

# About the Measure of Large Values of the Modulo of Trigonometric Sum

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*Dedicated with best wishes to Professor Klaus F. Roth on the occasion of his 80th birthday*

## Abstract

We study the connection between the additive structure of a finite set  $A \subset \mathbb{Z}$  and the measure of large values of a trigonometric sum.

**Keywords:** Inverse additive problems, trigonometric sum.

## 1 Introduction

Let  $\mathbb{Z}$  be the ring of integer rational numbers,  $\mathbb{R}$  - the field of real numbers,

$$\mathbb{T} = \mathbb{R}/\mathbb{Z}, e(\alpha) = e^{2\pi i\alpha}, i = \sqrt{-1}, [a, b] = \{a, a+1, \dots, b\}, a, b \in \mathbb{Z}, a < b.$$

For  $A \subset \mathbb{Z}$ ,  $|A| = \text{card } A$  - denotes the number of elements of a set  $A$ .

$$S_A(\alpha) = \sum_{a \in A} e(\alpha a), \alpha \in \mathbb{T},$$

Let

$$E(A, \lambda|A|) = \{\alpha \in \mathbb{T} : |S_A(\alpha)| \geq \lambda|A|\}.$$

In this paper, we study the following problem: find the set  $A$  for which mes  $E(A, \lambda|A|)$  has the maximal value assuming that  $|A|$  is given and  $A$  is a part of a short interval.

The problem has a rather long history (see [1], [2], [3] and [4]). The connection of this theme with problems of probability theory, harmonic analysis and coding theory is shown in [4], [5], [6] and [7]. Results in this direction you can also find in [9], [10] and [11]. There exists very strong connections of this field with Roth three element progression problem ([3], pp. 140-142).

In this paper, we shall prove the following

**Theorem 1** *Let  $A \subset [-N, N]$ ,  $|A| = k \geq N + 1$ ,  $N$  - sufficiently large integer.*

*Then*

$$\max_A \text{mes} E(A, \lambda|A|) = \mu^*(|A|) .$$

where  $\lambda = \frac{2\sqrt{2}}{\pi} = 0.90032^1$  is attained if and only if  $A$  is an arithmetic progression.

Let us stress that the conditions of Theorem 1 lead to the fact that the difference of arithmetic progression  $A$  is equal to 1 and only in the case  $k = N + 1$  is equal to 1 or 2.

An example: if  $|A| = N + 1$  then

$$\mu^*(N + 1) = \text{mes} \left\{ \alpha \in \mathbb{T} : \left| \sum_{a=0}^N e(\alpha a) \right| \geq \frac{2\sqrt{2}}{\pi}(N + 1) \right\} = \frac{2\theta}{N + 1} + O\left(\frac{1}{N^3}\right),$$

where  $\theta$  is a solution of the equation

$$\frac{\sin \pi\theta}{\pi\theta} = \frac{2\sqrt{2}}{\pi}$$

so that

$$\pi\theta = 0.775$$

Let us review existing results in more detail and explain existing and possible applications.

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<sup>1</sup>The result stated here for  $\lambda = 0.90032$  is, in fact, valid for  $\lambda = 0.75$ . This last result was obtained with the help of computations too massive to be included in this paper.

The problem of finding the maximal measure  $\mu_{\max} = \sup_A \mu$  for sets of  $\alpha$  for which  $|S_A(\alpha)|$  is bigger than some given number which is less than the trivial estimate equal to  $k$  and of finding the sets  $A$  with this condition was first formulated in [3], page 144.

The case when

$$|S_A(\alpha)| \geq (1 - \varepsilon)|A|$$

and  $\varepsilon = o(1)$  was studied in [4] by A. Yudin.

The case when  $\varepsilon$  is some positive constant was studied in [7] by A. Besser.

Nevertheless, the constant achieved appeared to be very small ( $\varepsilon = \frac{1}{20000}$ ) and further progress connected with too large technical difficulties.

That's why in [9] it was proposed to add an additional condition and to study only those sets included in a segment of not very big length. This approach was illustrated in [1] by a simple example. We explain now that an advance in this direction includes important applications giving a strong justification for the result of this paper.

