REGULAR SPLIT EMBEDDING PROBLEMS OVER FUNCTION FIELDS OF ONE VARIABLE OVER AMPLE FIELDS*

by

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Introduction

Separably closed fields, Henselian fields, PAC fields, PRC fields, and PpC fields enjoy a common feature: each of them is existentially closed in the corresponding field of formal power series. We have called a field K with this property **ample***. Alternatively, a field K is ample, if each absolutely irreducible curve C over K with a simple K-rational point has infinitely many K-rational points. The main result of [HaJ] reveals a remarkable property of K: Each finite constant split embedding problem over $K(x)$ has a rational solution. More precisely, [HaJ] gives an alternative proof to a result of Pop [Po1, Main Theorem A]:

THEOREM A: *Let K be an ample field and let L be a finite Galois extension of K. Suppose that $\mathcal{G}(L/K)$ acts on a finite group G. Then, there is a field F with the following properties:*

(a) *F is a Galois extension of $K(x)$ which contains L.*

(b) *There is an isomorphism $\alpha\colon G\rtimes\mathcal{G}(L/K) \to \mathcal{G}(F/K(x))$ such that $\mathrm{res}_L \circ\alpha = \mathrm{pr}$.*

(c) *F has an L-rational place $\varphi\colon F \to L \cup \{\infty\}$.*

Among others, this result settled Problem 24.41 of [FrJ]: Every PAC Hilbertian field is ω-free.

Previous proofs of this result used analytical methods (complex analytical methods in characteristic 0 [FrV] and rigid analytical methods in the general case [Po1]). In contrast, our approach in [HaJ] was elementary, algebraic, and together with [HaV], self-contained. Indeed, we took an axiomatic approach: Let F/E be a Galois extension of arbitrary fields. Suppose that $\mathcal{G}(F/E)$ acts on a finite group G. Suppose that this action extends to a "proper action" on an appropriate "patching data" $(E, F_i, Q_i, Q; G_i, G)_{i\in I}$. Then, the split embedding problem $G\rtimes\mathcal{G}(F/E) \to \mathcal{G}(F/E)$ has a solution.

In this note we use our approach via algebraic patching to give an elementary proof of a generalization of Theorem A, due to Pop [Po2, Theorem 2.7]:

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* Pop, who introduces this type of fields in [Po1], calls them 'large'. Since this name has been used earlier with a different meaning, we have modified it to 'ample' [HaJ, Definition 6.3 and the attached footenote].

THEOREM B: *Let E be a function field of one variable over an ample field K. Suppose that E/K is separable. Let F be a finite Galois extension of E. Denote the algebraic closure of K in F by L. Suppose that $\mathcal{G}(F/E)$ acts on a finite group G. Then there exists a finite field $\hat{F}$ with the following properties:*

(a) *$\hat{F}$ is a Galois extension of E which contains F;*

(b) *There is an isomorphism $\alpha\colon G \rtimes \mathcal{G}(F/E) \to \mathcal{G}(\hat{F}/E)$ such that $\mathrm{res}_F \circ \alpha = \mathrm{pr}$.*

(c) *$\hat{F}$ is a regular extension of L.*

Group theoretic and Galois theoretic manipulations reduce the proof of Theorem B to the case where $E = K(x)$ and x is, as always, transcendental over K (Proposition 1.4). Moreover, we may extend L if necessary, so that F has an L-rational place $\varphi\colon F \to L \cup \{\infty\}$ and $\varphi(x) \in K$. As usual, we replace K at this point by $K((t))$, if necessary, to assume that K is complete under an ultra-metric absolute value, its residue field is infinite, and L/K is an unramified extension. Let $\Gamma = \mathcal{G}(L(x)/E)$ and $G_1 = \mathcal{G}(F/L(x))$. The existence of φ implies that the extension $\mathcal{G}(F/E) \to \Gamma$ splits. As in [HaJ], we then construct a patching data $(L(x), F_i, Q_i, Q; G_i, G \rtimes G_1)_{i\in I}$, on which Γ acts properly such that $1 \in I$ and $F_1 = F$. The "compound" $\hat{F}$ of this patching data is a Galois extension of E, and there exists an isomorphism $\alpha\colon (G \rtimes G_1) \rtimes \Gamma \to \mathcal{G}(\hat{F}/E)$ such that $\mathrm{res}_{\hat{F}/L(x)} \circ \alpha = \mathrm{pr}_\Gamma$. Moreover, let α_0 be the restriction of α to $G \rtimes G_1$. Based on an observation of [HaV], we find that $\hat{F}$ contains F, $\alpha_0(G \rtimes G_1) = \mathcal{G}(\hat{F}/L(x))$, and $\mathrm{res}_{\hat{F}/F} \circ \alpha_0 = \mathrm{pr}_{G_1}$. As $(G \rtimes G_1) \rtimes \Gamma = G \rtimes (G_1 \rtimes \Gamma) = G \rtimes \mathcal{G}(F/E)$, the field $\hat{F}$ is a solution to the original embedding problem $\mathrm{pr}\colon G \rtimes \mathcal{G}(F/E) \to \mathcal{G}(F/E)$.

1. Generalities on split embedding problems

Let K_0 be a field. Let x be a transcendental element over K_0, let E_0 be a finite extension of $K_0(x)$, and let E be a finite Galois extension of E_0. Assume that $\mathcal{G}(E/E_0)$ acts on a finite group G; let $G \rtimes \mathcal{G}(E/E_0)$ be the semidirect product and let $\mathrm{pr}\colon G \rtimes \mathcal{G}(E/E_0) \to \mathcal{G}(E/E_0)$ be the corresponding projection. We call

$$(1) \qquad G \rtimes \mathcal{G}(E/E_0) \xrightarrow{\mathrm{pr}} \mathcal{G}(E/E_0).$$

a **split K_0-embedding problem**. A **solution field** to problem (1) is a finite Galois extension F of E_0 containing E for which there exists an isomorphism $\alpha\colon G \rtimes \mathcal{G}(E/E_0) \to \mathcal{G}(F/E_0)$ such that $\mathrm{res}_E \circ \alpha = \mathrm{pr}$.

Let K be the algebraic closure of K_0 in E. We say that (1) is **regular** if E/K is regular, that is, E/K is separable. For instance, if E_0/K_0 is separable, then (1) is regular. We say that the solution is **regular** if F is regular over K.

Clearly, only a regular embedding problem may have a regular solution.

LEMMA 1.1: *In the above notation let (1) be a split K_0-embedding problem. Let E' be a finite Galois extension of E_0 that contains E, let $\mathrm{res}\colon \mathcal{G}(E'/E_0) \to \mathcal{G}(E/E_0)$ be the restriction map, and let $h\colon G' \to G$ be an epimorphism of finite groups. Assume that $\mathcal{G}(E'/E_0)$ acts on G' such that*

$$(2) \qquad h(\sigma^\gamma) = h(\sigma)^{\mathrm{res}(\gamma)} \qquad \text{for each } \gamma \in \mathcal{G}(E'/E_0) \text{ and } \sigma \in G'.$$

Consider the corresponding split K_0-embedding problem

$$(1') \qquad G' \rtimes \mathcal{G}(E'/E_0) \xrightarrow{\mathrm{pr}'} \mathcal{G}(E'/E_0).$$

(a) *If (1′) has a solution, then (1) has a solution;*

(b) *if (1′) has a regular solution and E_0/K_0 is separable, then (1) has a regular solution.*

Proof: Let F' be a solution of (1′) and let $\alpha'\colon G' \rtimes \mathcal{G}(E'/E_0) \to \mathcal{G}(F'/E_0)$ be an isomorphism such that $\mathrm{res}_{E'} \circ \alpha' = \mathrm{pr}'$.

