P-adically Projective Groups as Absolute Galois Groups

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1 Introduction

We address one of the major problems of Galois theory: the characterization of absolute Galois groups among all profinite groups. Specifically, we consider a profinite group G equipped with a subset $\mathcal{G}$ of subgroups each of which is isomorphic to an absolute Galois group. The problem is to characterize those pairs for which G is isomorphic to an absolute Galois group $\mathrm{Gal}(K)$ of a field K that satisfies a local-global principle for points on smooth varieties with respect to the fixed fields of the groups in $\mathcal{G}$.

In [12] we extend the pairs $(G, \mathcal{G})$ to, so-called, *group structures* of a general nature. We prove that a proper profinite group structure **G** is projective if and only if **G** is the absolute Galois group structure of a proper field-valuation structure with block approximation.

The introduction to [12] contains an extensive historical background on the subject.

In this work we apply the general setup of [12] to a classical situation. Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Thus, each $\mathbb{F} \in \mathcal{F}$ is either the field $\mathbb{R}$ of real numbers or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime number p. We assume $\mathcal{F}$ is *closed under Galois isomorphism*: if $\mathbb{F}$ and $\mathbb{F}'$ are finite extensions of $\mathbb{Q}_p$, $\mathbb{F} \in \mathcal{F}$, and $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(\mathbb{F})$, then $\mathbb{F}' \in \mathcal{F}$.

Theorem 1.1 (main theorem)**.** Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 which is closed under Galois isomorphism and let G be a profinite group. Then G is

isomorphic to the absolute Galois group of a P$\mathcal{F}$C field K if and only if G is $\mathcal{F}$-projective and Subgr(G, Gal($\mathbb{F}$)) is strictly closed in Subgr(G) for each $\mathbb{F} \in \mathcal{F}$.   □

The notions appearing in **Theorem 1.1.** For each $\mathbb{F} \in \mathcal{F}$ let AlgExt(K, $\mathbb{F}$) be the set of all algebraic extensions F of K (within a fixed algebraic closure $\tilde{K}$ of K) which are elementarily equivalent to $\mathbb{F}$. We say that K is P$\mathcal{F}$C (*pseudo-$\mathcal{F}$-closed*) if it satisfies the following local-global principle: let V be a smooth absolutely irreducible variety over **K**. Suppose $V(F) \neq \varnothing$ for each $F \in \bigcup_{\mathbb{F} \in \mathcal{F}} \text{AlgExt}(K, \mathbb{F})$. Then $V(K) \neq \varnothing$.

The notation Subgr(G) stands for the space of all closed subgroups of G. Subgr(G) is the inverse limit of the discrete finite spaces Subgr(G/N), where N ranges over all open normal subgroups of G. Thus, Subgr(G) is a profinite space. We refer to its topology as *strict* and write Subgr(G, Gal($\mathbb{F}$)) = $\{\Gamma \in \text{Subgr}(G) \mid \Gamma \cong \text{Gal}(\mathbb{F})\}$.

Finally, G is $\mathcal{F}$-*projective* if it satisfies the following local-global principle: let $\alpha : B \to A$ be an epimorphism of finite groups and $\varphi : G \to A$ a homomorphism. Suppose for each $\mathbb{F} \in \mathcal{F}$ and each $\Gamma \in \text{Subgr}(G, \text{Gal}(\mathbb{F}))$ there is a homomorphism $\gamma_\Gamma : \Gamma \to B$ satisfying $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Then there is a homomorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \varphi$.

Theorem 1.1 generalizes several known special cases.

(1) First suppose $\mathcal{F}$ is the empty set. Then a P$\mathcal{F}$C field is a PAC field [7, Chapter 11] and an $\mathcal{F}$-projective group is a projective group. **Theorem 1.1** is due in this case to Lubotzky-v.d.Dries [7, Corollary 20.16] and Ax [7, Theorem 11.6.2].

(2) If $\mathcal{F} = \{\mathbb{R}\}$, then a P$\mathcal{F}$C field is a PRC field and an $\mathcal{F}$-projective group is a real projective group G such that the set of all involutions of G is closed. See [10, page 450, Theorem] for **Theorem 1.1** in this case.

(3) If $\mathcal{F} = \{\mathbb{Q}_p\}$, then a P$\mathcal{F}$C field is a P$p$C field and an $\mathcal{F}$-projective group is a p-adically projective G such that the set of all subgroups of G which are isomorphic to Gal($\mathbb{Q}_p$) is strictly closed in Subgr(G). See [11, page 148, Theorem] for **Theorem 1.1** in this case.

(4) The general case is announced in [26]. Unfortunately, the proofs of [26] are extremely sketchy and difficult to read. The current work is based on ideas of [26] and applies results of [12].

(5) Koenigsmann [22] proves **Theorem 1.1** in the case that AlgExt(K, $\mathbb{F}$) is finite. Ershov [5, 6] proves that Gal(K) is $\mathcal{F}$-projective (in a stronger sense) when (K, $\mathcal{F}$) is a multivalued P$\mathcal{F}$C field satisfying certain conditions.

None of those papers goes as far as we do in this work and equips K in **Theorem 1.1** with a set of valuations satisfying the "block approximation theorem." We refer the reader to **Section 11**, in particular to **Theorem 11.3**, for the exact result. Here it suffices to say that each minimal $F \in \text{AlgExt}(K, \mathcal{F})$ is the Henselian closure of K at a valuation v_F and the family $\{v_F \mid F \in \text{AlgExt}(K, \mathcal{F})\}$ satisfies a very strong independence-density property.

In the rest of the introduction we describe the structure of the proof of **Theorem 1.1** and its stronger versions (Theorems **10.4** and **11.3**) which we actually prove.

Denote the set of all separable algebraic extensions of a field K (within a fixed separable closure K_s of K) by $\mathrm{AlgExt}(K)$. The map $F \mapsto \mathrm{Gal}(F)$ is a bijection of $\mathrm{AlgExt}(K)$ onto $\mathrm{Subgr}(\mathrm{Gal}(K))$. It transfers the strict topology of $\mathrm{Subgr}(\mathrm{Gal}(K))$ to the *strict topology* of $\mathrm{AlgExt}(K)$. We prove that $\mathrm{AlgExt}(K, \mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(K)$ for each $\mathbb{F} \in \mathcal{F}$. In particular, this holds if K is P$\mathcal{F}$C. In this case the existence of points on smooth varieties over K translates into $\mathcal{F}$-projectivity.

Conversely, let G be a profinite group. Suppose G is $\mathcal{F}$-projective and $\mathrm{Subgr}(G, \mathrm{Gal}(\mathbb{F}))$ is strictly closed in $\mathrm{Subgr}(G)$ for each $\mathbb{F} \in \mathcal{F}$. Put $\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$. Let $\mathcal{G} = \mathrm{Subgr}(G, \mathcal{C})$ be the set of all $H \in \mathrm{Subgr}(G)$ which are isomorphic to some Γ in $\mathcal{C}$. Denote the set of all maximal elements in $\mathcal{G}$ by $\mathcal{G}_{\max}$. For each $\Gamma \in \mathcal{C}$ we construct a finite quotient $\bar{\Gamma}$ such that the set $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a "system of big quotients" of $\mathcal{C}$ in a sense made precise in **Section 7**. We use it to prove that G is "strongly $\mathcal{G}$-projective" (**Proposition 7.5**). Thus, every $\mathcal{G}$-embedding problem for G which is locally solvable is globally solvable (**Section 7**). In particular, there is a homomorphism κ of G into the free product $B^* = \circledast_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ isomorphically into a conjugate of some $\Gamma \in \mathcal{C}$. By a theorem of Geyer we may identify B^* with the absolute Galois group of an algebraic extension K_0 of $\mathbb{Q}$. Denote the fixed field of $\kappa(G)$ in $\tilde{\mathbb{Q}}$ by K_1. By **Proposition 7.5**, $\mathbf{G} = (G, \mathcal{G}_{\max})$ is a "proper projective group structure" (**Section 6**). Let $\mathrm{Gal}(\mathbf{K_1}) = (\mathrm{Gal}(K_1), \mathrm{Subgr}(\mathrm{Gal}(K_1), \mathcal{C}))$. Then κ extends to a Galois cover $\kappa : \mathbf{G} \to \mathrm{Gal}(\mathbf{K_1})$ of group structures (proof of **Theorem 11.3**). By the main result of [**12**], there are a field K and an isomorphism $\varphi : G \to \mathrm{Gal}(K)$ such that $\mathrm{res} \circ \varphi = \kappa$. Moreover, every $F \in \mathrm{AlgExt}(K, \mathcal{F})$ is either real closed or elementarily equivalent to some $\mathbb{F} \in \mathcal{F}$. In particular, F has a "P-adic valuation" v_F. The system of fields $F \in \mathrm{AlgExt}(K, \mathcal{F})$ and valuations v_F satisfies a strong version of the weak approximation theorem which we call the "block approximation theorem." In particular, K is P$\mathcal{F}$C (**Theorem 11.3**).

2   The étale and the strict topologies of $\mathrm{Subgr}(G)$

Let G be a profinite group. Denote the collection of all closed (resp., open, open normal) subgroups of G by $\mathrm{Subgr}(G)$ (resp., $\mathrm{Open}(G)$, $\mathrm{OpenNormal}(G)$). We introduce two topologies on $\mathrm{Subgr}(G)$ and relate them to each other.

For each $H, N \in \mathrm{Open}(G)$ with $N \lhd G$ let

$$\mathcal{V}(H, N) = \{A \in \mathrm{Subgr}(G) \mid AN = HN\}. \tag{2.1}$$

The collection of all $\mathcal{V}(H, N)$ is a basis for a topology on $\mathrm{Subgr}(G)$ which we call the

strict topology. When G is finite, the strict topology is the discrete topology. In general, $\text{Subgr}(G) \cong \varprojlim \text{Subgr}(G/N)$ with N ranging over all open normal subgroups of G. Thus, $\text{Subgr}(G)$ is a profinite space under the strict topology. We use the adverb "strictly" as a replacement for "in the strict topology." For example, for a subset $\mathcal{G}$ of $\text{Subgr}(G)$ we say $\mathcal{G}$ is *strictly open* (resp., *closed*, *compact*, *Hausdorff*) if it is open (resp., closed, compact, Hausdorff) in the strict topology. Likewise, for a function f from a topological space X into $\text{Subgr}(G)$, we say f is *strictly continuous* if f is continuous when $\text{Subgr}(G)$ is equipped with the strict topology. We denote the strict closure of a subset $\mathcal{G}$ of $\text{Subgr}(G)$ by $\text{StrictClosure}(\mathcal{G})$.

If $U_1, \ldots, U_m \in \text{Open}(G)$, then $U = \bigcap_{i=1}^{m} U_i$ is open and $\text{Subgr}(U)$ is $\bigcap_{i=1}^{m} \text{Subgr}(U_i)$. Therefore $\{\text{Subgr}(U) \mid U \in \text{Open}(G)\}$ is a basis for a topology on $\text{Subgr}(G)$ which we call the *étale topology*. As above, for a subset $\mathcal{G}$ of $\text{Subgr}(G)$ we say $\mathcal{G}$ is *étale open* (*closed*, *compact*, *Hausdorff*, etc.) if $\mathcal{G}$ is open (closed, compact, Hausdorff, etc.) in the étale topology. Likewise, for a function f from a topological space X into $\text{Subgr}(G)$ we say f is *étale continuous* if f is continuous when $\text{Subgr}(G)$ is equipped with the étale topology. Note that the étale topology of $\text{Subgr}(G)$ is weaker than the strict topology. Thus, every étale open subset of $\text{Subgr}(G)$ is also strictly open [**12**, Remark 1.2].

The *envelope* of a subset $\mathcal{G}$ of $\text{Subgr}(G)$ is the set of all $H_0 \in \text{Subgr}(G)$ which are contained in some $H \in \mathcal{G}$. We denote it by $\text{Env}(\mathcal{G})$ and use it to relate the strict topology and the étale topology of $\text{Subgr}(G)$ to each other.

Lemma 2.1. A subset $\mathcal{G}$ of $\text{Subgr}(G)$ is étale compact if and only if $\text{Env}(\mathcal{G})$ is strictly closed. $\square$

Proof. Suppose first $\text{Env}(\mathcal{G})$ is strictly closed, hence strictly compact. Let U_i, $i \in I$, be open subgroups of G with $\mathcal{G} \subseteq \bigcup_{i \in I} \text{Subgr}(U_i)$. Then $\text{Env}(\mathcal{G}) \subseteq \bigcup_{i \in I} \text{Subgr}(U_i)$. Since each of the sets $\text{Subgr}(U_i)$ is strictly open, I has a finite subset I_0 with $\text{Env}(\mathcal{G}) \subseteq \bigcup_{i \in I_0} \text{Subgr}(U_i)$. Thus, $\mathcal{G} \subseteq \bigcup_{i \in I_0} \text{Subgr}(U_i)$. Therefore, $\mathcal{G}$ is étale compact.

Conversely, suppose $\mathcal{G}$ is étale compact. Consider $A \in \text{Subgr}(G) \smallsetminus \text{Env}(\mathcal{G})$ and $H \in \mathcal{G}$. Then $A \not\leq H$. Hence, there is $N_H \in \text{OpenNormal}(G)$ with $A \not\leq HN_H$. Thus, A is not in the étale open neighborhood $\text{Subgr}(HN_H)$ of H.

The collection of all $\text{Subgr}(HN_H)$ covers $\mathcal{G}$. Since $\mathcal{G}$ is étale compact, there are $H_1, \ldots, H_n \in \mathcal{G}$ and $N_1, \ldots, N_n \in \text{OpenNormal}(G)$ with $\mathcal{G} \subseteq \bigcup_{i=1}^{n} \text{Subgr}(H_i N_i)$ and $A \notin \bigcup_{i=1}^{n} \text{Subgr}(H_i N_i)$. In addition, $\bigcup_{i=1}^{n} \text{Subgr}(H_i N_i)$ is strictly closed. Hence,

$$\text{StrictClosure}(\mathcal{G}) \subseteq \bigcup_{i=1}^{n} \text{Subgr}\,(H_i N_i). \tag{2.2}$$

Thus, A belongs to the strictly open set $\mathrm{Subgr}(G) \smallsetminus \bigcup_{i=1}^{n} \mathrm{Subgr}(H_i N_i)$ which is disjoint from $\mathrm{Env}(\mathcal{G})$. Therefore, A is not in $\mathrm{StrictClosure}(\mathrm{Env}(\mathcal{G}))$. It follows that $\mathrm{Env}(\mathcal{G})$ is strictly closed. ∎

Corollary 2.2. Let $\mathcal{G}$ be an étale compact subset of $\mathrm{Subgr}(G)$. Then $\mathrm{StrictClosure}(\mathcal{G})$ is contained in $\mathrm{Env}(G)$. □

For a profinite group G, a closed subgroup H, and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ let $\mathcal{G}^H = \{\Gamma^h \mid \Gamma \in \mathcal{G}, \ h \in H\}$. Put $\mathrm{Con}(\mathcal{G}) = \mathrm{Env}(\mathcal{G}^G) = \mathrm{Env}(\mathcal{G})^G$.

Lemma 2.3. Let $\mathcal{G}$ be an étale compact subset of $\mathrm{Subgr}(G)$. Then each of the sets $\mathcal{G}^G$, $\mathrm{Env}(\mathcal{G})$, and $\mathrm{Con}(\mathcal{G})$ is étale compact. □

Proof. The set $\mathcal{G}^G$ is the image of the compact space $\mathcal{G} \times G$ under the étale continuous map $(\Gamma, g) \mapsto \Gamma^g$. Hence, $\mathcal{G}^G$ is étale compact.

By Lemma 2.1, $\mathrm{Env}(\mathcal{G})$ is strictly closed, hence étale compact [12, Remark 1.2]. Therefore, by the first paragraph, $\mathrm{Con}(\mathcal{G}) = \mathrm{Env}(\mathcal{G})^G$ is étale compact. ∎

Lemma 2.4. Let H be a closed subgroup of G. Then $\mathrm{\acute{E}taleClosure}(\{H\}) = \{B \in \mathrm{Subgr}(G) \mid H \leq B\}$. □

Proof. First suppose $B \in \mathrm{\acute{E}taleClosure}(\{H\})$. Then H belongs to each étale open neighborhood of B. In other words, if $U \in \mathrm{Open}(G)$ and $B \leq U$, then $H \leq U$. Hence, $H \leq B$.