For the set  $A$  which satisfies  $|2A| < \frac{10}{3}|A| - 5$  was shown in [18] that we were led to a study of trigonometric sums for which the condition

$$|S_A(\alpha)| \geq 0.56|A| \tag{1}$$

was valid for some specially chosen  $\alpha$ . We see that the constant in (1) and the constant 0.75 in Theorem 1 are already very close (near).

As to the sets in the short intervals or, in other words, sets with a large density, we may mention that most problems for sum-tree sets are of that nature.

Now we will describe the connection with problems of information transfer. The code word  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1})$ , where  $\forall i \varepsilon_i \in \{+1, -1\}$  is transmitted with the aid of a signal

$$F_n(t) = \sum_{j=0}^{n-1} \varepsilon_j e(jt) .$$

Technical conditions ask for the value

$$\max_{t \in \mathbb{T}} |F_n(t)|$$

be as small as possible.

We have

$$F_n(t) = 2 \sum_{k:\varepsilon_k=1} e(kt) - D_n(t) ,$$

where  $D_n(t) = \sum_{k=0}^{n-1} e(kt)$  denotes Dirichlet's kernel. The behavior of  $D_n(t)$  is known. So, the study of  $|F_n(z)|$  leads us to the study of a trigonometric sum

$$\sum_{k:\varepsilon_n=1} e(kt) . \tag{2}$$

We know that “almost all” code words

$$(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1})$$

have almost the same number of  $+1$  and  $-1$ . Thus the number of summands in (2) is near  $n/2$  and the conditions of Theorem 1 are satisfied. See [15] for the connection of this work with information theory problems.

In this paragraph we will introduce some definitions and formulate and prove lemmas needed for the proof of Theorem 1. In §2 we will complete the proof of Theorem 1.

**Remark.** Note that Theorem 1 is an assertion about the *measure* of a set. Therefore, each inequality that will follow, is true for every  $\alpha$  excepting eventually a set of measure zero.

Let  $N \in \mathbb{Z}$ ,  $N > 0$  and  $\alpha \in [0, 1)$ .

**Definition.** The segment

$$[\beta, \gamma] \subset \mathbb{T} ,$$

where

$$\beta, \gamma \in \mathbb{T} = [0, 1)(\text{mod } 1)$$

is the set of numbers  $\rho$ , for which

$$\rho \in [\beta, \gamma)(\text{mod } 1) ,$$

if

$$\exists \beta_0 \in \beta + \mathbb{Z} , \gamma_0 \in \gamma + \mathbb{Z} , \beta_0 < \gamma_0 \leq \beta_0 + 1$$

and

$$\beta_0 \leq \rho \leq \gamma_0 .$$

**Example 1** The interval  $[\frac{1}{3}, \frac{2}{3}] (\text{mod } 1)$  is defined by a set of numbers  $\rho \in \mathbb{R}$  for which

$$1/3 \leq \rho \leq 2/3, \rho \in \mathbb{R} .$$

**Example 2** The interval  $[\frac{4}{5}, \frac{1}{5}] (\text{mod } 1)$  is defined by a set

$$\left\{ \frac{4}{5} \leq \rho \leq 1 \right\} \cup \left\{ 0 < \rho \leq \frac{1}{5} \right\} , \rho \in \mathbb{R} .$$

**Lemma 1** Let  $\alpha \in \mathbb{T}$  and

$$L = \{e(\alpha x) : x \in [a, b], a, b \in \mathbb{Z}, a < b\} ,$$

$$A \subset [a, b], |A| = m, 2 \leq m \leq b - a + 1 .$$

Let  $A_0 \subseteq [a, b]$  be an extremal set i.e. , for which  $|A_0| = m$ ,  $A_0 \subset L$  we have

$$\max_{\substack{A \subset [a, b], \\ |A|=m}} \left| \sum_{a \in A} e(\alpha a) \right| = \left| \sum_{a \in A_0} e(\alpha a) \right| .$$

Then  $A_0$  is a set such that a minimal segment  $[\beta, \gamma]$  containing the set  $\alpha A_0$  does not include points of the set  $\alpha([a, b] \setminus A_0)$ . For almost all points  $\alpha$  of the set  $\mathbb{T}$  .