By (2), there is a commutative diagram of group epimorphisms

$$
\text{(3)}\qquad
\begin{array}{ccc}
G'\ltimes\mathcal{G}(E'/E_0) & \xrightarrow{\mathrm{pr}'} & \mathcal{G}(E'/E_0) \\
\Big\downarrow{\scriptstyle (h,\mathrm{res})} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\
G\ltimes\mathcal{G}(E/E_0) & \xrightarrow{\mathrm{pr}} & \mathcal{G}(E/E_0).
\end{array}
$$

Let C be the kernel of the map $(h,\mathrm{res})\colon G'\ltimes\mathcal{G}(E'/E_0)\to G\ltimes\mathcal{G}(E/E_0)$ and let F be the fixed field of $\alpha'(C)$ in F', that is, $\alpha'(C)=\mathcal{G}(F'/F)$. As $C\triangleleft G'\ltimes\mathcal{G}(E'/E_0)$, the extension F/E_0 is Galois. We have

$$
\mathrm{res}_E\,\mathcal{G}(F'/F)=\mathrm{res}_{E'/E}\circ\mathrm{res}_{E'}\circ\alpha'(C)=\mathrm{res}_{E'/E}\circ\mathrm{pr}'(C).
$$

Hence by (3), $\mathrm{res}_E\,\mathcal{G}(F'/F)=1$. Therefore, $E\subseteq F$. The isomorphism α' induces an isomorphism $\alpha\colon G\ltimes\mathcal{G}(E/E_0)\to\mathcal{G}(F/E_0)$ such that $\mathrm{res}_E\circ\alpha=\mathrm{pr}$.

$$
\text{(4)}\qquad
\begin{array}{ccccc}
 & & F & \text{---} & F' \\
 & & | & & | \\
E_0 & \text{---} & E & \text{---} & E' \\
| & & | & & | \\
K_0 & \text{---} & K & \text{---} & K'
\end{array}
\qquad
\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{\alpha'} & \mathcal{G}(F'/F) & \xrightarrow{\mathrm{res}} & \mathcal{G}(E'/E) \\
\downarrow & & \downarrow & & \downarrow \\
G'\ltimes\mathcal{G}(E'/E_0) & \xrightarrow{\alpha'} & \mathcal{G}(F'/E_0) & \xrightarrow{\mathrm{res}} & \mathcal{G}(E'/E_0) \\
\Big\downarrow{\scriptstyle (h,\mathrm{res})} & & \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\
G\ltimes\mathcal{G}(E/E_0) & \xrightarrow{\alpha} & \mathcal{G}(F/E_0) & \xrightarrow{\mathrm{res}} & \mathcal{G}(E/E_0) \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}
$$

This proves (a).

Let K be the algebraic closure of K_0 in E and let K' be the algebraic closure of K_0 in E'. Assume that E_0/K_0 is separable. Then so is F/K_0, and hence also F/K.

By diagram (4), $\mathrm{res}_{E'}\,\mathcal{G}(F'/F)=\mathcal{G}(E'/E)$. Hence $F\cap E'=E$. It follows that $F\cap K'=F\cap E'\cap K'=E\cap K'=K$. Thus, if F'/K' is regular, so is F/K. ∎

LEMMA 1.2: *Let F be a (regular) solution of a split K_0-embedding problem*

$$G \rtimes \mathcal{G}(E/E_0) \xrightarrow{\mathrm{pr}} \mathcal{G}(E/E_0).$$

Let E_0' be an intermediate field of E/E_0, and let K_0' be the algebraic closure of K_0 in E_0'. Then, the subgroup $\mathcal{G}(E/E_0')$ of $\mathcal{G}(E/E_0)$ defines a split K_0'-embedding problem

$$G \rtimes \mathcal{G}(E/E_0') \xrightarrow{\mathrm{pr}'} \mathcal{G}(E/E_0')$$

and F is its (regular) solution.

Proof: If $\alpha\colon G \rtimes \mathcal{G}(E/E_0) \to \mathcal{G}(F/E_0)$ is an isomorphism such that $\mathrm{res}_E \circ \alpha = \mathrm{pr}$, then $\alpha(G \rtimes \mathcal{G}(E/E_0')) = \mathcal{G}(F/E_0')$. ∎

LEMMA 1.3: *Let Γ be a subgroup of a finite group Δ. Suppose that Γ acts on a finite group G. Then there exist a finite group G' and an epimorphism $h\colon G' \to G$ such that Δ acts on G' and $h(\sigma^\gamma) = h(\sigma)^\gamma$ for each $\gamma \in \Gamma$ and $\sigma \in G'$.*

Proof:

PART A: *A free group.* We first omit the requirement that G' be finite; in fact, we now require that it be a finitely generated free group. Choose a set X of generators of G. Let $Y = X \times \Delta$ and let $\hat{G}$ be the free group on Y. The group Δ acts on the set Y by multiplication from the right on the second factor. This action extends to an action of Δ on the group $\hat{G}$. Choose a system of representatives Δ_0 for the left cosets of Γ in Δ, that is, $\Delta = \bigcup_{\delta \in \Delta_0} \delta\Gamma$. Define a map $\hat{h}\colon Y \to G$ by

$$\hat{h}(x, \delta\gamma) = x^\gamma \text{ for } x \in X,\ \delta \in \Delta_0,\ \text{and } \gamma \in \Gamma.$$

This map extends to an epimorphism $\hat{h}\colon \hat{G} \to G$. For all $\gamma, \gamma' \in \Gamma$ we have $\hat{h}((x, \delta\gamma')^\gamma) = \hat{h}(x, \delta\gamma'\gamma) = x^{\gamma'\gamma} = \hat{h}(x, \delta\gamma')^\gamma$. Hence $\hat{h}(\hat{\sigma})^\gamma = \hat{h}(\hat{\sigma}^\gamma)$ for each $\hat{\sigma} \in \hat{G}$ and each $\gamma \in \Gamma$, as required.

PART B: *A finite group.* As $\hat{G}$ if finitely generated, the collection $\mathcal{F}$ of all epimorphisms of $\hat{G}$ onto G is finite. Therefore $N = \bigcap_{f \in \mathcal{F}} \mathrm{Ker}\, f$ is a normal subgroup of $\hat{G}$ of finite index. As $\hat{h} \in \mathcal{F}$, we have $N \leq \operatorname{Ker} \hat{h}$. Hence $G' = \hat{G}/N$ is a finite group and $\hat{h}$ induces an epimorphism $h\colon G' \to G$.

