

# FROBENIUS SUBGROUPS OF FREE PROFINITE PRODUCTS

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ABSTRACT. We solve an open problem of Herfort and Ribes: Profinite Frobenius groups of certain type do occur as closed subgroups of free profinite products of two profinite groups. This also solves a question of Pop about prosolvable subgroups of free profinite products.

## 1. INTRODUCTION

Herfort and Ribes show in [8, Theorem 3.2] that a closed solvable subgroup of the free product of a family of profinite groups  $\{A_x\}_{x \in X}$  must be one of the following:

- (1) a conjugate of a subgroup of one of the free factors  $A_x$ ;
- (2) isomorphic to  $\hat{\mathbb{Z}}_\sigma \rtimes \hat{\mathbb{Z}}_{\sigma'}$ , where  $\sigma$  and  $\sigma'$  are disjoint sets of prime numbers;
- (3) free pro- $\mathcal{C}$  product of two copies of the group of order 2, for some full class of  $\mathcal{C}$  of finite groups;
- (4) a profinite Frobenius group of the form  $\hat{\mathbb{Z}}_\sigma \rtimes C$  with Frobenius kernel  $\hat{\mathbb{Z}}_\sigma$ , where  $C$  is a finite cyclic group.

In [8, Section 4] they show that each group of the one of the first three types does occur as a closed subgroup of a free profinite product, namely, of two finite groups. As for the Frobenius groups, Herfort and Ribes state in [8] and show in [9] that they occur as closed subgroup of free *prosolvable* products of two finite groups. They (implicitly) leave open the question whether the above Frobenius groups occur as closed subgroups of free profinite products of, say, two finite groups.

This problem has been explicitly posed in [12] (see [12, Open Question 9.5.5]).

The main result of this paper is an affirmative answer to this question.

The proof uses the classification of finite simple groups (by analyzing automorphisms of simple groups and subgroups stabilized by them). If  $C$  has prime power order, the proof is much simpler and does not require the classification (Sylow's theorem is the main tool in that case). This had been essentially done by the authors several years ago in an unpublished work.

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See Remark 4.6. A. Zalesski and P. Zalesskii also found an independent proof in this case.

We thank P. Zalesskii for reminding us that our result also answers a question of Pop [10]. That paper characterizes closed prosolvable groups of free profinite products of profinite groups and the question is the following. Is there a free profinite product  $G = \coprod_{i \in I} G_i$  and a closed prosolvable subgroup  $H$  of  $G$  such that

- (1) there is no prime  $\ell$  such that  $H \cap G_i^\sigma$  is a pro- $\ell$  group for all  $i \in I$  and all  $\sigma \in G$ ;
- (2)  $H \leq G_i^\sigma$  for no  $i \in I$  and no  $\sigma \in G$ ?

A Frobenius subgroup of a free product of certain two finite groups provides such an example (Example 5.3).

## 2. FINITE AND PROFINITE FROBENIUS SUBGROUPS

Recall that the notions of order and index extend from finite group to profinite groups; instead of natural numbers these are supernatural numbers ([12, Section 2.3] or [3, Section 22.8]). In particular, a profinite group has Sylow  $p$ -subgroups for each prime  $p$  ([12, Corollary 2.3.6] or [3, Section 22.9]).

A profinite group  $F$  is a *Frobenius group* if it is a semidirect product  $F = C \rtimes K$  of nontrivial profinite groups  $C, K$ , where  $C$  acts on  $K$  so that  $[c, k] \neq 1$  for every  $1 \neq c \in C, 1 \neq k \in K$ . One then calls  $K$  the *Frobenius kernel* and  $C$  a *Frobenius complement* of  $F$ .

Since we deal only with a special type of Frobenius groups, we adopt the following notation. Let  $C$  be a finite cyclic group. A  $C$ -group is a profinite group with a distinguished subgroup isomorphic to  $C$ ; we identify this subgroup with  $C$ . A  $C$ -homomorphism of  $C$ -groups  $G \rightarrow H$  is a continuous homomorphism  $G \rightarrow H$  that maps the copy of  $C$  in  $G$  identically onto the copy of  $C$  in  $H$ . If  $A \triangleleft G$  such that  $A \cap C = 1$ , then  $G/A$  is a  $C$ -group (we identify  $C$  with  $CA/A$ ) and  $G \rightarrow G/A$  is a  $C$ -epimorphism.

We call a profinite  $C$ -group  $F$  a  $C$ -Frobenius group if  $F$  is a Frobenius group with complement  $C$  and procyclic kernel. The following properties are easy to verify:

**Lemma 2.1.** *Let  $C \neq 1$  be a finite cyclic group acting on a procyclic group  $K \neq 1$ . Then  $F = C \rtimes K$  is a  $C$ -Frobenius group if and only if for each prime  $p$  dividing the order of  $K$ , the order of  $C$  divides  $p - 1$  and  $C$  acts faithfully on the  $p$ -primary part of  $K$ . If  $F$  is a  $C$ -Frobenius group then:*

- (a) Every prime divisor of  $|C|$  is strictly smaller than any prime divisor of  $|K|$ .
- (b)  $K$  is of odd order.
- (c) Any quotient group  $\overline{F}$  of  $F$  is either a quotient of  $C$  or a  $\overline{C}$ -Frobenius group, where  $\overline{C}$  is the image of  $C$  in  $\overline{F}$ .

- (d) Let  $\hat{K} \rightarrow K$  be an epimorphism of procyclic groups of orders divisible by the same primes. Suppose that  $C$  acts on  $\hat{K}$  such that  $\hat{K} \rightarrow K$  is  $C$ -equivariant. Then  $C \rtimes \hat{K}$  is also a  $C$ -Frobenius group.
- (e) Every subgroup of  $F$  is a conjugate of  $C_1K_1$ , where  $C_1 \leq C$  and  $K_1 \leq K$ .
- (f) A subgroup of  $F$  is normal if and only if it is either a subgroup of  $K$  or of the form  $C_1K$ , where  $C_1 \leq C$ . In particular, a minimal normal subgroup of  $F$  is a minimal normal subgroup of  $K$ .
- (g) Let  $C_1 \leq C$ ,  $K_1 \leq K$ , and  $f \in F$ . Then

$$C_1^f K_1 \cap K = K_1 \quad \text{and} \quad C_1^f K_1 \cap C = \begin{cases} C_1 & \text{if } f \in CK_1 \\ 1 & \text{if } f \notin CK_1. \end{cases}$$

