Normal subgroups of profinite groups of finite cohomological dimension

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Abstract

We study a profinite group G of finite cohomological dimension with (topologically) finitely generated closed normal subgroup N. If G is pro-p and N is either free as a pro-p group or a Poincaré group of dimension 2 or analytic pro-p, we show that G/N has virtually finite cohomological dimension $\mathrm{cd}(G) - \mathrm{cd}(N)$. Some other cases when G/N has virtually finite cohomological dimension are considered too.

If G is profinite, the case of N projective or the profinite completion of the fundamental group of a compact surface is considered.

Introduction

The purpose of this paper is to study profinite groups of finite cohomological dimension having finitely generated normal subgroups. (In this paper, subgroups are always closed and homomorphisms are continuous; *finitely generated* is meant in the topological sense.) Our work is inspired by Bieri [2], [3] where the case of discrete groups of finite cohomological dimension 2 is treated. Bieri's approach is based on a spectral sequence argument that helps to study the cohomology groups $H^i(G, \mathbb{Z}[G])$ for $i \geq 1$. The latter can be viewed as a higher dimensional analogue of ends of groups, a theory that by now has not been developed in the profinite and in the pro-p cases. Our approach is close to [2] in the sense that we study spectral sequences and different in the sense that we do not rely on profinite (pro-p) analogues of the above cohomology groups.

We say that a profinite group G has *virtually* a given property if there is an open subgroup H of G which has the given property. Let $\mathrm{vcd}(G)$, $\mathrm{vcd}_p(G)$ denote the virtual cohomological dimension and the virtual cohomological p-dimension of G, respectively.

Our first result is about projective normal subgroups in profinite groups of finite cohomological dimension.

Theorem 1. *Let G be a profinite group of finite cohomological dimension $\mathrm{cd}(G) = n$ and N be a finitely generated normal projective subgroup. Then $\mathrm{vcd}_p(G/N) \leq n$ for every prime p. Moreover, denote by $N(p)$ the maximal pro-p quotient of N and let $\bar{T}$ be a finite p-subgroup of G/N. Then $\mathrm{rank}(N(p)) > |\bar{T}|$ (= the order of $\bar{T}$), provided $N(p)$ is not procyclic. If $N(p)$ is procyclic, then $\bar{T}$ is cyclic. In particular, $N(p) = 1$ implies $\bar{T} = 1$.*

*Partially supported by research grants from CNPq, Brazil.
†Partially supported by a grant from FAPESP, Brazil.

In particular, in the pro-p case we have

Theorem 2. *Let G be a pro-p group of finite cohomological dimension $\mathrm{cd}(G) = n$ and N be a non-trivial finitely generated normal subgroup that is free as a pro-p group. Then G/N has virtually finite cohomological dimension $n - 1$.*

Zelmanov [16] has shown that a torsion pro-p group (i.e., group whose elements are of finite order) is locally finite (every finitely generated subgroup of it is finite). This allows us to obtain more precise information as to when G is finitely generated of cohomological dimension 2. We note that it is a long standing open problem whether a torsion pro-p group has finite exponent.

Corollary 3. *Let G be a pro-p group of finite cohomological dimension $\mathrm{cd}(G) \leq 2$ and N be a non-trivial finitely presented (as a pro-p group) normal subgroup. Then either G/N is torsion or N is free as a pro-p group with G/N virtually free pro-p.*

In particular, if G is finitely generated, either N is open in G or N is free as a pro-p group with G/N virtually free pro-p group.

Using the description of finitely generated virtually free pro-p groups obtained in [5] we can deduce

Corollary 4. *Let G be a finitely generated pro-p group of cohomological dimension 2 and N be a non-trivial free finitely generated normal subgroup of G. Then G is the fundamental group of a finite graph of finitely generated free pro-p groups. Furthermore, G is finitely presented. If N is procyclic, then vertex and edge groups are procyclic.*

In fact, our proof starts with considering pro-p case first, where in general cohomology techniques work better. The proofs are based on examining the differentials in the Lyndon-Hochschild-Serre spectral sequence in pro-p cohomology for the extensions $1 \to N/\Phi(N) \to G/\Phi(N) \to G/N \to 1$ and $1 \to N \to G \to G/N \to 1$ where $\Phi(N)$ is the Frattini subgroup of N. Theorem 2 is a corollary of the following more general result.

Theorem 5. *Let G be a pro-p group of finite cohomological dimension and N a non-trivial finitely generated normal subgroup of G. Let $k = \mathrm{cd}(N)$. Suppose that the inflation $H^k(N/\Phi(N), \mathbb{F}_p) \to H^k(N, \mathbb{F}_p)$ is surjective. Then G/N has virtually finite cohomological dimension $\mathrm{cd}(G) - \mathrm{cd}(N)$.*

As corollaries we also obtain the following two results.

Theorem 6. *Let G be a pro-p group of cohomological dimension $\mathrm{cd}(G) = n$ and N be a normal subgroup of G, such that N is a Poincaré group of dimension 2. Then G/N is virtually of finite cohomological dimension $n - 2$.*

The profinite version of this theorem (when N is a profinite surface group) can be found in Section 5.

Theorem 7. *Let G be a pro-p group of cohomological dimension n and let N be a normal subgroup of G such that N is analytic pro-p of dimension $k \geq 1$. Then G/N has virtually finite cohomological dimension $n - k$.*

We briefly compare the case of discrete groups of finite cohomological dimension with the profinite one. By [2, Thm. B], if G is a discrete finitely generated group with $\mathrm{cd}(G) = 2$ and N a normal finitely presented subgroup, then N is either free or of finite index in G; this is a discrete counterpart of Corollary 3. Further, a discrete counterpart of Corollary 5.10 is given in [3, Thm. 8.8(b)] and a higher dimensional version of Corollary 3 for a discrete Poincaré duality group in [3, Prop. 9.22]. We do not know of the existence of any version of Theorem 5 for discrete groups beyond the cases stated above.

The paper is structured as follows. In Section 1 the basic properties and definitions used in the paper are collected. Section 2 is dedicated to Theorem 5. The cases of N free pro-p, a Demushkin group and analytic pro-p are treated in Sections 3 and 4, respectively. The profinite case is treated in Section 5, where some other consequences of Theorem 1 are obtained.

When the paper was submitted the fourth author learned that similar results were announced (but never written) by O.V. Melnikov at Ukranian Mathematical Congress in Kiev (2001).

1 Preliminaries

In this section we collect all the basic properties and definitions of homological nature about profinite and pro-p groups that will be needed in this paper.

1. By definition, a pro-p group G has cohomological dimension $\mathrm{cd}(G) \leq n$ if for every discrete G-module A and all $k > n$ we have $H^k(G, A) = 0$. By [10, Cor. 7.1.6]

$$\mathrm{cd}(G) \leq n \text{ if and only if } H^{n+1}(G, \mathbb{F}_p) = 0,$$

   where $\mathbb{F}_p$ is the cyclic group with p elements.

   By definition, the cohomological p-dimension $\mathrm{cd}_p(G)$ of a profinite group G is the smallest non-negative integer n such that $H^k(G, A)_p = 0$ for all $k > n$ and every discrete $[[\widehat{\mathbb{Z}}\, G]]$-module A, where $[[\widehat{\mathbb{Z}}\, G]]$ denotes the complete group ring. The cohomological dimension $\mathrm{cd}(G)$ of G is the supremum of $\mathrm{cd}_p(G)$ over all primes p.