If $\delta \in \operatorname{Aut}(\hat{G})$, then $\{f \circ \delta |\ f \in \mathcal{F}\} = \mathcal{F}$, and hence $\delta(N) = N$. Therefore each $\delta \in \Delta$ induces a unique automorphism δ of G' such that $(\sigma N)^\delta = \sigma^\delta N$ for each $\sigma \in \hat{G}$. It follows that Δ acts on the group G'. Moreover, for each $\gamma \in \Gamma$

$$h((\sigma N)^\gamma) = h(\sigma^\gamma N) = \hat{h}(\sigma^\gamma) = \hat{h}(\sigma)^\gamma = h(\sigma N)^\gamma.$$

Hence $h(\sigma^\gamma) = h(\sigma)^\gamma$ for each $\gamma \in \Gamma$ and each $\sigma \in G'$. ∎

PROPOSITION 1.4: *Suppose that every (regular) split K_0-embedding problem*

$$(5') \qquad G' \rtimes \mathcal{G}(E'/K_0(x)) \xrightarrow{\mathrm{pr}'} \mathcal{G}(E'/K_0(x))$$

has a (regular) solution. Then every (regular) split K_0-embedding problem

$$(5) \qquad G \rtimes \mathcal{G}(E/E_0) \xrightarrow{\mathrm{pr}} \mathcal{G}(E/E_0)$$

has a (regular) solution.

Proof: There are two cases to consider:

CASE A: *E_0/K_0 is separable.* Replace x by another transcendental element (separating transcendence basis) of E_0/K_0 to assume that $E_0/K_0(x)$ is separable. Let E' be the Galois closure of E over $K_0(x)$. Then $\mathcal{G}(E'/E_0)$ acts on G via the restriction map $\mathcal{G}(E'/E_0) \to \mathcal{G}(E/E_0)$. Lemma 1.3 gives a finite group G', an epimorphism $h\colon G' \to G$, and an action of $\mathcal{G}(E'/K_0(x))$ on G' such that $h(\sigma^\gamma) = h(\sigma)^\gamma$ for each $\gamma \in \mathcal{G}(E'/E_0)$ and $\sigma \in G'$. This action defines a split K_0-embedding problem (5′). As E'/K_0 is separable, (5′) is regular. By assumption, it has a regular solution. By Lemma 1.2, this solution is also a regular solution of

$$G' \rtimes \mathcal{G}(E'/E_0) \xrightarrow{\mathrm{pr}'} \mathcal{G}(E'/E_0).$$

By Lemma 1.1, (5) has a regular solution.

CASE B: E_0/K_0 *is not separable.* In this case $\mathrm{char}(K) = p > 0$. Let $K_0' = K_0^{1/q}$, where q is a power of p, and put $E_0' = EK_0'$ and $E' = EK_0'$. If q is sufficiently large, then E_0'/K_0' is a separable extension; assume this is the case.

As E_0'/E_0 is purely inseparable, E'/E_0' is a Galois extension and the restriction $\mathrm{res}\colon \mathcal{G}(E'/E_0') \to \mathcal{G}(E/E_0)$ is an isomorphism. Thus (5) induces a split K_0'-embedding problem

$$G \rtimes \mathcal{G}(E'/E_0') \longrightarrow \mathcal{G}(E'/E_0'). \tag{6}$$

The map $y \mapsto y^q$ gives an isomorphism of K_0' onto K_0, and hence the assumptions of our proposition are satisfied with K_0' instead of K_0. Therefore, by Case A, (6) has a regular solution F'. Again, as E_0'/E_0 is linearly disjoint from the separable closure of E_0 over E_0, there exists a unique Galois extension F/E_0 such that $F' = FE_0'$. In particular, the restriction $\mathcal{G}(F'/E_0') \to \mathcal{G}(F/E_0)$ is an isomorphism, and hence F is a solution of (5).

Suppose now that (5) is regular, that is, E is regular over $K = E \cap \widetilde{K_0}$. Let $K' = KK_0'$. Then K'/K is a purely inseparable extension and $EK' = EK_0' = E'$. Hence E'/K' is regular, that is, (6) is regular. By our construction F'/K' is regular.

As both F/E and E/K are separable, so is F/K. Therefore the algebraic closure M of K in F is separable over K and F/M is separable. But $F \subseteq F'$, and K' is algebraically closed in F', hence $M \subseteq K'$. As K'/K is a purely inseparable, we have $M = K$. Conclude that F is a regular extension of K. ∎

2. Embedding problems under existentially closed extensions

Consider a field extension $\hat{K}_0/K_0$ such that K_0 is **existentially closed** in $\hat{K}_0$. That is, *each algebraic subset A of $\mathbb{A}^n$ that has a $\hat{K}_0$-rational point also has a K_0-rational point.*

In particular, $\hat{K}_0/K_0$ is regular. Furthermore, if K is a finite extension of K_0 and $\hat{K} = K\hat{K}_0$, then K is existentially closed in $\hat{K}$. Indeed, let $\omega_1, \ldots, \omega_d$ be a linear basis of K/K_0. So if $f \in K[X_1, \ldots, X_n]$, there are unique $f_1, \ldots, f_d \in K_0[X_1, \ldots, X_n]$ such that $f = \sum_{i=1}^d \omega_i f_i$. As $\hat{K}_0/K_0$ is regular, $\omega_1, \ldots, \omega_d$ is also a basis of $\hat{K}/\hat{K}_0$. It follows that the equation $f(X_1, \ldots, X_n) = 0$ has a solution in K^n (resp., $\hat{K}^n$) if and only if

$$\sum_{i=1}^d \omega_i f_i(\sum_{i=1}^d \omega_i X_{1i}, \ldots, \sum_{i=1}^d \omega_i X_{ni}) = 0$$

has a solution in K^{nd} (resp., $\hat{K}^{nd}$). The latter equation can be written as a system of equations over K_0 (resp., over $\hat{K}_0$). Thus $f(X_1, \ldots, X_n) = 0$ has a solution in K^n if and only if it has a solution in $\hat{K}^n$.

Consider a regular split K_0-embedding problem

(1) $$H \rtimes \mathcal{G}(E/K_0(x)) \xrightarrow{\mathrm{pr}} \mathcal{G}(E/K_0(x)).$$

Assume that x is transcendental over $\hat{K}_0$ and put $\hat{E} = E\hat{K}_0$. Then E is linearly disjoint from $\hat{K}_0$ over K_0 [FrJ, Lemma 9.9] and therefore $\mathrm{res}_{\hat{E}/E} : \mathcal{G}(\hat{E}/\hat{K}_0(x)) \to \mathcal{G}(E/K_0(x))$ is an isomorphism. Thus $\mathcal{G}(\hat{E}/\hat{K}_0(x))$ acts on H via $\mathrm{res}_{\hat{E}/E}$. This gives rise to a regular split $\hat{K}_0$-embedding problem

(2) $$H \rtimes \mathcal{G}(\hat{E}/\hat{K}_0(x)) \xrightarrow{\mathrm{pr}} \mathcal{G}(\hat{E}/\hat{K}_0(x)).$$

Let K be the algebraic closure of K_0 in E. Then $\hat{K} = K\hat{K}_0$ is the algebraic closure of $\hat{K}_0(x)$ in $\hat{E}$ [FrJ, Lemma 9.3]. Furthermore, let φ: $E \to K \cup \{\infty\}$ be a K-place unramified over $K(x)$. As $\hat{K}$ and E are linearly disjoint over K, the place φ extends to a $\hat{K}$-rational place $\hat{\varphi}$ of $\hat{E}$, unramified over $\hat{K}(x)$.