Lemma 1 is obvious from a geometrical point of view. Let  $m$  be an integer for which  $1 \leq m \leq N$ . Take all subsets of  $L$  of cardinality  $m$ . It is clear that

the subset of  $L$  the sum of vectors of which has the maximal length - is all the vectors of  $L$  from a certain angle.

Note that  $A_0$  exists because  $|A|$  is finite.

The case  $A_0 = [a, b]$  is trivial. Look at the set

$$E = \{e(\alpha x) - e(\alpha x^*) : x \in A_0, x^* \in [a, b] \setminus A_0\} .$$

Let us suppose, first, that there exist two vectors,  $x$  and  $x^*$ ,  $x \in A_0$ ,  $x^* \in [a, b] \setminus A_0$  such that the projection of  $e(\alpha x) - e(\alpha x^*)$  on the vector  $S_{A_0}(\alpha)$  is positive, i.e. the angle between these two vectors is less than  $\frac{\pi}{2}$  (we identify here  $\mathbb{C}$  with  $\mathbb{R}^2$  in the usual way).

We have for  $A' = (A \cup \{x^*\}) \setminus \{x\}$ , and we have

$$|S_{A'}(\alpha)| > |S_A(\alpha) \text{ for } S_{A'}(\alpha) = S_{A_0}(\alpha) - (e(\alpha x) - e(\alpha x^*)) ,$$

contradiction.

We have to study now the case in which projection of all  $e(\alpha x) - e(\alpha x^*)$  on  $S_{A_0}(\alpha)$  are positive. Then the projection of any  $e(\alpha x)$  on  $S_{A_0}(\alpha)$  is bigger than a projection of an  $e(\alpha x^*)$ . It means that the angle between  $e(\alpha x^*)$  and  $S_{A_0}(\alpha)$  is larger than the angle between any  $e(\alpha x)$  and  $S_{A_0}(\alpha)$ . In other words, all vectors  $e(\alpha x^*)$  are situated outside of the angle where all vectors  $e(\alpha x)$  are situated. ■

We will now need the following definitions and notations.

$$z = (z_1, \dots, z_k) \in \mathbb{C}^k ,$$

$$\|z\| = \left( \sum_{j=1}^k |z_j|^2 \right)^{1/2} ,$$

$$(u, v) = \sum_{j=1}^k u_j \bar{v}_j ,$$

$$\overline{a + bi} = a - bi$$

$$e_A = (x_{-N}, \dots, x_0, \dots, x_N) , \text{ where } x_i = \begin{cases} 1 & \text{if } i \in A \\ -1 & \text{if } i \notin A \end{cases}$$

$$e_\alpha = (e(-N\alpha), e(-(N-1)\alpha), \dots, e(N\alpha)) .$$

Note that  $e_0 = \underbrace{(1, 1, \dots, 1)}_{2N+1}$  and let  $K_H(\alpha) = \sum_{k=-N}^N \cos 2\pi k\alpha$ .

**Lemma 2** *Let  $H = 2N + 1$ . In the conditions of Theorem 1 we have*

$$1) E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right) \subset \left\{ \alpha \subseteq \mathbb{T} : |D_H(\alpha)| \geq \frac{H}{60} \right\} \cup \left\{ \alpha \in \mathbb{T} : |D_H(2\alpha)| \geq \frac{H}{60} \right\} .$$

$$2) \text{ For every } j = 1, 2, \dots, 2N \text{ we have } \frac{j}{H} \notin E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right) .$$

**Proof of Lemma 2.** Let

$$e_A = c_0 e_0 + c_\alpha e_\alpha + c_{-\alpha} e_{-\alpha} + e_\perp , \quad (3)$$

where  $e_\perp \in \mathbb{C}^H$ , which is orthogonal to vectors  $e_0, e_\alpha, e_{-\alpha}$ .