Conversely, suppose $H \leq B$. Then, H belongs to each basic étale open neighborhood $\mathrm{Subgr}(U)$ of B. Therefore, $B \in \mathrm{\acute{E}taleClosure}(\{H\})$. ∎

Lemma 2.5. Let $\varphi : G \to H$ be an epimorphism of profinite groups and G_0 a closed subgroup of G. The set $\{B \in \mathrm{Subgr}(G) \mid \varphi(G_0) \leq \varphi(B)\}$ is étale closed. □

Proof. By [12, Remark 1.1(b)], the map $\varphi : \mathrm{Subgr}(G) \to \mathrm{Subgr}(H)$ induced by φ is étale continuous. By Lemma 2.4, the set $\{C \in \mathrm{Subgr}(H) \mid \varphi(G_0) \leq C\}$ is étale closed. Its inverse image in $\mathrm{Subgr}(G)$ is $\{B \in \mathrm{Subgr}(G) \mid \varphi(G_0) \leq \varphi(B)\}$, so it is étale closed. ∎

Lemma 2.6. Let $\mathcal{G}$ be an étale compact subset of $\mathrm{Subgr}(G)$. Then each $A \in \mathcal{G}$ is contained in a maximal element of $\mathcal{G}$. □

Proof. By Zorn's lemma, it suffices to prove that each ascending chain $\mathcal{G}_0$ in $\mathcal{G}$ is bounded by an element of $\mathcal{G}$. Consider $B_1, \ldots, B_n \in \mathcal{G}_0$. Then the B_i are comparable. Assume $B_1 \leq B_2 \leq \cdots \leq B_n$. By Lemma 2.4, $B_n \in \bigcap_{i=1}^{m} \mathrm{\acute{E}taleClosure}(\{B_i\})$. Thus,

$$\mathcal{G} \cap \bigcap_{i=1}^{n} \mathrm{\acute{E}taleClosure}\left(\{B_i\}\right) \neq \emptyset. \tag{2.3}$$

Since $\mathcal{G}$ is étale compact, $\mathcal{G} \cap \bigcap_{B \in \mathcal{G}_0} \text{ÉtaleClosure}(\{B\})$ is nonempty. Each element of the latter set is a bound of $\mathcal{G}_0$.   ∎

If a subset $\mathcal{G}$ of $\text{Subgr}(G)$ contains groups A and B with $A < B$, then $\mathcal{G}$ is not étale Hausdorff. Thus, removing all nonmaximal elements from $\mathcal{G}$ is the only way to make $\mathcal{G}$ étale Hausdorff while preserving the essential information stored in $\mathcal{G}$. We denote the set of all maximal elements of $\mathcal{G}$ by $\mathcal{G}_{\max}$.

Lemma 2.7. Let $\mathcal{G}$ be an étale compact subset of $\text{Subgr}(G)$. Then $\mathcal{G}_{\max}$ is étale compact.
$\square$

Proof. By Lemma 2.6, $\text{Env}(\mathcal{G}) = \text{Env}(\mathcal{G}_{\max})$. Hence, by Lemma 2.1, $\mathcal{G}_{\max}$ is étale compact.
∎

3   Relatively projective groups

Pairs $(G, \mathcal{G})$ consisting of a profinite group and a subset $\mathcal{G}$ of $\text{Subgr}(G)$ which satisfy a local global principle for finite embedding problems naturally arise from pairs $(K, \mathcal{X})$ consisting of a field K and a set $\mathcal{X}$ of separable algebraic extensions of K which satisfy a local global principle for points on absolutely irreducible varieties (Section 4). We prove in Proposition 4.1 that G is "$\mathcal{G}$-projective" in a sense we now explain.

Let G be a profinite group and $\mathcal{G}$ a subset of $\text{Subgr}(G)$. An *embedding problem* for G is a pair

(1) $(\varphi : G \to A,\ \alpha : B \to A)$,

where φ is a homomorphism and α is an epimorphism of profinite groups. The embedding problem is *finite* if A and B are finite. We call (1) a $\mathcal{G}$-*embedding problem* if it is *locally solvable*; that is,

(2) for each $\Gamma \in \mathcal{G}$ there exists a homomorphism $\gamma_\Gamma : \Gamma \to B$ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$.

We say (1) is a *rigid $\mathcal{G}$-embedding problem* if

(3) for each $\Gamma \in \mathcal{G}$ there is $B_0 \in \text{Subgr}(B)$ such that $\alpha : B_0 \to \varphi(\Gamma)$ is an isomorphism.

A *solution* of (1) is a homomorphism $\gamma : G \to B$ with $\gamma \circ \alpha = \varphi$. We say G is $\mathcal{G}$-*projective* if every finite $\mathcal{G}$-embedding problem for $\mathcal{G}$ is solvable. Our definition generalizes the definition of a "projective group." Indeed, "G is $\varnothing$-projective" means "G is projective." We refer to G as *relatively projective* if G is $\mathcal{G}$-projective for a subset $\mathcal{G}$ of $\text{Subgr}(G)$.

If $\mathcal{G} \subseteq \mathcal{G}' \subseteq \text{Subgr}(G)$ and G is $\mathcal{G}$-projective, then G is $\mathcal{G}'$-projective. Moreover, if $\mathcal{G}$ is étale compact, then by Lemma 2.6, each group in $\mathcal{G}$ is contained in a group of $\mathcal{G}_{\max}$.

Hence, under the assumption that $\mathcal{G}$ is étale compact, G is $\mathcal{G}$-projective if and only if G is $\mathcal{G}_{\max}$-projective. Hence, G is $\mathcal{G}$-projective if and only if G is $\mathcal{G}^G$-projective.

Suppose (1) is a rigid $\mathcal{G}$-embedding problem and Γ and B_0 are as in (3). Put $A_0 = \varphi(\Gamma)$ and $\gamma_\Gamma = (\alpha|_{A_0})^{-1} \circ \varphi|_\Gamma$. Then $\gamma_\Gamma : \Gamma \to B$ is a homomorphism satisfying $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Thus, every rigid $\mathcal{G}$-embedding problem is a $\mathcal{G}$-embedding problem. The next lemma establishes a sort of converse to this statement.

Lemma 3.1. Let G be a profinite group and $\mathcal{G}$ an étale compact subset of $\mathrm{Subgr}(G)$. Let (1) be a finite $\mathcal{G}$-embedding problem for G. Then

(a) there exists a commutative diagram

$$
\begin{array}{ccc}
 & G & \\
 & \Big\downarrow{\widehat{\varphi}} & \\
\widehat{B} \xrightarrow{\ \widehat{\alpha}\ } & \widehat{A} & \varphi \\
\beta\Big\downarrow & \Big\downarrow{\bar{\varphi}} & \\
B \xrightarrow{\ \alpha\ } & A &
\end{array}
\tag{3.1}
$$

in which $\widehat{\varphi}$ is an epimorphism and $(\widehat{\varphi} : G \to \widehat{A},\ \widehat{\alpha} : \widehat{B} \to \widehat{A})$ is a finite rigid $\mathcal{G}$-embedding problem,

(b) if every finite rigid $\mathcal{G}$-embedding problem (1) for G in which φ is an epimorphism is solvable, then G is $\mathcal{G}$-projective. $\qquad\square$

Proof of (a). Consider $\Gamma \in \mathcal{G}$. Choose a homomorphism γ_Γ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Then $\mathrm{Ker}(\gamma_\Gamma)$ is an open subgroup of Γ. Choose $N_\Gamma \in \mathrm{OpenNormal}(G)$ with $N_\Gamma \leq \mathrm{Ker}(\varphi)$ and $\Gamma \cap N_\Gamma \leq \mathrm{Ker}(\gamma_\Gamma)$. Then $\mathrm{Subgr}(\Gamma N_\Gamma)$ is an étale open neighborhood of Γ in $\mathrm{Subgr}(G)$ and γ_Γ extends to a homomorphism $\gamma'_\Gamma : \Gamma N_\Gamma \to B$ with kernel $\mathrm{Ker}(\gamma_\Gamma)N_\Gamma$.

Since $\mathcal{G}$ is étale compact, there are $\Gamma_1, \ldots, \Gamma_m \in \mathcal{G}$ with $\mathcal{G} \subseteq \bigcup_{i=1}^m \mathrm{Subgr}(\Gamma_i N_{\Gamma_i})$. Then $N = \bigcap_{i=1}^m N_{\Gamma_i} \in \mathrm{OpenNormal}(G)$ and $N \leq \mathrm{Ker}(\varphi)$. Let $\widehat{A} = G/N$, $\widehat{\varphi} : G \to \widehat{A}$ the quotient map, and $\bar{\varphi} : \widehat{A} \to A$ the map induced by φ. Then $\varphi = \bar{\varphi} \circ \widehat{\varphi}$. Now consider the fiber product $\widehat{B} = B \times_A \widehat{A}$ with the projection maps $\widehat{\alpha} : \widehat{B} \to \widehat{A}$ and $\beta : \widehat{B} \to B$ on the coordinates. Since α is surjective, so is $\widehat{\alpha}$.

For each i put $N_i = N_{\Gamma_i}$ and $\gamma_i = \gamma'_{\Gamma_i} : \Gamma_i N_i \to B$. Then there is a homomorphism $\widehat{\gamma}_i : \Gamma_i N_i \to \widehat{B}$ satisfying $\widehat{\alpha} \circ \widehat{\gamma}_i = \widehat{\varphi}|_{\Gamma_i N_i}$ and $\beta \circ \widehat{\gamma}_i = \gamma_i$ [7, Proposition 22.2.1]. Put $\widehat{B}_i = \widehat{\gamma}_i(\Gamma_i N_i)$.

Claim 1. $\widehat{\alpha} : \widehat{B}_i \to \widehat{\varphi}(\Gamma_i N_i)$ is an isomorphism.

It suffices to prove that $\widehat{\alpha}$ is injective on $\widehat{B}_i$. Indeed, consider $b \in \widehat{B}_i$ with $\widehat{\alpha}(b) = 1$. Choose $g \in \Gamma_i N_i$ with $\widehat{\gamma}_i(g) = b$. Then $\widehat{\varphi}(g) = \widehat{\alpha}(\widehat{\gamma}_i(g)) = 1$, so $g \in N \leq N_i$. Thus, $\beta(b) = \gamma_i(g) = 1$. Therefore, $b = 1$, as desired.

Now consider $\Gamma \in \mathcal{G}$. Choose i with $\Gamma \leq \Gamma_i N_i$. Then $\widehat{\varphi}(\Gamma) \leq \widehat{\varphi}(\Gamma_i N_i)$. By **Claim 1**, $\widehat{\alpha}$ maps $\widehat{\alpha}^{-1}(\widehat{\varphi}(\Gamma)) \cap \widehat{B}_i$ isomorphically onto $\widehat{\varphi}(\Gamma)$. Hence, $(\widehat{\varphi} : G \to \widehat{A}, \ \widehat{\alpha} : \widehat{B} \to \widehat{A})$ is a finite $\mathcal{G}$-embedding problem for G satisfying the rigidity condition.

Proof of (b). Consider a finite $\mathcal{G}$-embedding problem (1) for G. Let $(\widehat{\varphi} : G \to \widehat{A}, \ \widehat{\alpha} : \widehat{B} \to \widehat{A})$ be the embedding problem given by (a). By assumption, it has a solution $\widehat{\gamma}$. Then $\gamma = \beta \circ \widehat{\gamma}$ solves (1). $\blacksquare$

4   Pseudo closed fields

Let K be a field and $\mathcal{X}$ a subset of the set $\mathrm{SepAlgExt}(K)$ of all separable algebraic extensions of K. By an *absolutely irreducible variety over* K we mean a nonempty geometrical integral scheme of finite type over K. We say K is *pseudo-$\mathcal{X}$-closed* (abbreviated *P$\mathcal{X}$C*) if it satisfies the following condition.

(1) Every smooth absolutely irreducible variety over K, with an F-rational point for each $F \in \mathcal{X}$, has a K-rational point.

If V is an arbitrary absolutely irreducible variety over K, then the Zariski open subset V_{simp} of all simple points of V is also an absolutely irreducible variety over K. Hence, (1) is equivalent to the following condition.

(2) Every absolutely irreducible variety over K, with a simple F-rational point for each $F \in \mathcal{X}$, has a K-rational point.

Note that K is P$\varnothing$C if and only if K is PAC [**7**, Chapter 11].

Under a mild topological assumption on $\mathcal{X}$, the P$\mathcal{X}$C property of K results in a relative projectivity of $\mathrm{Gal}(K)$. The topology in question is the étale topology of the space $\mathrm{SepAlgExt}(K)$. This space stands in a bijective correspondence with $\mathrm{Subgr}(\mathrm{Gal}(K))$. Thus, $\mathrm{SepAlgExt}(K)$ inherits the étale topology from that of $\mathrm{Subgr}(\mathrm{Gal}(K))$. Basic étale open subsets of $\mathrm{SepAlgExt}(K)$ are $\mathrm{SepAlgExt}(L)$ with L/K finite and separable.

If $\mathcal{X} \subseteq \mathcal{X}' \subseteq \mathrm{SepAlgExt}(K)$ and K is P$\mathcal{X}$C, then K is P$\mathcal{X}'$C. Denote the set of all minimal fields in $\mathcal{X}$ by $\mathcal{X}_{\min}$. If $\mathcal{X}$ is étale compact, then by **Lemma 2.6**, each field in $\mathcal{X}$ contains a minimal field in $\mathcal{X}$. Hence, K is P$\mathcal{X}$C if and only if K is P$\mathcal{X}_{\min}$C.

Proposition 4.1. Let K be a field and $\mathcal{X}$ a subset of $\mathrm{SepAlgExt}(K)$. Put $\mathcal{G} = \{\mathrm{Gal}(K') \mid K' \in \mathcal{X}\}$. Suppose $\mathcal{X}$ is étale compact and K is P$\mathcal{X}$C. Then $\mathrm{Gal}(K)$ is $\mathcal{G}$-projective. $\square$

Proof. By **Lemma 2.3**, $\mathcal{G}^G$ is étale compact. If we prove that $\mathrm{Gal}(K)$ is $\mathcal{G}^G$-projective, it will follow that $\mathrm{Gal}(K)$ is $\mathcal{G}$-projective. We may therefore assume $\mathcal{G}$ is $\mathrm{Gal}(K)$-invariant.

By Lemma 3.1, it suffices to solve every finite rigid $\mathcal{G}$-embedding problem

(3) $(\varphi : \text{Gal}(K) \to A, \ \alpha : B \to A)$,

where φ is an epimorphism.

Let L be the fixed field of $\text{Ker}(\varphi)$ in K_s. Then identify A with $\text{Gal}(L/K)$ and φ with $\text{res}_{K_s/L}$. Next use [12, Lemma 6.2] to construct a finitely generated regular extension E of K and a finite Galois extension F of E containing L with these properties:

(4a) $B = \text{Gal}(F/E)$ and α is the restriction map $\text{res}_{F/L} : \text{Gal}(F/E) \to \text{Gal}(L/K)$;

(4b) if L_0 is a field between K and L, F_0 is a field between E and F which contains L_0, and $\text{res}_{F/L} : \text{Gal}(F/F_0) \to \text{Gal}(L/L_0)$ is an isomorphism, then F_0 is a purely transcendental extension of L_0.

Since E/K is finitely generated and regular, one may view E as the function field of an absolutely irreducible smooth affine variety V over K [7, Corollary 10.2.2].

Now let $\{L_i \mid i \in I\} = \{K' \cap L \mid K' \in \mathcal{X}\}$. By rigidity, choose for each $i \in I$ a field F_i between E and F containing L_i such that $\text{res}_{F/L} : \text{Gal}(F/F_i) \to \text{Gal}(L/L_i)$ is an isomorphism. By (4b), F_i is a purely transcendental extension of L_i. Hence, $V(L_i) \neq \varnothing$. Therefore, $V(K') \neq \varnothing$ for each $K' \in \mathcal{X}$. Since K is P$\mathcal{X}$C, V has a K-rational point, which, by assumption, is simple. By [19, Corollary A2], E has a valuation which is trivial on K and with K as its residue field. Lemma 7.4 in [12] gives an algebraic extension E' of E such that $\text{res}_{E_s'/K_s} : \text{Gal}(E') \to \text{Gal}(K)$ is an isomorphism. Denote its inverse by γ'. Then $\gamma = \text{res}_{E_s'/F} \circ \gamma'$ solves (3).   ∎

Again, let K be a field and $\mathcal{X}$ a subset of $\text{SepAlgExt}(K)$. For each algebraic extension L of K let $\mathcal{X}_L = \{K'L \mid K' \in \mathcal{X}\}$. If $[L : K] < \infty$, then L is P$\mathcal{X}_L$C [15, Lemma 7.2]. The same result holds for arbitrary L if $\mathcal{X}$ is strictly closed [15, Lemma 7.4]. Here we prove that L is P$\mathcal{X}_L$C under the weaker condition that $\mathcal{X}$ is étale compact.

Proposition 4.2 (extension theorem)**.** Let K be a field, $\mathcal{X}$ an étale compact subset of $\text{SepAlgExt}(K)$, and L a separable algebraic extension of K. Suppose K is P$\mathcal{X}$C. Then $\mathcal{X}_L$ is étale compact and L is P$\mathcal{X}_L$C.   □

Proof. The map $K' \mapsto K'L$ from $\mathcal{X}$ to $\mathcal{X}_L$ is étale continuous. Since $\mathcal{X}$ is étale compact, so is $\mathcal{X}_L$.