**Lemma 2.2.** *Let  $F = CK$  be a finite  $C$ -Frobenius group (with Frobenius kernel  $K$ ). Suppose  $F$  acts transitively on a set  $\Delta$ . Then*

- (1) *There is  $L \in \Delta$  such that its  $F$ -stabilizer is  $C_1K_1$  with  $C_1 \leq C$  and  $K_1 \leq K$ . Fix such  $L$ . Then*
- (2)  *$C_1$  is the  $C$ -stabilizer of  $L$ , and hence also of every  $L^c$ , with  $c \in C$ .*
- (3) *Every point of  $\Delta \setminus \{L^c \mid c \in C\}$  has a trivial  $C$ -stabilizer.*

*Proof.* (a) Let  $L \in \Delta$ . Its  $F$ -stabilizer  $F_1$  is, by Lemma 2.1(e), a conjugate of  $C_1K_1$  for some  $C_1 \leq C$  and  $K_1 \leq K$ . Replacing  $L$  by a conjugate we may assume that  $F_1 = C_1K_1$ .

(b),(c) By Lemma 2.1(g)

$$(L^f)^c = L^f \iff c \in (C_1K_1)^f \iff c \in C_1^f K_1 \cap C = \begin{cases} C_1 & \text{if } f \in CK_1 \\ 1 & \text{if } f \notin CK_1 \end{cases}$$

and  $L^{CK_1} = L^{K_1C} = L^C$ . □

### 3. INTRAVARIANT SUBGROUPS

*Definition 3.1.* Let  $H \leq G$  be groups and let  $A$  be a group acting on  $G$  from the right. We say that  $H$  is  *$A$ -invariant in  $G$*  if for every  $x \in A$  there is  $g \in G$  such that  $H^x = H^g$ . We say that  $H$  is an *invariant subgroup of  $G$*  if it is  $\text{Aut}(G)$ -invariant in  $G$ .

We point out that Sylow subgroups and their normalizers are invariant subgroups. In the rest of this section we exhibit further families of invariant subgroups of finite simple groups.

Recall that an *almost simple group* is a group  $G$  with a unique minimal normal subgroup  $S$  which is a nonabelian simple group. Thus,  $S \triangleleft G \leq \text{Aut}(S)$ . We refer the reader to [1, 2, 5, 6] for the basic facts about automorphisms of finite simple groups.

We recall some facts about automorphisms of the finite simple groups and most especially about Chevalley groups. The most complicated cases to deal with are PSL and PSU.

Our first result is [7, 3.22].

**Lemma 3.2.** *Let  $S$  be a finite nonsolvable group. Let  $x \in \text{Aut}(S)$ . Then  $C_S(x) \neq 1$ .*

The next result is the Borel-Tits Theorem [6, Theorem 3.1.3].

**Lemma 3.3.** *Let  $S$  be a simple Chevalley group and  $U$  a nontrivial unipotent subgroup. Let  $A = \text{Aut}(S)$ . Then there exists a proper parabolic subgroup  $P$  of  $S$  such that  $U$  is contained in the unipotent radical of  $P$  and  $N_A(P) \geq N_A(U)$ .*

We remark that  $P$  is proper, since the unipotent radical of  $P$  is normal in  $P$ , while  $S$  is simple.

We also require:

**Lemma 3.4.** *Let  $S \triangleleft G$  be finite groups. Let  $R$  be an  $r$ -subgroup of  $G$  for some prime  $r$ .*

- (1) *If  $r$  divides  $|S|$ , then  $R$  normalizes some Sylow  $r$ -subgroup of  $S$ .*
- (2) *If  $r$  does not divide  $|S|$  but another prime  $p$  does divide  $|S|$ , then  $R$  normalizes a Sylow  $p$ -subgroup of  $S$ .*

*Proof.* Let  $Q$  be a Sylow  $r$ -subgroup of  $G$  containing  $R$ . Then  $R$  normalizes  $Q \cap S$ , which gives (1).

To prove (2), let  $P$  be a Sylow  $p$ -subgroup of  $S$ . By Sylow's theorem,  $G = SN_G(P)$ . Since  $r$  does not divide  $|S|$ ,  $N_G(P)$  contains a Sylow  $r$ -subgroup of  $G$ , say,  $Q^g$ , with  $g \in G$ . Then  $R \leq Q \leq N_G(P^{g^{-1}})$ , that is,  $R$  normalizes  $P^{g^{-1}}$ .  $\square$

We now examine PSL and PSU more closely. Let  $S = \text{PSL}(d, p^e)$ , where  $p$  is a prime and  $S$  is simple. Let  $\sigma$  be the Frobenius automorphism and  $\tau$  the graph automorphism (which we may view as the inverse transpose map) of  $S$ . Then

$$\Omega = \text{PGL}(d, p^e) \cup \{\sigma, \tau\}$$

generates  $A = \text{Aut}(S)$ . Moreover,  $\text{PGL}(d, p^e)\langle\sigma\rangle$  is of index 2 in  $A$ .

We will use the following elementary result from linear algebra.

**Lemma 3.5.** *Let  $F$  be a field. Let  $A \in M_n(F)$  such that its minimal polynomial  $g$  is irreducible of degree  $d$ . Then  $K = F[A]$  is a finite field extension of  $F$  and for every  $A' \in K$  we have  $\det A' = N_F^K(A')^{n/d}$ , where  $N_F^K: K^\times \rightarrow F^\times$  is the norm.*

*Proof.* As  $g$  is irreducible,  $K \cong F[X]/(g)$  is a field extension of  $F$  of degree  $d$ .

For each  $m \in \mathbb{N}$  and  $A' \in K$  let  $A'_m: K^m \rightarrow K^m$  be the  $F$ -linear map given by  $A'_m(x_1, \dots, x_m) = (A'x_1, \dots, A'x_m)$ . By definition,  $N_F^K(A')$  is the determinant of  $A'_1: K \rightarrow K$ . Hence  $N_F^K(A')^m = \det A'_m$ .

Now, the matrix of  $A_1$  with respect to the basis  $I, A, \dots, A^{d-1}$  of  $K$  is the companion matrix  $C$  of  $g$ . Thus the matrix of  $A_m$  with respect to some basis of  $K^m$  is the direct sum  $C^{(m)}$  of  $m$  copies of  $C$ . But  $A$  is similar  $C^{(m)}$ , where

$m = \frac{n}{d}$  (this is the rational canonical form of  $A$ ), hence  $A$  is the matrix of  $A_m$  with respect to a suitable basis  $\mathcal{B}$  of  $K^m$ .