2. A profinite group has cohomological dimension ≤ 1 if and only if it is projective (see Proposition 7.6.7 in [10]). Projective pro-p groups are free pro-p, so pro-p groups of cohomological dimension ≤ 1 are free pro-p groups [10, Thm. 7.7.4], [15, Cor. 11.2.3, 11.2.4].

3. Let H be a subgroup of a profinite group G. Then $\mathrm{cd}_p(H) \leq \mathrm{cd}_p(G)$. If either $\mathrm{cd}_p(G) < \infty$ and H is open in G or $(G : H)$ is prime to p then $\mathrm{cd}_p(H) = \mathrm{cd}_p(G)$ [10, Thm. 7.3.1]. For example for H a p-Sylow subgroup of G i.e. a subgroup of G such that $\#H$ is a p-number and $(G : H)$ is a p'-number $\mathrm{cd}_p(H) = \mathrm{cd}_p(G)$. For every prime p there are p-Sylow subgroups in G and they are all conjugate in G.

4. For a profinite group G with no subgroups of order p and an open subgroup H of G we have $\mathrm{cd}_p(G) = \mathrm{cd}_p(H)$ [10, Thm. 7.3.7, p. 274]; thus $\mathrm{vcd}_p(G) = \mathrm{cd}_p(G)$. In particular, if G is torsion-free, of finite virtual cohomological dimension $\mathrm{vcd}(G) = n$, then $\mathrm{cd}(G) = n$.

5. In general, if N is a normal subgroup of a profinite group G with both $\mathrm{cd}_p(N)$ and $\mathrm{cd}_p(G/N)$ finite, then $\mathrm{cd}_p(G) \leq \mathrm{cd}_p(N) + \mathrm{cd}_p(G/N)$, but the inequality can be strict [15, Prop. 11.3.1]. Still, if N is pro-p and $H^k(N, \mathbb{F}_p)$ is finite for $k = \mathrm{cd}_p(N)$ or N is inside the center of G, there is equality, i.e., $\mathrm{cd}_p(G) = \mathrm{cd}_p(N) + \mathrm{cd}_p(G/N)$ [10, Prop. 7.4.2].

6. A pro-p group is a Poincaré group of dimension n if all of the following conditions hold:
   1. $H^i(G, \mathbb{F}_p)$ is finite for every i;
   2. $H^n(G, \mathbb{F}_p) \simeq \mathbb{F}_p$;
   3. $H^i(G, \mathbb{F}_p) = 0$ for $i > n$;
   4. For every $0 \leq i \leq n$ the cup product
   $$H^i(G, \mathbb{F}_p) \times H^{n-i}(G, \mathbb{F}_p) \to H^n(G, \mathbb{F}_p)$$
   is a non-degenerated bilinear form.

   A Poincaré group is finitely generated, since $H^1(G, \mathbb{F}_p)$ is finite. There is only one Poincaré group of dimension 1, the procyclic group $\mathbb{Z}_p$.

7. Poincaré groups of dimension 2 are called Demushkin groups.

8. An example of Poincaré group of dimension n is a uniformly powerful pro-p group with a minimal set of generators of cardinality n. By a result of Lazard, for such groups the cohomology algebra $\bigoplus_{i \geq 0} H^i(G, \mathbb{F}_p)$ is the exterior algebra of $H^1(G, \mathbb{F}_p)$ [15, Thm. 11.6.1], [7].

9. In general, a pro-p group G has the structure of an analytic pro-p group if and only if it has an open subgroup which is uniformly powerful pro-p group [4, Thm. 9.34]. Hence an analytic pro-p group has an open characteristic subgroup which is uniformly powerful pro-p. The direct product $\mathbb{Z}_p^n$ is uniformly powerful.

2 The pro-p case in full generality

In this section G is a pro-p group such that $\mathrm{cd}(G) = n$ is finite, N is a normal finitely generated subgroup of G and $\Phi(N)$ is the Frattini subgroup of N. Set $\bar{N} = N/\Phi(N)$ and $\bar{G} = G/\Phi(N)$. Then $\bar{N}$ is abelian and $\bar{G}/\bar{N} \simeq G/N$. From now on we identify $\bar{G}/\bar{N}$ with G/N. Denote by $\mathbb{F}_p$ the field of p elements.

Let $E_t^{r,s}$ and $\bar{E}_t^{r,s}$ be, respectively, the groups of the Lyndon-Hochschild-Serre spectral sequences such that $E_2^{r,s} = H^r(G/N, H^s(N, \mathbb{F}_p))$ and $\bar{E}_2^{r,s} = H^r(G/N, H^s(\bar{N}, \mathbb{F}_p))$. Observe first that the canonical projection $N \to \bar{N}$ induces an isomorphism of G/N-modules $H^1(N, \mathbb{F}_p) \simeq H^1(\bar{N}, \mathbb{F}_p)$. Therefore $E_2^{r,s} = \bar{E}_2^{r,s}$ for $s = 0, 1$.

We shall first assume that the exact sequence

$$1 \longrightarrow \bar{N} \longrightarrow \bar{G} \longrightarrow G/N \longrightarrow 1 \qquad (\dagger)$$

is central and splits, i.e., gives a decomposition of $\bar{G}$ as $\bar{N} \times G/N$. In the next lemma we show that the differentials of the spectral sequence associated to this decomposition are trivial for every $t \geq 2$.

Lemma 2.1. *Let $\bar{M}, \bar{N}$ be pro-p groups and let $\bar{G} = \bar{M} \times \bar{N}$. Let $\bar{E}_t^{r,s}$ be the Lyndon-Hochschild-Serre spectral sequence such that $\bar{E}_t^{r,s} = H^r(\bar{M}, H^s(\bar{G}/\bar{M}, \mathbb{F}_p))$. Then the differential map $\bar{d}_t^{r,s}\colon \bar{E}_t^{r,s} \to \bar{E}_t^{r+t,s+1-t}$ is zero for all $t \geq 2$.*

Proof. This is a very special case of the fact that the Lyndon-Hochschild-Serre spectral sequence with second term $H^p(G, H^q(H, B))$ that converges to $H^{p+q}(G \times H, B)$ degenerates for all profinite groups G and H and any discrete H-module B, regarded as a $(G \times H)$-module via the trivial G-action [6]. $\square$

In the general case, $(\dagger)$ is virtually central and splits:

Lemma 2.2. *If N is finitely generated, there exists an open subgroup U of G containing $\Phi(N)$ such that $NU/\Phi(N)$ is the direct product of $N/\Phi(N)$ and $U/\Phi(N)$.*

Proof. As $\bar{G}$ is Hausdorff, we can find for every $a \in \bar{N} \smallsetminus \{1\}$ an open subgroup U_a in $\bar{G}$ that does not contain a. Let U be the preimage in G of $\bigcap_{a \in \bar{N} \setminus \{1\}} U_a$. Then $U \cap N = \Phi(N)$. As $\bar{N}$ is finite, U is open in G. Furthermore, G acts on $\bar{N}$ via conjugation and this gives a homomorphism $\theta\colon G \to \mathrm{Aut}(\bar{N})$. The kernel of θ is an open subgroup in G. Replacing U, if necessary, by $U \cap \operatorname{Ker} \theta$, we may assume that the elements of $U/\Phi(N)$ commute with the elements of $N/\Phi(N)$. $\square$

Proof of Theorem 5

By Lemma 2.2 we may replace G by an open subgroup containing N to assume that $G/\Phi(N) \simeq N/\Phi(N) \times G/N$. As $N/\Phi(N)$ is a finite abelian p-group, all cohomology groups $H^i(N/\Phi(N), \mathbb{F}_p)$ are finite. By assumption, the inflation $H^k(N/\Phi(N), \mathbb{F}_p) \to H^k(N, \mathbb{F}_p)$ is surjective, hence $H^k(N, \mathbb{F}_p)$ is finite. Substituting G by another open subgroup containing N we may assume that G acts trivially on $H^k(N/\Phi(N), \mathbb{F}_p)$ and $H^k(N, \mathbb{F}_p)$. Also, $H^k(N, \mathbb{F}_p)$ is a direct sum of m copies of $\mathbb{F}_p$, for some m.