Using scalar multiplication of both sides of (3) on each of the vectors  $e_0, e_\alpha, e_{-\alpha}$  we will obtain for the numbers  $c_0, c_\alpha, c_{-\alpha}$  the following system of equations

$$\begin{cases} 2|A| - H = c_0 H + c_\alpha K_H(\alpha) + c_{-\alpha} K_H(\alpha) \\ 2\bar{S}_A(\alpha) - K_H(\alpha) = c_0 K_H(\alpha) + c_\alpha H + c_{-\alpha} K_H(2\alpha) \\ 2S_A(\alpha) - K_H(\alpha) = c_0 K_H(\alpha) + c_\alpha K_H(2\alpha) + c_{-\alpha} H \end{cases} \quad (4)$$

Let

$$G = \begin{pmatrix} H & K_H(\alpha) & K_H(\alpha) \\ K_H(\alpha) & H & K_H(2\alpha) \\ K_H(\alpha) & K_H(2\alpha) & H \end{pmatrix}$$

be the matrix of (4). A simple computation shows that

$$\begin{aligned} \det G &= H^3 + 2K_H^2(\alpha)K(2\alpha) - H(2K_H^2(\alpha) + K_H^2(2\alpha)) \\ &= H(H^2 - K_H^2(2\alpha)) - 2K_H^2(\alpha)(H - K_H(2\alpha)) = \\ &= (H - K_H(2\alpha))(H(H + K_H(2\alpha)) - 2K_H^2(\alpha)) . \end{aligned}$$

Let us now find those  $\alpha \in \mathbb{T}$  for which the matrix  $G$  is non-singular. If  $\det G = 0$  we get

$$H - K_H(2\alpha) = 0 \quad (5)$$

or

$$H(H + K_H(2\alpha)) - 2K_H^2(\alpha) = 0 . \quad (6)$$

From equation (5) it follows that

$$\alpha = \alpha_0 = 0 \quad (7)$$

or

$$\alpha = \alpha_1 = \frac{1}{2} .$$

We now analyze equation (6) and show that it has only one solution (7). We have

$$H^2 + HK_H(2\alpha) - 2K_H^2(\alpha) \geq H^2 - H|K_H(2\alpha)| - 2K_H^2(\alpha) .$$

In a neighborhood of  $\alpha = \frac{1}{2}$  we have  $|K_H(\alpha)| \leq C$ , and  $K_H(2\alpha) \geq -\frac{2}{3\pi}H$ . Thus, near  $\alpha = \frac{1}{2}$ , equation (6) has no solutions.

If  $\alpha^* \neq 0$ ,  $|\alpha^*|$  is small and  $\alpha^*$  is the solution of (6) then the vectors  $e_0, e_{\alpha^*}, e_{-\alpha^*}$  are linearly dependent, i.e. for each  $j \in [-N, N]$

$$\begin{aligned} e_0 &= x_{\alpha^*}e_{\alpha^*} + x_{-\alpha^*}e_{-\alpha^*} \\ 1 &= x_{\alpha^*}e(j\alpha^*) + x_{-\alpha^*}e(-j\alpha^*) , \end{aligned}$$

and this is impossible.

Thus the matrix of system (4) is not degenerate everywhere, excepting the case  $\alpha \in \{0, 1/2\}$ . This means that if  $\alpha \notin \{0, 1/2\}$  the system (4) has a solution for  $c_0, c_\alpha, c_{-\alpha}$  and only one. Because we are interested in a value of a measure of  $E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right)$  elimination of two points does not matter.

We now find the eigenvalues of matrix  $G$ . Solving the characteristic equations

$$\begin{vmatrix} H - \lambda & K_H(\alpha) & K_H(\alpha) \\ K_H(\alpha) & H - \lambda & K_H(2\alpha) \\ K_H(\alpha) & K_H(2\alpha) & H - \lambda \end{vmatrix} = 0$$

we obtain three solutions (eigenvalues)

$$\begin{aligned}\lambda_1 &= H - K_H(2\alpha) \\ \lambda_2 &= H + \frac{1}{2} \left( K_H(2\alpha) + \sqrt{K_H^2(2\alpha) + 8K_H^2(\alpha)} \right) \\ \lambda_3 &= H - \frac{1}{2} \left( -K_H(2\alpha) + \sqrt{K_H^2(2\alpha) + 8K_H^2(\alpha)} \right)\end{aligned}\tag{8}$$