In this setup we prove:

LEMMA 2.1: *Assume that (2) has a solution field $\hat{F}$ such that $\hat{\varphi}$ extends to a $\hat{K}$-rational place of $\hat{F}$ unramified over $\hat{K}(x)$. Then (1) has a solution field F such that φ extends to a K-rational place of F unramified over $K(x)$.*

Proof: We may assume that $\varphi(x) = \infty$, otherwise replace x by another generator of $K(x)$ over K.

By assumption, there exists an isomorphism $\alpha: H \rtimes \mathcal{G}(\hat{E}/\hat{K}_0(x)) \to \mathcal{G}(\hat{F}/\hat{K}_0(x))$ such that $\mathrm{res}_{\hat{E}} \circ \alpha = \mathrm{pr}$.

So, there exist polynomials $f \in \hat{K}_0[X,Z]$, $g \in \hat{K}[X,Y]$, and elements $z, y \in \hat{F}$ such that the following conditions hold:

(3a) $\hat{F} = \hat{K}_0(x,z)$, $f(x,Z) = \mathrm{irr}(z, \hat{K}_0(x))$; we may therefore identify $\mathcal{G}(f(x,Z), \hat{K}_0(x))$ with $\mathcal{G}(\hat{F}/\hat{K}_0(x))$;

(3b) $\hat{F} = \hat{K}(x,y)$, $g(x,Y) = \mathrm{irr}(y, \hat{K}(x))$; therefore $g(X,Y)$ is absolutely irreducible; by Lemma 2.2 below we may assume that $g(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X)$ with $a_i \in \hat{K}[X]$ and $\deg a_i(X) \leq \deg a_1(X) \geq 1$, for $i = 1, \ldots, d$.

All of these objects depend on only finitely many parameters from $\hat{K}_0$. So, let $u_1, \ldots, u_n$ be elements of $\hat{K}_0$ such that the following conditions hold:

(4a) $F = K_0(\mathbf{u}, x, z)$ is a Galois extension of $K_0(\mathbf{u}, x)$, the coefficients of $f(X,Z)$ lie in $K_0[\mathbf{u}]$, $f(x,Z) = \mathrm{irr}(z, K_0(\mathbf{u}, x))$, and $\mathcal{G}(f(x,Z), K_0(\mathbf{u}, x)) = \mathcal{G}(f(x,Z), \hat{K}_0(x))$;

(4b) $F = K(\mathbf{u}, x, y)$ and the coefficients of g lie in $K[\mathbf{u}]$; hence $g(x,Y) = \mathrm{irr}(y, K(\mathbf{u}, x))$;

As $\hat{K}_0/K_0$ is regular over K_0, so is $K_0(\mathbf{u})$. Thus, $\mathbf{u}$ generates an absolutely irreducible variety $U = \mathrm{Spec}(K_0[\mathbf{u}])$ over K_0. The variety U has a nonempty Zariski open subset U' such that for each $\mathbf{u}' \in U'$ the K_0-specialization $(\mathbf{u}, x) \to (\mathbf{u}', x)$ extends to an E-homomorphism $'$: $E[\mathbf{u}, x, z, \mathbf{y}] \to E[\mathbf{u}', x, z', \mathbf{y}']$ such that the following conditions hold:

(5a) $f'(x, z') = 0$, the discriminant of $f'(x,Z)$ is not zero, and $F' = K_0(\mathbf{u}', x, z')$ is the splitting field of $f'(x,Z)$ over $K_0(\mathbf{u}', x)$; in particular $F'/K_0(\mathbf{u}', x)$ is Galois;

(5b) $g'(X,Y)$ is absolutely irreducible and $g'(x,y') = 0$; so $g'(x,Y) = \mathrm{irr}(y', K(\mathbf{u}', x))$; furthermore, $g'(X,Y) = Y^d + a_1'(X)Y^{d-1} + \cdots + a_d'(X)$ with $a_i' \in K[X]$ and $\deg a_i'(X) \leq \deg a_1'(X) \geq 1$, for $i = 1, \ldots, d$.

To achieve the absolute irreducibility of g' we have used the Bertini-Noether theorem [FrJ, Prop. 8.8]. Since K_0 is existentially closed in $\hat{K}_0$ and since $\mathbf{u} \in U'(\hat{K}_0)$, we can choose $\mathbf{u}' \in U'(K_0)$. By (5a), the homomorphism $'$ induces an embedding

$$\varphi^*\colon \mathcal{G}(f'(x,Z), K_0(x)) \to \mathcal{G}(f(x,Z), K_0(\mathbf{u},x))$$

which commutes with the restriction to $\mathcal{G}(K(x)/K_0(x))$ [La, p. 248]. Observe that $K(x)$ is linearly disjoint from $K_0(\mathbf{u})$ over K_0.

$$\begin{array}{ccccc}
F' & & F & \text{---} & \hat{F} \\
| & & | & & | \\
E & \text{---} & E(\mathbf{u}) & \text{---} & \hat{E} \\
| & & | & & | \\
K(x) & \text{---} & K(\mathbf{u},x) & \text{---} & \hat{K}(x) \\
| & & | & & | \\
K_0(x) & \text{---} & K_0(\mathbf{u},x) & \text{---} & \hat{K}_0(x) \\
| & & | & & | \\
K_0 & \text{---} & K_0(\mathbf{u}) & \text{---} & \hat{K}_0
\end{array}$$

Hence, by (5b),

$$\begin{aligned}
|\mathcal{G}(f'(x,Z), K_0(x))| &= [F' : K_0(x)] = \deg(g'(x,Z))[K(x) : K_0(x)] \\
&= \deg(g(x,Z))[K(\mathbf{u},x) : K_0(\mathbf{u},x)] \\
&= [F : K_0(\mathbf{u},x)] = |\mathcal{G}(f(x,Z), K_0(\mathbf{u},x))|.
\end{aligned}$$

It follows that φ^* is an isomorphism. Hence $(\varphi^*)^{-1} \circ \alpha$ solves embedding problem (1).