We now consider the following two Lyndon-Hochschild-Serre spectral sequences $E_2^{r,s} = H^r(G/N, H^s(N, \mathbb{F}_p))$ and $\bar{E}_2^{r,s} = H^r(G/N, H^s(N/\Phi(N), \mathbb{F}_p))$. We show that for $k = \mathrm{cd}(N)$ and $n = \mathrm{cd}(G)$ we have $E_2^{n+1-k,k} = E_\infty^{n+1-k,k}$. As $E_\infty^{n+1-k,k}$ is a subquotient of $H^{n+1}(G, \mathbb{F}_p) = 0$, we deduce that $E_2^{n+1-k,k} = 0$. Thus $\bigoplus_{i=1}^m H^{n+1-k}(G/N, \mathbb{F}_p) = H^{n+1-k}(G/N, H^k(N, \mathbb{F}_p)) = E_2^{n+1-k,k} = 0$. In particular, $H^{n+1-k}(G/N, \mathbb{F}_p) = 0$. Thus G/N is of finite cohomological dimension that is not larger than $n - k$. As $H^k(N, \mathbb{F}_p)$ is finite, we may use the additive formula $\mathrm{cd}(G/N) = \mathrm{cd}(G) - \mathrm{cd}(N)$ of (1.5).

We consider in more detail the spectral sequence $E_2^{r,s}$. First note that it has only $k+1$ non-trivial lines, i.e., $E_2^{r,s} = 0$ for $s \geq k+1$ and hence $E_{k+2}^{r,s} = E_\infty^{r,s} = 0$ for all r, s. We show that the differential $d_t^{n+1-k,k}\colon E_t^{n+1-k,k} \to E_t^{n+1+t-k,k-t+1}$ is zero for $2 \leq t \leq k+1$. Since

$k+t-1 \geq k+1$, we have $E_t^{n+1-k-t,k+t-1} = 0$, hence the differential that enters $E_t^{n+1-k,k}$ is zero and the above property will imply that $E_2^{n+1-k,k} = E_3^{n+1-k,k} = \cdots = E_{k+2}^{n+1-k,k} = E_\infty^{n+1-k,k}$.

Finally, we prove that $d_t^{n+1-k,k} = 0$, $t \geq 2$. The essential case is $t = 2$. By Lemma 2.1 and the commutative diagram

$$\begin{array}{ccc} \bar{E}_2^{n+1-k,k} & \xrightarrow{\bar{d}_2^{n+1-k,k}=0} & \bar{E}_2^{n+3-k,k-1} \\ \Big\downarrow \pi_2^{n+1-k,k} & & \Big\downarrow \pi_2^{n+3-k,k-1} \\ E_2^{n+1-k,k} & \xrightarrow{d_2^{n+1-k,k}} & E_2^{n+3-k,k-1} \end{array}$$

it is sufficient to show that $\pi_2^{n+1-k,k}$ is an epimorphism.

Viewing the inflation map $\theta : H^k(N/\Phi(N), \mathbb{F}_p) \to H^k(N, \mathbb{F}_p)$ as a map of finite dimensional vector spaces, it has a right inverse λ, which is a map of trivial G/N-modules. Then the induced map

$$\pi_2^{n+1-k,k} \colon H^{n+1-k}(G/N, H^k(N/\Phi(N), \mathbb{F}_p)) \to H^{n+1-k}(G/N, H^k(N, \mathbb{F}_p))$$

has a right inverse induced by λ.

Now, let $t > 2$. Assume we have proven that $d_i^{n+1-k,k} = 0$ for $2 \leq i \leq t-1$. Then $E_t^{n+1-k,k} = E_2^{n+1-k,k}$. By Lemma 2.1 all the differentials of $\bar{E}_t$ for $t \geq 2$ are trivial and hence $\bar{E}_t^{n+1-k,k} = \bar{E}_2^{n+1-k,k}$ and $\pi_t^{n+1-k,k} = \pi_2^{n+1-k,k}$ is surjective. Finally, we consider the commutative diagram

$$\begin{array}{ccc} \bar{E}_t^{n+1-k,k} & \xrightarrow{\bar{d}_t^{n+1-k,k}=0} & \bar{E}_t^{n+1+t-k,k-t+1} \\ \Big\downarrow \pi_t^{n+1-k,k} & & \Big\downarrow \pi_t^{n+1+t-k,k-t+1} \\ E_t^{n+1-k,k} & \xrightarrow{d_t^{n+1-k,k}} & E_t^{n+1+t-k,k-t+1} \end{array}$$

As $\pi_t^{n+1-k,k}$ is surjective, we deduce that $d_t^{n+1-k,k} = 0$, as required. □

Corollary 2.3. *Let G be a pro-p group of finite cohomological dimension $\mathrm{cd}(G) = n$, N a finitely generated normal subgroup of G such that $H^*(N, \mathbb{F}_p)$ is generated as a ring by $H^1(N, \mathbb{F}_p)$. Then G/N has virtually finite cohomological dimension $\mathrm{cd}(G) - \mathrm{cd}(N)$.*

Proof. First note that all $H^i(N, \mathbb{F}_p)$ are finite dimensional as images of $\wedge^i H^1(N, \mathbb{F}_p)$. Consider the commutative diagram for $k = \mathrm{cd}(N)$

$$\begin{array}{ccc} \wedge^k H^1(N/\Phi(N), \mathbb{F}_p) & \xrightarrow{\alpha_1} & \wedge^k H^1(N, \mathbb{F}_p) \\ \Big\downarrow \beta_1 & & \Big\downarrow \beta_2 \\ H^k(N/\Phi(N), \mathbb{F}_p) & \xrightarrow{\alpha_2} & H^k(N, \mathbb{F}_p) \end{array}$$

where the horizontal maps are induced by the projection $N \to N/\Phi(N)$ and the vertical maps are given by the $\cup$-product. By assumption, β_2 is an epimorphism; obviously α_1 is an isomorphism. Hence α_2 is an epimorphism and we may apply Theorem 5. □

3 N is a free group

In this section we discuss the case when N is a non-trivial free subgroup of a pro-p group G with $\mathrm{cd}(G) = n < \infty$.

Theorem 2 follows immediately from Theorem 5, since for $k = 1$ θ is always surjective (actually, an isomorphism).

In particular, if $n = 2$, then G/N is virtually free. Hence, by Serre's result [12] or its generalization (1.4), G/N is either free or has elements of finite order.

Under some additional conditions we can get more than the general result Theorem 2 for the quotient group G/N. This is the content of the next three propositions. In particular, we discuss the cases where rank(N) is bounded above by p. In the case N of rank 1, Proposition 3.3 below is a special case of Theorem 1 for pro-p groups.