Solutions of the system (4) are

$$(c_0, c_\alpha, c_{-\alpha})^t = G^{-1}(2|A| - H, 2\bar{S}_A(\alpha) - K_H(\alpha), 2S_A(\alpha) - K_H(\alpha))^t$$

or  $\vec{c} = G^{-1}\vec{h}$ . From (3) and (4) we get

$$(\vec{h}, \vec{c}) = (G\vec{c}, \vec{c}) = \|e_A\|^2 - \|e_\perp\|^2 \leq H\tag{9}$$

and

$$|(\vec{h}, \vec{c})| = |(\vec{h}, G^{-1}\vec{h})| \geq \lambda_{\min}(G^{-1})\|\vec{h}\|^2\tag{10}$$

where  $\lambda_{\min}(G^{-1})$ -minimal eigenvalue of the matrix  $G^{-1}$ . From (9) and (10) we get

$$\lambda_{\min}(G^{-1})\|\vec{h}\|^2 \leq H$$

or

$$\|\vec{h}\|^2 < \frac{H}{\lambda_{\min}(G^{-1})} = \lambda_{\max}(G)H,\tag{11}$$

where  $\lambda_{\max}(G)$  is the maximum eigenvalue of the matrix  $G$ .

The inequality (10) and the equality  $\lambda_{\min}(G^{-1}) = \lambda_{\max}(G)^{-1}$  are valid because of extremal properties of hermitian matrices (see [16]).

The inequality (11) may be rewritten in the following way:

$$(2|A| - H)^2 + 2|2S_A(\alpha) - K_H(\alpha)|^2 \leq \lambda_{\max}(G)H\tag{12}$$

We shall now prove the second conclusion of Lemma 2.

Let  $\alpha = \frac{j}{H}$ ,  $j \neq 0$ . Then

$$K_H(\alpha) = K_H(2\alpha) = 0, \quad \lambda_1 = \lambda_2 = \lambda_3 = H.$$

From (12) and  $|S_A(\alpha)| > \frac{3}{4}|A|$  we get that

$$\begin{aligned} (2|A| - H)^2 + 8|S_A(\alpha)|^2 &\leq H^2 \Rightarrow \\ (2|A| - H)^2 + 8\left(\frac{3|A|}{4}\right)^2 &\leq H^2 \Rightarrow \\ 4|A|^2 - 4|A|H + \frac{9|A|^2}{2} &\leq 0 \end{aligned}$$

and thus

$$|A| \leq \frac{8(2N+1)}{17},$$

which contradicts  $|A| \geq N+1$ . We get  $|S_A(\alpha)| < \frac{2\sqrt{2}}{\pi}|A|$  for all  $\alpha = \frac{j}{H}$ ,  $j \neq 0$ .

In order to prove the first assertion of Lemma 2 we shall estimate  $\lambda_{\max}(G)$  using (8). We get

$$\lambda_{\max}(G) \leq H + 2 \max(|D_H(\alpha)|, |D_H(2\alpha)|).$$

From (12), we get

$$(2|A| - H)^2 + 2|2S_A(\alpha) - K_H(\alpha)|^2 \leq H(H + 2 \max(|D_H(\alpha)|, |D_H(2\alpha)|)).$$

Let  $\alpha \in E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right)$ . If

$$2|S_A(\alpha)| - |K_H(\alpha)| < 0$$

then

$$|D_H(\alpha)| > 2|S_A(\alpha)| > 2\frac{2\sqrt{2}}{\pi}|A| = \frac{4\sqrt{2}}{\pi}|A| \geq \frac{2\sqrt{2}}{\pi}H,$$

and thus  $\alpha \in \left(-\frac{1}{H}, \frac{1}{H}\right)$ , i.e.  $\alpha$  satisfies the condition 1 of Lemma 2.