As  $A' \in K = F[A]$ , there is  $h \in F[X]$  such that  $A' = h(A)$ . Clearly  $A'_m = h(A)_m = h(A_m)$  and the matrix of  $h(A_m)$  with respect to  $\mathcal{B}$  is  $h(A) = A'$ . Hence  $N_F^K(A')^m = \det A'_m = \det(A')$ .  $\square$

**Lemma 3.6.** *Let  $F$  be a finite field. Let  $B \in M_d(F)$  be semisimple. Let  $a \in F$ . Then there is  $B' \in M_n(F)$  such that  $BB' = B'B$  and  $\det(B') = a$ .*

*Proof.* Without loss of generality  $B$  is in rational canonical form. Thus  $B = \text{diag}(B_1, \dots, B_r)$ , where each  $B_i \in M_{n_i}(F)$  has minimal polynomial irreducible of degree  $n_i$ . Then by Lemma 3.5  $K = F[A]$  is a finite field extension of  $F$ . Hence the norm  $N_F^K: K^\times \rightarrow F^\times$  is surjective. Therefore, again by Lemma 3.5, there is  $B'_i \in M_{n_i}(F)$  such that  $B'_i \in F[B_i]$ , whence  $B'_i$  commutes with  $B_i$ , and  $\det(B'_1) = a$  and  $\det(B'_i) = 1$  for  $i > 1$ . Then  $B = \text{diag}(B'_1, \dots, B'_r)$  commutes with  $B$  and  $\det(B') = a$ .  $\square$

**Lemma 3.7.** *Let  $S = \text{PSL}(d, p^e)$  and  $H = \text{PGL}(d, p^e)$ . Put  $A = \text{Aut}(S)$ .*

- (1) *Any parabolic subgroup  $P$  of  $S$  whose normalizer contains an element outside of  $H\langle\sigma\rangle$  is intravariant.*
- (2) *If  $h \in H$  is a semisimple element (i.e. has order prime to  $p$ ), then  $\langle h \rangle$  is  $A$ -intravariant in  $H$ .*
- (3) *If  $h \in H$  is semisimple, then  $C_S(h)$  and  $N_S(\langle h \rangle)$  are intravariant subgroups of  $S$ .*

*Proof.* (1) We may conjugate  $P$  in  $S$  and thus assume that  $P$  contains the standard Borel subgroup of  $S$  that consists of upper triangular matrices of determinant 1. Clearly, diagonal matrices and  $\sigma$  normalize  $P$ . Hence  $P$  is an intravariant subgroup of  $S$  if and only if  $N_A(P)$  contains an element outside of  $H\langle\sigma\rangle$ .

(2) Let  $x \in A$ . We have to show that there is  $s \in H$  such that  $\langle h \rangle^x = \langle h \rangle^s$ . We may assume that  $x \in \Omega$ . If  $x \in H$ , the assertion is trivial. So assume that either  $x = \sigma$  or  $x = \tau$ . Put  $F = \mathbb{F}_{p^e}$ . Lift  $h$  to a semisimple element of  $\text{GL}(d, F)$ . It suffices to show that there is  $m \in \mathbb{Z}$  such that  $h^x, h^m$  are conjugate in  $\text{GL}(d, F)$ , i.e., similar over  $F$ .

Since every square matrix is similar over  $F$  to its transpose,  $h^\tau$  is similar to  $h^{-1}$ . On the other hand,  $h^\sigma$  is similar over  $F$  to  $h^p$ . Indeed, consider  $\text{GL}(d, F)$  as a subgroup of  $\text{GL}(d, \overline{F})$  and extend  $\sigma$  to the Frobenius automorphism of  $\text{GL}(d, \overline{F})$ . There is  $z \in \text{GL}(d, \overline{F})$  such that  $h^z \in \text{GL}(d, \overline{F})$  is diagonal. Then clearly  $(h^z)^\sigma = (h^z)^p$ . Therefore  $(h^\sigma)^{z^\sigma} = (h^z)^\sigma = (h^z)^p = (h^p)^z$ . Hence  $h^\sigma, h^p$  are similar over  $\overline{F}$ . Therefore they are similar over  $F$ .

(3) Let  $x \in A$ . By (2) there is  $s \in H$  such that  $\langle h \rangle^x = \langle h \rangle^s$ . By Lemma 3.6 there is  $z \in H$  such that  $zh = hz$  and  $\det(z) \equiv \det(s) \pmod{(F^\times)^d}$ . Replace  $s$  by  $z^{-1}s$  to get that  $s \in S$ . As  $C_S(h) = S \cap C_H(h)$ ,  $N_S(h) = S \cap N_H(h)$ , and  $S^x = S = S^s$ , we get  $C_S(h)^x = C_S(h)^s$  and  $N_S(h)^x = N_S(h)^s$ .  $\square$

**Proposition 3.8.** *Let  $S = \text{PSL}(n, p^e)$  be simple. Then every  $x \in \text{Aut}(S)$  normalizes a nontrivial proper intravariant subgroup of  $S$ .*

*Proof.* Let  $H = \mathrm{PGL}(n, p^e)$ . If  $C_H(x)$  is not a  $p$ -group, we may choose  $1 \neq h \in C_H(x)$  of order prime to  $p$ . Thus,  $h$  is semisimple and  $C_S(h)^x = C_S(h^x) = C_S(h)$ . By Lemma 3.7(3),  $C_S(h)$  is an intravariant subgroup of  $S$ . By Lemma 3.2,  $C_S(h) \neq 1$ ; clearly  $C_S(h) \neq S$ . So we may assume that  $C_H(x)$  is a  $p$ -group (i.e. consists of unipotent elements). In particular,  $C_H(x) \leq S$ , whence  $C_S(x) = C_H(x)$ .

If  $x \in \mathrm{Aut}(S) \setminus H\langle\sigma\rangle$ , then, by Lemma 3.3,  $x$  normalizes some proper parabolic subgroup of  $S$  which is intravariant by Lemma 3.7(1).

If  $x$  is in  $H$ , then  $x \in C_H(x) \leq S$ , so  $x$  is contained in some Sylow  $p$ -subgroup of  $S$ , which is intravariant.

