the contrary that $R_3(A_5) < 3k_5 - \sqrt{3.241k_5}$. Using (25) we get

$$\begin{aligned} R_3(A_0) &\leqslant R_3(A_5) + 3(k_0 - k_5) - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\ &< 3k_5 - \sqrt{3.241k_5} + 3(k_0 - k_5) - \sqrt{6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\ &\leqslant 3k_0 - \sqrt{3.241k_5 + 6(k_0 - k_5) - 6(n_0 + n_1 + n_2) - 9} \\ &\leqslant 3k_0 - \sqrt{3.241k_0 - 6(n_0 + n_1 + n_2) - 9} \\ &\leqslant 3k_0 - \sqrt{3.241k_0 - 10.8\sqrt{k_0} - 9}, \end{aligned}$$

which contradicts inequality (21), if $k = |A|$ is sufficiently large.

Choose a proper subset $A_5 \subset A_0$ such that (26) and (27) are true and $k_5 = |A_5|$ is minimal. The choice of u, v, w and the minimality of k_5 imply that A_5 is normal and connected. Let

$$H(A_5) : \begin{cases} -2\alpha + 1 \leqslant x \leqslant 2\alpha - 1, & x \text{ odd}, \\ -2\beta + 1 \leqslant y \leqslant 2\beta - 1, & y \text{ odd}, \\ -2\gamma + 2 \leqslant x + y \leqslant 2\gamma \end{cases} \qquad (28)$$

be the covering polygon of A_5. Then $H(A_5)$ lies on $a = 2\alpha$ lines parallel to $(x = 0)$, on $b = 2\beta$ lines parallel to $(y = 0)$, on $c = 2\gamma$ lines parallel to $(x + y = 1)$ and $2 \leqslant c \leqslant a+b-2$. We will use now inequality (27) and assertion (c) of Corollary 1. We may assume without loss of generality that $a \leqslant b \leqslant c$. We get that

$$|H(A_5)| < 1.081|A_5|, \quad a > 0.8\sqrt{k_5}, \quad b < 1.75a \quad \text{and} \quad c < 0.75(a+b).$$

Define $A^* = A_5$ and $H(a,b,c) = H(A_5)$. Using (21) and (26), we conclude that

$$\begin{aligned} k - k_5 = (k - k_0) + (k_0 - k_5) &\leqslant 2\sqrt{k} + 0.08k_0 + 10.8\sqrt{k_0} + 3 \\ &\leqslant 0.08k + 12.8\sqrt{k} + 3 \end{aligned}$$

and thus $|A^*| = |A_5| = k_5 \geqslant 0.92k - 12.8\sqrt{k} - 3$. Theorem 2 is proved, if k is sufficiently large. ■

6. Remarks

We use now the notations of Section 1 for finite sets of *integers*. It is a natural question whether it is possible to describe the structure of sets of integers $A \subseteq \mathbb{Z}$ such that $R_3(A) \geqslant 3k - 1.8\sqrt{k}$.

We propose the following:

Conjecture. *Let $A \subseteq \mathbb{Z}$ be a finite set of integers. Assume that $|A| = k$ and*

$$R_3(A) = |\mathrm{Diff}(A)| \geqslant 3k - 1.8\sqrt{k}. \tag{29}$$

Then there is a two dimensional set of odd lattice points on the plane $\bar{A} \subseteq \mathbb{Z}^2$ with the following properties:

(a) $|\bar{A}| = |A| = k$,
(b) $3k - 1.8\sqrt{k} \leqslant R_3(A) \leqslant R_3(\bar{A}) \leqslant 3k - \sqrt{3k}$,
(c) *the canonical projection $\pi : \bar{A} \to \mathbb{Z}, \pi(x,y) = x$ has the image $\pi(\bar{A}) = A$.*

Inequality (29) for integers is similar to condition (7) for sets of lattice points in the plane and in a subsequent paper we will show that it is possible to apply Theorem 2 in order to study the structure of such sets of integers.

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