Extend φ to a place φ' of F'. Then φ' extends the specialization $x \to \infty$. By Lemma 2.2 below and (5b), φ' totally decomposes in $F'/K(x)$, that is, φ' is unramified and K-rational. ∎

LEMMA 2.2 ([GeJ, Lemma 9.2 and Lemma 9.3]): *Let K be an arbitrary field and consider a Galois extension F of $K(x)$ of degree d which is regular over K. Then the K-place $x \to \infty$ of $K(x)$ totally decomposes in F if and only if there exists $y \in F$ such that* $\mathrm{irr}(y, K(x)) = Y^d + a_1(x)Y^{d-1} + \cdots + a_d(x)$ *with $a_i \in K[x]$ such that* $\deg a_i(X) \leq \deg a_1(X) \geq 1$, *for $i = 1, \ldots, d$.*

3. Split embedding problems and patching data

In this section we fix a finite Galois extension E/E_0 with Galois group Γ. Assume that Γ properly acts on a patching data

$$\mathcal{E} = (E, F_i, Q_i, Q; G_i, G)_{i\in I}. \tag{1}$$

We explain these notions [HJ, Definition 1.1 and Definition 1.4]:

Definition 3.1: Patching data with a proper action. Let I be a finite set with $|I| \geq 2$. A **patching data** (1) consists of fields $E \subseteq F_i$, $Q_i \subseteq Q$ and finite groups $G_i \leq G$, $i \in I$, such that

(2a) F_i/E is a Galois extension with group G_i, $i \in I$;

(2b) $F_i \subseteq Q_i'$, where $Q_i' = \bigcap_{j\neq i} Q_j$, $i \in I$;

(2c) $\bigcap_{i\in I} Q_i = E$;

(2d) $G = \langle G_i |\, i \in I\rangle$;

(2e) Let $n = |G|$. For all $B \in \mathrm{GL}_n(Q)$ and $i \in I$ there exist $B_i \in \mathrm{GL}_n(Q_i)$ and $B_i' \in \mathrm{GL}_n(Q_i')$ such that $B = B_iB_i'$.

A **proper action** of Γ on $\mathcal{E}$ is a triple that consists of an action of Γ on the group G, an action of Γ on the field Q, and an action of Γ on the set I such that the following conditions hold:

(3a) The action of Γ on Q extends the action of Γ on E;

(3b) $F_i^\gamma = F_{i^\gamma}$, $Q_i^\gamma = Q_{i^\gamma}$, and $G_i^\gamma = G_{i^\gamma}$, for all $i \in I$ and $\gamma \in \Gamma$;

(3c) $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $a \in F_i$, $\tau \in G_i$, $i \in I$, and $\gamma \in \Gamma$.

The action of Γ on G defines a semidirect product $G\rtimes\Gamma$ such that $\tau^\gamma = \gamma^{-1}\tau\gamma$ for all $\tau \in G$ and $\gamma \in \Gamma$. ∎

For each $i \in I$ let $P_i = F_iQ_i$ be the compositum of F_i and Q_i in Q.

Remark 3.2: Identifications. (a) Identify Γ with a subgroup of $\mathrm{Aut}(Q/E_0)$ by (3a). Furthermore, if L/E_0 is a Galois extension such that $E \subseteq L \subseteq Q$, then the restriction $\mathrm{res}_{Q/L}\colon \mathrm{Aut}(Q/E_0) \to \mathcal{G}(L/E_0)$ maps Γ onto a subgroup $\bar\Gamma$ of $\mathcal{G}(L/E_0)$. Moreover, $\mathrm{res}_{L/E}\colon \mathcal{G}(L/E_0) \to \mathcal{G}(E/E_0)$ maps $\bar\Gamma$ onto Γ. Hence $\bar\Gamma$ is isomorphic to Γ. Again, identify $\bar\Gamma$ with Γ. Thus both restrictions $\mathrm{res}_{L/E}\colon \mathcal{G}(L/E_0) \to \mathcal{G}(E/E_0)$ and $\mathrm{res}_{Q/L}\colon \Gamma \to \mathcal{G}(L/E)$ map Γ identically onto itself. In particular, $\mathcal{G}(L/E_0) = \mathcal{G}(L/E)\rtimes\Gamma$.

(b) Conditions (2b) and (2c) imply that $F_i \cap Q_i = E$. Hence P_i/Q_i is a Galois extension with Galois group isomorphic (via the restriction of automorphisms) to $G_i = \mathcal{G}(F_i/E)$. Identify $\mathcal{G}(P_i/Q_i)$ with G_i via this isomorphism. If L/E is a Galois extension such that $LQ_i = P_i$, then the restriction of G_i to L is isomorphic to G_i; again, identify this group with G_i. ∎

Consider the Q-algebra

$$N = \mathrm{Ind}_1^G Q = \Big\{ \sum_{\theta \in G} a_\theta \theta \,|\, a_\theta \in Q \Big\},$$

where addition and multiplication are defined componentwise. Thus Q embeds diagonally in N. For each $i \in I$, consider the Q-subalgebra

$$N_i = \mathrm{Ind}_{G_i}^G P_i = \Big\{ \sum_{\theta \in G} a_\theta \theta \in N \,|\, a_\theta \in P_i,\ a_\theta^\tau = a_{\theta\tau} \text{ for all } \theta \in G,\ \tau \in G_i \Big\}.$$

Let $F = \bigcap_{i \in I} N_i$.

We know [HJ, Proposition 1.5] that F/E_0 is a Galois extension of fields and there is an isomorphism $\psi\colon G \rtimes \Gamma \to \mathcal{G}(F/E_0)$. In fact, the proof of [HJ, Proposition 1.5] explicitly describes this isomorphism, or, equivalently, the action of $G \rtimes \Gamma$ on F. Indeed, G acts on N by

$$\Big(\sum_{\theta \in G} a_\theta \theta\Big)^\sigma = \sum_{\theta \in G} a_\theta \sigma^{-1}\theta = \sum_{\theta \in G} a_{\sigma\theta}\theta, \qquad \sigma \in G, \tag{4}$$

and Γ acts on N by

$$\Big(\sum_{\theta \in G} a_\theta \theta\Big)^\gamma = \sum_{\theta \in G} a_\theta^\gamma \theta^\gamma \qquad a_\theta \in Q,\ \gamma \in \Gamma; \tag{5}$$

these two actions combine to an action of $G \rtimes \Gamma$ on N. The restriction of this action to F is the required action.

The homomorphism $\pi\colon N \to Q$ given by $\big(\sum_{\theta \in G} a_\theta \theta\big)^\pi = a_1$ fixes E and hence also E_0. Therefore $F' = F^\pi$ (the **compound** of $\mathcal{E}$) is a Galois extension of E_0 with $\mathcal{G}(F'/E_0) \cong \mathcal{G}(F/E_0) \cong G \rtimes \Gamma$ and and π defines an action of $G \rtimes \Gamma$ on F' by

$$(a^\pi)^g = (a^g)^\pi, \qquad a \in F,\ g \in G \rtimes \Gamma.$$

Let us describe this action, using (4) and (5). Let $a = \sum_{\theta\in G} a_\theta\theta \in F$. For each $\gamma \in \Gamma$ we have

$$a^{\gamma\pi} = \Big(\sum_{\theta\in G} a_\theta^\gamma \theta^\gamma\Big)^\pi = a_1^\gamma = a^{\pi\gamma}.$$

Furthermore, for each $i \in I$ and each $\sigma \in G_i = \mathcal{G}(P_i/Q_i)$ we have $F' \subseteq P_i$ and

$$a^{\sigma\pi} = \Big(\sum_{\theta\in G} a_{\sigma\theta}\theta\Big)^\pi = a_\sigma = a_1^\sigma = a^{\pi\sigma}.$$