Next let V be a smooth absolutely irreducible variety over L with $V(K'L) \neq \varnothing$ for each $K' \in \mathcal{X}$. Choose a finite subextension K_1 of L/K over which V is already defined. Denote the set of all finite subextensions of L/K_1 by $\mathcal{E}$. For each $E \in \mathcal{E}$ let $\mathcal{T}_E = \{K' \in \mathcal{X} \mid V(K'E) \neq \varnothing\}$.

Claim 1. $\mathcal{T}_E$ is étale open in $\mathcal{X}$.

Indeed, let $K' \in \mathcal{T}_E$. Then $V(K'E) \neq \varnothing$. Hence, K'/K has a finite subextension K_0'/K with $V(K_0'E) \neq \varnothing$. The open neighborhood $\mathcal{X} \cap \mathrm{SepAlgExt}(K_0')$ of K' in $\mathcal{X}$ is contained in $\mathcal{T}_E$. Therefore, $\mathcal{T}_E$ is open in $\mathcal{X}$.

Since $L = \bigcup_{E \in \mathcal{E}} E$, we have $\bigcup_{E \in \mathcal{E}} \mathcal{T}_E = \mathcal{X}$. Since $\mathcal{X}$ is étale compact, $\mathcal{E}$ has a finite subset $\mathcal{E}_0$ with $\bigcup_{E \in \mathcal{E}_0} \mathcal{T}_E = \mathcal{X}$. Let F be the union of all $E \in \mathcal{E}_0$. Then $\mathcal{T}_E \subseteq \mathcal{T}_F$ for each $E \in \mathcal{E}_0$, so $\mathcal{X} = \mathcal{T}_F$. Thus, $V(K'F) \neq \varnothing$ for each $K' \in \mathcal{X}$. By [**15**, Lemma 7.4], F is $P\mathcal{X}_F C$. Hence, $V(F) \neq \varnothing$, so $V(L) \neq \varnothing$. Consequently, L is $P\mathcal{X}_L C$. ∎

5   Strongly projective groups

Consider a profinite group G and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$. Suppose G is $\mathcal{G}$-projective. If $\mathcal{G}$ is empty, then G is projective, so every embedding problem for G is solvable [**7**, Lemma 22.3.2]. Unfortunately, we are able to solve an arbitrary $\mathcal{G}$-embedding problem in the general case only if we impose a strong condition on the global solution of each finite embedding problem: the solution has to map every local group $\Gamma \in \mathcal{G}$ into a subset of $\mathrm{Subgr}(B)$, given in advance, which is closed under conjugations and taking subgroups. In addition, we have to assume that every $\Gamma \in \mathcal{G}$ is maximal in $\mathcal{G}$ and $1 \notin \mathrm{\acute{E}taleClosure}(\mathcal{G})$. **Section 6** shows that when these conditions are fulfilled, $G, \mathcal{G}$ naturally give rise to a proper projective group structure $\mathbf{G} = (G, X, G_x)_{x \in X}$ in the sense of [**12**, Section 4]. By [**12**, Proposition 4.2], every embedding problem for $\mathbf{G}$ is solvable.

Consider again a profinite group G and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$. A $\mathcal{G}$-embedding problem for G with *local data* is a triple

(1) $(\varphi : G \to A,\ \alpha : B \to A, \mathcal{B})$,

where A and B are profinite groups, $\mathcal{B}$ is a strictly closed subset of $\mathrm{Subgr}(B)$ which is closed under conjugations and taking closed subgroups, φ is a homomorphism, and α is an epimorphism. In addition, we assume α is $\mathcal{B}$-*rigid*. That is,

(2) $\varphi(\mathcal{G}) \subseteq \alpha(\mathcal{B})$ and α is injective on each $B_0 \in \mathcal{B}$.

Call (1) *finite* if B is finite.

Occasionally we construct $\mathcal{B}$ as above in the following way. Let $\mathcal{B}_0$ be a subset of $\mathrm{Subgr}(B)$. Then $\mathcal{B} = \mathrm{Con}(\mathcal{B}_0)$ is the set of all subgroups of B which are contained in B_0^b for some $B_0 \in \mathcal{B}_0$ and $b \in B$. By **Lemma 2.1**, $\mathcal{B}$ is strictly compact (hence closed) if $\mathcal{B}_0$ is étale compact. In particular, this is the case if $\mathcal{B}_0$ is finite.

A *solution* of (1) is a homomorphism $\gamma : G \to B$ with $\gamma(\mathcal{G}) \subseteq \mathcal{B}$. Call G *strongly $\mathcal{G}$-projective* if every finite $\mathcal{G}$-embedding problem (1) for G with local data has a solution. If in addition $\mathcal{G}$ is étale compact, then G is $\mathcal{G}$-projective.

Indeed, let $(\varphi : G \to A,\ \alpha : B \to A)$ be a rigid $\mathcal{G}$-embedding problem in the sense of **Section 3**, (3). For each $\Gamma \in \mathcal{G}$ choose $B_\Gamma \in \mathrm{Subgr}(B)$ such that $\alpha : B_\Gamma \to \varphi(\Gamma)$ is an

isomorphism. Let $\mathcal{B} = \mathrm{Con}(B_\Gamma \mid \Gamma \in \mathcal{G})$. Then (1) is a $\mathcal{G}$-embedding problem for G with local data and α is $\mathcal{B}$-rigid. By assumption, there exists a homomorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \varphi$. By **Lemma 3.1**(b), G is $\mathcal{G}$-projective. Moreover, by Lemmas **2.6** and **2.7**, $\mathcal{G}$ is étale compact and G is strongly $\mathcal{G}_{\max}$-projective.

Example 5.1 free product of finitely many profinite groups. Consider a free product $G = \bar{\Huge*}{}_{i=1}^{n}\, G_i$ of finitely many profinite groups. Put $\mathcal{G} = \{G_1, \ldots, G_n\}$ and $\mathcal{B} = \mathrm{Con}(G_1, \ldots, G_n)$. Then G is strongly $\mathcal{G}$-projective.

Indeed, let (1) be a finite embedding problem for G with local data. Then φ maps each G_i onto a subgroup A_i of A and there is $B_i \in \mathcal{B}$ which α maps isomorphically onto A_i. Then $\gamma_i = (\alpha|_{B_i})^{-1} \circ (\varphi|_{G_i})$ is an epimorphism of G_i onto B_i. Extend $\gamma_1, \ldots, \gamma_n$ to a homomorphism $\gamma : G \to B$. Then γ solves embedding problem (1).

Remark 5.2. Suppose $\mathcal{G}$ is étale compact and G is a strongly $\mathcal{G}$-projective group. An obvious modification of **Lemma 3.1** proves (1) is solvable even if α is not necessarily rigid but satisfies the weaker condition instead:

(3) for each $\Gamma \in \mathcal{G}$ there are $B_0 \in \mathcal{B}$ and a homomorphism $\gamma_0 : \Gamma \to B_0$ with $\alpha \circ \gamma_0 = \varphi|_\Gamma$. However, we do not use (3) in the definition of strong projectivity because all embedding problems which we use in this work satisfy the condition "α is $\mathcal{B}$-rigid."

Lemma 5.3. Let G be a strongly $\mathcal{G}$-projective group with $\mathcal{G} \subseteq \mathrm{Subgr}(G)$. Then every embedding problem with local data (1) such that A is finite and $\mathrm{rank}(B) \leq \aleph_0$ is solvable.
$\square$

Proof. Put $N_0 = \mathrm{Ker}(\alpha)$ and identify A with B/N_0 and α with the quotient map $B \to B/N_0$. Choose a descending sequence $N_i \in \mathrm{OpenNormal}(B)$ with $N_i \leq \mathrm{Ker}(\alpha)$, $i = 1, 2, 3, \ldots$, and $\bigcap_{i=1}^{n} N_i = 1$. For $j \geq i$ let $\alpha_{ji} : B/N_j \to B/N_i$ and $\beta_i : B \to B/N_i$ be the quotient maps. For each i, $\mathcal{B}/\mathcal{N}_i = \beta_i(\mathcal{B})$ is closed under conjugation and taking subgroups. The map α is injective on each $B_0 \in \mathcal{B}$, so $\alpha_{i+1,i}$ is injective on B_0/N_{i+1}. Therefore, we may inductively construct a sequence of homomorphisms $\gamma_i : G \to B/N_i$ satisfying $\gamma_0 = \varphi$, $\gamma_i(\mathcal{G}) \subseteq \mathcal{B}/\mathcal{N}_i$, and $\alpha_{i+1,i} \circ \gamma_{i+1} = \gamma_i$, $i = 1, 2, 3, \ldots$.

The γ_i's define a homomorphism $\gamma : G \to B$ with $\beta_i \circ \gamma = \gamma_i$, $i = 0, 1, 2, \ldots$. Since $\mathcal{B}$ is strictly closed, $\mathcal{B} = \varprojlim \mathcal{B}/\mathcal{N}_i$. Hence, $\gamma(\mathcal{G}) \subseteq \mathcal{B}$. Therefore, γ is a solution of (1). $\blacksquare$

Free products of finitely many profinite groups have some nice properties.

Lemma 5.4 [13, Proposition 2 and Theorem B′]**.** Let $G = \bar{\Huge*}{}_{i \in I}\, G_i$ be the free profinite product of finitely many profinite groups G_i. Then $G_i^g \cap G_j \neq 1$ implies $i = j$, $g \in G_i$. $\square$

Lemma 5.4 carries over to strongly $\mathcal{G}$-projective groups.

Proposition 5.5. Let G be a profinite group and $\mathcal{G}$ an étale compact subset of $\mathrm{Subgr}(G)$ which is closed under conjugation. Suppose G is strongly $\mathcal{G}$-projective. Then

  (a) $\Gamma_1 \cap \Gamma_2 = 1$ for all distinct $\Gamma_1, \Gamma_2 \in \mathcal{G}_{\max}$;

  (b) $N_G(\Gamma) = \Gamma$ for each nontrivial $\Gamma \in \mathcal{G}_{\max}$. $\square$

Proof. Consider an epimorphism $\varphi : G \to A$ with A finite. Write $\varphi(\mathcal{G}) = \{A_i \mid i \in I\}$ with I finite. For each $i \in I$ choose an isomorphic copy B_i of A_i. Choose a large positive integer e and put $B = \widehat{F}_e * \text{\Large⊛}_{i \in I} B_i$. Then there is an epimorphism $\alpha : B \to A$ which maps B_i isomorphically onto A_i. Let $\mathcal{B} = \mathrm{Con}(B_1, \ldots, B_n)$. Then, (1) is a $\mathcal{G}$-embedding problem for G with local data. By Lemma 5.3, there is a homomorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \varphi$ and $\gamma(\mathcal{G}) \subseteq \mathcal{B}$.

Proof of (a). Assume $\Gamma_1 \cap \Gamma_2 \neq 1$. Choose $N_0 \in \mathrm{OpenNormal}(G)$ with $\Gamma_1 N_0 \neq \Gamma_2 N_0$. Consider $N \in \mathrm{OpenNormal}(G)$ with $N \leq N_0$. Put $A = G/N$ and let $\varphi : G \to G/N$ be the quotient map. Then let $B, B_i, \mathcal{B}, \alpha$, and γ be as above. In particular, $\gamma(\Gamma_i) \in \mathcal{B}$, $i = 1, 2$. Hence, there are $j, k \in I$ and $b_j, b_k \in B$ with $\gamma(\Gamma_1) \leq B_j^{b_j}$ and $\gamma(\Gamma_2) \leq B_k^{b_k}$. Also, $\alpha(\gamma(\Gamma_1) \cap \gamma(\Gamma_2)) \subseteq \varphi(\Gamma_1) \cap \varphi(\Gamma_2) \neq 1$, hence $\gamma(\Gamma_1) \cap \gamma(\Gamma_2) \neq 1$, so $B_j^{b_j} \cap B_k^{b_k} \neq 1$. By Lemma 5.4, $B_j^{b_j} = B_k^{b_k}$. Consider $\Gamma_N \in \mathcal{G}$ with $\varphi(\Gamma_N) = \alpha(B_j^{b_j})$. Then, $\varphi(\Gamma_i) \leq \varphi(\Gamma_N)$, $i = 1, 2$. It follows that the set $\mathcal{G}_N = \{\Gamma \in \mathcal{G} \mid \varphi(\Gamma_1), \varphi(\Gamma_2) \leq \varphi(\Gamma)\}$ is nonempty. By Lemma 2.5, $\mathcal{G}_N$ is étale closed. If $N_1, \ldots, N_m$ are open normal subgroups of G and $N = \bigcap_{j=1}^{m} N_j$, then $\mathcal{G}_N \subseteq \bigcap_{j=1}^{m} \mathcal{G}_{N_j}$. Hence, since $\mathcal{G}$ is weakly compact, $\bigcap_{N \in \mathrm{OpenNormal}(G)} \mathcal{G}_N \neq \varnothing$. Each Γ in this intersection belongs to $\mathcal{G}$ and satisfies $\Gamma_1 \cap \Gamma_2 \leq \Gamma$. Since Γ_1 and Γ_2 are maximal in $\mathcal{G}$, $\Gamma_1 = \Gamma = \Gamma_2$, in contradiction to assumption.

Proof of (b). Let $g \in G$ with $\Gamma^g = \Gamma$. Choose $N_0 \in \mathrm{OpenNormal}(G)$ with $\Gamma \not\leq N_0$. Consider $N \in \mathrm{OpenNormal}(G)$ with $N \leq N_0$. Put $A = G/N$ and let $\varphi : G \to G/N$ be the quotient epimorphism. Then let $B, B_i, \mathcal{B}$, and α be as in the first paragraph of the proof. In particular, there are $i \in I$ and $b \in B$ with $\gamma(\Gamma) \leq B_i^b$. Also, $\gamma(\Gamma)^{\gamma(g)} = \gamma(\Gamma)$ and $\gamma(\Gamma) \neq 1$, so $B_i^b \cap B_i^{b\gamma(g)} \neq 1$. By Lemma 5.4, $\gamma(g) \in B_i^b$. Choose $\Gamma_N \in \mathcal{G}$ with $\varphi(\Gamma_N) = \alpha(B_i^b)$. Then $\varphi(\Gamma) \leq \varphi(\Gamma_N)$ and $\varphi(g) \in \varphi(\Gamma_N)$.

  Again, by Lemma 2.5, the nonempty set

$$\mathcal{G}'_N = \left\{ \Gamma' \in \mathcal{G} \mid \varphi(\Gamma) \leq \varphi(\Gamma'), \ \varphi(g) \in \varphi(\Gamma') \right\} \tag{5.1}$$

is étale closed. Since $\mathcal{G}$ is étale compact, there is Γ' which belongs to all $\mathcal{G}'_N$. It satisfies $\Gamma \leq \Gamma'$ and $g \in \Gamma'$. Since Γ is maximal in $\mathcal{G}$, we have $\Gamma = \Gamma'$. Therefore, $g \in \Gamma$.   ■

Lemma 5.6. Let G be a profinite group and $\mathcal{G}$ an étale compact subset of $\mathrm{Subgr}(G)$ which is closed under conjugation. Suppose $1 \notin \mathrm{StrictClosure}(\mathcal{G})$ and G is strongly $\mathcal{G}$-projective.

Then

    (a) G is strongly $\mathcal{G}_{\max}$-projective,

    (b) $\mathcal{G}_{\max}$ is étale compact Hausdorff,

    (c) $N_G(\Gamma) = \Gamma$ for each $\Gamma \in \mathcal{G}_{\max}$.   □

Proof of (a). By **Lemma 2.7**, $\mathcal{G}_{\max}$ is étale compact. Suppose (1) is a finite $\mathcal{G}_{\max}$-embedding problem with local data for G. In particular, $\varphi(\mathcal{G}_{\max}) \subseteq \alpha(\mathcal{B})$ and α is injective on each $B_0 \in \mathcal{B}$. We prove (1) is a $\mathcal{G}$-embedding problem with local data for G. To this end let $\Gamma_0 \in \mathcal{G}$. By **Lemma 2.6**, Γ_0 is contained in some $\Gamma \in \mathcal{G}_{\max}$. Choose $B_1 \in \mathcal{B}$ with $\alpha(B_1) = \varphi(\Gamma)$. Let $B_0 = B_1 \cap \alpha^{-1}(\varphi(\Gamma_0))$. Then $B_0 \in \mathcal{B}$, and α maps B_0 isomorphically onto $\varphi(\Gamma_0)$, as needed.

    Since G is strongly $\mathcal{G}$-projective, there is a homomorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \varphi$ and $\gamma(\mathcal{G}) \subseteq \mathcal{B}$, so $\gamma(\mathcal{G}_{\max}) \subseteq \mathcal{B}$. Thus, G is strongly $\mathcal{G}_{\max}$-projective.

Proof of (b). By (a) and **Proposition 5.5**(a), $\Gamma_1 \cap \Gamma_2 = 1$ for all distinct $\Gamma_1, \Gamma_2 \in \mathcal{G}_{\max}$. Hence, by [12, Corollary 1.4], $\mathcal{G}_{\max}$ is étale Hausdorff.