Lemma 3.1. *If (†) is central and splits, then* $d_2^{n-1,1}\colon E_2^{n-1,1} \to E_2^{n+1,0}$ *is zero.*

Proof. We have noticed that $E_2^{n-1,1} \simeq \bar{E}_2^{n-1,1}$ and $E_2^{n+1,0} \simeq \bar{E}_2^{n+1,0}$, so it suffices to show that $\bar{d}_2^{n-1,1}\colon \bar{E}_2^{n-1,1} \to \bar{E}_2^{n+1,0}$ is zero. This is the content of Lemma 2.1, with $\bar{M} = \bar{G}/\bar{N} = G/N$. □

Proposition 3.2. *Assume that N is a non-trivial finitely generated free pro-p group and that (†) gives a direct product decomposition of the middle group. Then* $\mathrm{cd}(G/N) = n - 1$.

Proof. By Lemma 3.1, $d_2^{n-1,1} = 0$. Therefore $E_3^{n+1,0} = \mathrm{Ker}(E_2^{n+1,0} \to E_2^{n+3,-1} = 0) = E_2^{n+1,0}$. On the other hand, $E_2^{r,s}$ has only two non-trivial lines, i.e. $E_2^{r,s} = 0$ for every $s \geq 2$, because $\mathrm{cd}(N) = 1$. This implies $E_3^{n+1,0} = E_\infty^{n+1,0}$, and hence $E_2^{n+1,0} = E_3^{n+1,0} = E_\infty^{n+1,0}$.

By ([10, Prop. A2.2, p. 406]), there is an injection $E_\infty^{n+1,0} \hookrightarrow H^{n+1}(G, \mathbb{F}_p)$. Since $\mathrm{cd}(G) = n$, we have $H^{n+1}(G, \mathbb{F}_p) = 0$ and hence $E_\infty^{n+1,0} = 0$. By the preceding paragraph, $H^{n+1}(G/N, \mathbb{F}_p) = E_2^{n+1,0}$ is trivial. Hence $\mathrm{cd}(G/N)$ is finite. Also, $H^1(N, \mathbb{F}_p) \simeq H^1(\bar{N}, \mathbb{F}_p)$ is finite. So (1.5) implies that $\mathrm{cd}(G/N) = \mathrm{cd}(G) - \mathrm{cd}(N) = n - 1$, as desired. □

Proposition 3.3. *Let G be a pro-p group of* $\mathrm{cd}(G) = n$ *and let N be a normal subgroup of G isomorphic to $\mathbb{Z}_p$. Then G/N has virtual cohomological dimension $n-1$ and all finite subgroups of G/N are cyclic. If $N \not\subset \Phi(G)$, then* $\mathrm{cd}(G/N) = n - 1$.
Moreover, if $n = 2$, the following conditions are equivalent:

(i) $N \not\subset \Phi(G)$.

(ii) G/N is a free pro-p group.

(iii) $G \simeq N \rtimes F$, for some free pro-p group F acting on N.

Proof. First, $\mathrm{vcd}(G/N) = n - 1$ follows from Theorem 2. Next, we show that every finite subgroup $\bar{T}$ of G/N is cyclic. Indeed, the preimage T of $\bar{T}$ in G is virtually procyclic and of finite cohomological dimension. By (1.4) T is of finite cohomological dimension 1, hence is free. As T is virtually procyclic and free it is procyclic itself. Hence $\bar{T}$ is cyclic.

It follows from $N \not\subset \Phi(G)$ that there exists a subgroup H of G such that $(G : H) = p$ and $N \not\subset H$. Thus $(N : H \cap N) = (G : H) = p$. Therefore $H \cap N = \Phi(N)$. It follows from this that the exact sequence (†) splits. Furthermore, $\bar{G}$ acts trivially on $\bar{N} \simeq \mathbb{F}_p$. Hence, by Proposition 3.2, $\mathrm{cd}(G/N) = n - 1$, as required.

In the case $n = 2$, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) follow trivially. □

Proposition 3.4. *Let G be a pro-p group of cohomological dimension n and let N be a normal free subgroup such that $2 \leq \operatorname{rank}(N) \leq p$. Then $\operatorname{cd}(G/N) = n-1$. Moreover, if $n = 2$ then G/N is free and $G \simeq N \rtimes (G/N)$.*

Proof. Suppose that G/N has torsion element c of order p. Then the preimage L of $\langle c \rangle$ in G is virtually free and of finite cohomological dimension. In particular, L is torsion free. By (1.4), L is free. If $\operatorname{rank}(L) \geq 2$, by Schreier's formula [10, Thm. 3.6.2], $\operatorname{rank}(N) \geq p+1$, a contradiction. If $\operatorname{rank}(L) = 1$ then L and hence N are procyclic, a contradiction. Thus G/N is torsion-free. By Theorem 2, $\operatorname{vcd}(G/N) = n-1$. By (1.4), $\operatorname{cd}(G/N) = n-1$. □

Proof of Corollary 3

If $\operatorname{cd}(G) = 1$, by Theorem 2, G/N is finite.

The case $\operatorname{cd}(G) = 2$, $\operatorname{cd}(N) = 1$ has been done in Theorem 2.

Assume that $\operatorname{cd}(G) = \operatorname{cd}(N) = 2$. We shall now show that every element of G/N has finite order. It follows from this that if G is finitely generated, then by [16] G/N is finite. Note that by [10, Prop. 4.2.6] every subgroup of finite index in a finitely generated pro-p group is open, hence N is open, as desired.

Assume, by contrary, that there is $x \in G$ such that its class xN in G/N is not of finite order. Therefore the subgroup $\overline{\langle xN \rangle}$ of G/N, generated by xN, is isomorphic to $\mathbb{Z}_p$. Let $S = \overline{\langle x \rangle}N$. Then $S/N \simeq \mathbb{Z}_p$. Since N is finitely presented, $H^2(N, \mathbb{F}_p)$ is finite by Thm. 7.8.3 (p. 290) of [10]. Since N has cohomological dimension 2, we may apply (1.5) to S. Hence $\operatorname{cd}(S) = \operatorname{cd}(N) + \operatorname{cd}(S/N) = 3$, contradicting $\operatorname{cd}(S) \leq \operatorname{cd}(G) = 2$ of (1.3). □

Proof of Corollary 4

By Theorem 2, G/N has virtual cohomological dimension 1, i.e. is virtually free. By [5], G/N is the fundamental group of a finite graph of finite p-groups $(\mathcal{G}_N, \Gamma)$. Hence G is the fundamental group of the finite graph of pro-p groups $(\mathcal{G}, \Gamma)$, where every edge and vertex group is a finite extension of N. Since G is torsion free, it follows by (1.2) and (1.4) that every edge and vertex group is finitely generated free pro-p. □

4 N is a Demushkin group or analytic pro-p

Lemma 4.1. *Let N be a pro-p group. Assume that the cup product map*

$$\overbrace{H^1(N, \mathbb{F}_p) \times \cdots \times H^1(N, \mathbb{F}_p)}^{n} \to H^n(N, \mathbb{F}_p)$$

is surjective. Then the inflation

$$\theta \colon H^n(N/\Phi(N), \mathbb{F}_p) \to H^n(N, \mathbb{F}_p)$$

is surjective. In particular, θ is surjective if N is either a Demushkin pro-p group or a uniformly powerful pro-p group of dimension n.