This gives the following result (in which F stands for F'; the original F will not be used henceforth):

PROPOSITION 3.3: *Let F be the compound of $\mathcal{E}$. Then F/E_0 is Galois and there is an isomorphism ψ: $G\rtimes\Gamma \to \mathcal{G}(F/E_0)$ that maps Γ and the G_i identically onto themselves (under the identification of Remark 3.2).*

COROLLARY 3.4: *Assume that $1 \in I$ and the following condition holds:*

(6a) $1^\gamma = 1$ *for all* $\gamma \in \Gamma$; *and*

(6b) $G = H\rtimes G_1$, *where* $H = \langle G_i \mid i \in I,\ i \neq 1\rangle \leq G$; *let* ρ: $G \to G_1$ *be the canonical projection.*

Then

(a) F_1, Q_1, *and* G_1 *are* Γ*-invariant; put* $\Delta = G_1\rtimes\Gamma$;

(b) F_1/E_0 *is a Galois extension;*

(c) $\mathcal{G}(F_1/E_0) = \Delta$, *that is, the action of* Γ *on* $G_1 = \mathcal{G}(F_1/E)$ *by conjugation in* $\mathcal{G}(F_1/E_0)$ *coincides with the action induced from the given action of* Γ *on* G;

(d) $F_1 \subseteq F$ *and* $\mathrm{res}_{F/F_1} : \mathcal{G}(F/E) \to \mathcal{G}(F_1/E)$ *is* ρ: $G \to G_1$;

(e) *The following diagram is commutative*

$$\begin{array}{ccc} G\rtimes\Gamma & \xrightarrow{\ \psi\ } & \mathcal{G}(F/E_0) \\ {\scriptstyle \rho\rtimes id_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \Delta = G_1\rtimes\Gamma & =\!=\!= & \mathcal{G}(F_1/E_0). \end{array}$$

Proof: (a) This follows from (3b) by (6a).

(b) By assumption, F_1/E is Galois. By (a), F_1 is Γ-invariant, that is, every element of $\mathcal{G}(E/E_0)$ extends to an automorphism of F_1. Hence each E_0-isomorphism of F_1 into $\widetilde{E_0}$ maps F_1 onto itself.

(c) Remark 3.2 asserts that $\mathcal{G}(F_1/E_0)$ is a semidirect product of G_1 with Γ. By (3c), $a^{\gamma^{-1}\tau\gamma} = a^{\tau^\gamma}$ for all $a \in F_1$ and $\gamma \in \Gamma$.

(d) We have $F_1 \subseteq \bigcap_{i\in I} P_i$, because $P_1 = F_1 Q_1$ and, by (2b), $F_1 \subseteq Q_i \subseteq P_i$, for each $1 \neq i \in I$. Hence, if $\sigma \in G_i$ and $1 \neq i \in I$, then, since $F_1 \subseteq Q_i$, we have $\rho(\sigma) = 1 = \mathrm{res}_{P_i/F_1}(\sigma)$. If $\sigma \in G_1 = \mathcal{G}(P_1/Q_1)$ then $\rho(\sigma) = \sigma = \mathrm{res}_{P_1/F_1}(\sigma)$, by our identifications. Hence by [HV, Lemma 3.6(c)] we have $F_1 \subseteq F$ and $\mathrm{res}_{F/F_1} = \rho$.

(e) It suffices to verify the commutativity on the elements of Γ and the G_i's, since they generate $G \rtimes \Gamma$. Therefore the result follows from (d) and Proposition 3.3. ∎

4. Split embedding problems over functions fields of one variable over ample fields

In this section we present the main result. We first consider the special case of complete field and then deduce the general case from it.

PROPOSITION 4.1: *Let K/K_0 be a finite unramified Galois extension of complete fields under a nontrivial ultra-metric absolute value such that the residue field $\bar{K}_0$ is infinite. Let*

$$(1) \qquad H \rtimes \mathcal{G}(F_1/K_0(x)) \xrightarrow{\mathrm{pr}} \mathcal{G}(F_1/K_0(x))$$

be a split K_0-embedding problem. Suppose that $K \subseteq F_1$. Let φ be a K-rational K-place of F_1, unramified over $K(x)$, such that $\varphi(x) \in K_0 \cup \{\infty\}$. Then (1) has a solution field F such that φ extends to a K-rational place of F unramified over $K(x)$.

Proof: Put $E_0 = K_0(x)$, $E = K(x)$, and let $\Gamma = \mathcal{G}(K/K_0) = \mathcal{G}(E/E_0)$. We may assume that $H \neq 1$.

We break up the proof into several parts. The idea of the proof is to extend (E, F_1) to a patching data $\mathcal{E} = (E, F_i, Q_i, Q; G_i, G)_{i\in I}$ with $1 \in I$ on which Γ properly acts; its compound F will be the required solution field.

PART A: *Completion of $(E, |\ |)$.* Extend $|\ |$ to an absolute value on E by the formula $\sum_{i=1}^{n} a_i x^i = \max\{|a_0|, \ldots, |a_n|\}$. Then, the residue $\bar{x}$ of x is transcendental over $\bar{K}$ and the residue fields satisfy $\bar{E}_0 = \bar{K}_0(x)$ and $\bar{E} = \bar{K}(\bar{x})$ [HJ, Remark 3.2(b)]. Since K/K_0 is unramified, $[K : K_0] = [\bar{K} : \bar{K}_0] = [\bar{E} : \bar{E}_0]$. Let $(\hat{E}_0, |\ |)$ be the completion of $(E_0, |\ |)$. Then $\hat{E} = \hat{E}_0 K$ is the completion of E with respect to $|\ |$. Moreover, $[K : K_0] = [\bar{E} : \bar{E}_0] \leq [\hat{E} : \hat{E}_0] \leq [K : K_0]$. Hence, $[\hat{E} : \hat{E}_0] = [K : K_0]$ and therefore $\hat{E}_0 \cap K = K_0$. So, we may identify $\mathcal{G}(\hat{E}/\hat{E}_0)$ with Γ via restrictions to E and K. Since the extension of $|\ |$ from $\hat{E}_0$ to $\hat{E}$ is unique, each $\gamma \in \Gamma$ preserves the absolute value on $\hat{E}$. In particular, each $\gamma \in \Gamma$ is a continuous automorphism of $\hat{E}$.

PART B: *Construction of the Q_i's.* Write H as

$$(2) \qquad H = \{\tau_j \mid j \in J\}$$

with the index set J of the same cardinality as that of H. Put $I_2 = J \times \Gamma$ and let Γ act on I_2 by $(j,\gamma')^\gamma = (j,\gamma'\gamma)$. Identify $(j,1) \in I_2$ with j, for each $j \in J$. Then

(3) every $i \in I_2$ can be uniquely written as $i = j^\gamma$ with $j \in J$ and $\gamma \in \Gamma$.

Let $I = \{1\} \mathbin{\dot\cup} I_2$ and extend the action of Γ on I_2 to an action on I by $1^\gamma = 1$ for each $\gamma \in \Gamma$.