So we may assume that  $x \in H\langle\sigma\rangle \setminus H$ . Write  $x = yz = zy$ , where  $y \in \langle x \rangle$  has order prime to  $p$  and  $z \in \langle x \rangle$  has order a power of  $p$ . The restriction to  $\langle y \rangle$  of the projection  $H\langle\sigma\rangle \rightarrow \langle\sigma\rangle$  is injective, because its kernel  $H \cap \langle y \rangle$  is contained in the  $p$ -group  $C_H(x)$ . Thus, writing  $y$  as  $\sigma^j h$ , with some  $h \in H$ , we see that  $y$  and  $\sigma^j$  have the same order. By Shintani descent [5, p. 81] they are conjugate by an element of  $H$ . Since  $C_H(\sigma^j)$  contains the Jordan matrix  $J_n(1)$ , it follows that  $C_H(y)$  contains a regular unipotent element (a conjugate of  $J_n(1)$ ).

As  $z$  commutes with  $y$ , it normalizes  $C_H(y)$ . Its order is a power of  $p$ , and so  $z$  normalizes some Sylow  $p$ -subgroup  $T$  of  $C_H(y)$ . But  $y$  centralizes  $T$  and so  $x = yz$  normalizes  $T$ . Since  $T$  contains a regular unipotent element,  $T$  is contained in a unique Sylow  $p$ -subgroup  $P$  of  $H$ . As  $T = T^x \leq P^x$ , we have  $P^x = P$ . Thus  $x$  normalizes this intravariant subgroup.  $\square$

The same result holds for  $S = \mathrm{PSU}(d, p^e)$ .

**Lemma 3.9.** *Let  $S = \mathrm{PSU}(d, p^e)$  be simple. If  $x \in \mathrm{Aut}(S)$ , then  $x$  normalizes a nontrivial proper intravariant subgroup of  $S$ .*

*Proof.* In this case  $\mathrm{Aut}(S) = H\langle\sigma\rangle$ , where  $H = \mathrm{PGU}(d, p^e)$  and  $\sigma$  is the Frobenius automorphism of order  $2e$ . Let  $x \in \mathrm{Aut}(S)$ . If  $C_H(x)$  contains a nontrivial semisimple element  $h$ , then  $C_S(h)$  is an intravariant subgroup normalized by  $x$ . Otherwise,  $C_H(x)$  is unipotent and by Lemma 3.3,  $x$  normalizes some (proper) parabolic subgroup of  $S$ . In this case, all parabolics are intravariant, whence the result.  $\square$

The analogous result is true for almost all of the families of simple groups. However, it is not true for  $S = \Omega^+(8, q)$ . Take  $x \in S$  of order  $(q^4 - 1)/\mathrm{gcd}(2, q - 1)$ . It is not difficult to see that  $x$  is contained in no maximal subgroup of  $\mathrm{Aut}(S)$  not containing  $S$ , whence it cannot normalize any nontrivial intravariant subgroup of  $S$ . If we only consider cyclic groups of automorphisms of simple groups which do not contain any inner automorphisms, then the result is true. We will revisit this topic in a future paper. It is not required for the results of this paper.

The outer automorphism groups of the other simple groups are less complicated. It is convenient to use the notation  $\mathrm{PSL}^\epsilon(d, q)$  with  $\epsilon = \pm$ —here  $\mathrm{PSL}^+$  means  $\mathrm{PSL}$  and  $\mathrm{PSL}^-$  means  $\mathrm{PSU}$ .

**Lemma 3.10.** *Let  $G$  be an almost simple finite group with socle  $S$ . Suppose that  $G/S$  is a Frobenius group with a cyclic Frobenius kernel  $K$  and a cyclic Frobenius complement  $C$ . If  $|C| > 3$ , then  $S \cong \text{PSL}^\epsilon(d, q)$  and the Frobenius kernel is contained in the subgroup of diagonal automorphisms of  $\text{Out}(S)$ . Moreover,  $d \geq 5$  and  $q > 5$ .*

*Proof.* We have  $G/S \leq \text{Out}(S)$ . If  $S$  is an alternating or sporadic group, then  $\text{Out}(S)$  and hence also  $G/S$  has exponent 2, a contradiction to  $G/S$  being Frobenius. So  $S$  is a Chevalley group.

Let  $r$  be a prime dividing  $|K|$ . By Lemma 2.1,  $|C|$  divides  $r - 1$ . As  $|C| > 3$ , we have  $r \geq 5$ . As  $K$  is cyclic,  $C$  normalizes its Sylow  $r$ -subgroup. If  $S$  is not  $\text{PSL}^\epsilon(d, q)$ , then the Sylow  $r$ -subgroup of  $K$  consists of field automorphisms. However, the normalizer in  $\text{Out}(S)$  of a field automorphism is its centralizer. Thus,  $C$  centralizes a nontrivial subgroup of  $K$ , a contradiction to  $G/S = C \rtimes K$  being Frobenius. This proves that  $S \cong \text{PSL}^\epsilon(d, q)$ . Arguing similarly, it follows that  $K$  consists of diagonal automorphisms and so  $r \mid \gcd(d, q - \epsilon 1)$ , whence  $d \geq 5$  and  $q > 5$ .  $\square$

We can now show:

**Corollary 3.11.** *Let  $G$  be an almost simple finite group with socle  $S$ . Suppose that  $G/S$  is a Frobenius group with cyclic Frobenius kernel  $K$  and cyclic Frobenius complement  $C$ . Let  $D$  be any cyclic subgroup of  $G$  with  $DS/S = C$ . If  $C$  has prime power order, assume the same is true of  $D$ . Then  $D$  normalizes a proper nontrivial invariant subgroup  $H$  of  $S$ .*

*Proof.* If  $|C|$ , and hence also  $|D|$ , is a prime power, then  $D$  normalizes a nontrivial Sylow subgroup of  $S$  by Lemma 3.4. Otherwise,  $|C| > 3$ , and hence by Lemma 3.10,  $S$  is either PSL or PSU. Now apply Proposition 3.8 and Lemma 3.9.  $\square$

#### 4. LIFTING FROBENIUS GROUPS

To show our main result we need some preparations.

