The Domatic Number Problem on Some Perfect Graph Families

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Abstract

An extremely simple, linear time algorithm is given for constructing a domatic partition in totally balanced hypergraphs. This simplifies and generalizes previous algorithms for interval and strongly chordal graphs. On the other hand, the domatic number problem is shown to be NP-complete for several families of perfect graphs, including chordal and bipartite graphs.

Keywords: algorithms, computational complexity, domatic number, domatic partition, strongly chordal graphs, chordal graphs, bipartite graphs, classes of perfect graphs, totally balanced hypergraphs.

1 Introduction

A *dominating set* in an undirected graph $G = (V, E)$ is a subset of the vertices, $S \subseteq V$, such that every vertex $v \in V - S$ is adjacent to a vertex in S. The *domatic number* of G, denoted $DN(G)$, is the maximum number k such that V can be partitioned into k dominating sets. The *domatic number problem* (DNP) is to decide for a graph G and

a constant k if $DN(G) \geq k$. The *domatic partition problem* (DPP) is to construct a partition of the vertices of G into $DN(G)$ dominating sets.

The domatic partition problem arises in communication networks [4]. The network is modeled by an undirected graph in which edges represent communication links and vertices represent cities. A *transmitting group* is a set of cities which, acting as transmitting stations, can transmit messages to every city in the network. Hence, a transmitting group in the network is a dominating set in the graph. Finding a maximum number of disjoint transmitting groups in a communication network is equivalent to the domatic partition problem in the corresponding graph.

It is easy to see that $DN(G) \leq \min_{v \in V} \delta(v) + 1$, where $\delta(v)$ is the degree of v. Graphs for which $DN(G) = \min\limits_{v \in V} \delta(v) + 1$ are called *domatically full*. Farber has shown that strongly chordal (SC) graphs are domatically full [6]. Hence, DNP is trivial on SC graphs. Farber's result is in fact a special case of a theorem due to Berge [2], which states that for a balanced hypergraph in which Δ is the minimum degree of an edge, there exists Δ disjoint transversal sets. (Definitions of all terms mentioned in this section are given in Section 2.) Berge's theorem, however, does not indicate *how to find* such partition in a balanced hypergraph or in the special case of SC graphs.

Rao and Rangan [14] gave a linear time algorithm which finds a domatic partition for interval graphs. Recently, Peng and Chang [13] described a linear time algorithm which constructs a domatic partition in a SC graph, given a strong elimination ordering of that graph. In section 3 we give a very simple linear time algorithm for finding a maximum number of disjoint transversal sets in a totally balanced hypergraph. In particular, this algorithm can find in linear time a domatic partition of a strongly chordal graph. The algorithm is extremely simple, and in addition to applying in more generality to totally balanced hypergraphs, both the algorithm and its proof are substantially simpler than those described in [13]. Since interval graphs and path graphs are SC, the algorithm solves the domatic partition problem for such graphs, simplifying previously known algorithms.

DNP is NP-complete for general graphs (cf. [7]). Bonuccelli proved that DNP is NP-complete for circular-arc graphs [3]. For DNP on proper circular-arc graphs he described an $O(|V|^2 \log(|V|))$ algorithm. In section 4 we show that DNP is NP-complete for split graphs, and therefore for chordal and co-chordal graphs, for every fixed $k \geq 3$. Since SC graphs are chordal, this result delineates a sharp border between polynomial and NP-complete cases of DNP. We also prove that DNP is NP-complete for bipartite graphs,

and thus also for comparability graphs, for every fixed $k \geq 3$.

2 Definitions and Background

Let $G = (V, E)$ be an undirected graph. The (closed) neighborhood of vertex v, denoted $N[v]$, is the set consisting of v together with all the vertices adjacent to v. A *strong elimination ordering* (SEO) of G is an ordering $v_1, v_2, \ldots, v_n$ of V with the property that for each i, j, k and l, if $i < j,\ k < l,\ v_k, v_l \in N[v_i]$, and $v_k \in N[v_j]$, then $v_l \in N[v_j]$. A graph is *strongly chordal* if it admits a strong elimination ordering. Strongly chordal graphs were first defined and characterized by Farber [5], who also gave a polynomial time recognition algorithm for them.