Proof. Put $\bar{N} = N/\Phi(N)$. Consider the commutative diagram

$$\begin{array}{ccc} H^1(\bar{N},\mathbb{F}_p)\times\cdots\times H^1(\bar{N},\mathbb{F}_p) & \xrightarrow{\bar{\alpha}} & H^n(\bar{N},\mathbb{F}_p) \\ \downarrow{\scriptstyle\theta^1} & & \downarrow{\scriptstyle\theta} \\ H^1(N,\mathbb{F}_p)\times\cdots\times H^1(N,\mathbb{F}_p) & \xrightarrow{\alpha} & H^n(N,\mathbb{F}_p) \end{array}$$

where the horizontal maps are the cup product maps and θ^1 comes from the inflation. By assumption, α is an epimorphism. Obviously, θ^1 is an isomorphism. Hence θ is an epimorphism.

The assumption on the cup product is satisfied if N is either a Demushkin group (1.7) or uniformly powerful (1.8). □

Proof of Theorem 6

Follows from Theorem 5 together with Lemma 4.1. □

Proof of Theorem 7

By (1.9) an analytic pro-p group has a characteristic subgroup that is uniformly powerful. Hence we can assume that N is uniformly powerful. Then Theorem 5 together with Lemma 4.1 completes the proof. □

5 The profinite case

For a profinite group G denote by $G(p)$ the maximal pro-p quotient of G.

We first state a technical lemma.

Lemma 5.1. *Let G be a profinite group. Let K be the kernel of the epimorphism of G onto its maximal pro-p quotient $G(p)$ and let G_p be a p-Sylow subgroup of G. Then:*

(i) $G_pK = G$.

(ii) If K is abelian, then its order is prime to p and $G_p \simeq G(p)$.

(iii) Let N be a normal subgroup of G containing K. Then N/K is the maximal pro-p quotient of N.

Proof. Condition (i) follows from [10, Exer. 2.3.3(b)]; condition (iii) follows from [10, Lemma 3.4.1(d)]. We prove (ii). For every prime ℓ let K_ℓ be the unique ℓ-Sylow subgroup of K. Then $K \simeq \prod_\ell K_\ell$. Hence $K \simeq L \times K_p$, where $L = \prod_{\ell\neq p} K_\ell$.

The epimorphism $G \to G(p)$ induces the epimorphism $G/L \to G(p)$ with kernel K/L. Both $G(p)$ and $K/L \simeq K_p$ are pro-p, hence so is G/L. By the maximality, $K = L$, that is, K is of order prime to p.

By (i), $G \to G(p)$ maps G_p onto $G(p)$. As the orders of K and G_p are coprime, $K\cap G_p = 1$. Thus $G_p \to G(p)$ is an isomorphism. □

The following straightforward generalization of (1.3) reduces the computation of the virtual cohomological dimension to pro-p groups.

Lemma 5.2. *Let G be a profinite group, let p be a prime, and let S be a p-Sylow subgroup of G. Then* $\mathrm{vcd}_p(G) = \mathrm{vcd}(S)$.

Proof. By (1.3), $\mathrm{cd}_p(G) = \mathrm{cd}(S)$.

Suppose that $\mathrm{vcd}(S) \leq n$. Then there is an open subgroup S_0 of S such that $\mathrm{cd}(S_0) \leq n$. There is an open normal subgroup G_0 of G such that $S \cap G_0 \subseteq S_0$. As $S \cap G_0$ is a p-Sylow subgroup of G_0, we have $\mathrm{cd}_p(G_0) = \mathrm{cd}(S \cap G_0) \leq \mathrm{cd}(S_0) \leq n$. Therefore $\mathrm{vcd}_p(G) \leq n$.

Conversely, suppose that $\mathrm{vcd}_p(G) \leq n$. Then there is an open subgroup G_0 of G such that $\mathrm{cd}_p(G_0) \leq n$. As $S \cap G_0$ is an open subgroup of S and $\mathrm{cd}(S \cap G_0) \leq \mathrm{cd}_p(G_0) \leq n$, we have $\mathrm{vcd}(S) \leq n$. ☐

The next lemma generalizes [10, Prop. 7.7.7].

Lemma 5.3. *Let G be a profinite group and p a prime. Let K be the kernel of the epimorphism $G \to G(p)$. Assume that* $\mathrm{cd}_p(K) \leq 1$. *Then* $\mathrm{cd}(G(p)) \leq \mathrm{cd}_p(G)$.

Proof. Put $n = \mathrm{cd}_p(G)$. We may assume that $n < \infty$. Then $H^{n+1}(G, \mathbb{F}_p) = 0$. The assumption on K implies that $H^i(K, \mathbb{F}_p) = 0$ for all $i \geq 2$. Furthermore, as K has no non-trivial pro-p quotients ([10, Lemma 3.4.1(e)]), we also have $H^1(K, \mathbb{F}_p) = \mathrm{Hom}(K, \mathbb{F}_p) = 0$. By [10, Cor. 7.2.5(a)] the inflation $H^{n+1}(G/K, \mathbb{F}_p) \to H^{n+1}(G, \mathbb{F}_p) = 0$ is an isomorphism. Thus $H^{n+1}(G/K, \mathbb{F}_p) = 0$. As G/K is pro-p, this implies $\mathrm{cd}(G/K) \leq n$. ☐

The next two propositions prove Theorem 1 and slightly more.

Proposition 5.4. *Let G be a profinite group and N a finitely generated normal subgroup. Assume that* $\mathrm{vcd}_p(G) = n < \infty$ *and* $\mathrm{vcd}_p(N) \leq 1$. *Then* $n - 1 \leq \mathrm{vcd}_p(G/N) \leq n$. *If the maximal pro-p quotient $N(p)$ of N is not trivial, then* $\mathrm{vcd}_p(G/N) = n - 1$.

Proof. Replacing G by a sufficiently small open subgroup G_0 and N by $N \cap G_0$ we may assume that $\mathrm{cd}_p(G) = n$ and $\mathrm{cd}_p(N) \leq 1$. (If $N(p)$ is infinite, then so is $(N \cap G_0)(p)$.)

A p-Sylow subgroup of G/N is of the form S/N, where S is a subgroup of G containing N. By (1.3), $\mathrm{cd}_p(S) = \mathrm{cd}_p(G) = n$. Let K be the kernel of the epimorphism of S onto its maximal pro-p quotient. Then K is a subgroup of N and $\mathrm{cd}_p(K) \leq \mathrm{cd}_p(N) \leq 1$. By Lemma 5.3, $\mathrm{cd}(S/K) \leq n$.

By Lemma 5.1(iii), $N/K = N(p)$. Therefore by Lemma 5.3, $\mathrm{cd}(N/K) \leq \mathrm{cd}_p(K) \leq 1$, that is, by (1.2), N/K is a free pro-p group. We have two cases. If $N(p) = 1$, that is, $N = K$, then $\mathrm{cd}(S/N) \leq n$. If $N(p) = N/K \neq 1$, we may apply Theorem 2 to the normal subgroup N/K of S/K. Then $\mathrm{vcd}(S/N) = \mathrm{cd}(S/K) - 1 \leq n - 1$. Again, replacing G by an open subgroup G_0 and N by $N \cap G_0$ we may actually assume that $\mathrm{vcd}(S/N) = \mathrm{cd}(S/N)$. On the other hand, by (1.5) in both cases $\mathrm{cd}(S/N) = \mathrm{cd}_p(S/N) \geq \mathrm{cd}_p(S) - \mathrm{cd}_p(N) = n - \mathrm{cd}_p(N) \geq n - 1$. By Lemma 5.2, $\mathrm{vcd}_p(G/N) = \mathrm{cd}(S/N)$, so we are done. ☐

Proposition 5.5. *Let G be a profinite group and N a finitely generated normal subgroup. Assume that* $\mathrm{cd}_p(G) < \infty$ *and* $\mathrm{cd}_p(N) \leq 1$. *Let $\bar{T}$ be a finite p-subgroup of G/N. Then*

(i) If $N(p)$ is procyclic, then $\bar{T}$ is cyclic. In particular, $N(p) = 1$ implies $\bar{T} = 1$.