**Lemma 4.1.** *Let  $\rho_1: F_1 \rightarrow F_3$  and  $\rho_2: F_2 \rightarrow F_3$  be  $C$ -epimorphisms of  $C$ -Frobenius groups. Then  $F_1 \times_{F_3} F_2$  contains a  $C$ -Frobenius group mapped onto  $F_1$ .*

*Proof.* Let  $K_i$  be the Frobenius kernel of  $F_i$ , and  $K_i^{(p)}$  its Sylow  $p$ -subgroup, respectively, for  $i = 1, 2, 3$  and for each prime  $p$ . Put  $F = F_1 \times_{F_3} F_2$ . We identify  $C$  with  $C \times_{F_3} C$  and thus  $F$  is a  $C$ -group and the coordinate projections  $F \rightarrow F_1, F \rightarrow F_2$  are  $C$ -epimorphisms. Clearly every  $1 \neq c \in C$  acts fixed-point-freely on the abelian subgroup  $K = K_1 \times_{F_3} K_2$  of  $F$ . For each prime  $p$  we find below a cyclic  $p$ -subgroup  $\langle k_p \rangle$  of  $K$  normalized by  $C$  and mapped by  $F \rightarrow F_1$  onto  $K_1^{(p)}$ . Then  $C(\prod_p \langle k_p \rangle)$  is a  $C$ -Frobenius group mapped onto  $F_1$ .

Fix a generator  $k_1$  of  $K_1^{(p)}$  and put  $k_3 = \rho_1(k_1)$ . By Sylow's theorem,  $\rho_1(K_1^{(p)}) = K_3^{(p)} = \rho_2(K_2^{(p)})$ . So there is  $k_2 \in K_2^{(p)}$  such that  $k_3 = \rho_2(k_2)$ . If  $k_3 = 1$ , take  $k_2 = 1$ ; otherwise  $k_2$  generates  $K_2^{(p)}$ . Then  $k = (k_1, k_2) \in F$  and  $\langle k \rangle \leq F$  is normalized by  $C$ , that is,  $(k_1^c, k_2^c) \in \langle k \rangle$ , where  $c$  is a generator of  $C$ .

Indeed, if  $k_2 = 1$ , the assertion is clear. So assume that  $k_2 \neq 1$  and hence  $k_3 \neq 1$ .

Let  $i = 1, 2, 3$ . Writing  $K_i^{(p)}$  additively, it is a quotient of  $\mathbb{Z}_p$  so that  $k_i$  is the class of the generator 1 of  $\mathbb{Z}_p$  and  $\rho_1: K_1^{(p)} \rightarrow K_3^{(p)}$ ,  $\rho_2: K_2^{(p)} \rightarrow K_3^{(p)}$  are the quotient maps. As the order of  $c$  is prime to  $p$ , there is a unique  $m_i \in (\mathbb{Z}_p)^\times$  such that  $c$  acts on  $K_i^{(p)}$  by multiplication by the image of  $m_i$ . But  $\rho_1, \rho_2$  are  $C$ -equivariant, hence  $m_1 = m_3 = m_2$ . Thus  $(k_1^c, k_2^c) = (m_1 k_1, m_2 k_2) = m_1 k \in \langle k \rangle$ .  $\square$

**Lemma 4.2.** *Let  $G$  be a finite group and  $A$  a minimal normal subgroup of  $G$ . Assume that  $A$  is an elementary abelian  $p$ -group and  $G/A$  is a  $C$ -Frobenius group. Then  $G$  contains a  $C$ -Frobenius group  $H$  such that  $G = HA$ .*

*Proof.* Put  $F = G/A$  and let  $K$  be the Frobenius kernel of  $F$ . Let  $A \leq M \triangleleft G$  such that  $K = M/A$ . So  $G = C \rtimes M$ . For each prime  $r$  let  $K_r$  be the Sylow  $r$ -subgroup of  $K$  and let  $M_r$  be a Sylow  $r$ -subgroup of  $M$  such that  $M_r A/A = K_r$ . As  $A$  acts trivially on itself,  $F$  acts on  $A$ , and  $A$  is an irreducible  $F$ -module.

Assume first that  $K$  acts nontrivially on  $A$ . Then there is a prime  $r$  such that  $K_r$  acts nontrivially on  $A$ . Thus  $A^{K_r} \neq A$ . As  $K_r$  is normal in  $F$ ,  $A^{K_r}$  is an  $F$ -submodule of  $A$ , and hence  $A^{K_r} = 0$ . Therefore  $K_r$  has no nontrivial fixed points on  $A$ . In particular,  $r \neq p$  and  $K_r \neq 1$ . Thus  $M_r$  maps modulo  $A$  isomorphically onto  $K_r$  and  $C_A(M_r) = 1$ . As  $M_r \cap A = 1$ , we have  $N_A(M_r) = C_A(M_r)$ . So  $N_A(M_r) = 1$ . By Sylow's theorem,  $G = AN_G(M_r)$ . Thus,  $N_G(M_r)$  is a complement to  $A$  in  $G$ , and hence isomorphic to  $F$ . In particular  $N_G(M_r)$  is a  $C_1$ -Frobenius group for some cyclic subgroup  $C_1$ , isomorphic to  $C$ . Note that  $A$  (after scalars extension) is the direct sum of eigenspaces  $A_\lambda$  of  $K_r$  corresponding to eigenvalues  $\lambda$  of  $r$ -power order. Since generators  $k$  of  $K_r$  and  $c$  of  $CA/A$  satisfy  $kc = ck^n$ , where  $n = |C|$  is prime to  $r$ , it follows that  $C$  maps  $A_\lambda$  onto  $A_{\lambda^n}$ . Therefore  $C$  permutes freely the  $A_\lambda$ 's. Thus,  $A$  is a free  $C$ -module. In particular,  $H^1(C, A) = 0$  and so any two complements to  $A$  in  $AC$  are conjugate. Thus,  $C$  is conjugate to  $C_1$  and so is contained in a complement  $H$  to  $A$ .

Next assume that  $K$  acts trivially on  $A$ . As  $K$  is cyclic,  $M$  is abelian. It suffices to find, for each prime  $r$  dividing  $|K|$ , a cyclic  $r$ -subgroup  $L_r$  of  $M$ , normalized by  $C$ , such that  $L_r A/A = K_r$ . Indeed, then  $H = (\prod_r L_r)C$  is a  $C$ -Frobenius group mapped onto  $F$ .