A 0-1 matrix is *balanced* if it does not contain as a submatrix an (edge-vertex) incidence matrix of an odd cycle, and is *totally balanced* (TB) if it does not contain as a submatrix an incidence matrix of any cycle of length greater than three. A 0-1 matrix is called Γ-*free* if it does not contain the submatrix $\Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The following relation holds between Γ-free and TB matrices:

Theorem 1 (Anstee-Farber [1],Lubiw [11], Hoffman Kolen Sakarovich [10]) *A 0-1 matrix is totally balanced if and only if one can order its rows and columns such that a Γ-free matrix is obtained.*

The *neighborhood matrix* of a graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is an $n \times n$ 0-1 matrix $M(G)$ s.t. $M_{ij} = 1$ iff $v_i \in N[v_j]$. Farber proved:

Theorem 2 (Farber [5]) *G is strongly chordal if and only if $M(G)$ is totally balanced.*

A *hypergraph* is an ordered pair $H = (V, E)$ such that V is a finite set of elements called *vertices* and E is a collection of subsets of V whose elements are called *edges* of H. Let $M(H)$ be the edge-vertex incidence matrix of a hypergraph H. H is called *balanced* (resp., *TB*) if $M(H)$ is balanced (resp., TB). A *transversal* in a hypergraph is a set of vertices meeting all of the edges. Berge proved the following duality theorem pertaining to transversals in hypergraphs:

Theorem 3 (Berge [2]) *In a balanced hypergraph H, the maximum number of pairwise disjoint transversals equals the minimum cardinality of an edge.*

Restating the hypergraph notions in matrix terminology, we define: A *dominating set* of columns in a 0-1 matrix M is a set of columns such that the entries of every row corresponding to these columns are not all zero. A *domatic partition* of M is a partition of its columns into dominating sets. Thus, Berge's theorem says that if M is balanced and Δ is the minimum number of ones in a row of M, there is a domatic partition of M into Δ dominating sets. Since a domatic partition of a graph G is equivalent to a domatic partition in $M(G)$, Farber [6] using Theorems 3 and 2 observed:

Theorem 4 *Strongly chordal graphs are domatically full.*

Lubiw [11] described polynomial algorithms to obtain a Γ-free ordering of a 0-1 matrix or determine that no such ordering exists, and to obtain a SEO of a SC graph. Paige and Tarjan [12] improved Lubiw's algorithms, giving an $O(e \log(n+m) + n + m)$ algorithm for the Γ-free ordering of an $m \times n$ matrix with e nonzeros. Their implementation can be used to recognize a SC graph and to obtain a SEO of it in $O(|E|\log(|V|))$. For dense matrices, Spinrad [15] recently gave an $O(nm)$ algorithm for constructing a Γ-free ordering.

3 An algorithm for totally balanced hypergraphs

We now present an algorithm which constructs a domatic partition of a TB matrix. In particular, using Theorem 4, applying the algorithm to the neighborhood matrix of a SC graph G gives a domatic partition of G. The algorithm also provides a simpler proof of Berge's Theorem, for the special case of TB hypergraphs.

We assume that the TB matrix A is in Γ-free form, and that Δ is the minimum number of ones in a row of A. The algorithm constructs a partition of the columns into Δ sets. If column k is assigned to the i-th set then we say that column k is given *color* i, and that color i *dominates* all rows j with $A_{jk} = 1$. If color i does not dominate row j we shall say that color i is *missing* for row j.

The algorithm scans rows in decreasing order, and in each row it scans the non-zero positions from right to left. When position ij is scanned, if column j is uncolored and some color c is still missing for row i, the algorithm gives color c to column j. Upon termination, uncolored columns may be colored arbitrarily. A formal description of the algorithm follows. (Initially all columns are uncolored).

```
algorithm DP(A);
begin
    for row i = m, m − 1, ..., 1 do
    for column j = n, n − 1, ..., 1 do
        if A_ij = 1, column j is uncolored
        and there is a color c which does not dominate row i then
        give column j color c.
end
```

Let e be the number of ones in the $m \times n$ TB matrix.