(ii) If $N(p)$ is not procyclic, then its rank (as a free pro-p group) is larger than the order of $\bar{T}$.

Proof. We have $\bar{T} = T/N$, where T is a subgroup of G containing N. Replace G by T to assume that $G/N = \bar{T}$.

Let K be the kernel of the epimorphism $G \to G(p)$. Then K is a subgroup of N and hence $\mathrm{cd}_p(K) \le \mathrm{cd}_p(N) \le 1$. By Lemma 5.3, G/K has finite cohomological dimension. Hence G/K is torsion free.

As in the proof of Proposition 5.4, $N/K = N(p)$ is a free pro-p group. It is of finite index $(G : N)$ in G/K. If $N(p) = 1$, then $G/N = G/K$ is finite and torsion free, and hence trivial. If $N(p) \neq 1$, by (1.4), G/K is a free pro-p group. By Schreier's formula [10, Thm. 3.6.2(b)], $\mathrm{rank}(N(p)) - 1 = (G : N)(\mathrm{rank}(G/K) - 1)$. So if $N(p)$ is of rank 1, that is, procyclic, then so is G/K, and hence its finite quotient G/N is cyclic. If $\mathrm{rank}(N(p)) > 1$, then $\mathrm{rank}(N(p)) > (G : N)$, as desired. □

Under some additional conditions we may improve on Theorem 1 stating bounds for $\mathrm{vcd}(G/N)$. Part of the proposition is about pro-$\mathcal{C}$ groups for a class of finite groups $\mathcal{C}$ i.e. groups that are inverse limits of surjective systems of groups from the class $\mathcal{C}$.

Proposition 5.6. *Let G be a profinite group of finite cohomological dimension $\mathrm{cd}(G) = n$ and N be a finitely generated normal projective subgroup. Then $n - 1 \le \mathrm{vcd}(G/N) \le n$ in all of the following cases:*

(a) G/N has an open torsion free subgroup;

(b) for all but finitely many primes p, a p-Sylow subgroup $\bar{S}_p$ of G/N is torsion free.

(c) N is pro-π, where π is a finite set of primes;

(d) for all but finitely many primes p, $N(p)$ is not infinite procyclic;

(e) N is a free pro-$\mathcal{C}$ group of rank > 1, where $\mathcal{C}$ is a class of finite groups containing $\mathbb{Z}/p\mathbb{Z}$ for every prime p and closed under quotients, subdirect products and extensions.
Moreover, if $N(p) \neq 1$ for every p such that $\mathrm{cd}_p(G) = n$, then in all the above cases $\mathrm{vcd}(G/N) = n - 1$.

Proof. Let p be a prime. By Lemma 5.3, $\mathrm{cd}_p(N(p)) \le 1$; by (1.2), $N(p)$ is a free pro-p group. In particular, it is torsion free. So $N(p)$ is finite if and only if $N(p) = 1$.

(a) Let $\bar{U}$ be an open torsion free subgroup of G/N. By (1.4), $\mathrm{vcd}(\bar{U}) = \mathrm{cd}(\bar{U})$. The preimage U of $\bar{U}$ in G is open, hence by (1.3), $\mathrm{cd}(U) = \mathrm{cd}(G) = n$. Apply Proposition 5.4 to U to get $n - 1 \le \mathrm{cd}(\bar{U}) \le n$ in general, and $\mathrm{cd}(\bar{U}) = n - 1$ under the additional assumption.

(b) For each prime p choose a p-Sylow subgroup $\bar{S}_p$ of G/N. Let π be the set of primes p with $\bar{S}_p$ not torsion free. Let $p \in \pi$. By Lemma 5.2, $\mathrm{vcd}(\bar{S}_p) = \mathrm{vcd}_p(G/N)$. By Proposition 5.4, $\mathrm{vcd}_p(G/N) \le \mathrm{cd}_p(G) \le n$. So there is an open subgroup $\bar{U}_p$ of $\bar{S}_p$ such that $\mathrm{cd}(\bar{U}_p) \le n$. In particular, $\bar{U}_p$ is torsion free. Since π is finite, there exists an open normal subgroup $\bar{U}$ of G/N such that $\bar{U} \cap \bar{S}_p \le \bar{U}_p$ for all $p \in \pi$. Since $\bar{U}$ is a normal subgroup and $\bar{U}_p$ is torsion free, $\bar{U}$ has no p-torsion for every prime $p \in \pi$. Therefore $\bar{U}$ is p-torsion free for every prime p and so $\bar{U}$ is torsion free. By (a) we get the desired conclusions.

(c) If $p \notin \pi$, then $N(p) = 1$, and hence by Proposition 5.5(i), $\bar{S}_p$ is torsion free. By (b) we get the desired conclusions.

(d) Let d be an integer larger than the minimal number of generators of N such that for every $p > d$, $N(p)$ is not infinite procyclic.

For a prime number $p > d$, Proposition 5.5(ii) implies that a finite p-subgroup $\bar{T}$ of G/N has order smaller than the rank of $N(p)$ which is bounded above by d and hence by p. Thus $\bar{T}$ has to be trivial. So the assertion follows from (b).

(e) Let $r > 1$ be the rank of N. By assumption, the direct product A of r copies of a group of order p is in $\mathcal{C}$. It is an epimorphic image of N, and hence an epimorphic image of $N(p)$. Therefore $\mathrm{rank}(N(p)) \geq \mathrm{rank}(A) = r$. So the assertion follows from (d). □

Proposition 5.7. *Let G be a profinite group and A an abelian normal subgroup of G. If G/A is torsion free, then* $\mathrm{cd}(G/A) \leq \mathrm{cd}(G)$.

Proof. We may assume that $\mathrm{cd}(G)$ is finite.

Fix a prime p. Let S/A be a p-Sylow subgroup of G/A and S/K be a maximal pro-p quotient of S. Then $K \subseteq A$ is an abelian group. By Lemma 5.1(ii), S/K is isomorphic to a p-Sylow subgroup of S. As $(G : S) = (G/A : S/A)$ is prime to p, the latter subgroup is a p-Sylow subgroup of G. By (1.3), $\mathrm{cd}(S/K) = \mathrm{cd}_p(G) \leq \mathrm{cd}(G) < \infty$.