Now let's assume

$$2|S_A(\alpha)| - |K_H(\alpha)| > 0$$

Then using  $|2S_A(\alpha) - K_H(\alpha)| \geq 2|S_A(\alpha)| - |D_H(\alpha)|$  and (12), we get

$$\begin{aligned} (2|A| - H)^2 + 2(2|S_A(\alpha)| - |D_H(\alpha)|)^2 \\ \leq H(H + 2 \max(|D_H(\alpha)|, |D_H(2\alpha)|)). \end{aligned}$$

It follows from here that

$$\max(|D_H(\alpha)|, |D_H(2\alpha)|) \geq \frac{H}{60}.$$

■

Lemma 2 shows that the values  $\alpha$  in the sums of the form

$$\sum_{a \in [h_1, h_2]} e(\alpha a)$$

are small and we can look at these sums as integral sums and replace them by integrals with an error which is sufficiently small.

## 2

The proof of Theorem 1 follows from steps  $A, B, C$  below.

- A.  $E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right) \subset \left(-\frac{1}{H}, \frac{1}{H}\right)$ , then the set is an arithmetic progression with a difference one.
- B. The set  $E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right)$  is included in a neighborhood of 0, defined in a Lemma 2; then  $E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right) \subset \left(-\frac{1}{H}, \frac{1}{H}\right)$
- C. We assume that there is  $\alpha$  such that  $\alpha \in E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right)$  and  $\alpha$  belongs to a neighborhood of  $1/2$ .

**Case A.** The condition in  $A$  is such that we look on trigonometric sums for which

$$\alpha \in \left(-\frac{1}{H}, \frac{1}{H}\right).$$

Choose  $\alpha^* \in E\left(A, \frac{2\sqrt{2}}{\pi}|A|\right)$ , where  $A$  is any set of cardinality  $|A| = m \geq N + 1$ . According to Lemma 1  $\max_{A, |A|=m} |S_A(\alpha^*)|$  is achieved if the set  $A$  is an arithmetic progression with a difference equal to 1. Denote it by  $A_0$ .

We see that for almost all  $\alpha \in E(A, \frac{2\sqrt{2}}{\pi}|A|)$  and for every set  $A$  such that  $|A| = |A_0|$  we have

$$|S_{A_0}(\alpha)| \geq |S_A(\alpha)| .$$

We conclude that in this case  $mesE\left(A_0, \frac{2\sqrt{2}}{\pi}|A_0|\right) \geq mesE\left(A, \frac{2\sqrt{2}}{\pi}|A|\right)$ .

**Case B.** We will show first that if

$$\frac{1}{H} < |\alpha| \leq \frac{100}{H}$$

then a.e. we have

$$|S_A(\alpha)| < \frac{2\sqrt{2}}{\pi}|A| . \quad (13)$$

In fact we will prove (13) only in case  $\alpha \in (\frac{1}{H}, \frac{2}{H})$ . This case is the most difficult and other cases may be studied analogously and they are much simpler.

The use of Lemma 1 shows that  $|S_A(\alpha)|$  in the case we study achieves its maximum value if  $A$  is a union of two segments:

$$A = [-N, -(k+1)] \cup [k+1, N] .$$

Thus

$$|S_A(\alpha)| \leq 2 \left| \sum_{s=k+1}^N e(\alpha s) \right| = 2 \left| \frac{\sin \pi(N-k)\alpha}{\sin \pi \alpha} \right| .$$

According to the condition of Theorem 1 we have  $|A| \geq N+1$  and therefore  $|A| = 2(N-k) \geq N+1$ ,  $1 - \frac{k}{N} > \frac{1}{2}$ .