Theorem 5 *The above algorithm constructs a domatic partition in a Γ-free matrix in $O(e + m + n)$ steps.*

Proof. Validity: Suppose columns j and k have color c before scanning row i and $A_{ij} = A_{ik} = 1$, $j < k$. Say column j was colored by color c during the scanning of row l, $l > i$ Clearly $A_{lj} = 1$, since while scanning a row we color only columns with one in that row. Also $A_{lk} = 1$ or else the matrix is not Γ-free. But then, if column j is colored during the scan of row l, column k must already have a color. If that color is c then column j cannot be assigned the same color while scanning row l, a contradiction.

Hence, before scanning row i, no two columns j and k such that $A_{ij} = 1$ and $A_{ik} = 1$ have the same color. Since row i has at least Δ ones, when the algorithm scans this row there are always at least as many uncolored columns j with $A_{ij} = 1$ as there are missing colors.

Complexity: By one pass on the nonzero entries of a row we can determine which colors do not dominate it yet. Another pass suffices to distribute the missing colors to uncolored columns from right to left. ■

4 NP-completeness results

We now show that DNP is NP-complete for several families of graphs. For more information on these families, see [9] and the references thereof.

A graph $G = (V, E)$ is a *split graph* if there is a partition of the vertex set $V = V' + V''$ where V' induces a clique in G and V'' induces an independent set. (Throughout, $+$ denotes a union of disjoint, non-empty sets).

Theorem 6 *The domatic number problem for split graphs is NP-complete, for every fixed $k \geq 3$.*

Proof. First, we prove the claim for $k = 3$: Clearly the problem is in NP. We give a reduction from the NP-complete 3-coloring problem: Given a graph $G = (V, E)$, is it 3-colorable? [8]. We can assume without loss of generality that G is not 1- or 2-colorable. Given a graph G for the 3-coloring problem, form a new graph $\tilde{G}$ by adding a new vertex on each of the original edges, and adding edges to form a clique on the original vertices. Precisely, $\tilde{G} = (\tilde{V}, \tilde{E})$ where $\tilde{V} = V' + V''$, $V' = V$, $V'' = \{v_{ij} | ij \in E\}$, and $\tilde{E} = E' + E''$. E' is a clique on V' and $E'' = \{iv_{ij}, jv_{ij} | ij \in E\}$. The construction of $\tilde{G}$ is clearly polynomial.

Observe that $\tilde{G}$ is a split graph, since V' induces a clique and V'' induces an independent set. Since $\min_{v \in \tilde{V}} \delta(v) = 2$, $DN(\tilde{G}) \leq 3$. We claim that G is 3-colorable if and only if $DN(\tilde{G}) = 3$.

Suppose G is 3-colorable. The 3-coloring induces a partition $V = V_1 + V_2 + V_3$ where V_i is the set of vertices colored by color i, and each V_i induces an independent set in G. Form a partition of $\tilde{V} = \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3$ as follows: Assign each $v \in V'$ to $\tilde{V}_i$ if $v \in V_i$. For $v_{ij} \in V''$, if $i \in V_\ell$, $j \in V_m$, then assign v_{ij} to the third class $\tilde{V}_k$, $k \neq \ell$, $k \neq m$. Each $\tilde{V}_i$ clearly dominates V', since $\tilde{V}_i$ is represented in V' and V' induces a clique. Also, each triangle $\{k, j, v_{kj}\}$ contains one representative from each $\tilde{V}_i$, so each $\tilde{V}_i$ also dominates V''.

For the converse, suppose $DN(\tilde{G}) = 3$. Given a partition into 3 dominating sets, color vertices by their set numbers in the partition. Every triangle $\{k, j, v_{kj}\}$ is 3-colored, which implies that the same coloring on $V' = V$ is a proper 3-coloring of G.