By Claim A of the proof of [HJ, Proposition 5.2], K has a subset $\{c_i |\ i \in I_2\}$ such that

(4) $c_i^\gamma = c_{i^\gamma}$ and $|c_i| = |c_i - c_j| = 1$ for $i \neq j$ and $\gamma \in \Gamma$.

As $\bar{K}_0$ is infinite, we may choose $c_1 \in K_0$ such that $\bar{c}_1 \neq 0, \infty$ and $\bar{c}_1 \notin \{\bar{c}_i |\ i \in I_2\}$. It follows that (4) holds for all $i, j \in I$.

For each $i \in I$ let $w_i = \frac{1}{x - c_i}$. Let $R = K\{w_i |\ i \in I\}$ be the closure of $K[w_i |\ i \in I]$ in $\hat{E}$ and let $Q = \mathrm{Quot}(R)$. For each $i \in I$ let

$$Q_i = Q_{I \smallsetminus \{i\}} = \mathrm{Quot}(K\{w_j |\ j \neq i\}) \quad \text{and} \quad Q_i' = Q_{\{i\}} = \mathrm{Quot}(K\{w_i\}).$$

By [HJ, Proposition 3.10], $Q_i' = \bigcap_{j \neq i} Q_j$ and $E = K(x) = \bigcap_{i \in I} Q_i$. By (4), each $\gamma \in \Gamma$ satisfies $w_i^\gamma = w_{i^\gamma}$ and therefore maps $K[w_i |\ i \in I]$ onto itself. Since the action of γ on $\hat{E}$ is continuous, γ leaves R, and hence also Q, invariant. We identify Γ with its image in $\mathrm{Aut}(Q)$. In addition, $Q_i^\gamma = Q_{i^\gamma}$ and $(Q_i')^\gamma = Q_{i^\gamma}'$ for each $i \in I$.

PART C: *Without loss of generality $F_1 \subseteq Q_1'$ and $\varphi(w_1) = 0$.* To show this it suffices to construct a K-embedding θ: $F_1 \to Q_1'$ such that $\theta(E_0) = E_0$, $\theta(E) = E$, and $\varphi \circ \theta^{-1}(w_1) = 0$. Indeed, the isomorphism θ: $F_1 \to \theta(F_1)$ ensures that the assumptions and the conclusions of our proposition hold for (F_1, φ) if and only if they hold for $(\theta(F_1), \varphi \circ \theta^{-1})$.

We construct θ as above in two steps.

As φ maps w_1 into $K_0 \cup \{\infty\}$, there is a K_0-automorphism ω of $E_0 = K_0(w_1)$ such that $\varphi \circ \omega^{-1}(w_1) = 0$. Extend ω to a K-automorphism of E and then to an isomorphism of fields $F_1 \to F_1'$. Apply it to assume that $\varphi(w_1) = 0$.

Let F_1^* be the completion of F_1 at φ, and let $E^* \subseteq F_1^*$ be the corresponding completion of E. Then $[F_1^* : E^*] = e(F_1/E)\, f(F_1/E) = 1$. But $E^* = K((w_1))$. Hence $F_1 \subseteq F_1^* = K((w_1))$.

Let $z \in K((w_1))$ be a primitive element for F_1/E. For $c \neq 0$ in K_0 let μ_c be the automorphism of $K((w_1))$ mapping $f(w_1) = \sum_{i=m}^{\infty} a_i w_1^i$ to $f(cw_1) = \sum_{i=m}^{\infty} (a_i c^i) w_1^i$. Note that μ_c leaves $E = K(w_1)$ and $E_0 = K_0(w_1)$ invariant, and $\varphi \circ \mu_c^{-1}(w_1) = \varphi(c^{-1}w_1) = 0$. By [Ar, Theorem 2.14]* there is $c \in K^\times$ such that z as a Laurent series in w_1 converges at c. Thus $\mu_c(z) \in Q_1'$ and hence $\mu_c(F_1) \subseteq Q_1'$.

PART D: *Groups.* As $F_1 \subseteq Q$ is a Galois extension of E_0, it is Γ-invariant. Let $G_1 = \mathcal{G}(F_1/E)$. Identify $\Gamma \leq \mathrm{Aut}(Q/E_0)$ with its image in $\mathcal{G}(F_1/E_0)$. Then $\mathcal{G}(F_1/E_0) = G_1 \rtimes \Gamma$, where Γ acts on G_1 by conjugation in $\mathcal{G}(F_1/E_0)$. Thus

(5) $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $\gamma \in \Gamma$, $a \in F_1$ and $\tau \in G_1$.

The given action of $\mathcal{G}(F_1/E_0)$ on H induces an action of its subgroups G_1 and Γ on H. Let $G = H \rtimes G_1$ with respect to this action. Then

$$H \rtimes \mathcal{G}(F_1/E_0) = H \rtimes (G_1 \rtimes \Gamma) = (H \rtimes G_1) \rtimes \Gamma = G \rtimes \Gamma.$$

Let $i \in I_2$. Use (3) to write $i = j^{\gamma'}$ with unique $j \in J$ and $\gamma' \in \Gamma$. Then define $\tau_i = \tau_j^{\gamma'}$ and observe that

(6a) $\tau_i^\gamma = \tau_{i^\gamma}$ for all $i \in I_2$ and $\gamma \in \Gamma$.

By (2),

(6b) $H = \langle \tau_i |\, i \in I_2 \rangle$.

For each $i \in I_2$ let $G_i = \langle \tau_i \rangle \leq H$. Thus

(6c) $G = \langle G_i |\, i \in I \rangle$ and $H = \langle G_i |\, i \in I_2 \rangle$;

(6d) $G_i^\gamma = G_{i^\gamma}$ for all $i \in I$ and $\gamma \in \Gamma$;

(6e) $|I| \geq 2$.

PART E: *Patching data.* For each $j \in J$, [HJ, Proposition 5.1] gives a cyclic extension F_j/E with Galois group $G_j = \langle \tau_j \rangle$ such that $F_j \subseteq Q_j'$. For an arbitrary $i \in I_2$ there exist unique $j \in J$ and $\gamma \in \Gamma$ such that $i = j^\gamma$ (by (3)). Let $F_i = F_j^\gamma$. As γ acts on Q and leaves E invariant, F_i is a Galois extension of E and $F_i \subseteq Q_i'$.

The isomorphism γ: $F_j \to F_i$ gives an isomorphism $\mathcal{G}(F_j/E) \cong \mathcal{G}(F_i/E)$ which maps each $\tau \in \mathcal{G}(F_j/E)$ onto $\gamma^{-1} \circ \tau \circ \gamma \in \mathcal{G}(F_i/E)$. We can therefore identify G_i with

* Although Artin uses analysis to prove that an algebraic power series converges, one can give an algebraic proof of this result, in the style of the proof of Hensel's Lemma.

$\mathcal{G}(F_i/E)$ such that τ_i coincides with $\gamma^{-1}\circ\tau_j\circ\gamma$. This means that $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $a\in F_j$ and $\tau\in G_j$.