Proof of (c). By assumption, each $\Gamma \in \mathcal{G}_{\max}$ is nontrivial. Hence, by **Proposition 5.5**(b), $N_G(\Gamma) = \Gamma$.   ■

    Our next goal is to prove under the assumptions of **Lemma 5.6** that $\mathcal{G}_{\max}$ is a profinite space in the étale topology. By definition, a *profinite space* X is an inverse limit of discrete finite spaces. In particular, X has a basis consisting of open-closed sets. Conversely, every compact Hausdorff space which has a basis consisting of open-closed sets is profinite. (See also [30, Theorem 1.1.12] for the connection with the condition "X is totally disconnected.")

Lemma 5.7. Let X be a compact Hausdorff space and G a profinite group which acts continuously on X. Suppose X/G has a basis consisting of open-closed sets. Then X is profinite.   □

Proof. Let $x \in X$ and W an open neighborhood of x. We have to find an open-closed neighborhood of x in W.

Part A: G is finite. Let $S = \{\sigma \in G \mid x^\sigma = x\}$. Write $G = \bigcup_{i=1}^m S\sigma_i$ with $\sigma_1 = 1$. Then $x^{\sigma_1}, \dots, x^{\sigma_m}$ are the distinct conjugates of x. Since X is Hausdorff, there are open neighborhoods $V_1, \dots, V_m$ of x in W such that $V_1^{\sigma_1}, \dots, V_m^{\sigma_m}$ are disjoint. Put $V = \bigcap_{i=1}^m \bigcap_{\sigma \in S} V_i^\sigma$. This is an S-invariant open neighborhood of x in W and $V^{\sigma_1}, \dots, V^{\sigma_m}$ are disjoint.

    The quotient map $\pi : X \to X/G$ is continuous and open. In particular, $\pi(V)$ is an open neighborhood of $\pi(x)$ in X/G. By assumption, there is an open-closed neighborhood

$\bar{U}$ of $\pi(x)$ in X/G with $\bar{U} \subseteq \pi(V)$. Then, $U = \pi^{-1}(\bar{U})$ is an open-closed G-invariant neighborhood of x in X and $U \subseteq \pi^{-1}(\pi(V)) = \bigcup_{i=1}^{m} V^{\sigma_i}$. Therefore $U = \bigcup_{i=1}^{m} U \cap V^{\sigma_i}$. Since the sets $U \cap V^{\sigma_i}$ are open, they are also closed in U, and hence in X. Thus $U \cap V = U \cap V^{\sigma_1}$ is an open-closed neighborhood of x contained in W.

Part B: G is arbitrary. The action $X \times G \to X$ is continuous and $x^1 \in W$. Hence, x has an open neighborhood V and G has an open normal subgroup N with $V^N \subseteq W$. Let $\nu : X \to X/N$ be the quotient map. Then $\nu(V)$ is an open neighborhood of $\nu(x)$ in X/N. Since X is compact Hausdorff, so is X/N [2, Theorem 3.1(1)]. The finite group G/N acts on X/N continuously and $X/G = (X/N)/(G/N)$. Thus, by Part A, X/N is profinite. Therefore, $\nu(x)$ has an open-closed neighborhood $\bar{U}$ in X/N with $\bar{U} \subseteq \nu(V)$. Therefore $U = \nu^{-1}(\bar{U})$ is an open-closed neighborhood of x in X and $U \subseteq V^N \subseteq W$, as desired. ∎

Proposition 5.8. Let G be a profinite group and $\mathcal{G}$ an étale compact G-invariant subset of $\mathrm{Subgr}(G)$. Suppose $1 \notin \mathrm{StrictClosure}(\mathcal{G})$ and G is strongly $\mathcal{G}$-projective. Then $\mathcal{G}_{max}$ is étale profinite. □

Proof. By Lemma 5.6, G is strongly $\mathcal{G}_{max}$-projective and $\mathcal{G}_{max}$ is étale Hausdorff compact. We may therefore replace $\mathcal{G}$ by $\mathcal{G}_{max}$, if necessary, to assume $\mathcal{G} = \mathcal{G}_{max}$ and prove that the étale topology of $\mathcal{G}$ has a basis consisting of étale open-closed sets.

Let $\pi : \mathrm{Subgr}(G) \to \mathrm{Subgr}(G)/G$ be the quotient map modulo conjugation. Put a bar over each group in $\mathrm{Subgr}(G)$ and each subset of $\mathrm{Subgr}(G)$ to denote their images under π. By Lemma 5.7, it suffices to prove that the étale topology of $\bar{\mathcal{G}}$ has a basis consisting of étale open-closed sets. Thus, given $\Gamma_0 \in \mathcal{G}$ and $H \in \mathrm{Open}(G)$ with $\Gamma_0 \leq H$, it suffices to find a G-invariant étale open-closed subset $\mathcal{U}_0$ of $\mathrm{Subgr}(G)$ with $\bar{\Gamma}_0 \in \bar{\mathcal{U}}_0 \subseteq \overline{\mathrm{Subgr}(H)}$. The construction of $\mathcal{U}_0$ breaks up into three parts.

Part A: an open normal subgroup of G. Since $1 \notin \mathrm{ÉtaleClosure}(\mathcal{G})$, there is $N_0 \in \mathrm{Open\ Normal}(G)$ which contains no $\Gamma \in \mathcal{G}$. Consider the étale open subset $\mathcal{H} = \bigcup_{g \in G} \mathrm{Subgr}(H^g)$ of $\mathrm{Subgr}(G)$ and the étale closed subset $\mathcal{H}' = \mathcal{G} \setminus \mathcal{H}$ of $\mathcal{G}$. Both $\mathcal{H}$ and $\mathcal{H}'$ are G-invariant. Since $\mathcal{G}$ is étale compact and Hausdorff, so is $\mathcal{H}'$. Each $\Gamma \in \mathcal{H}'$ is not contained in H, so $\Gamma \neq \Gamma_0$. Since $\mathcal{G} = \mathcal{G}_{max}$, $\Gamma_0 \not\leq \Gamma$. Therefore, there is $N_\Gamma \in \mathrm{OpenNormal}(G)$ with $N_\Gamma \Gamma_0 \not\leq N_\Gamma \Gamma$ and $N_\Gamma \Gamma \not\leq N_\Gamma H$.

The set $\mathrm{Subgr}(N_\Gamma \Gamma)$ is an étale open neighborhood of Γ in $\mathrm{Subgr}(G)$. Since $\mathcal{H}'$ is étale compact, there are $\Delta_1, \ldots, \Delta_m \in \mathcal{H}'$ with $\mathcal{H}' \subseteq \bigcup_{i=1}^{m} \mathrm{Subgr}(N_{\Delta_i} \Delta_i)$. Then $N = N_0 \cap \bigcap_{i=1}^{m} N_{\Delta_i}$ is an open normal subgroup of G, $N\Gamma \neq N$ for each $\Gamma \in \mathcal{G}$, and $N\Gamma_0 \not\leq N\Gamma$, $N\Gamma \not\leq NH$ for each $\Gamma \in \mathcal{H}'$.

Part B: $\mathcal{G}$-embedding problem for G with local data. Put $A = G/N$ and let $\varphi : G \to A$ be the quotient map. By Part A,

(4a) $\varphi(\Gamma) \neq 1$ for each $\Gamma \in \mathcal{G}$,

(4b) $\varphi(\Gamma_0) \not\leq \varphi(\Gamma)$ and $\varphi(\Gamma) \not\leq \varphi(H)$ for each $\Gamma \in \mathcal{H}'$.

Now choose $\Gamma_1, \ldots, \Gamma_n \in \mathcal{H}'$ such that $\varphi(\Gamma_1), \ldots, \varphi(\Gamma_n)$ represent the conjugacy classes in A of the maximal elements of $\varphi(\mathcal{H}')$. Let B_0 be an isomorphic copy of $\varphi(H)$ and B_i an isomorphic copy of $\varphi(\Gamma_i)$, $i = 1, \ldots, n$. Choose a positive integer $e \geq \mathrm{rank}(A)$. Put $B = \widehat{F}_e * \boxed{*}_{i=0}^n B_i$. Then B is finitely generated and there is an epimorphism $\alpha : B \to A$ which maps B_0 isomorphically onto $\varphi(H)$ and B_i isomorphically onto $\varphi(\Gamma_i)$, $i = 1, \ldots, n$. Let $\mathcal{B} = \mathrm{Con}(B_0, \ldots, B_n)$. Then

(5) $(\varphi : G \to A, \ \alpha : B \to A, \mathcal{B})$

is a $\mathcal{G}$-embedding problem for G with local data. By Lemma 5.3, there is a homomorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \varphi$ and $\gamma(\mathcal{G}) \subseteq \mathcal{B}$.

Part C: partition of $\mathcal{G}$. For each i let B_i' be an identical copy of B_i. Let $B' = \prod_{i=0}^n B_i'$ be the direct product of $B_0, \ldots, B_n$. Let $\beta : B \to B'$ be the epimorphism which maps $\widehat{F}_e$ to 1 and each B_i identically onto B_i'. Put $\gamma' = \beta \circ \gamma$. For each i put $\mathcal{U}_i = \{\Gamma \in \mathcal{G} \mid \gamma'(\Gamma) \leq B_i'\}$. Then $\mathcal{U}_i$ is a G-invariant étale open subset of $\mathcal{G}$. Therefore, $\bar{\mathcal{U}}_i$ is an étale open subset of $\bar{\mathcal{G}}$.

Claim 1. $\bar{\mathcal{G}} = \bigcup_{i=0}^m \bar{\mathcal{U}}_i$.

Indeed, since $\gamma(\mathcal{G}) \subseteq \mathcal{B}$, there are for each $\Gamma \in \mathcal{G}$ an i between 0 and m and a $b \in B$ with $\gamma(\Gamma) \leq B_i^b$. Hence, $\gamma'(\Gamma) \leq B_i'$ and $\bar{\Gamma} \in \bar{\mathcal{U}}_i$.

Moreover, by (4a), $\alpha(\gamma(\Gamma)) = \varphi(\Gamma) \neq 1$. Hence, $\gamma(\Gamma) \neq 1$. Since β is injective on B_i^b, we have $\gamma'(\Gamma) \neq 1$. Hence, $\gamma'(\Gamma) \not\leq B_j'$, so $\bar{\Gamma} \notin \bar{\mathcal{U}}_j$ for all $j \neq i$.

It follows that each $\bar{\mathcal{U}}_i$ is an étale open-closed subset of $\bar{\mathcal{G}}$; in particular, so is $\bar{\mathcal{U}}_0$.

Claim 2. $\bar{\Gamma}_0 \in \bar{\mathcal{U}}_0$.

Indeed, $\gamma(\Gamma_0) \leq B_i^b$ with $0 \leq i \leq n$ and $b \in B$ (Claim 1). Assume $i \geq 1$. Then $\varphi(\Gamma_0) = \alpha(\gamma(\Gamma_0)) \leq \alpha(B_i)^{\alpha(b)} = \varphi(\Gamma_i^g)$ for some $g \in G$ and $\Gamma_i^g \in \mathcal{H}'$ (by the choice of Γ_i). Hence, by (4b), $\varphi(\Gamma_0) \not\leq \varphi(\Gamma_i^g)$. This contradiction proves that $i = 0$, $\gamma'(\Gamma) \leq B_0'$, and $\bar{\Gamma}_0 \in \bar{\mathcal{U}}_0$.

Claim 3. $\bar{\mathcal{U}}_0 \subseteq \bar{\mathcal{H}}$.

Indeed, consider $\Gamma \in \mathcal{U}_0$. By Part B, $\gamma(\Gamma) \leq B_i^b$ with $0 \leq i \leq n$ and $b \in B$. If $i \geq 1$, then $\gamma'(\Gamma) \leq B_i'$ and $\bar{\Gamma} \in \bar{\mathcal{U}}_i$, in contradiction to Claim 1. Hence, $\gamma(\Gamma) \leq B_0^b$. Therefore, $\varphi(\Gamma^g) \leq \varphi(H)$ for some $g \in G$ with $\varphi(g) = \alpha(b)^{-1}$. Since $\mathcal{H}'$ is G-invariant, (4b) implies $\Gamma \notin \mathcal{H}'$. Consequently, $\Gamma \in \mathcal{H}$, as desired.

Finally, observe that $\bar{\mathcal{H}} = \overline{\mathrm{Subgr}(H)}$ to conclude the proof of the proposition. ∎

6 Projective group structures

The crucial step of going from solvability of finite $\mathcal{G}$-embedding problems for a profinite group G to solvability of arbitrary $\mathcal{G}$-embedding problems occurs in the category of "profinite group structures." We recall the definition of this concept from [**12**, Section 2].

A *profinite group structure* is a data $\mathbf{G} = (G, X, G_x)_{x \in X}$, where G is a profinite group, X is a profinite space on which G acts continuously from the right, and G_x is a closed subgroup of G, $x \in X$. These objects must satisfy the following conditions:

(1a) the map $x \mapsto G_x$ from X into $\mathrm{Subgr}(G)$ is étale continuous,

(1b) $G_{x^g} = G_x^g$ for all $x \in X$ and $g \in G$,

(1c) $\{g \in G \mid x^g = x\} \leq G_x$ for each $x \in X$.

The structure $\mathbf{G}$ is *finite* if both G and X are finite.

A *morphism* of group structures $\varphi : (G, X, G_x)_{x \in X} \to (H, Y, H_y)_{y \in Y}$ is a couple consisting of a homomorphism $\varphi : G \to H$ and a continuous map $\varphi : X \to Y$ such that $\varphi(G_x) \leq H_{\varphi(x)}$ and $\varphi(x^g) = \varphi(x)^{\varphi(g)}$ for all $x \in X$ and $g \in G$. The morphism φ is an *epimorphism* if $\varphi(G) = H$, $\varphi(X) = Y$, and for each $y \in Y$, there is $x \in X$ with $\varphi(G_x) = H_y$. We call φ a *cover* if $\varphi(G) = H$, $\varphi(X) = Y$, $\varphi : G_x \to H_{\varphi(x)}$ is an isomorphism for each $x \in X$, and $\varphi(x) = \varphi(x')$ implies $x^k = x'$ for some $k \in \mathrm{Ker}(\varphi)$.

An *embedding problem* for $\mathbf{G}$ is a pair $(\varphi : \mathbf{G} \to \mathbf{A}, \ \alpha : \mathbf{B} \to \mathbf{A})$, where $\mathbf{A}$ and $\mathbf{B}$ are profinite group structures, φ is a morphism, and α is a cover. A *solution* of the problem is a morphism $\gamma : \mathbf{G} \to \mathbf{B}$ satisfying $\alpha \circ \gamma = \varphi$. The problem is *finite* if both $\mathbf{B}$ and $\mathbf{A}$ are finite. We say $\mathbf{G}$ is *projective* if every finite embedding problem for $\mathbf{G}$ is solvable. Then every embedding problem for $\mathbf{G}$ is solvable [**12**, Proposition 4.2].

Lemma 6.1. Let $\mathbf{G} = (G, X, G_x)_{x \in X}$ be a projective group structure. Put $\mathcal{G} = \{G_x \mid x \in X\}$. Then G is strongly $\mathcal{G}$-projective. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof. By definition, $\mathcal{G}$ is the image of the compact space X under the étale continuous map $x \mapsto G_x$. Hence, $\mathcal{G}$ is étale compact.

Now consider a finite $\mathcal{G}$-embedding problem

(2) $(\varphi : G \to A, \ \alpha : B \to A, \mathcal{B})$

for G with local data. Replace A by $A_0 = \varphi(G)$, B by $B_0 = \alpha^{-1}(\varphi(G))$, and $\mathcal{B}$ by $\mathcal{B}_0 = \mathcal{B} \cap \mathrm{Subgr}(B_0)$, if necessary, to assume φ is surjective. By [**12**, Lemma 3.8], $\varphi : G \to A$ extends to an epimorphism φ of $\mathbf{G}$ onto a finite group structure $\mathbf{A} = (A, I, A_i)_{i \in I}$. Choose a set of representatives I_0 for the A-orbits of I. For each $i \in I_0$ there exists $x \in X$ with $\varphi(x) = i$ and $\varphi(G_x) = A_i$. The rigidity condition (2) of Section 5 gives $B' \in \mathcal{B}$ which α maps isomorphically onto A_i. Hence, by [**12**, Lemma 4.5], there is a group structure $\mathbf{B} = (B, J, B_j)_{j \in J}$ and $\alpha : B \to A$ extends to a cover $\alpha : \mathbf{B} \to \mathbf{A}$ with $B_j \in \mathcal{B}$ for all $j \in J$. In particular, $\varphi : J \to I$ is an epimorphism with finite fibers, so J is finite, hence $\mathbf{B}$ is finite.