Since  $M$  is abelian,  $M_r$  is normalized by  $C$ . So if  $M_r$  is cyclic, take  $L_r = M_r$ . This is the case if  $r \neq p$ , since  $M_r \cong K_r$ . So let  $r = p$  with  $M_p$  not cyclic and  $K_p \neq 1$ . Thus,  $|C|$  divides  $p - 1$ . Hence every vector space

over  $\mathbb{Z}/p\mathbb{Z}$  on which  $C$  acts is the direct sum of 1-dimensional  $C$ -modules. Since  $C$  acts irreducibly on  $A$ , this implies that  $A$  is 1-dimensional. As  $K_p$  is cyclic and  $M_p$  not,  $M_p$  is of rank 2, whence its Frattini quotient  $V$  is of dimension 2. In particular, the image  $\overline{A}$  of  $A$  in  $V$  is of dimension 1. Thus  $V = \overline{A} \oplus \overline{B}$  for some 1-dimensional  $C$ -module  $\overline{B}$ . Choose a cyclic subgroup  $L_p$  of  $M_p$  such that its image in  $V$  is  $\overline{B}$ . Then,  $L_p$  is normalized by  $C$  and  $M_p = L_p A$ , whence  $L_p A/A = K_p$ .  $\square$

**Lemma 4.3.** *Let  $G$  be a finite group and let  $A \triangleleft G$ . Assume that  $A = \prod_{i=1}^t Q_i$  and the conjugation in  $G$  transitively permutes the  $Q_i$ . Let  $G_1 = N_G(Q_1)$  and for each right coset  $Z$  of  $G_1$  in  $G$  let  $\widehat{Z} \in Z$ . Let  $U_1$  be a  $G_1$ -invariant subgroup in  $Q_1$ . Then  $U := \prod_Z U_1^{\widehat{Z}} \leq A$  is  $G$ -invariant in  $A$ . Moreover, if  $U_1$  is not normal in  $Q_1$  then  $U$  is not normal in  $A$ .*

*Proof.* We have  $\{Q_i\}_{i=1}^t = \{Q_1^{\widehat{Z}}\}_{Z \in G/G_1}$ . Thus  $A = \prod_{Z \in G/G_1} Q_1^{\widehat{Z}}$  and hence  $U \leq A$ .

To show the invariance, let  $g \in G$ . For each  $Z \in G/G_1$  we have  $Q_1^{\widehat{Z}g\widehat{Z}g^{-1}} = Q_1$ , hence  $\widehat{Z}g\widehat{Z}g^{-1} \in G_1$ . Thus there is  $b_Z \in Q_1$  such that  $U_1^{\widehat{Z}g\widehat{Z}g^{-1}} = U_1^{b_Z}$ . Put  $a_Z = \widehat{Z}g^{-1}b_Z\widehat{Z}g$ , then  $a_Z \in Q_1^{\widehat{Z}g}$  and  $U_1^{\widehat{Z}g} = (U_1^{\widehat{Z}g})^{a_Z}$ . Thus  $a := \prod_{Z \in G/G_1} a_Z \in A$  and  $U^g = U^a$ .

The last assertion is clear.  $\square$

**Lemma 4.4.** *Assume, in the situation of the preceding lemma, that  $G$  is a  $C$ -group such that  $G/A$  is a  $C$ -Frobenius group. Let  $C_1 = N_C(Q_1) = C \cap G_1$  and assume that  $G_1/A = C_1 K_1$  for some subgroup  $K_1$  of the Frobenius kernel of  $G/A$ . Assume that  $U_1$  is  $C_1$ -invariant. Then we may choose the  $\widehat{Z} \in Z$  so that  $U$  is  $C$ -invariant.*

*Proof.* It suffices to choose the representatives  $\widehat{Z}$  so that

$$(1) \quad U_1^{\widehat{Z}c} = U_1^{\widehat{Z}} \text{ for every } c \in C.$$

But to achieve (1), it suffices to achieve it only for a representative  $Z$  of each  $C$ -orbit in  $G/G_1$  (when  $G$  and  $C \leq G$  act on  $G/G_1$  by multiplication from the right). Let  $C_Z$  be the  $C$ -stabilizer of  $Z$  and let  $R_Z$  be a set of representatives of  $C/C_Z$  in  $C$ . Suppose we have found  $\widehat{Z} \in Z$  such that  $C_Z$  normalizes  $U_1^{\widehat{Z}}$ . Then put  $\widehat{Z}cr = \widehat{Z}r$  for all  $c \in C_Z$  and  $r \in R_Z$  to get (1).

As  $G$  acts transitively on  $G/G_1$  with  $A$  acting trivially, this induces a transitive action of  $F = G/A$  on  $G/G_1$  and the  $F$ -stabilizer of  $G_1$  is  $C_1 K_1$ . By Lemma 2.2 either  $Z = G_1 c$  for some  $c \in C$  and  $C_Z = C_1$  or  $C_Z = 1$ . In the former case we may take  $\widehat{Z} = c$  to make  $C_Z$  normalize  $U_1^{\widehat{Z}}$ . In the latter case we may choose  $\widehat{Z}$  arbitrarily.  $\square$

We now prove the main step:

**Theorem 4.5.** *Let  $G$  be a finite group with a normal subgroup  $A$  and a cyclic subgroup  $C$  such that  $C \cap A = 1$  and  $G/A$  is a  $C$ -Frobenius group. Then  $G$  contains a  $C$ -Frobenius group  $H$  such that  $G = HA$ .*

*Proof.* Put  $F = G/A$  and let  $K$  be its Frobenius kernel. We identify  $CA/A$  with  $C$ .

We prove the result by induction on the order of  $A$ . We may assume that  $A \neq 1$ . We divide the proof into several parts.

**Part A** *We may assume that  $A$  is a minimal normal subgroup of  $G$ .* Let  $B$  be a minimal normal subgroup  $G$  contained in  $A$ . By induction,  $G/B$  satisfies the theorem with  $A/B \triangleleft G/B$ . Thus there is a subgroup  $H_0$  of  $G$  containing  $B$  and  $C$  such that  $H_0A = G$  and  $H_0/B$  is a  $C$ -Frobenius group. If  $B \neq A$ , apply the induction to  $H_0$  with  $B \triangleleft H_0$  to get a  $C$ -Frobenius subgroup  $H$  of  $H_0$  such that  $H_0 = HB$ . Then  $HA = H_0A = G$ . So we may assume that  $B = A$ .

**Part B** Thus  $A$  is the direct product of copies of a finite simple group. If  $A$  is an elementary abelian  $p$ -group with  $p$  prime, then we are done by Lemma 4.2. So we may assume that  $A = Q_1 \times \dots \times Q_t$  where  $Q_i = Q$  is a nonabelian simple group.