In our case  $\alpha \in (\frac{1}{H}, \frac{2}{H}) = (\frac{1}{2N+1}, \frac{2}{2N+1})$ . Take  $\alpha = \frac{\theta}{N+0.5}$  and we will get

$$\frac{1}{2} < \theta < 1 .$$

Now we have

$$\begin{aligned} \frac{|S_A(\alpha)|}{|A|} &= \frac{2}{|A|} \left| \frac{\sin \pi \left(1 - \frac{k}{N}\right) \theta}{\sin \pi \frac{\theta}{N}} \right| = \\ &= \frac{|\sin \pi \left(1 - \frac{k}{N}\right) \theta|}{N \left(1 - \frac{k}{N}\right) |\sin \pi \frac{\theta}{N}|} = \frac{\sin \pi \left(1 - \frac{k}{N}\right) \theta}{\pi \left(1 - \frac{k}{N}\right) \theta} \left(1 + O\left(\frac{1}{N}\right)\right) \end{aligned}$$

Because of  $\frac{1}{2} < \theta < 1$  and  $\frac{1}{2} < (1 - \frac{k}{N}) < 1$  we have to find

$$\max_{\frac{1}{4} < x < 1} \frac{\sin \pi x}{\pi x}$$

A simple computation shows that for every  $\frac{\pi}{4} \leq x \leq \pi$  we have:

$$\frac{\sin x}{x} \leq \frac{\sin \pi/4}{\pi/4} = \frac{2\sqrt{2}}{\pi}$$

Thus, for every  $\alpha \in (\frac{1}{H}, \frac{2}{H})$  and  $N$  sufficiently large we have:

$$|S_A(\alpha)| \leq \frac{2\sqrt{2}}{\pi} |A| .$$

It is clear that both parts of this inequality may be equal for a set of zero measure and thus (13) is proved.

**Case C.** Let  $\alpha = \frac{1}{2} + \beta$ ,  $|\beta| \leq \frac{100}{N}$  and

$$\left| S_A \left( \frac{1}{2} + \beta \right) \right| \geq \frac{2\sqrt{2}}{\pi} |A|, |A| \geq N + 1 .$$

We will show that in this case

$$|\beta| \leq \frac{0.13}{N} .$$

Using Lemma 1 we can change the set  $A$  on a set which is a union of a finite number of segments.

Take

$$A_0 = \{a \in A : a \equiv 0(\text{mod } 2)\}, \quad A_1 = \{a \in A : a \equiv 1(\text{mod } 2)\}$$

and

$$2\pi\gamma = \arg S_A \left( \frac{1}{2} + \beta \right) .$$

We have

$$\left| S_A \left( \frac{1}{2} + \beta \right) \right| = S_A \left( \frac{1}{2} + \beta \right) e^{-2\pi i \gamma} = \sum_{a \in A_0} \cos 2\pi(a\beta - \gamma) - \sum_{a \in A_1} \cos 2\pi(a\beta - \gamma) .$$

Summing these two last sums on all even and odd numbers of the segment  $[-N, N]$  we will get the following estimate.

$$\left| S_A \left( \frac{1}{2} + \beta \right) \right| \leq \frac{1}{2} \sum_{x \in [-N, N]} |\cos 2\pi(\beta x - \gamma)| + O(1) = (1/2)\Sigma_1 + O(1) .$$

According to Lemma 2, the value of  $\beta$  is small and we can approximate  $\Sigma_1$ , with the aid of an integral

$$\Sigma_1 = \int_{-N}^N |\cos 2\pi(\beta x - \gamma)| dx = \frac{1}{\beta} \int_{-N\beta - \gamma}^{N\beta - \gamma} |\cos 2\pi t| dt .$$

Let us use the Cauchy inequality

$$\Sigma_1 \leq \frac{1}{\beta} \sqrt{2N\beta} \sqrt{N\beta + \frac{1}{4\pi}}$$

and thus

$$\left| S_A \left( \frac{1}{2} + \beta \right) \right| \leq \frac{1}{2} \frac{1}{\beta} \sqrt{2N\beta} \sqrt{N\beta + \frac{1}{4\pi}} + O(1) .$$

Using the estimate

$$\left| S_A \left( \frac{1}{2} + \beta \right) \right| \geq \frac{2\sqrt{2}}{\pi} |A| \geq \frac{2\sqrt{2}}{\pi} (N + 1) ,$$

we obtain

$$|\beta| \leq \frac{0.13}{N} .$$

Let  $A$  be an extremal set,

$$A \subseteq [-N, N]$$

for which the trigonometric sum is large in a neighborhood of  $\alpha = 1/2$ . Then

$$|S_A(\alpha)| \geq \frac{2\sqrt{2}}{\pi} |A|, \tag{14}$$

where  $\alpha = 1/2 + \beta$ . As was shown earlier

$$|\beta| \leq \frac{0.13}{N}. \tag{15}$$

Lemma 2, [1], pp.35 shows that the cardinality of the set of points  $\{\alpha a\}$ ,  $a \in A$  which belong to a suitable segment of the length  $\frac{1}{2}$  in  $\mathbb{R}/\mathbb{Z}$  is no less than  $0.95|A|$ .