Since $\mathbf{G}$ is projective, there is a morphism $\gamma : \mathbf{G} \to \mathbf{B}$ with $\alpha \circ \gamma = \varphi$. Its group component $\gamma : G \to B$ solves embedding problem (2). Consequently, G is strongly $\mathcal{G}$-projective. ∎

Lemma 6.2. Let G be a profinite group and $\mathcal{G}$ an étale compact subset of Subgr(G). Suppose $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$ is a partition of $\mathcal{G}$ into finitely many disjoint open-closed subsets. Then there exists an open normal subgroup N of G such that if $\varphi : G \to A$ is an epimorphism with $\text{Ker}(\varphi) \leq N$ and $\Gamma, \Gamma' \in \mathcal{G}$ satisfy $\varphi(\Gamma) \leq \varphi(\Gamma')$, then there is $i \in I$ with $\Gamma, \Gamma' \in \mathcal{G}_i$. □

Proof. Let $\Gamma \in \mathcal{G}$. There Γ belongs to a unique $\mathcal{G}_i$. Since $\mathcal{G}_i$ is open in $\mathcal{G}$, there is an open normal subgroup N_Γ of G with $\mathcal{G} \cap \text{Subgr}(\Gamma N_\Gamma) \subseteq \mathcal{G}_i$. Thus, $\mathcal{G} \subseteq \bigcup_{\Gamma \in \mathcal{G}} \text{Subgr}(\Gamma N_\Gamma)$. Since $\mathcal{G}$ is étale compact, there are $\Gamma_1, \ldots, \Gamma_m \in \mathcal{G}$ with $\mathcal{G} \subseteq \bigcup_{j=1}^{m} \text{Subgr}(\Gamma_j N_{\Gamma_j})$.

For each $1 \leq j \leq m$ there is a unique $i(j) \in I$ with $\mathcal{G} \cap \text{Subgr}(\Gamma_j N_{\Gamma_j}) \subseteq \mathcal{G}_{i(j)}$. Put $N = \bigcap_{j=1}^{m} N_{\Gamma_j}$. Let $\varphi : G \to A$ be an epimorphism with $\text{Ker}(\varphi) \leq N$ and let $\Gamma, \Gamma' \in \mathcal{G}$ with $\varphi(\Gamma) \leq \varphi(\Gamma')$. Choose j between 1 and m with $\Gamma' \in \text{Subgr}(\Gamma_j N_{\Gamma_j})$. Then $\Gamma' \in \mathcal{G}_{i(j)}$. Hence, $\Gamma \leq \Gamma' \text{Ker}(\varphi) \leq \Gamma' N \leq \Gamma' N_{\Gamma_j} \leq \Gamma_j N_{\Gamma_j}$. Therefore, $\Gamma \in \mathcal{G}_{i(j)}$, as desired. ∎

Let $\mathbf{G} = (G, X, G_x)_{x \in X}$ be a group structure. Put $\mathcal{G} = \{G_x \mid x \in X\}$. We say $\mathbf{G}$ is *proper* if the map $x \mapsto G_x$ of X onto $\mathcal{G}$ is an étale homeomorphism.

Proposition 6.3. Let $\mathbf{G} = (G, X, G_x)_{x \in X}$ be a proper group structure. Let $\mathcal{G} = \{G_x \mid x \in X\}$. Suppose G is strongly $\mathcal{G}$-projective. Then $\mathbf{G}$ is projective. □

Proof. Consider a finite embedding problem

(3) $(\varphi : \mathbf{G} \to \mathbf{A}, \ \alpha : \mathbf{B} \to \mathbf{A})$

for $\mathbf{G}$ with $\mathbf{A} = (A, I, A_i)_{i \in I}$. The solution of this problem breaks up into three parts.

Part A: a partition of $\mathcal{G}$. Consider the partition $X = \bigcup_{i \in I} \varphi^{-1}(i)$ into open-closed sets. For each $i \in I$ let $\mathcal{G}_i = \{G_x \mid \varphi(x) = i\}$. Since the map $x \mapsto G_x$ is an étale homeomorphism, $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$ is a partition of $\mathcal{G}$ into étale open-closed sets. **Lemma 6.2** gives an open normal subgroup N of G such that if $\widehat{\varphi} : G \to \widehat{A}$ is an epimorphism with $\text{Ker}(\widehat{\varphi}) \leq N$, then

(4) $x, y \in X$ and $\widehat{\varphi}(G_x) \leq \widehat{\varphi}(G_y)$ imply $\varphi(x) = \varphi(y)$.

By [12, Lemma 3.8] there are a morphism $\bar{\varphi} : \widehat{\mathbf{A}} \to \mathbf{A}$ of finite group structures and an epimorphism $\widehat{\varphi} : \mathbf{G} \to \widehat{\mathbf{A}}$ such that $\varphi = \bar{\varphi} \circ \widehat{\varphi}$ and $\text{Ker}(\widehat{\varphi}) \leq N$. In particular, (4) holds.

The fiber product $\widehat{\mathbf{B}} = (\widehat{B}, \widehat{J}, \widehat{B}_j)_{j \in \widehat{J}} = \mathbf{B} \times_{\mathbf{A}} \widehat{\mathbf{A}}$ fits into a commutative diagram

$$
\begin{array}{ccc}
 & & \mathbf{G} \\
 & & \downarrow{\widehat{\varphi}} \\
\widehat{\mathbf{B}} & \xrightarrow{\widehat{\alpha}} & \widehat{\mathbf{A}} \\
\downarrow{\beta} & & \downarrow{\bar{\varphi}} \\
\mathbf{B} & \xrightarrow{\alpha} & \mathbf{A}
\end{array}
\tag{6.1}
$$

in which $\widehat{\alpha}$ is a cover [**12**, Lemma 2.12(c)]. Let $\widehat{\mathcal{B}} = \mathrm{Con}\,(\{\widehat{B}_j \mid j \in \widehat{J}\})$. Then $(\widehat{\varphi} : G \to \widehat{A},\ \widehat{\alpha} : \widehat{B} \to \widehat{A}, \widehat{\mathcal{B}})$ is an embedding problem for G with local data. Since G is strongly $\mathcal{G}$-projective, there exists a homomorphism $\widehat{\gamma} : G \to \widehat{B}$ such that $\widehat{\alpha} \circ \widehat{\gamma} = \widehat{\varphi}$ and

(5) for each $x \in X$ there is $j \in \widehat{J}$ with $\widehat{\gamma}(G_x) \leq \widehat{B}_j$.

Part B: the map $\widehat{\gamma} : X \to \widehat{J}$. Consider the open normal subgroup $K = \mathrm{Ker}(\widehat{\gamma})$ of G. For each $y \in X$, the open subgroup $G_y K$ of G contains $S_y = \{\sigma \in G \mid y^\sigma = y\}$. Also, $V_y = \{x \in X \mid G_x \leq G_y K\}$ of y is an open neighborhood of y in X which is $G_y K$-invariant. Indeed, if $\sigma \in G_y$, $\kappa \in K$, and $x \in V_y$, then

$$G_{x^{\sigma\kappa}} = G_x^{\sigma\kappa} \leq (G_y K)^{\sigma\kappa} = (G_y^\sigma K^\sigma)^\kappa = (G_y K)^\kappa = G_y K, \tag{6.2}$$

whence $x^{\sigma\kappa} \in V_y$.

By [**12**, Lemma 3.6], there are $y_1, \ldots, y_m \in X$ and open-closed subsets $X_1, \ldots, X_m$ of X such that the following hold for each k between 1 and m:

(6a) X_k is $G_{y_k} K$-invariant and $y_k \in X_k \subseteq V_{y_k}$,

(6b) $X = \bigcup_{k=1}^m \bigcup_{\tau \in T_k} X_k^\tau$, where $G = \bigcup_{\tau \in T_k} G_{y_k} K\tau$ and $1 \in T_k \subseteq G$.

Define $\widehat{\gamma} : X \to \widehat{J}$ as follows. For each k between 1 and m use (5) to choose $\widehat{\gamma}(y_k) \in \widehat{J}$ with $\widehat{\gamma}(G_{y_k}) \leq \widehat{B}_{\widehat{\gamma}(y_k)}$. Then let

(7) $\widehat{\gamma}(y^\tau) = \widehat{\gamma}(y_k)^{\widehat{\gamma}(\tau)}$ for all $y \in X_k$ and $\tau \in T_k$.

By (6b), $\widehat{\gamma} : X \to \widehat{J}$ is well defined. In addition,

(8) $\widehat{\gamma}$ is constant on each X_k^τ with $\tau \in T_k$.

Hence, by (6b), $\widehat{\gamma}$ is continuous.

Taking $\tau = 1$ in (7) gives $\widehat{\gamma}(y) = \widehat{\gamma}(y_k)$ for all $y \in X_k$. Hence, by (7),

(9) $\widehat{\gamma}(y^\tau) = \widehat{\gamma}(y)^{\widehat{\gamma}(\tau)}$ for all $y \in X_k$ and $\tau \in T_k$.

We claim that

(10) $\widehat{\gamma}(G_x) \leq \widehat{B}_{\widehat{\gamma}(x)}$ for every $x \in X$.

Indeed, $x = y^\tau$, where $y \in X_k$, $\tau \in T_k$. By (6a), $y \in V_{y_k}$, that is, $G_y \leq G_{y_k} K$, so $\widehat{\gamma}(G_y) \leq \widehat{\gamma}(G_{y_k}) \leq \widehat{B}_{\widehat{\gamma}(y_k)} = \widehat{B}_{\widehat{\gamma}(y)}$. By (9), $\widehat{\gamma}(y)^{\widehat{\gamma}(\tau)} = \widehat{\gamma}(x)$. Hence, $\widehat{\gamma}(G_x) = \widehat{\gamma}(G_y^\tau) = \widehat{\gamma}(G_y)^{\widehat{\gamma}(\tau)} \leq \widehat{B}_{\widehat{\gamma}(y)}^{\widehat{\gamma}(\tau)} = \widehat{B}_{\widehat{\gamma}(y)^{\widehat{\gamma}(\tau)}} = \widehat{B}_{\widehat{\gamma}(x)}$, as claimed.

We know that $\widehat{\alpha} \circ \widehat{\gamma} = \widehat{\varphi}$ on G. But we do not know that $\widehat{\alpha} \circ \widehat{\gamma} = \widehat{\varphi}$ on X. Therefore, we define $\gamma = \beta \circ \widehat{\gamma} : G \to B$ and $\gamma = \beta \circ \widehat{\gamma} : X \to J$ and prove directly that $\gamma : \mathbf{G} \to \mathbf{A}$ is a morphism which solves embedding problem (3).

Part C: the morphism $\gamma : \mathbf{G} \to \mathbf{B}$. An application of β on (8), (9), and (10) implies:

(11a) γ is constant on each X_k^τ with $\tau \in T_k$, so, by (6b), $\gamma : X \to J$ is continuous,

(11b) $\gamma(y^\tau) = \gamma(y)^{\gamma(\tau)}$ for all $y \in X_k$ and $\tau \in T_k$,

(11c) $\gamma(G_x) \leq B_{\gamma(x)}$ for every $x \in X$.

Claim 1. $\alpha \circ \gamma = \varphi$.

That $\alpha \circ \gamma = \varphi$ on G follows from the equality $\widehat{\alpha} \circ \widehat{\gamma} = \widehat{\varphi}$ on G. Consider therefore $x \in X$. Since $\widehat{\varphi} : \mathbf{G} \to \widehat{\mathbf{A}}$ is an epimorphism, there is $y \in X$ such that $\widehat{\varphi}(y) = \widehat{\alpha}(\widehat{\gamma}(x))$ and $\widehat{\varphi}(G_y) = \widehat{A}_{\widehat{\alpha}(\widehat{\gamma}(x))}$. By (10),

$$\widehat{\varphi}\big(G_x\big) = \widehat{\alpha}\big(\widehat{\gamma}\big(G_x\big)\big) \leq \widehat{\alpha}\big(\widehat{B}_{\widehat{\gamma}(x)}\big) = \widehat{A}_{\widehat{\alpha}(\widehat{\gamma}(x))} = \widehat{\varphi}\big(G_y\big). \tag{6.3}$$

By (4), $\varphi(x) = \varphi(y)$. In addition,

$$\varphi(y) = \bar{\varphi}\big(\widehat{\varphi}(y)\big) = \bar{\varphi}\big(\widehat{\alpha}\big(\widehat{\gamma}(x)\big)\big) = \alpha\big(\beta\big(\widehat{\gamma}(x)\big)\big) = \alpha\big(\gamma(x)\big). \tag{6.4}$$

Hence, $\varphi(x) = \alpha(\gamma(x))$, as claimed.

Claim 2. $\gamma(G_{y_k})$ is contained in the stabilizer $S_{\gamma(y_k)}$ of $\gamma(y_k)$ in B.

Let $j = \gamma(y_k)$. By Claim 1, $\alpha(j) = \varphi(y_k)$. By [12, Remark 2.1], $G_{y_k} = S_{y_k}$. Hence,

$$(12) \quad \alpha(\gamma(G_{y_k})) = \varphi(G_{y_k}) = \varphi(S_{y_k}) \leq S_{\varphi(y_k)} = S_{\alpha(j)}.$$

By (11c), $\gamma(G_{y_k}) \leq B_j$. Since $\alpha : \mathbf{B} \to \mathbf{A}$ is a cover, $\alpha : B_j \to A_{\alpha(j)}$ is an isomorphism that maps S_j onto $S_{\alpha(j)}$ [12, Lemma 2.2]. Therefore, by (12), $\gamma(G_{y_k}) \leq S_{\gamma(y_k)}$.

Claim 3. γ preserves the action.

We prove first that $\gamma(y^\sigma) = \gamma(y)^{\gamma(\sigma)}$ for all $y \in X_k$ and $\sigma \in G$. To this end we use (6b) to write $\sigma = \lambda\tau$ with $\lambda \in G_{y_k}K$ and $\tau \in T_k$. Then $\gamma(\lambda) \in \gamma(G_{y_k}K) = \gamma(G_{y_k})$. Hence, by Claim 2, $\gamma(y_k)^{\gamma(\lambda)} = \gamma(y_k)$. Whence, by (11a), $\gamma(y)^{\gamma(\lambda)} = \gamma(y)$. By (6a), $y^\lambda \in X_k^\lambda = X_k$. Hence, by (11a), $\gamma(y^\lambda) = \gamma(y)$, and by (11b), $\gamma((y^\lambda)^\tau) = \gamma(y^\lambda)^{\gamma(\tau)}$. Therefore, $\gamma(y^\sigma) = \gamma((y^\lambda)^\tau) = \gamma(y^\lambda)^{\gamma(\tau)} = \gamma(y)^{\gamma(\tau)} = \gamma(y)^{\gamma(\lambda)\gamma(\tau)} = \gamma(y)^{\gamma(\sigma)}$.

Now consider $x \in X_k^{\tau'}$ with $\tau' \in T_k$. Write $x = y^{\tau'}$ with $y \in X_k$. Let $g \in G$. By the preceding paragraph, $\gamma(x^g) = \gamma(y^{\tau'g}) = \gamma(y)^{\gamma(\tau'g)} = \gamma(y)^{\gamma(\tau')\gamma(g)} = \gamma(y^{\tau'})^{\gamma(g)} = \gamma(x)^{\gamma(g)}$, as claimed.

Thus, γ is a solution of embedding problem (3). ∎

Let G be a profinite group and $\mathcal{G}$ an étale profinite G-invariant subset of Subgr(G). Suppose $N_G(\Gamma) = \Gamma$ for each $\Gamma \in \mathcal{G}$. Choose a homeomorphic copy X of $\mathcal{G}$ and a homeomorphism $x \mapsto G_x$ of X onto $\mathcal{G}$. The action of G on $\mathcal{G}$ induces an action on X making $\mathbf{G} = (G, X, G_x)_{x \in X}$ a proper group structure. In this case we also refer to $(G, \mathcal{G})$ as a *proper group structure*. We call $(G, \mathcal{G})$ *projective* if $\mathbf{G}$ is projective.

Proposition 6.4. Let G be a profinite group and $\mathcal{G}$ an étale compact G-invariant subset of Subgr(G). Suppose $1 \notin \text{StrictClosure}(\mathcal{G})$ and G is strongly $\mathcal{G}$-projective. Then $(G, \mathcal{G}_{\max})$ is a proper projective group structure. □

Proof. By Lemma 5.6, $N_G(\Gamma) = \Gamma$ for each $\Gamma \in \mathcal{G}_{\max}$ and G is strongly $\mathcal{G}_{\max}$-projective. By Proposition 5.8, $\mathcal{G}_{\max}$ is étale profinite. It follows that $(G, \mathcal{G}_{\max})$ is a proper group structure. By Proposition 6.3, $(G, \mathcal{G}_{\max})$ is projective. ■

Remark 6.5 (relatively projective groups). Let G and $\mathcal{G}$ be as in Proposition 6.4. Then $\mathcal{G}_{\max}$ is étale profinite. Let Γ_1, Γ_2 be distinct groups in $\mathcal{G}_{\max}$. By Proposition 5.5, $\Gamma_1 \cap \Gamma_2 = 1$. Choose étale open-closed neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of Γ_1 and Γ_2 in $\mathcal{G}_{\max}$, respectively, with $\mathcal{G} = \mathcal{U}_1 \bigcup \mathcal{U}_2$. By [12, Lemma 2.3], the union of all Γ in $\mathcal{U}_i$ is a closed subset of G. Thus, $\mathcal{G}$ is separated in the sense of [9, Definition 3.1]. In addition, G is strongly $\mathcal{G}_{\max}$-projective. Consequently, G is projective relative to $\mathcal{G}$ in the sense of [9, Definition 4.2].