**Part C** *We may assume that  $G$  acts faithfully on  $A$ , that is,  $G \rightarrow \text{Aut}(A)$  is an embedding.* Indeed, otherwise let  $B$  be a minimal normal subgroup of  $G$  contained in the kernel of this map. Then  $A \cap B = 1$ , and hence  $B$  is isomorphic to a subgroup of  $G/A$ . In particular,  $B$  is solvable. Moreover, the image of  $B$  in  $G/A$  is a minimal normal subgroup, and hence is contained in  $K$  by Lemma 2.1(f). Thus,  $AB \cap C = 1$ . This allows us to identify  $C$  with  $CAB/AB$ .

The quotient  $G/AB$  of  $G/A$  either equals to  $C$  (that is,  $G = ABC$ ) or is a  $C$ -Frobenius group with  $|G/AB| < |G/A|$ . In both cases there is a solvable subgroup  $G_0$  of  $G$  containing  $C$  and  $B$ , such that  $G_0A/B = G/B$ : In the first case take  $G_0 = CB$ . In the second case proceed by subinduction on  $|G/A|$ ; by the hypothesis, there is a subgroup  $G_0$  of  $G$  containing  $C$  and  $B$ , such that  $G_0A/B = G/B$  and  $G_0/B$  is  $CB/B$ -Frobenius. In particular,  $G_0/B$  is solvable, hence so is  $G_0$ . In both cases, as  $G_0A = G$ , we may replace  $G$  by  $G_0$  and  $A$  by  $A \cap G_0$ . As  $G_0$  is solvable but  $A$  not,  $|A \cap G_0| < |A|$ . So the existence of  $H$  follows by induction.

**Part D** *Reduction to invariance* Our aim is to construct a subgroup  $U$  of  $A$  such that  $U$  is  $A$ -invariant in  $G$ ,  $C$  normalizes  $U$ , and  $A$  does not normalize  $U$ . Then  $N_G(U)A = G$  and  $N_G(U) \cap A$  is a proper subgroup of  $A$ . Hence by induction hypothesis  $N_G(U)$  contains a  $C$ -Frobenius subgroup  $H$  with  $HA = G$ .

**Part E** *Division into three cases.* So let  $G_1 = N_G(Q_1)$ . Then  $A \leq G_1$ ; put  $F_1 = G_1/A \leq F$ . Note that  $F = G/A$  acts transitively on the set  $\Delta := \{Q_1, \dots, Q_t\}$ , and  $F_1$  is the stabilizer of  $Q_1$ . By Lemma 2.1(e) we may replace  $Q_1$  by some conjugate  $Q_i$  to assume that  $F_1 = C_1K_1$ , where  $C_1 \leq C$  and  $K_1 \leq K$ . We divide the rest of the proof into three cases.

*Case I:  $C_1$  centralizes  $Q_1$ .* In this case let  $U_1 \neq 1$  be a Sylow subgroup of  $Q_1$ . Then  $U_1$  is  $C_1$ -invariant, invariant in  $Q_1$ , and not normal in  $Q_1$ . By

Lemma 4.3 and Lemma 4.4 there is a  $C$ -invariant  $G$ -invariant subgroup  $U$  of  $A$  which is not normal in  $A$ . (Notice that case  $C_1 = 1$  is included here.)

*Case II:*  $C_1, K_1 \neq 1$ . In this case  $F_1$  is a  $C_1$ -Frobenius group. Let  $\phi: G_1 \rightarrow \text{Aut}(Q_1)$  be the map induced by the action of  $G_1$  on  $Q_1$  and let  $\bar{G}_1 = \phi(G_1)$ . Then  $\phi(A) = \phi(Q_1) = Q_1$ , hence  $\phi$  induces a surjection  $\bar{\phi}: F_1 = \bar{G}_1/A \rightarrow \bar{G}_1/Q_1 \leq \text{Out}(Q_1)$ . We first claim that  $\bar{\phi}$  is an isomorphism, that is,  $F_1 \rightarrow \text{Out}(Q_1)$  is injective.

It suffices to show that  $K_1 \rightarrow \text{Out}(Q_1)$  is injective, since the kernel of  $F_1 \rightarrow \text{Out}(Q_1)$  is a normal subgroup of the Frobenius group  $F_1$ , which does not intersect its Frobenius kernel  $K_1$  and hence is trivial by Lemma 2.1(f).

So let  $g \in G$  with image in  $K_1$  act on  $Q_1$  as an inner automorphism. As  $F = G/A$  is a Frobenius group, for every  $\sigma \in G$  we have  $\sigma g \sigma^{-1} = g^m a$  for some  $m \in \mathbb{N}$  and  $a \in A$ . It follows that  $g$  acts as an inner automorphism, namely  $(g^\sigma)^m a^\sigma$ , on  $Q_1^\sigma$ . Thus  $g$  acts as an inner automorphism on  $A$ . As  $G$  acts faithfully on  $A$ , this means that  $g \in A$ . Therefore the image of  $g$  in  $K_1$  is trivial. This proves the claim.

Thus  $\bar{G}_1/Q_1 \cong F_1$  is a  $C_1$ -Frobenius group. By Corollary 3.11,  $C_1$  normalizes some nontrivial invariant proper subgroup  $U_1$  of  $Q_1$ . So again we are done by Lemma 4.3 and Lemma 4.4.

*Case III:*  $K_1 = 1$  and  $C_1$  does not centralize  $Q_1$ . In particular,  $C$  does not centralize  $Q_1$ . Notice that  $G_1 = C_1 A \leq CA$ . For each  $g \in G$  let  $\Delta_g = \{Q_1^{\sigma g} \mid \sigma \in CA\} \subseteq \Delta$ . Then  $\Delta_{g_1} \cap \Delta_{g_2} \neq \emptyset \implies CA_{g_1} = CA_{g_2} \implies \Delta_{g_1} = \Delta_{g_2}$ . Thus  $\Delta$  is the disjoint union of the distinct  $\Delta_g$ . Therefore if we define  $\tilde{Q}_g = \langle Q_1^{\sigma g} \mid \sigma \in CA \rangle$ , then  $A$  is the direct product of the distinct  $\tilde{Q}_g$ . The conjugation in  $G$  permutes the  $\Delta_g$  and hence also the  $\tilde{Q}_g$ . Put  $\tilde{G}_1 = N_G(\tilde{Q}_1)$ ; then  $\tilde{G}_1 = CA$ . Thus  $\tilde{C}_1 = C \cap \tilde{G}_1 = C$ .