Since S/K is a pro-p group of finite cohomological dimension, so is A/K. Furthermore, A/K is abelian. By [13, Ex. 1, p. 40], A/K is the direct product of finitely many copies of $\mathbb{Z}_p$. In particular, A/K is analytic (1.9). Also S/A is torsion free. This, together with Theorem 7, implies $\mathrm{cd}(S/A) = \mathrm{vcd}(S/A) = \mathrm{cd}(S/K) - \mathrm{cd}(A/K) \leq \mathrm{cd}(S/K) \leq \mathrm{cd}(G)$. By (1.3), $\mathrm{cd}_p(G/A) = \mathrm{cd}(S/A) \leq \mathrm{cd}(G)$. This holds for every prime p, and hence $\mathrm{cd}(G/A) \leq \mathrm{cd}(G)$. □

Lemma 5.8. *Let G be a virtually projective profinite group and let B be a profinite $[[\widehat{\mathbb{Z}}\,G]]$-module. Then $H_2(G,B)$ is periodic, i.e. for some $m \in \mathbb{N}$ we have $mH_2(G,B) = 0$.*

Proof. Let N be an open projective normal subgroup of G. Put $d = (G : N)$. The dual A of B is a discrete torsion $[[\widehat{\mathbb{Z}}\,G]]$-module. By [1, Satz 4.7] there is an exact sequence $H^2(G/N, A^N) \to H^2(G, A) \to H^1(G/N, H^1(N, A))$. By [10, Corollary 6.7.4], $dH^2(G/N, A^N) = dH^1(G/N, H^1(N, A)) = 0$. Therefore $d^2H^2(G, A) = 0$. It follows that $d^2H_2(G, B) = 0$. □

Theorem 5.9. *Let G be a profinite group of finite cohomological dimension* $\mathrm{cd}(G) = n$ *having central normal subgroup N. For every prime p such that G/N has a non-trivial p-Sylow subgroup we assume that* $\mathrm{cd}_p(N) = \mathrm{cd}_p(G) - 1$. *Then the commutator subgroup G' is projective.*

Proof. Let S/N be a p-Sylow subgroup of G/N and let G_p, N_p be some p-Sylow subgroups of S, N, respectively. Assume S/N is not trivial. Since $(G : S)$ is prime to p, G_p is a p-Sylow subgroups of G. Let S/K be the maximal pro-p quotient of S. As a subgroup of N, K is an abelian group. Hence, by Lemma 5.1(ii), $G_p \simeq S/K$ and $N_p \simeq N/K$.

Thus $\mathrm{cd}(S/K) = \mathrm{cd}(G_p) = \mathrm{cd}_p(G) < \infty$ and $\mathrm{cd}(N/K) = \mathrm{cd}(N_p) = \mathrm{cd}_p(N) < \infty$. Therefore, as in the proof of Proposition 5.7, N/K is an analytic pro-p group. Then by Theorem 7, $\mathrm{vcd}(S/N) = \mathrm{cd}(S/K) - \mathrm{cd}(N/K) = \mathrm{cd}_p(G) - \mathrm{cd}_p(N) = 1$ i.e., S/N is virtually free pro-p. By the preceding lemma, $H_2(S/N, \mathbb{Z}_p)$ is periodic.

This holds for every prime p. We now claim that $H_2(G/N,\widehat{\mathbb{Z}})$ is generated by its torsion. The corestriction $H_2(S/N,\mathbb{Z}_p)\to H_2(G/N,\mathbb{Z}_p)$ is an epimorphism (the dual statement of it for cohomology is the subject of [10, Cor. 6.7.7]). So $H_2(G/N,\mathbb{Z}_p)$ is periodic. Note that $\widehat{\mathbb{Z}}\simeq\prod_p\mathbb{Z}_p$ where the product is over all primes p. As profinite homology commutes with inverse limits [10, Prop. 6.5.7] it commutes with direct products. Hence $H_2(G/N,\widehat{\mathbb{Z}})\simeq H_2(G/N,\prod_p\mathbb{Z}_p)\simeq\prod_p H_2(G/N,\mathbb{Z}_p)$ is the direct product of torsion profinite groups, as claimed.

Consider the 5-term Lyndon-Hochschild-Serre exact sequence ([10, Corollary 7.2.6])

$$H_2(G,\widehat{\mathbb{Z}})\to H_2(G/N,\widehat{\mathbb{Z}})\to H_1(N,\widehat{\mathbb{Z}})_{G/N}\to H_1(G,\widehat{\mathbb{Z}})\to H_1(G/N,\widehat{\mathbb{Z}})\to 0.$$

Since G acts trivially on N and on $\widehat{\mathbb{Z}}$, we have $H_1(N,\widehat{\mathbb{Z}})_{G/N}=H_1(N,\widehat{\mathbb{Z}})$. Taking into account that $H_1(G,\widehat{\mathbb{Z}})=G/\overline{G'}$ and $H_1(N,\widehat{\mathbb{Z}})=N/\overline{N'}=N$ (cf. [10, Lemma 6.8.6]), one gets that the kernel of the natural homomorphism $N\to G/G'$ is the image of $H_2(G/N,\widehat{\mathbb{Z}})$, hence is generated by torsion. Since $\mathrm{cd}(N)\le\mathrm{cd}(G)<\infty$, N is torsion free, and so this kernel is trivial. Then $N\times G'$ is a subgroup of G and hence for every prime p we have $\mathrm{cd}_p(G)\ge\mathrm{cd}_p(N\times G')=\mathrm{cd}_p(N)+\mathrm{cd}_p(G')$ (1.5), whence $\mathrm{cd}_p(G')\le\mathrm{cd}_p(G)-\mathrm{cd}_p(N)$. So if there is a non-trivial p-Sylow subgroup of G/N, by assumption $\mathrm{cd}_p(G')\le 1$. Otherwise, $\mathrm{cd}_p(G/N)=0$ and by (1.5), $\mathrm{cd}_p(G/N)=\mathrm{cd}_p(G)-\mathrm{cd}_p(N)$, so $\mathrm{cd}_p(G')=0$. It follows that $\mathrm{cd}(G')\le 1$, i.e. G' is projective. □

In particular, by (1.2):

Corollary 5.10. *Let G be a pro-p group of finite cohomological dimension $\mathrm{cd}(G)=n$ having central normal subgroup N of cohomological dimension $\mathrm{cd}(N)=n-1$. Then the commutator subgroup G' is free.*

Proposition 5.11. *Let G be a profinite group of finite cohomological dimension n and N a normal subgroup of G. Suppose N is the profinite completion of the fundamental group Π of a compact surface of genus g, where $g>0$ if the surface is orientable and $g>1$ if not. Then $\mathrm{vcd}_p(G/N)\le n-2$ for every prime p. Moreover, if G/N is torsion free, then $\mathrm{cd}(G/N)=n-2$.*

For the reader's convenience we collect in the next lemma properties of a surface group.

Lemma 5.12. *Let N be the profinite completion of the fundamental group Π of a compact surface M of genus g, where $g>0$ if the surface is orientable and $g>1$ if not. Let p be a prime. Then:*

(i) The maximal pro-p quotient $N(p)$ of N is the pro-p completion of Π and $N(p)$ is a Demushkin group.

(ii) $\mathrm{cd}_p(N)\le 2$.

(iii) For every simple p-primary discrete N-module A, the natural maps $\Pi\to N$ and $\Pi\to N(p)$ induce isomorphisms

$$H^j(N,A)\to H^j(\Pi,A)\quad\textit{and}\quad H^j(N(p),A)\to H^j(\Pi,A),$$

for every $j\ge 0$. Consequently, the inflation $H^j(N(p),\mathbb{F}_p)\to H^j(N,\mathbb{F}_p)$ is an isomorphism for every $j\ge 0$.

Proof. (i) It is easy to see that $N(p)$ is the pro-p completion of Π. By [13, Exer. 2, p. 40] $N(p)$ is a Demushkin group.