The split graph DNP for fixed $k = 3$ can be reduced to the split graph DNP for every fixed $k > 3$ as follows: Given a split graph G as an input to the DNP for $k = 3$, add $k - 3$ new vertices to the clique in G and connect them to all other vertices. The new graph is a split graph, and it has a domatic number not less than k if and only if $DN(G) \geq 3$. ■

A graph is *triangulated* or *chordal* if every cycle of length at least four contains a chord. A graph is *co-chordal* if its complement is chordal. Since split graphs are chordal and co-chordal (cf. [9]) we conclude:

Corollary 7 *The domatic number problem is NP-complete for chordal and for co-chordal graphs.* ■

Example Using the proof of Theorem 6 one can construct a simple example of a chordal graph which is not domatically full: Consider the graph G in figure 1. Here the minimum degree is 2, and G is a split graph. However, no partition into three sets forms a domatic partition: In any partition, two vertices out of a, b, c, d will be in the same set, thereby forming a triangle which is not 3-colored.

ADD FIG 1 HERE

A graph $G = (V, E)$ is *bipartite* if its vertex set V can be partitioned into $V = V' + V''$ such that $E \subseteq V' \times V''$.

Theorem 8 *The domatic number problem is NP-complete for bipartite graphs, for every fixed $k \geq 3$.*

Proof. First, we prove that the bipartite graph DNP is NP-complete for $k = 3$: Membership in NP is easily established. The reduction will again be from 3-colorability. For an instance $G = (V, E)$ of 3-colorability, form a bipartite graph $\tilde{G}$, by adding a vertex on each edge of the original graph and connecting each of the original vertices to three special new vertices. Precisely, $\tilde{G} = (V', V'', \tilde{E})$ where $V' = V, V'' = \{v_{ij} | ij \in E\} + \{w_1, w_2, w_3\}$ and $\tilde{E} = \{iv_{ij}, jv_{ij} | ij \in E\} + \{iw_k | i \in V', k = 1, 2, 3\}$. The reduction is clearly polynomial.

If G is 3-colorable, let $V_1 + V_2 + V_3$ be a partition of V into three independent sets in G. Assign each $v \in V'$ in $\tilde{G}$ to $\tilde{V}_i$ if $v \in V_i$ in G. For each edge $ij \in E$, assign v_{ij} into the third set not assigned to either i or j. (Since each V_k is independent no edge $ij \in E$ has both endpoints in the same set so the third set is uniquely defined.) Finally assign w_i to $\tilde{V}_i$, $i = 1, 2, 3$. The resulting partition $\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3$ is domatic in $\tilde{G}$: Each vertex in V' is dominated by w_i and hence by $\tilde{V}_i$ for $i = 1, 2, 3$. For $v_{ij} \in V''$, the three vertices v_{ij}, i, j are assigned to three different sets, hence each set dominates v_{ij}. Finally, since this is a 3-coloring (and not 1− or 2-coloring), each set also dominates w_1, w_2, w_3.

For the converse, given a domatic 3-partition of $\tilde{V}$, in each triplet $\{i, j, v_{ij}\}$ the three sets are represented. Hence, in that partition induced on $V' = V$, no edge $ij \in E$ has both endpoints in the same set, and thus it is a proper 3-coloring of G.

The bipartite graph DNP for fixed $k = 3$ can be reduced to the bipartite graph DNP for every fixed $k > 3$ as follows: Given a bipartite graph $G = (V_1, V_2, E)$ as an

input to the DNP for $k = 3$, build a bipartite graph $G' = (V_1 + X_1, V_2 + X_2, E + F)$ where $|X_1| = |X_2| = k - 3$, $F = V_1 \times X_2 + V_2 \times X_1 + X_1 \times X_2$. We claim that $DN(G) \geq 3$ iff $DN(G') \geq k$. The 'only-if' part is clear. For the 'if' part, let c be a k-coloring of G' such that each color class is dominating. We shall denote color classes by superscripts. Let S_i be the set of colors which are not represented in X_i, $i = 1, 2$, Note that $|S_i| \geq 3$. If $j \in S_1 \cap S_2$ then $V_1^j + V_2^j$ is dominating in G. If $j \in S_1 - S_2, k \in S_2 - S_1$ then $V_1^j + V_2^j + V_1^k + V_2^k$ is dominating in G: All the vertices in V_2 are dominated by $V_1^j + V_2^j$ and all the vertices in V_1 are dominated by $V_1^k + V_2^k$. Since $|S_i| \geq 3$, $i = 1, 2$ $|S_1 \cap S_2| + \min\{|S_1 - S_2|, |S_2 - S_1|\} \geq 3$ and $V_1 + V_2$ can be partitioned into at least 3 dominating sets in G. ■

A *comparability graph* is an undirected graph whose edges can be oriented transitively.