We interpret the notions of a "morphism" and a "cover" of proper group structures in terms of the pairs $(G, \mathcal{G})$: let $(H, \mathcal{H})$ and $(G, \mathcal{G})$ be proper group structures. Then a morphism $\varphi : (H, \mathcal{H}) \to (G, \mathcal{G})$ is just a homomorphism $\varphi : H \to G$ which maps $\mathcal{H}$ into Con($\mathcal{G}$). In other words, for each $\Delta \in \mathcal{H}$ there is $\Gamma \in \mathcal{G}$ with $\varphi(\Delta) \leq \Gamma$.

The morphism φ is a *cover* if the following hold:

(13a) $\varphi(H) = G$, $\varphi(\mathcal{H}) = \mathcal{G}$,

(13b) φ is injective on each $\Delta \in \mathcal{H}$,

(13c) if $\Delta, \Delta' \in \mathcal{H}$ and $\varphi(\Delta) = \Delta'$, then there exists $\kappa \in \text{Ker}(\varphi)$ with $\Delta^{\kappa} = \Delta'$.

A *subgroup structure* of $(H, \mathcal{H})$ is a proper group structure $(H_0, \mathcal{H}_0)$ with $H_0 \leq H$ and $\mathcal{H}_0 \subseteq \mathcal{H}$. Specializing [12, Corollary 4.3] to proper group structures gives the following result.

Proposition 6.6. Let $\varphi : (H, \mathcal{H}) \to (G, \mathcal{G})$ be a cover of proper group structures. Suppose $(G, \mathcal{G})$ is projective. Then $(H, \mathcal{H})$ has a subgroup structure $(H_0, \mathcal{H}_0)$ which φ maps isomorphically onto $(G, \mathcal{G})$. □

7   Big quotients

Let G be a profinite group and $\mathcal{G}$ a subset of Subgr(G). We have already mentioned in Section 5 that if G is strongly $\mathcal{G}$-projective, then G is $\mathcal{G}$-projective. We show in this section that the converse is also true if there are only finitely many isomorphism types of groups in $\mathcal{G}$ and they have a "system of big quotients."

Let $\mathcal{C}$ be a finite set of finitely generated profinite groups. Each profinite group Δ which is isomorphic to a group in $\mathcal{C}$ is of *type* $\mathcal{C}$. A set $\mathcal{G}$ of profinite groups is said to be of *type* $\mathcal{C}$ if each $H \in \mathcal{G}$ is of type $\mathcal{C}$.

Let G be a profinite group and $\mathcal{G}$ a subset of Subgr(G). For each $\Gamma \in \mathcal{C}$ let $\mathcal{G}_\Gamma = \{H \in \mathcal{G} \mid H \cong \Gamma\}$. We prove an analog of Lemma 5.3 for $\mathcal{G}$-projective groups.

Lemma 7.1. Let G be a profinite group and $\mathcal{G}$ a subset of Subgr(G) of type $\mathcal{C}$. Suppose G is $\mathcal{G}$-projective and

$$(1) \quad (\varphi : G \to A, \ \alpha : B \to A)$$

is a $\mathcal{G}$-embedding problem with A finite and rank(B) $\leq \aleph_0$. Then (1) is solvable. $\qquad \square$

Proof. There exists a descending sequence $\mathrm{Ker}(\alpha) = N_0 \geq N_1 \geq N_2 \geq \cdots$ of open normal subgroups of B with trivial intersection. Identify A with B/N_0 and α with the quotient map $B \to B/N_0$. Let $\varphi_0 = \varphi$ and $\alpha_0 = \alpha$. For each i and j with $j \geq i \geq 0$ let $\alpha_i : B \to B/N_i$ and $\alpha_{ji} : B/N_j \to B/N_i$ be the quotient maps.

Claim 1. Let $i \geq 0$ and let $\varphi_i : G \to B/N_i$ be a homomorphism such that $(\varphi_i : G \to B/N_i, \ \alpha_i : B \to B/N_i)$ is a $\mathcal{G}$-embedding problem for G. Then there is a homomorphism $\varphi_{i+1} : G \to B/N_{i+1}$ such that $\alpha_{i+1,i} \circ \varphi_{i+1} = \varphi_i$ and $(\varphi_{i+1} : G \to B/N_{i+1}, \ \alpha_{i+1} : B \to B/N_{i+1})$ is a $\mathcal{G}$-embedding problem for G.

Once the claim has been proved, we may inductively construct for each $i \geq 0$ a homomorphism $\varphi_{i+1} : G \to B/N_{i+1}$ with $\alpha_{i+1,i} \circ \varphi_{i+1} = \varphi_i$. The maps φ_i define a $\gamma \in \mathrm{Hom}(G, B)$ with $\alpha \circ \gamma = \varphi$.

Without loss we prove the claim for $i = 0$. To this end note that for each j, $(\varphi : G \to A, \ \alpha_{j,0} : B/N_j \to A)$ is a finite $\mathcal{G}$-embedding problem for G. Indeed, given $\Gamma \in \mathcal{G}$, there is a homomorphism $\gamma' : \Gamma \to B$ with $\alpha \circ \gamma' = \varphi|_\Gamma$. Thus, $\alpha_{j,0} \circ (\alpha_j \circ \gamma') = \varphi|_\Gamma$, as desired.

For each $\beta \in \mathrm{Hom}(G, B/N_j)$ let

$$\beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) = \left\{ (\beta \circ \psi_\Gamma)_{\Gamma \in \mathcal{C}} \mid \psi_\Gamma \in \mathrm{Hom}(\Gamma, G) \text{ for each } \Gamma \in \mathcal{C} \right\}. \tag{7.1}$$

This is a subset of $\prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, B/N_j)$. Since $\mathcal{C}$ is finite, each $\Gamma \in \mathcal{C}$ is finitely generated, and B/N_j is finite, $\prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, B/N_j)$ is finite. Hence, the collection of subsets

$$\mathcal{H}_j = \left\{ \beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) \mid \beta \in \mathrm{Hom}\left(G, B/N_j\right), \ \alpha_{j,0} \circ \beta = \varphi \right\} \tag{7.2}$$

of $\prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, B/N_j)$ is finite. Since G is $\mathcal{G}$-projective, $\mathcal{H}_j$ is nonempty.

The map $\beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) \mapsto \alpha_{j+1,j} \circ \beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G)$ maps $\mathcal{H}_{j+1}$ into $\mathcal{H}_j$. Hence, $\varprojlim \mathcal{H}_j \neq \varnothing$. Thus, there are homomorphisms $\beta_j : G \to B/N_j$ with $\alpha_{j,0} \circ \beta_j = \varphi$ and

(2) $\alpha_{j+1,j} \circ \beta_{j+1} \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) = \beta_j \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G), j = 0, 1, 2, \ldots$.

In particular, $\alpha_{1,0} \circ \beta_1 = \varphi$.

We prove that $(\beta_1 : G \to B/N_1, \ \alpha_1 : B \to B/N_1)$ is a $\mathcal{G}$-embedding problem for G. To this end consider $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then $H \cong \Gamma$. Hence, by (2),

(3) $\alpha_{j+1,j} \circ \beta_{j+1} \circ \mathrm{Hom}(H, G) = \beta_j \circ \mathrm{Hom}(H, G), j = 0, 1, 2, \ldots$.

Use (3) to inductively construct homomorphisms $\eta_j : H \to B/N_j, j = 1, 2, \ldots$, with $\eta_1 = \beta_1|_H$ and $\alpha_{j+1,j} \circ \eta_{j+1} = \eta_j$. The η_j's define a homomorphism $\eta : H \to B$ with $\alpha_1 \circ \eta = \beta_1$, as needed. This concludes the proof of the claim.   ∎

Lemma 7.2. Let $\mathcal{C}$ be a finite set of finitely generated profinite groups, G a profinite group, and $\mathcal{G}$ a subset of $\mathrm{Subgr}(G)$ of type $\mathcal{C}$. Consider a finite $\mathcal{G}$-embedding problem with local data for G

(4) $(\varphi : G \to A, \ \alpha : B \to A, \mathcal{B})$.

Then there are

(5a)   a positive integer e,

(5b)   a finite set $\{\Delta_\lambda \mid \lambda \in \Lambda\}$ of groups of type $\mathcal{C}$,

(5c)   an epimorphism $\beta : B^* = \widehat{F}_e * \text{⫿}_{\lambda \in \Lambda} \Delta_\lambda \to B$ such that,

(6) $(\varphi : G \to A, \ \alpha \circ \beta : B^* \to A)$

is a $\mathcal{G}$-embedding problem for G with A finite, $\mathrm{rank}(B^*) \leq \aleph_0$, and $\beta(\Delta_\lambda) \in \mathcal{B}$ for each $\lambda \in \Lambda$.   □

Proof.   The proof has two parts.

Part A: free product.   For each $\Gamma \in \mathcal{C}$ let Λ_Γ be the set of all homomorphisms $\lambda : \Gamma \to B$ satisfying:

(7) $\lambda(\Gamma) \in \mathcal{B}$ and there is an embedding $\varepsilon : \Gamma \to G$ such that $\alpha \circ \lambda = \varphi \circ \varepsilon$.

Since Γ is finitely generated and B is finite, Λ_Γ is a finite set. For each $\lambda \in \Lambda_\Gamma$ choose an isomorphic copy Δ_λ of Γ and an isomorphism $\delta_\lambda : \Delta_\lambda \to \Gamma$. Let $\Lambda = \bigcup_{\Gamma \in \mathcal{C}} \Lambda_\Gamma$. Then $\{\Delta_\lambda \mid \lambda \in \Lambda\}$ is a finite set of groups of type $\mathcal{C}$. Put $e = \mathrm{rank}(B)$. Choose an epimorphism $\beta_e : \widehat{F}_e \to B$. Then consider the free product $B^* = \widehat{F}_e * \text{⫿}_{\lambda \in \Lambda} \Delta_\lambda$. Let $\beta : B^* \to B$ be the unique epimorphism whose restriction to $\widehat{F}_e$ is β_e and to Δ_λ is $\lambda \circ \delta_\lambda$. By (7), $\beta(\Delta_\lambda) \in \mathcal{B}$ for all $\lambda \in \Lambda$.

Part B: $\mathcal{G}$-embedding problem.   Let $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then, there is an isomorphism $\theta : \Gamma \to H$. The definition of embedding problems with local data (**Section 5**) gives a homomorphism η of H into B such that $\eta(H) \in \mathcal{B}$ and $\alpha \circ \eta = \varphi \circ \iota$, where ι is the inclusion

$H \to G$. Put $\lambda = \eta \circ \theta$ and $\varepsilon = \iota \circ \theta$. Then $\alpha \circ \lambda = \varphi \circ \varepsilon$, so $\lambda \in \Lambda_\Gamma$. Thus, $\delta_\lambda^{-1} \circ \theta^{-1}$ maps H onto the subgroup Δ_λ of B^*. Furthermore, $\alpha \circ \beta \circ (\delta_\lambda^{-1} \circ \theta^{-1}) = \alpha \circ (\lambda \circ \delta_\lambda) \circ (\delta_\lambda^{-1} \circ \theta^{-1}) = \alpha \circ \lambda \circ \theta^{-1} = \alpha \circ \eta \circ \theta \circ \theta^{-1} = \varphi \circ \iota$. Therefore, (6) is a $\mathcal{G}$-embedding problem for G. $\blacksquare$

Lemma 7.3. Let G be a profinite group and $\mathcal{G}$ a subset of $\mathrm{Subgr}(G)$ of type $\mathcal{C}$. For each $\Gamma \in \mathcal{C}$ let $\bar{\Gamma}$ be a finite quotient of Γ. Suppose $\mathcal{G}_\Gamma$ is strictly closed in $\mathrm{Subgr}(G)$. Then G has an open normal subgroup N satisfying: for each $\Gamma \in \mathcal{C}$ and each $H \in \mathcal{G}_\Gamma$ the group $\bar{\Gamma}$ is a quotient of $H/H \cap N = HN/N$. $\square$

Proof. Let $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then $H \cong \Gamma$, so H has an open normal subgroup M_H with $H/M_H \cong \bar{\Gamma}$. Choose $N_H \in \mathrm{OpenNormal}(G)$ with $H \cap N_H \leq M_H$. Let $\mathcal{U}_H = \{H' \in \mathcal{G}_\Gamma \mid H'N_H = HN_H\}$. Then $\mathcal{U}_H$ is a strictly open neighborhood of H in $\mathcal{G}_\Gamma$. By assumption, $\mathcal{G}_\Gamma$ is strictly compact. Hence, there are $H_{\Gamma,1}, \ldots, H_{\Gamma,m(\Gamma)} \in \mathcal{G}_\Gamma$ with $\mathcal{G}_\Gamma = \bigcup_{i=1}^{m(\Gamma)} \mathcal{U}_{H_{\Gamma,i}}$.

Let $N = \bigcap_{\Gamma \in \mathcal{C}} \bigcap_{i=1}^{m(\Gamma)} N_{H_{\Gamma,i}}$. Consider $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then there is i with $HN_{H_{\Gamma,i}} = H_{\Gamma,i}N_{H_{\Gamma,i}}$. By construction, $N \leq N_{H_{\Gamma,i}}$. This gives a sequence $H/H \cap N \to H/H \cap N_{H_{\Gamma,i}} \cong HN_{H_{\Gamma,i}}/N_{H_{\Gamma,i}} = H_{\Gamma,i}N_{H_{\Gamma,i}}/N_{H_{\Gamma,i}} \cong H_{\Gamma,i}/H_{\Gamma,i} \cap N_{H_{\Gamma,i}} \to H_{\Gamma,i}/M_{H_{\Gamma,i}} \cong \bar{\Gamma}$, where the arrows are epimorphisms. Therefore, $\bar{\Gamma}$ is a quotient of $H/H \cap N$. $\blacksquare$

Definition 7.4 (big quotients). For each $\Gamma \in \mathcal{C}$ let $\bar{\Gamma}$ be a finite quotient of Γ. We say $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a *system of big quotients* for $\mathcal{C}$ if it has the following property: let e be a nonnegative integer, J a finite set, and for each $j \in J$ let Δ_j be a profinite group of type $\mathcal{C}$. Consider the free product $B^* = \widehat{F}_e * \boxasterisk_{j \in J} \Delta_j$. Let $\Gamma \in \mathcal{C}$ and let Δ be a closed subgroup of B^* with epimorphisms $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$. Then Δ is conjugate to a closed subgroup of some Δ_j and γ is an isomorphism.

Proposition 7.5. Let $\mathcal{C}$ be a finite set of finitely generated groups, G a profinite group, and $\mathcal{G}$ a G-invariant subset of $\mathrm{Subgr}(G)$ of type $\mathcal{C}$. Suppose $\mathcal{C}$ has a system of finite big quotients, $\mathcal{G}_\Gamma$ is strictly closed in $\mathrm{Subgr}(G)$ for each $\Gamma \in \mathcal{C}$, and G is $\mathcal{G}$-projective. Then

(a) G is strongly $\mathcal{G}$-projective,

(b) there is a homomorphism $\delta : G \to \boxasterisk_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ injectively into a conjugate of some $\Gamma \in \mathcal{C}$,

(c) suppose, in addition, $1 \notin \mathcal{C}$; then $(G, \mathcal{G}_{\max})$ is a proper projective group structure. $\square$

Proof of (a). By assumption, $\mathcal{G} = \bigcup_{\Gamma \in \mathcal{C}} \mathcal{G}_\Gamma$ is strictly closed. By [12, Remark 1.2], $\mathcal{G}$ is étale compact. It remains to solve a finite $\mathcal{G}$-embedding problem for G with local data (4). Let $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ be a system of finite big quotients for $\mathcal{C}$.

Part A. $\bar{\Gamma}$ is a quotient of $\varphi(H)$ for each $\Gamma \in \mathcal{C}$ and each $H \in \mathcal{G}$. Lemma 7.3 gives $N \in \mathrm{OpenNormal}(G)$ such that $\bar{\Gamma}$ is a quotient of $H/H \cap N$ for all $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. We may

assume $N \leq \mathrm{Ker}(\varphi)$, otherwise replace N with $N \cap \mathrm{Ker}(\varphi)$. Put $A' = G/N$. Let $\varphi' : G \to A'$ be the quotient map and $\bar{\varphi} : A' \to A$ the map induced by φ. Then $\bar{\Gamma}$ is a quotient of $\varphi'(H)$ for all $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Also, $\varphi = \bar{\varphi} \circ \varphi'$. Put $B' = B \times_A A'$ and let $\alpha' : B' \to A'$ and $\beta : B' \to B$ be the canonical projections.