It follows that for all $i\in I$ and $\gamma\in\Gamma$ we have $F_i^\gamma = F_{i^\gamma}$. Moreover, $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $a\in F_i$ and $\tau\in G_i$; this extends (5).

By [HJ, Corollary 4.5], $\mathrm{GL}_n(Q) = \mathrm{GL}_n(Q_i)\,\mathrm{GL}_n(Q_i')$ for each $n\in\mathbb{N}$ and each $i\in I$. Thus $\mathcal{E} = (E, F_i, Q_i; G_i, G)_{i\in I}$ is a patching data on which Γ properly acts (Definition 3.1). By Corollary 3.4(e) the compound F of $\mathcal{E}$ is a solution of (1).

PART F: *Extension of φ.* Let $b\in K_0$ such that $|b|>1$ and put $z=\frac{b}{x}$. Let $K\{z\}$ be the ring of convergent power series in z over K with respect to the absolute value $|\ |_z$ given by $|\sum_{n=0}^\infty a_n z^n|_z = \max(|a_n|)$. Let $R_0 = K[w_i|\ i\in I]$. Observe that

$$w_i = \frac{1}{x-c_i} = \frac{z}{b-c_iz} = \frac{z}{b}\cdot\frac{1}{1-\frac{c_i}{b}z} = \frac{z}{b}\sum_{n=1}^{\infty}(\frac{c_i}{b})^n z^n \in K\{z\}, \qquad \text{for each } i\in I.$$

Thus $R_0\subseteq K\{z\}$. Moreover, $|w_i|_z = \frac{1}{|b|} < 1 = |w_i|$. By [HJ, Lemma 3.3] every $f\in R_0$ is of the form $f = a_0 + \sum_{i\in I}\sum_{n=1}^\infty a_{in}w_i^n$, where $a_{in}\in K$ and almost all of them are 0. Hence $|f|_z\le|f|$ and therefore $|f|_z\le|f|$. Therefore the inclusion $R_0\subseteq K\{z\}$ is a continuous R_0-homomorphism. As R is the completion of R_0 with respect to $|\ |_z$ [HJ, Lemma 3.3], this inclusion induces a continuous R_0-homomorphism $\lambda\colon R\to K\{z\}$. By [HJ, Proposition 3.9] there is $p\in R_0$ such that $\operatorname{Ker}\lambda = (p)$. It follows that $p=0$ and hence λ is injective.

Identify R with its image under λ to assume that $R\subseteq K\{z\}\subseteq K[[z]]$. The specialization $z\to 0$ extends to a K-rational place of $K((z))$ unramified over $E=K(z)$. Its restriction to F is a K-rational place ψ of F unramified over $E=K(z)$.

As $\psi(w_1) = 0 = \varphi(w_1)$, we have $\mathrm{res}_{E_0}\psi = \mathrm{res}_{E_0}\varphi$. Replace ψ by $\psi\circ\sigma$ for a suitable $\sigma\in\mathcal{G}(F/E_0)$, if necessary, to assume that $\mathrm{res}_{F_1}\psi=\varphi$. ∎

PROPOSITION 4.2: *Let K_0 be an ample field. Consider a (regular) split K_0-embedding problem*

$$\text{(7)}\qquad H\rtimes\mathcal{G}(E/K_0(x)) \xrightarrow{\mathrm{pr}} \mathcal{G}(E/K_0(x)).$$

Let K be the algebraic closure of K_0 in E. Then

(a) *(7) has a (regular) solution F.*

(b) *Suppose that E has a K-rational K-place φ unramified over $K(x)$ such that $\varphi(x) \in K_0 \cup \{\infty\}$. Then F has a K-rational K-place φ unramified over $K_0(x)$.*

Proof: We first prove (b) and then deduce (a) from (b).

Proof of (b): Let t be transcendental over E. Let $\hat{K}_0 = K_0((t))$, $\hat{K} = K((t))$ and $\hat{E} = E\hat{K}$. Then $\hat{K}/\hat{K}_0$ is a finite Galois extension of complete fields under the t-adic absolute value and the corresponding extension of residue fields is K/K_0 (these are infinite fields). In particular, $\hat{K}/\hat{K}_0$ is an unramified extension. Since the extension $\hat{K}/K$ is regular and free from E/K, the fields $\hat{K}$ and E are linearly disjoint over K. Hence φ extends to a $\hat{K}$-rational place $\hat{\varphi}$ of $\hat{E}$, and therefore $\hat{K}$ is the algebraic closure of $\hat{K}_0$ in $\hat{E}$. Furthermore, $\hat{\varphi}$ is unramified over $\hat{K}(x)$. Finally, $\mathcal{G}(\hat{E}/\hat{K}_0(x))$ is isomorphic to $\mathcal{G}(E/K_0(x))$ and acts on H via the restriction map. Thus

$$H \rtimes \mathcal{G}(\hat{E}/\hat{K}_0(x)) \xrightarrow{\mathrm{pr}} \mathcal{G}(\hat{E}/\hat{K}_0(x)) \tag{8}$$

is a split $\hat{K}_0$-embedding problem.

By Proposition 4.1, (8) has a solution field $\hat{F}$ such that $\hat{\varphi}$ extends to a $\hat{K}$-rational place of $\hat{F}$ unramified over $\hat{E}$. Since K_0 is ample, it is existentially closed in $\hat{K}_0$. Lemma 2.1 therefore asserts the existence of a solution field F of (7) and of a K-rational K-place of F unramified over $K(x)$.

Proof of (a): Only finitely many K_0-places of E are ramified over $K_0(x)$. Thus, there is a K_0-place φ of E unramified over $K_0(x)$ such that $\varphi(x) \in K_0$. Composing φ with an automorphism of E over $K_0(x)$, we may assume that the restriction of φ to $K(x)$ is a K-place. However, φ need not be K-rational. Nevertheless, the residue field K' of φ is a finite Galois extension of K_0 that contains K. Let $E' = EK'$. Then φ extends to a K'-rational place φ' of E', unramified over $K'(x)$. Furthermore, $E'/K_0(x)$ is a Galois extension and its Galois group $\mathcal{G}(E'/K_0(x))$ acts on H via the restriction

$\mathcal{G}(E'/K_0(x)) \to \mathcal{G}(E/K_0(x))$.

$$\begin{tikzcd} & E \arrow[r, dash] \arrow[d, dash] & E' \arrow[d, dash] \\ K_0(x) \arrow[r, dash] & K(x) \arrow[r, dash] & K'(x) \end{tikzcd}$$

The existence of φ' implies that E'/K' is regular.

By (b), the split embedding problem

$$(7') \qquad H \rtimes \mathcal{G}(E'/K_0(x)) \xrightarrow{\mathrm{pr}} \mathcal{G}(E'/K_0(x))$$

has a regular solution. Conclude from Lemma 1.1 that (7) has a solution which is regular, if (7) is regular. ∎

Combine Proposition 4.2 with Proposition 1.4 to get:

THEOREM 4.3: *Let K_0 be an ample field. Then every (regular) split K_0-embedding problem*

$$H \rtimes \mathcal{G}(E/E_0) \xrightarrow{\mathrm{pr}} \mathcal{G}(E/E_0)$$

has a (regular) solution.

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