Corollary 9 *The domatic number problem is NP-complete on comparability graphs for every fixed $k \geq 3$.*

Proof. Immediate, since every bipartite graph is a comparability graph. ■

A comparability graph which has exactly two transitive orientations is called *uniquely partially orderable.*

Corollary 10 *The domatic number problem is NP-complete on uniquely partially orderable (UPO) graphs. for every fixed $k \geq 3$.*

Proof. Follows since every connected bipartite graph is UPO, and since the proof of Theorem 8 still holds if we require the bipartite graph to be connected. ■

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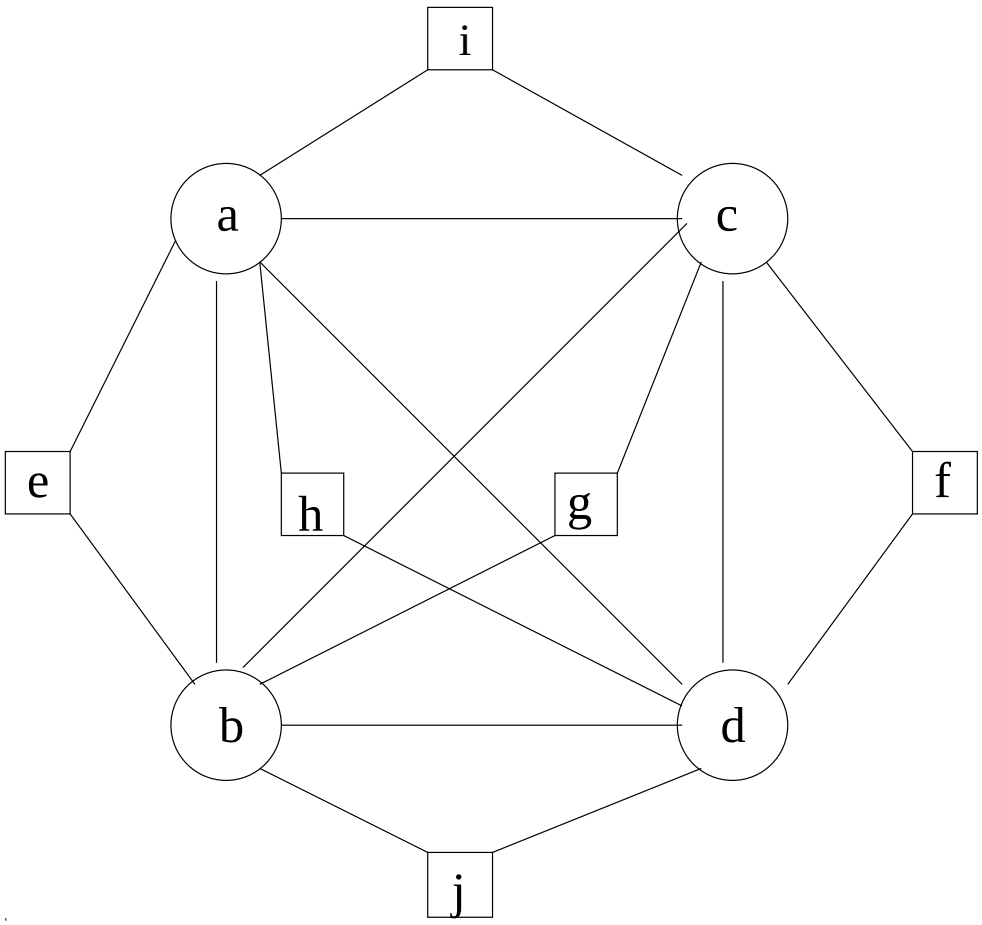

Figure 1: A chordal graph which is not domatically full.