Observe first that if M is the torus, (ii) and (iii) are trivially true. Thus we assume that $g > 1$ for the rest of the proof. We need a more convenient description of Π. By [8, p. 132–135], Π admits a presentation with one relation, more precisely:

If M is orientable, $\Pi = \langle x_1, \ldots, x_{2g} \mid r\rangle$, where $r = [x_1, x_2][x_3, x_4] \ldots [x_{2g-1}, x_{2g}]$. Therefore, as in [10, Example 9.2.11], we can write Π as the amalgamated free product $F_1 *_{\mathbb{Z}} F_2$, where F_1 is the free group on the generators $\{x_1, x_2\}$ and F_2 is the free group on the generators $\{x_3, x_4, \ldots, x_{2g}\}$. The amalgamated subgroup is $\mathbb{Z} \simeq \langle [x_1, x_2]\rangle \simeq \langle [x_3, x_4] \cdots [x_{2g-1}, x_{2g}]\rangle$.

For M non-orientable, Π has a presentation as $\langle x_1, \ldots, x_g \mid r\rangle$, where $r = x_1^2 x_2^2 \cdots x_g^2$. Therefore, $\Pi = F_1 *_{\mathbb{Z}} F_2$, where F_1 is the free group on $\{x_1\}$ and F_2 is the free group on $\{x_2, \ldots, x_g\}$. The amalgamated subgroup is $\mathbb{Z} \simeq \langle x_1^2\rangle \simeq \langle x_2^2 x_3^2 \cdots x_g^2\rangle$.

Observe that the profinite completion N of Π has exactly the same presentation and so $N \simeq \widehat{F}_1 *_{\widehat{\mathbb{Z}}} \widehat{F}_2$ is a profinite amalgamated free product of two finitely generated free profinite groups with a procyclic amalgamated subgroup. In fact, this profinite amalgamated free product is proper (see [11, Proposition 3.2]) in the sense that $\widehat{F}_1$, $\widehat{\mathbb{Z}}$ and $\widehat{F}_2$ are naturally embedded in N.

We now consider the Mayer-Vietoris sequence associated with the above description of Π and N as amalgamated free products (see [10, Proposition 9.2.13] in the profinite case and [3, Theorem 2.10] in the discrete case),

$$\begin{array}{ccccccc}
\ldots \to & H^{j-1}(\widehat{\mathbb{Z}}, A) & \to & H^j(N, A) & \to & H^j(\widehat{F}_1, A) \oplus H^j(\widehat{F}_2, A) & \to \ldots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\ldots \to & H^{j-1}(\mathbb{Z}, A) & \to & H^j(\Pi, A) & \to & H^j(F_1, A) \oplus H^j(F_2, A) & \to \ldots
\end{array}$$

where A is a simple p-primary discrete N-module and the vertical arrows are induced by the canonical maps $\mathbb{Z} \to \widehat{\mathbb{Z}}$, $F_i \to \widehat{F}_i$, $i = 1, 2$ and $\Pi \to N$.

Now, (ii) follows from the sequence on the top of the above diagram. On the other hand, for the finitely generated free groups the above vertical arrows are clearly isomorphisms for every j and so $H^j(N, A) \to H^j(\Pi, A)$ is also an isomorphism.

Applying the same argument to $N(p)$ we also get the isomorphisms $H^j(N(p), A) \to H^j(\Pi, A)$. $\square$

Proof of Proposition 5.11

The proof depends essentially on the fact that the pro-p completion of Π is a Demushkin group and an open subgroup of a Demushkin group is also a Demushkin group [13, Corollaire, p. 38].

For the profinite completion N of Π denote by K the kernel of the natural map of $N \to N(p)$. We shall see that $H^j(K, \mathbb{F}_p) = 0$ for all $j > 0$. Indeed, $\mathrm{cd}(K) \le \mathrm{cd}(N) = 2$, hence $H^i(K, \mathbb{F}_p) = 0$ for all $i \ge 3$. As K has no nontrivial p-quotients ([10, Lemma 3.4.1(e)]), also $H^1(K, \mathbb{F}_p) = \mathrm{Hom}(K, \mathbb{F}_p) = 0$. Hence it remains to be seen that $H^2(K, \mathbb{F}_p) = 0$.

Let $\mathcal{U}$ be the family of open subgroups of N containing K. Recall that K is the intersection of all $U_i \in \mathcal{U}$ and hence can be regarded as the inverse limit $K = \varprojlim U_i$. Therefore $H^2(K, \mathbb{F}_p) = \varinjlim H^2(U_i, \mathbb{F}_p)$, the direct limit.

Consider $U_2 \subsetneqq U_1$ in $\mathcal{U}$. Then U_2/K is an open subgroup of the Demushkin group U_1/K and by [13, Exer. 5(a), p. 41] the restriction $H^2(U_1/K, \mathbb{F}_p) \to H^2(U_2/K, \mathbb{F}_p)$ is the zero map.

On the other hand, any subgroup of finite index of Π is also a surface group. Therefore any open subgroup U of N is the profinite completion of the surface group $\Phi = U \cap \Pi$. Moreover for $U \in \mathcal{U}$, U/K is the closure of Φ in the pro-p completion $N/K = N(p)$ of Π, and hence U/K is the pro-p completion of Φ. Therefore, by the lemma above applied to U, U/K and Φ, the inflation $H^2(U/K, \mathbb{F}_p) \to H^2(U, \mathbb{F}_p)$ is an isomorphism. We then conclude, for $U_2 \subsetneqq U_1$ in $\mathcal{U}$, that the restriction $H^2(U_1, \mathbb{F}_p) \to H^2(U_2, \mathbb{F}_p)$ is the zero map.

Therefore, all maps in the direct limit $\varinjlim H^2(U_i, \mathbb{F}_p)$ are 0-maps. Hence $H^2(K, \mathbb{F}_p) = 0$.

Now let S/N be a p-Sylow subgroup of G/N. Observe that S/K is a pro-p group. As $H^i(K, \mathbb{F}_p) = 0$ for all $i > 0$, the spectral sequence in cohomology collapses, i.e., $H^i(S, \mathbb{F}_p) \simeq H^i(S/K, \mathbb{F}_p)$ for all $i \geq 0$. Then $\mathrm{cd}(S/K) \leq \mathrm{cd}_p(S) \leq \mathrm{cd}_p(G) \leq \mathrm{cd}(G)$.

Since $N(p) = N/K$ is a Demushkin pro-p group, we can apply Theorem 6 to S/K and $N(p)$. Then S/N has virtually finite cohomological dimension $\mathrm{vcd}(S/N) = \mathrm{cd}(S/K) - 2 \leq n - 2$. Thus, Lemma 5.2 implies that $\mathrm{vcd}_p(G/N) \leq n - 2$, as required.

Moreover, the additional assumption yields S/N torsion free. By (1.4), $\mathrm{cd}(S/N) = \mathrm{vcd}(S/N) \leq n - 2$. We show that for some prime p, $\mathrm{cd}(S/N) = n - 2$. Indeed, choose a prime p such that $\mathrm{cd}_p(G) = n$ and observe that a p-Sylow subgroup of S is also a p-Sylow subgroup of G. Then $\mathrm{cd}_p(S) = n$ and $(n-2)+2 \geq \mathrm{cd}_p(S/N) + \mathrm{cd}_p(N) \geq \mathrm{cd}_p(S) = n$. Hence $\mathrm{cd}_p(S/N) = n - 2$ and by (1.3) $\mathrm{cd}_p(G/N) = n - 2$, as needed. $\square$

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