Put $\mathcal{B}'_0 = \{B_0 \times_A \varphi'(H) \mid B_0 \in \mathcal{B}, H \in \mathcal{G}, \alpha(B_0) = \varphi(H)\}$ and $\mathcal{B}' = \mathrm{Con}(\mathcal{B}'_0)$. Then $\varphi'(\mathcal{G}) \subseteq \alpha'(\mathcal{B}')$. By definition, α is injective on each $B_0 \in \mathcal{B}$. Therefore, α' is injective on each $B'_0 \in \mathcal{B}'_0$, hence on each $B'_0 \in \mathcal{B}'$. Thus,

(8) $(\varphi' : G \to A', \alpha' : B' \to A', \mathcal{B}')$

is a finite $\mathcal{G}$-embedding problem with local data for $\mathcal{G}$.

Since $\beta(\mathcal{B}') \subseteq \mathcal{B}$, any solution γ' of (8) gives rise to a solution $\beta \circ \gamma'$ of embedding problem (4). Thus, replacing (4) with (8), if necessary, we may assume $\bar{\Gamma}$ is a quotient of $\varphi(H)$ for each $\Gamma \in \mathcal{C}$ and each $H \in \mathcal{G}$.

Part B: solving embedding problem (6). Lemma 7.2 gives a $\mathcal{G}$-embedding problem (6) for G with $\mathrm{rank}(B^*) \leq \aleph_0$, and $\beta(\Delta_\lambda) \in \mathcal{B}$ for each $\lambda \in \Lambda$. Since G is $\mathcal{G}$-projective, **Lemma 7.1** gives a homomorphism $\gamma^* : G \to B^*$ with $\alpha \circ \beta \circ \gamma^* = \varphi$. We claim that $\beta \circ \gamma^*$ solves (4).

Let $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Put $\Delta = \gamma^*(H)$. Then Δ is a subgroup of B^* as well as a quotient of Γ. Moreover, $\alpha(\beta(\Delta)) = \varphi(H)$. Therefore, by Part A, $\bar{\Gamma}$ is a quotient of Δ. By the definition of big quotients, Δ is conjugate to a closed subgroup of Δ_λ for some $\lambda \in \Lambda$, and γ^* is injective on H. Since $\beta(\Delta_\lambda) \in \mathcal{B}$, we have $\beta \circ \gamma^*(H) \in \mathcal{B}$.

Proof of (b). Consider embedding problem (4) with $B = A = 1$. The proof of (a) gives a homomorphism $\gamma^* : G \to \widehat{F}_e * {\Large *}_{\lambda \in \Lambda} \Delta_\lambda$ which maps each $H \in \mathcal{G}$ isomorphically into a conjugate of some Δ_λ. Define a homomorphism $\delta^* : \widehat{F}_e * {\Large *}_{\lambda \in \Lambda} \Delta_\lambda \to {\Large *}_{\Gamma \in \mathcal{C}} \Gamma$ which is the trivial map on $\widehat{F}_e$ and which maps each $\Delta_\lambda \in \mathcal{G}_\Gamma$ isomorphically onto Γ. Then $\delta = \delta^* \circ \gamma^*$ is a homomorphism of G into ${\Large *}_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ injectively into a conjugate of some $\Gamma \in \mathcal{C}$.

Proof of (c). Since $\Gamma \neq 1$ for each $\Gamma \in \mathcal{C}$ and $\mathcal{G}$ is strictly closed, $1 \notin \mathrm{StrictClosure}(\mathcal{G})$. By (a), G is strongly $\mathcal{G}$-projective. Therefore, by **Proposition 6.4**, $(G, \mathcal{G}_{\max})$ is a proper projective group structure. ∎

8   P-adically closed fields

We prove in the next section that any finite family of absolute Galois groups of P-adically closed fields has a system of big quotients. This section gives the necessary prerequisites for the proof.

Let p be a prime number. Denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$ by $\mathbb{Q}_{p,\mathrm{abs}}$. It is well defined up to an isomorphism.

Let (L, v) be a valued field. Call (L, v) *P-adic* if there is a prime number p satisfying these conditions:

(1a) the residue field $\bar{L}_v$ is finite, say with $q = p^f$ elements;

(1b) there is a $\pi \in L$ with a smallest positive value $v(\pi)$ in $v(L^\times)$; call π a *prime element* of (L, v);

(1c) there is a positive integer e with $v(p) = ev(\pi)$.

Refer to (p, e, f) as the *type* of (L, v) and to p as the *residue characteristic* of (L, v). We say (L, v) is *P-adically closed* if (L, v) admits no finite proper P-adic extension of the same type. Refer to a field L as *P-adically closed* if L admits a valuation v with (L, v) P-adically closed.

Remark 8.1 (comparison with former definitions). Prestel and Roquette [28] use "p-adically closed" instead of "P-adically closed of residue characteristic p." The same expression, "p-adically closed," is used in [11] for "P-adically closed field of type $(p, 1, 1)$."

The proposition below summarizes well-known facts about P-adically closed fields (see also [24, Section 1]). We use $F \equiv F'$ to denote elementary equivalence between fields and $(F, v) \equiv (F', v')$ to denote elementary equivalence between valued fields.

Proposition 8.2. (a) $\mathbb{Q}_p$ is P-adically closed of type $(p, 1, 1)$.

(b) Every P-adically closed valued field (L, v) is Henselian of characteristic 0.

(c) A field L is P-adically closed for at most one P-adic valuation.

(d) Suppose (K, v) is a P-adic field. Then (K, v) has a P-adically closed algebraic extension (L, w) of the same type. Call (L, w) a *P-adic closure* of (K, v). If (K, v) is discrete, then (L, w) is uniquely determined up to a K-isomorphism.

(e) In the notation of (d), L is minimal among all P-adically closed extensions of K.

(f) Let K be a subfield of a P-adically closed field L. Suppose K is algebraically closed in L. Then, K is a P-adically closed field of the same type as L and $K \equiv L$. Moreover, the restriction of the P-adic valuation of L to K is the P-adic valuation of K.

(g) A P-adic field (L, v) is P-adically closed if and only if (L, v) is Henselian and $v(L^\times)/nv(L^\times) \cong \mathbb{Z}/n\mathbb{Z}$ for every positive integer n.

(h) Suppose a field L is elementarily equivalent (in the language of fields) to a P-adically closed field F. Then L is a P-adically closed field of the same type as F. Moreover, let v (resp., w) be the P-adic valuation of L (resp., F). Then $(L, v) \equiv (F, w)$.

(i) Every finite extension of a P-adically closed field is a P-adically closed field of the same residue characteristic.

(j) Every P-adically closed field of residue characteristic p is elementarily equivalent to a finite extension of $\mathbb{Q}_{p,\mathrm{abs}}$ and also to a finite extension of $\mathbb{Q}_p$.

(k) Let L be a P-adically closed field. Then $\mathrm{Gal}(L)$ is a finitely generated prosolvable group.

(l) Let L be a P-adically closed field and $L_0 = L \cap \tilde{\mathbb{Q}}$. Then $\mathrm{res} : \mathrm{Gal}(L) \to \mathrm{Gal}(L_0)$ is an isomorphism.

(m) Let F be a P-adically closed field and F' an arbitrary field. Suppose $F' \equiv F$. Then $\mathrm{Gal}(F') \cong \mathrm{Gal}(F)$.

(n) Let F be a P-adically closed field and F' an arbitrary field. Suppose $\mathrm{Gal}(F') \cong \mathrm{Gal}(F)$. Then F' is a P-adically closed field of the same type as F. $\square$

Proof of (a). Let K be a finite proper extension of $\mathbb{Q}_p$. Then $[K : \mathbb{Q}_p] = ef$, where e is the ramification index and f is the residue degree [3, page 19, Proposition 3]. In particular, $e = v(p)$, where v is the unique normalized p-adic valuation of K. Also, the residue fields of $\mathbb{Q}_p$ and K are $\mathbb{F}_p$ and $\mathbb{F}_{p^f}$, respectively. Hence, $e > 1$ or $f > 1$. This proves (a).

Proof of (b). That (L, v) is Henselian is stated in [28, page 34, Theorem 3.1]. By (1a), $p = \mathrm{char}(\bar{L}_v)$ is a prime number. Hence, either $\mathrm{char}(L) = 0$ or $\mathrm{char}(L) = p$. By (1b) and (1c), $v(p) \neq 0$. Therefore, $\mathrm{char}(L) = 0$.

Proof of (c). Suppose v and v' are P-adic valuations of L. By (b), both are Henselian. Since their residue fields are not separably closed (by (1a)), Schmidt-Engler [16, Proposition 13.4] implies v is equivalent to v'.

Proof of (d). See [28, page 37, Theorem 3.2].

Proof of (e). Let (L_0, w_0) be a P-adically closed field with $K \subseteq L_0 \subseteq L$. Denote the unique P-adically closed valuation of L by w. By (b), both (L, w) and (L_0, w_0) are Henselian. Extend w_0 to a valuation w_1 of L. Then (L, w_1) is Henselian.

Assume w_1 is inequivalent to w. By (1a), both $\bar{L}_w$ and $\bar{L}_{w_1}$ are algebraic extensions of finite fields. Hence, none of them is the residue field of a nontrivial valuation of the other. This means w and w_1 are incomparable. Hence, by Schmidt-Engler [16, Proposition 13.4], $\bar{L}_w$ is separably closed, in contradiction to (1a). Therefore, w and w_1 are equivalent.

Thus, (L, w) extends (L_0, w_0) and (L_0, w_0) extends (K, v). Since (L, w) and (K, v) have the same type, also (L_0, w_0) has the same type. In particular, the residue characteristic of (L_0, w_0) is p. Since (L_0, w_0) is P-adically closed, $(L_0, w_0) = (L, w)$, as contended.

Proof of (f). Theorem 3.4 on [28, page 38] says K is P-adically closed of the same type as L. Moreover, the P-adic valuation of K is the restriction of the P-adic valuation of L. By [28, page 86, Theorem 5.1], $K \equiv L$.

Proof of (g). See [28, page 34, Theorem 3.1].

Proof of (h). Denote the P-adic valuation of F by v. Let (p, e, f) be the type of (F, v) and π a prime element F. Consider the Kochen operator

$$\gamma(X) = \frac{1}{\pi} \frac{X^q - X}{(X^q - X)^2 - 1},\tag{8.1}$$

with $q = p^f$. Put $\gamma(F) = \{\gamma(x) \mid x \in F \text{ and } x^q - x \neq \pm 1\}$. By [19, Lemma 4.1(iii)], $\gamma(F)$ is the valuation ring of v.

Since $L \equiv F$, $\gamma(L)$ is a valuation ring of L. Denote the corresponding valuation by w. Then $(L, w) \equiv (F, v)$. Since (F, v) satisfies (1), so does (L, w). Thus, (L, w) is P-adic.

Finally note that the conditions of (g) for a P-adic field to be P-adically closed are elementary in the language of valued fields. Consequently, (L, w) is P-adically closed.

Proof of (i). Let (L, v) be a P-adically closed field and L' a finite extension of L. Since L is Henselian, v uniquely extends to a valuation v of L' and (L', v) is Henselian. Since both $[\bar{L}'_v : \bar{L}_v]$ and $(v((L')^\times) : v(L^\times))$ are finite, (L', v) is a P-adic valued field and $v((L')^\times)/nv((L')^\times) \cong \mathbb{Z}/n\mathbb{Z}$ for every positive integer n. By (g), L' is P-adically closed.

Proof of (j). Let (L, v) be a P-adically closed field of residue characteristic p. By (b), $\mathrm{char}(L) = 0$. Put $L_0 = L \cap \tilde{\mathbb{Q}}$. By (f), L_0 is a P-adically closed field of the same type as L and $L_0 \equiv L$.

Let v_0 be the P-adic valuation of L_0. By (b), (L_0, v_0) is Henselian. Moreover, $v_0|_{\mathbb{Q}}$ is the p-adic valuation v_p of $\mathbb{Q}$. Hence, $\mathbb{Q}_{p,\mathrm{abs}} \subseteq L_0$. The relation $[K : \mathbb{Q}_{p,\mathrm{abs}}] = ef$ for finite extensions $K/\mathbb{Q}_{p,\mathrm{abs}}$ and the finiteness of the type of L_0 imply $[L_0 : \mathbb{Q}_{p,\mathrm{abs}}] < \infty$.

It follows that $F = L_0\mathbb{Q}_p$ is a finite extension of $\mathbb{Q}_p$ with $F \cap \tilde{\mathbb{Q}} = L_0$. By (i), F is P-adically closed. By (f), $F \equiv L_0$. Consequently, $F \equiv L$.

Proof of (k) and (l). Let L_0 and F be as in the proof of (j). By [14, Satz 3.6], $\mathrm{Gal}(F)$ is finitely generated (see also [18, page 2]). By [3, page 31, Corollary 1], $\mathrm{Gal}(F)$ is prosolvable. By Krasner's lemma, $\tilde{\mathbb{Q}}\mathbb{Q}_p = \tilde{\mathbb{Q}}_p$, so res : $\mathrm{Gal}(F) \to \mathrm{Gal}(L_0)$ is an isomorphism and $\mathrm{Gal}(L_0)$ is finitely generated. Since $L \equiv L_0$, every finite quotient G of $\mathrm{Gal}(L)$ is a finite quotient of $\mathrm{Gal}(L_0)$ (see [7, Remark 20.4.5(d)]). It follows from [7, Corollary 16.10.8] that the epimorphism res : $\mathrm{Gal}(L) \to \mathrm{Gal}(L_0)$ is an isomorphism. Consequently, $\mathrm{Gal}(L)$ is prosolvable and finitely generated.

Proof of (m). By (h), F' is a P-adically closed field. Let $F_0 = F \cap \tilde{\mathbb{Q}}$ and $F_0' = F' \cap \tilde{\mathbb{Q}}$. Then $F_0 \equiv F_0'$. Hence, $F_0 \cong F_0'$ [7, Lemma 26.6.3(b)]. By (l), $\mathrm{Gal}(F) \cong \mathrm{Gal}(F_0)$ and $\mathrm{Gal}(F') \cong \mathrm{Gal}(F_0')$. Therefore, $\mathrm{Gal}(F) \cong \mathrm{Gal}(F')$.

Proof of (n). Efrat [4, Theorem A] (in the case $p \neq 2$) and Koenigsmann [21, Theorm 4.1] (in general) construct a Henselian valuation ν' of F' with $\mathrm{char}(\bar{F}_{\nu'}') \neq 0$. It follows from [24, Assertion E9] that F' is P-adically closed. Moreover, if F is a finite extension of $\mathbb{Q}_p$, then so is F'. ∎

Lemma 8.3. For each prime number p the group $\mathrm{Gal}(\mathbb{Q}_p)$ is torsion-free. □

Proof. For $p \neq 2$ there is an $x \in \mathbb{Q}_p$ with $x^2 + p - 1 = 0$ (Hensel's lemma). For $p = 2$ there is an $x \in \mathbb{Q}_2$ with $x^2 + 7 = 0$ (use Hensel-Rychlik). Since both $p - 1$ and 7 are sums of squares (namely, $1^2 + \cdots + 1^2$), $\mathbb{Q}_p$ is not formally real. Therefore, by Artin-Schreier, $\mathrm{Gal}(\mathbb{Q}_p)$ is torsion-free. ∎

We summarize some well-known facts about real closed fields and algebraically closed fields of characteristic 0.

Remark 8.4 (algebraically closed and real closed fields). Let $\mathbb{F}$ be a finite extension of $\mathbb{R}$. Then either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Suppose $F \equiv \mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, then F is real closed and $\mathrm{Gal}(F) = \mathbb{Z}/2\mathbb{Z}$. If $\mathbb{F} = \mathbb{C}$, then F is algebraically closed and $\mathrm{Gal}(F)$ is trivial. Now let K be a subfield of F. Then $\mathrm{res} : \mathrm{Gal}(F) \to \mathrm{Gal}(F \cap \tilde{K})$ is an isomorphism and $F \equiv F \cap \tilde{K}$ [27, page 53, Corollary, 5.6 and page 51, Corollary 5.3]. Conversely, if F' extends F and $F' \equiv F$, then $F' \cap \tilde{F} = F$.

9   Construction of big quotients for classical groups

We say that a field $\mathbb{F}$ is *classical local of characteristic* 0 if $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ for some p. A profinite group G is *classical local of characteristic* 0 if G is isomorphic to the absolute Galois group of a classical local field of characteristic 0.

Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Put

$$\mathcal{C} = \big\{ \, \mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F} \big\}. \tag{9.1}$$

We have already mentioned that each $\Gamma \in \mathcal{C}$ is finitely generated and prosolvable (**Proposition 8.2(k)**). We use the next result together with **Lemma 8.3** to equip $\mathcal{C}$ with a system of big quotients.

Notation 9.1. Let p be a prime number and G a profinite group. Denote the maximal pro-p-quotient of G by $G(p)$. Let G_p be a p-Sylow subgroup of G.