We define  $U_1 = C_{\tilde{Q}_1}(\tilde{C}_1)$ ; thus  $U_1$  is a  $\tilde{C}_1$ -invariant subgroup of  $\tilde{Q}_1$ . By our assumption,  $U_1 \neq \tilde{Q}_1$ . By Lemma 3.2,  $U_1 \neq 1$ . As  $\tilde{G}_1 = \tilde{C}_1 A = \tilde{C}_1(\tilde{Q}_1 \times \prod_{\tilde{Q}_g \neq \tilde{Q}_1} \tilde{Q}_g)$  and  $U_1$  is  $\tilde{C}_1$ -invariant,  $U_1$  is  $\tilde{G}_1$ -invariant in  $\tilde{Q}_1$ . Now apply again Lemma 4.3 and Lemma 4.4 with  $\{\tilde{Q}_g \mid g \in G\}$  instead of  $\Delta$  and with  $\tilde{Q}_1, \tilde{G}_1, \tilde{C}_1$  instead of  $Q_1, G_1, C_1$ .

□

*Remark 4.6.* If  $C$  is an  $r$ -group for some prime  $r$ , we may considerably simplify the proof of Theorem 4.5, leaving out the classification of finite simple groups. Mainly, omit Part C and replace Part E by the following:

Let  $M$  be the normal subgroup of  $G$  such that  $M/A = K$ . By Lemma 3.4,  $C$  normalizes some nontrivial Sylow subgroup  $U$  of  $M$ . By Frattini argument,  $U$  is invariant in  $G$ .

## 5. PROFINITE GROUPS

**Theorem 5.1.** *Let  $G$  be a  $C$ -group and let  $G \rightarrow F$  be a  $C$ -epimorphism onto a  $C$ -Frobenius group  $F$ . Then  $G$  contains a  $C$ -Frobenius subgroup  $H$*

that maps onto  $F$ . Moreover, we may assume that  $F$  and  $H$  have precisely the same prime divisors.

*Proof.* The last assertion of the theorem follows immediately from the rest, because we may drop from the Frobenius kernel of  $H$  its  $p$ -primary components for those primes  $p$  that do not divide the order of the Frobenius kernel of  $F$ .

If necessary, replace  $G$  by a closed subgroup to assume that  $G$  is finitely generated. Then there is a sequence  $N_0 \geq N_1 \geq N_2 \geq \dots$  of open normal subgroups of  $G$  such that  $\bigcap_i N_i = \{1\}$ . Let  $A = \ker(G \rightarrow F)$ . Since  $\bigcap_i N_i A = A$ , without loss of generality  $C \cap N_i A = \{1\}$  for every  $i$ .

Put  $M_i = N_i \cap A$  for every  $i$ . Then  $F = G/M_0$  and  $G = \varprojlim_i G/M_i$ . For each  $i$  we have the following cartesian diagram of epimorphisms of  $C$ -groups

$$\begin{array}{ccc} G/M_{i+1} & \longrightarrow & G/N_{i+1} \\ \downarrow & & \downarrow \\ G/M_i & \longrightarrow & G/M_i N_{i+1} \end{array}$$

in which the groups on the right handed side are finite. By induction hypothesis  $G/M_i$  contains a  $C$ -Frobenius subgroup  $F_i$  that maps onto  $F$ . Its image in  $G/M_i N_{i+1}$  is a  $C$ -Frobenius subgroup. By Theorem 4.5 it lifts to a  $C$ -Frobenius subgroup of  $G/N_{i+1}$ . Hence by Lemma 4.1  $F_i$  lifts to a  $C$ -Frobenius subgroup  $F_{i+1}$  of  $G/N_{i+1}$ . Thus  $H = \varprojlim_i F_i$  is a  $C$ -Frobenius subgroup of  $G$  that maps onto  $F$ .  $\square$

**Corollary 5.2.** *Let  $\beta: B \rightarrow C$  be an epimorphism of a finite group  $B$  onto a finite nontrivial cyclic group  $C$  and let  $F$  be a  $C$ -Frobenius group with Frobenius kernel  $K \cong \hat{\mathbb{Z}}_\pi$  for some set  $\pi$  of primes. Then there is a  $C$ -embedding  $F \rightarrow B \amalg C$ .*

*Proof.* By Theorem 5.1 it suffices to construct a  $C$ -epimorphism  $B \amalg C \rightarrow F$ . Let  $k$  be a generator of  $K$ . The epimorphism  $B \rightarrow C$  given by  $b \mapsto \beta(b)^k$  together with the identity map  $C \rightarrow C$  define a  $C$ -homomorphism  $B \amalg C \rightarrow F$ . Its image  $\langle C, C^k \rangle$  contains  $C$  and  $(c^k)^{-1}c = k^{-1}k^c \in K$ , where  $c$  is a generator of  $C$ . Since  $F = CK$ , it suffices to show that  $k^{-1}k^c$  generates  $K$ .

If this assertion holds for each Sylow subgroup of  $K$  (and its generator instead of  $k$ ) then it holds for  $K$ . Thus we may assume that  $K \cong \mathbb{Z}_p$  for some prime  $p$ . Finally, we may replace  $K$  by its Frattini quotient.  $\bar{K} \cong \mathbb{Z}/p\mathbb{Z}$ .

As  $c \neq 1$ , we have  $k^{-1}k^c \neq 1$  and hence  $k^{-1}k^c$  generates  $K$ .  $\square$

*Example 5.3.* Let  $K = \mathbb{Z}_7$ . Then  $C = \mathbb{Z}/6\mathbb{Z}$  acts fixed-point freely on  $K$  and hence the semidirect product  $F = CK$  is a  $C$ -Frobenius group. By the corollary there is an embedding  $\lambda: F \rightarrow G_1 \amalg G_2$ , where  $G_1 = G_2 = C$ , such that  $\lambda(C) = G_2$ . Let  $H = \lambda(F)$ . Then  $H$  is prosolvable, but

- (1)  $H \cap G_2 = G_2$  is of order 6, and hence is not an  $\ell$ -group for some prime  $\ell$ ;

(2)  $H$  is infinite, and hence  $H \leq G_i^\sigma$  for no  $i \in I$  and no  $\sigma \in G$ .

This answers the question of Pop mentioned in the introduction.

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