Because  $\alpha = 1/2 + \beta$  and  $\beta$  is small in view of (15) we see that those  $a \in A$  for which  $\{\alpha a\} \in \gamma$  are or all even or all odd. Suppose, for example, that they are even. If  $|S_A(\beta + \frac{1}{2})| \geq \frac{2\sqrt{2}}{\pi}|A|$ , then from (15) we get  $|S_A(\beta)| \geq \frac{2\sqrt{2}}{\pi}|A|$ . These inequalities lead us to an equality

$$|S_A(\beta)|^2 + |S_A(\beta + \frac{1}{2})|^2 = |S_{A_0}(\beta) + S_{A_1}(\beta)|^2 + |S_{A_0}(\beta) - S_{A_1}(\beta)|^2 \quad (16)$$

$$= 2|S_{A_0}(\beta)|^2 + 2|S_{A_1}(\beta)|^2 \geq \frac{16}{\pi^2}|A|^2 \quad (17)$$

We will estimate now the measure of the set of solutions of inequality (14).

Let us stress once more that our conditions (14) imply that the maximal values of  $|S_{A_0}(\beta)|$  and  $|S_{A_1}(\beta)|$  are achieved as it follows from Lemma 1 only in a case when  $A_0$  consists of an interval of even numbers and  $A_1$  consists of an interval of odd numbers.

We see that if (16) takes place then  $A = A_0 \cup A_1$  and  $|A_0| \geq 0.95|A|$ . We have

$$|S_A(\alpha)| = |S_A(1/2 + \beta)| = \left| \sum_{a \in A_0} e^{2\pi i \beta a} - \sum_{a \in A_1} e^{2\pi i \beta a} \right| = |S_{A_0}(\beta) - S_{A_1}(\beta)|.$$

The modulus of the sum

$$S_{A_0}(\alpha) + S_{A_1}(\alpha)$$

will increase if  $A_0$  consists of a segment. In fact, the norm of the sum of two vectors can only grow, if we will increase the norm of the biggest of summands.

Then

$$|S_{A_0}(\beta)| \leq \left| \sum_{0 \leq a \leq |A_0| - 1} e^{2\pi i \beta 2a} \right|$$

and

$$|S_{A_1}(\beta)| \leq \left| \sum_{0 \leq a \leq |A_1|-1} e^{2\pi i \beta 2a} \right|.$$

We see that the measure of solutions of the inequality (14) is not greater than the measure of solutions of the inequality

$$\frac{\sin^2 2\pi\beta|A_0|}{\sin^2 2\pi\beta} + \frac{\sin^2 2\pi\beta|A_1|}{\sin^2 2\pi\beta} \geq \frac{8}{\pi^2}|A|^2.$$

And let now stress that the function

$$\sin^2 2\pi\beta|A_0| + \sin^2 2\pi\beta|A_1|$$

under the conditions (15),  $|A_0| \geq 0.95|A|$ , has a maximal value only if  $|A_0| = |A_1|$ . So, the measure of the set of solutions of inequality (14) is not greater than

$$mes\{\beta : \left| \frac{\sin 2\pi\beta|A|}{\sin 2\pi\beta} \right| \geq \frac{2\sqrt{2}}{\pi}|A|\},$$

and it is the measure of the set

$$\{\alpha : \left| \sum_{0 \leq a \leq N} e^{2\pi i \alpha a} \right| \geq \frac{2\sqrt{2}}{\pi}|A|\},$$

i.e. of arithmetic progression of the length  $N + 1$ . ■.

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