Proposition 9.2. Let p, l be prime numbers and let $\mathbb{F}$ be a finite extension of $\mathbb{Q}_p$. Then $\mathbb{F}$ has a finite Galois extension $\mathbb{F}'$ with the following properties.

(a) Let Δ be a quotient of $\mathrm{Gal}(\mathbb{F})$ which has $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ as a quotient. Then, Δ_p is not a free pro-p-group and Δ_l is not a free pro-l-group.

(b) Let p' be a prime number, L an algebraic extension of $\mathbb{Q}_{p'}$, and $\gamma : \mathrm{Gal}(\mathbb{F}) \to \mathrm{Gal}(L)$ an epimorphism. Suppose there is an epimorphism $\beta : \mathrm{Gal}(L) \to \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$. Then $p = p'$, γ is an isomorphism, and $[\mathbb{F} : \mathbb{Q}_p] = [L : \mathbb{Q}_p]$. $\qquad\qquad\square$

Proof. By Proposition 8.2(k), $\mathrm{Gal}(\mathbb{F})$ is finitely generated.

Construction of $\mathbb{F}'$. Denote the compositum of all extensions of $\mathbb{F}$ of degree at most $\max(p - 1, l - 1)$ by E_0. In particular, E_0 contains the roots of unity ζ_p and ζ_l of orders p and l, respectively.

By [11, Lemma 11.1], E_0 has a finite extension E_1 with this property:

(1) $\mathrm{rank}(\mathrm{Gal}(E_1'/E_0)(p)) = \mathrm{rank}(\mathrm{Gal}(E_0)(p))$ for every Galois extension E_1' of E_0 which contains E_1.

Since $\zeta_p \in E_0$, $\mathrm{Gal}(E_0)(p)$ is not a free pro-p-group [20, page 96, Satz 10.3]. By [11, Lemma 11.2], E_0 has a proper finite p-extension $E_{2,p}$ with this property:

(2a) for every Galois extension E_2' of E_0 containing $E_{2,p}$, the group $\mathrm{Gal}(E_2'/E_0)$ is not a free pro-p-group.

Similarly, E_0 has a proper finite l-extension $E_{2,l}$ satisfying this:

(2b) for every Galois extension E_2' of E_0 containing $E_{2,l}$, the group $\mathrm{Gal}(E_2'/E_0)$ is not a free pro-l-group.

Put $E_2 = E_{2,p}E_{2,l}$.

Since $\mathrm{Gal}(\mathbb{Q}_p)$ is finitely generated, $\mathbb{Q}_p$ has only finitely many extensions of degree $[\mathbb{F} : \mathbb{Q}_p]$. Let $L_1, \ldots, L_k$ be all extensions of $\mathbb{Q}_p$ satisfying this:

(3) $[L_j : \mathbb{Q}_p] = [\mathbb{F} : \mathbb{Q}_p]$, $\mathrm{Gal}(L_j)$ is a quotient of $\mathrm{Gal}(\mathbb{F})$, but $\mathrm{Gal}(L_j) \not\cong \mathrm{Gal}(\mathbb{F})$, $j = 1, \ldots, k$ (k may be 0).

For each j choose a finite Galois extension F_j of $\mathbb{F}$ such that $\mathrm{Gal}(F_j/\mathbb{F})$ is not a quotient of $\mathrm{Gal}(L_j)$ [7, Proposition 16.10.7]. Let $\mathbb{F}'$ be the compositum of all extensions of $\mathbb{F}$ of degree at most $m = \max([E_i : \mathbb{F}], [F_j : \mathbb{F}])_{i=1,2;\ j=1,\ldots,k}$. Then $\mathbb{F}'$ is a finite Galois extension of $\mathbb{F}$ which contains $E_1, E_2, F_1, \ldots, F_k$.

Proof of (a). Let Δ be as in (a). Then $\Delta \cong \mathrm{Gal}(M/\mathbb{F})$ for some Galois extension M of $\mathbb{F}$. Since $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ is a quotient of Δ, there is a Galois extension $\mathbb{F}''$ of $\mathbb{F}$ in M with $\mathrm{Gal}(\mathbb{F}''/\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$. In particular, $\mathbb{F}''$ is the compositum of extensions of $\mathbb{F}$ of degree at most m, so $\mathbb{F}'' \subseteq \mathbb{F}'$. Since $[\mathbb{F}'' : \mathbb{F}] = [\mathbb{F}' : \mathbb{F}]$, we have $\mathbb{F}' = \mathbb{F}'' \subseteq M$.

Assume $\mathrm{Gal}(M/\mathbb{F})_p$ is a free pro-p-group. Then $\mathrm{Gal}(M/E_0)_p$ is also a free pro-p-group [7, Corollary 22.7.7], so $\mathrm{cd}_p\,\mathrm{Gal}(M/E_0) \leq 1$ [29, page 235, Theorem 6.5]. Therefore, by [29, page 255, Theorem 3.2], $\mathrm{Gal}(M/E_0)(p)$ is pro-p-free. This contradiction to (2a) proves that $\mathrm{Gal}(M/\mathbb{F})_p$ is not a free pro-p-group. Similarly, $\mathrm{Gal}(M/\mathbb{F})_l$ is not a free pro-l-group.

Proof of (b). Let p', L, γ, and β be as in (b). Denote the fixed field of $\mathrm{Ker}(\gamma)$ (resp., $\mathrm{Ker}(\beta \circ \gamma)$) in $\tilde{\mathbb{Q}}_p$ by N (resp., F'). Then F' is a Galois extension of $\mathbb{F}$ in N satisfying $\mathrm{Gal}(F'/\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$. In particular, F' is a compositum of extensions of $\mathbb{F}$ of degree at most m. Hence, $F' \subseteq \mathbb{F}'$. Since $[F' : \mathbb{F}] = [\mathbb{F}' : \mathbb{F}]$, we have $F' = \mathbb{F}'$.

By construction, $E_0 \subset E_{2,p} \subseteq \mathbb{F}' \subseteq N$ and $E_{2,p}/E_0$ is a proper p-extension, so p divides $[N : E_0]$. Let $E_0^{(p)}$ be the maximal pro-p-extension of E_0. Then $\mathrm{Gal}(N \cap E_0^{(p)}/E_0)$ is the maximal pro-p-quotient of $\mathrm{Gal}(N/E_0)$. Also, $E_{2,p} \subseteq N \cap E_0^{(p)}$. Hence, by (2a), $\mathrm{Gal}(N \cap E_0^{(p)}/E_0)$ is not a free pro-p-group. It follows from [29, page 255] that $\mathrm{cd}_p\,\mathrm{Gal}(N/E_0) > 1$.

Let L_0 be the fixed field of $\gamma(\mathrm{Gal}(E_0))$ in $\tilde{\mathbb{Q}}_{p'}$. Then $\mathrm{Gal}(N/E_0) \cong \mathrm{Gal}(L_0)$. Hence, by the preceding paragraph, $\mathrm{cd}_p\,\mathrm{Gal}(L_0) > 1$. This implies $p^\infty \nmid [L_0 : \mathbb{Q}_{p'}]$ [29, pages 291-292]. Also, L_0 is the compositum of all extensions of L of degree at most $\max(p-1, l-1)$. In particular, $\zeta_p \in L_0$. By (1) applied to N instead of to E_1' and by [23, Satz 4],

(4) $\mathrm{rank}(\mathrm{Gal}(L_0)(p)) = \mathrm{rank}(\mathrm{Gal}(N/E_0)(p)) = \mathrm{rank}(\mathrm{Gal}(E_0)(p)) = 2 + [E_0 : \mathbb{Q}_p]$.

In particular, $\mathrm{rank}(\mathrm{Gal}(L_0)(p)) \geq 3$. Hence, $p' = p$ and $\mathrm{rank}(\mathrm{Gal}(L_0)(p)) = 2 + [L_0 : \mathbb{Q}_p]$ [23, Satz 4]. It follows from (4) that $[E_0 : \mathbb{Q}_p] = [L_0 : \mathbb{Q}_p]$. Since $[E_0 : \mathbb{F}] = [L_0 : L]$, this implies $[\mathbb{F} : \mathbb{Q}_p] = [L : \mathbb{Q}_p]$.

Finally assume $\mathrm{Gal}(L) \not\cong \mathrm{Gal}(\mathbb{F})$. By assumption, $\mathrm{Gal}(L)$ is a quotient of $\mathrm{Gal}(\mathbb{F})$. Hence, $L = L_j$ with $1 \leq j \leq m$. By construction, $\mathrm{Gal}(F_j/\mathbb{F})$ is not a quotient of $\mathrm{Gal}(L)$. Since $\mathbb{F} \subseteq F_j \subseteq \mathbb{F}'$, this implies $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ is not a quotient of $\mathrm{Gal}(L)$, in contradiction to our assumption. Thus, $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(L)$. Consequently, by [7, Proposition 16.10.6(a)], γ is an isomorphism. ∎

The following result gives sufficient conditions for a finite set $\mathcal{C}$ of finitely generated profinite groups to have a system of big quotients (Definition 7.4).

Proposition 9.3. Let $\mathcal{C}$ be a finite set of finitely generated profinite groups. Suppose each $\Gamma \in \mathcal{C}$ is finite or prosolvable. For each infinite $\Gamma \in \mathcal{C}$ let $\bar{\Gamma}$ be a finite quotient of Γ, and for each finite $\Gamma \in \mathcal{C}$ let $\bar{\Gamma} = \Gamma$. Suppose there exists a prime number l such that for every infinite $\Gamma \in \mathcal{C}$ and every profinite group Δ with epimorphisms $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$ the following hold:

(5a) Γ_l is torsion-free;

(5b) Δ_l is not a free pro-l-group;

(5c) there is a prime number $p \neq l$ such that Δ_p is not a free pro-p-group;

(5d) if Δ is isomorphic to a subgroup of some $\Gamma' \in \mathcal{C}$, then γ is an isomorphism.

Then $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a system of big quotients for $\mathcal{C}$.   □

Proof. Let $\{\Delta_j \mid j \in J\}$ be a finite collection of profinite groups of type $\mathcal{C}$ and let e be a positive integer. Put $B^* = \widehat{F}_e * \text{⧓}_{j \in J} \Delta_j$. Let Γ be a group in $\mathcal{C}$ and Δ a closed subgroup of B^* with epimorphisms $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$.

First suppose Δ is finite. By [13, page 160, Theorem 1], Δ is conjugate to a closed subgroup of some Δ_j. By (5d), γ is an isomorphism.

Now suppose Δ is prosolvable. Let l be a prime number as in the proposition. By the first paragraph and (5a) (applied to Δ_j instead of to Γ), no element of B^* has order l. In particular, Δ_l is torsion-free. By [25, Theorem 2(2)], Δ is conjugate to a subgroup of some Δ_j. Again, by (5d), γ is an isomorphism.   ■

Lemma 9.4. Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Put $\mathcal{C} = \{\text{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$. Then $\mathcal{C}$ has a system of big quotients.   □

Proof. Let S be the set of all residue characteristics of $\mathbb{F} \in \mathcal{F}$. Choose a prime number l not in $S \cup \{2\}$. By Proposition 8.2(k) and Remark 8.4, each $\Gamma \in \mathcal{C}$ is finitely generated and prosolvable. Moreover, Γ_l is torsion-free (Lemma 8.3). Omit $\mathbb{C}$ from $\mathcal{F}$, if necessary, to assume $1 \notin \mathcal{C}$. For each $\Gamma \in \mathcal{C}$ choose $\mathbb{F} \in \mathcal{F}$ with $\Gamma \cong \text{Gal}(\mathbb{F})$. If Γ is finite (i.e., $\mathbb{F} = \mathbb{R}$), choose $\bar{\Gamma} = \Gamma$. If Γ is infinite and $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$, let $\bar{\Gamma} = \text{Gal}(\mathbb{F}'/\mathbb{F})$, where $\mathbb{F}'$ is the finite extension of $\mathbb{F}$ given by Proposition 9.2. We apply Lemma 8.3 to prove that $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a system of big quotients for $\mathcal{C}$ in the sense of Definition 7.4.

Since each $\Gamma \in \mathcal{C}$ is an absolute Galois group, Γ_l is torsion-free. Let $\mathbb{F} \in \mathcal{F}$, $\Gamma = \text{Gal}(\mathbb{F})$, Δ a profinite group, and $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$ epimorphisms. Let p be the residue characteristic of $\mathbb{F}$. By Proposition 9.2(a), Δ_p is not a free pro-p-group and Δ_l is not a free pro-l-group. Finally, suppose Δ is isomorphic to a subgroup of some $\Gamma' \in \mathcal{C}$. Identify Δ with $\text{Gal}(L)$, where L is an algebraic extension of $\mathbb{Q}_{p'}$. By Proposition 9.2, γ is an isomorphism. Thus, all parts of condition (4) hold. By Proposition 9.3, $\mathcal{C}$ has a system of big quotients.   ■

10   Spaces of classically local fields

Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 and let K be a field. For each $\mathbb{F} \in \mathcal{F}$ let $\text{AlgExt}(K, \mathbb{F})$ be the set of all algebraic extensions of K which are elementarily equivalent to $\mathbb{F}$. Put $\text{AlgExt}(K, \mathcal{F}) = \bigcup_{\mathbb{F} \in \mathcal{F}} \text{AlgExt}(K, \mathbb{F})$. Call a field K *pseudo-$\mathcal{F}$-closed* (abbreviated $P\mathcal{F}C$) if it is pseudo-$\text{AlgExt}(K, \mathcal{F})$-closed; that is, $V(K) \neq \varnothing$ for each smooth absolutely irreducible variety V satisfying $V(\mathbb{F}) \neq \varnothing$ for all $\mathbb{F} \in \text{AlgExt}(K, \mathcal{F})$.

We call a profinite group G *(strongly)* $\mathcal{F}$*-projective* if G is (strongly) $\mathcal{G}$-projective, where $\mathcal{G} = \bigcup_{\mathbb{F} \in \mathcal{F}} \{H \in \mathrm{Subgr}(G) \mid H \cong \mathrm{Gal}(\mathbb{F})\}$.

Our first main result is that "K is P$\mathcal{F}$C" implies "Gal(K) is strongly $\mathcal{F}$-projective." The only still missing ingredient of the proof is the strict closedness of $\mathrm{AlgExt}(K, \mathbb{F})$.

Lemma 10.1. Let K be a field and $\mathbb{F}$ a finite extension of $\mathbb{Q}_p$ or of $\mathbb{R}$. Then $\mathrm{AlgExt}(K, \mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(K)$.   □

Proof. We prove the theorem in the case where $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$. The same proof applies to the case where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. We only have to replace the references to Proposition 8.2 by references to Remark 8.4. By Proposition 8.2(f) and Remark 8.4, $\mathbb{F} \equiv \mathbb{F} \cap \tilde{\mathbb{Q}}$. Hence, we may replace $\mathbb{F}$ by $\mathbb{F} \cap \tilde{\mathbb{Q}}$.

So, assume without loss that $\mathbb{F}$ is a finite extension of $\mathbb{Q}_{p,\mathrm{alg}}$. By [7, Lemma 20.6.3], $\mathrm{AlgExt}(\mathbb{Q}, \mathbb{F}) = \{\mathbb{F}^\sigma \mid \sigma \in \mathrm{Gal}(\mathbb{Q})\}$. Thus, $\mathrm{AlgExt}(\mathbb{Q}, \mathbb{F})$ is the image of the strictly continuous map $\mathrm{Gal}(\mathbb{Q}) \to \mathrm{AlgExt}(\mathbb{Q})$ given by $\sigma \mapsto \mathbb{F}^\sigma$. Since both spaces are profinite, $\mathrm{AlgExt}(\mathbb{Q}, \mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(\mathbb{Q})$.

Now put $\mathcal{X} = \mathrm{AlgExt}(K, \mathbb{F})$. Consider $\mathsf{F} \in \mathrm{StrictClosure}(\mathcal{X})$. Then, for every finite Galois extension N of K, $\mathcal{W}_N = \{E \in \mathcal{X} \mid E \cap N = \mathsf{F} \cap N\} \neq \varnothing$. By Proposition 8.2(m), $\mathrm{Gal}(E) \cong \mathrm{Gal}(\mathbb{F})$ for each $E \in \mathcal{X}$. Therefore,

(1) every finite quotient of $\mathrm{Gal}(\mathsf{F})$ is a finite quotient of $\mathrm{Gal}(\mathbb{F})$.

Conversely, let $\mathsf{F}_0 = \mathsf{F} \cap \tilde{\mathbb{Q}}$. Observe that the map $\varphi : \mathrm{AlgExt}(K) \to \mathrm{AlgExt}(\mathbb{Q})$ given by $L \mapsto L \cap \tilde{\mathbb{Q}}$ is strictly continuous. It maps $\mathrm{AlgExt}(K, \mathbb{F})$ into $\mathrm{AlgExt}(\mathbb{Q}, \mathbb{F})$ (by Proposition 8.2(f)). Hence, $\mathsf{F}_0 = \varphi(\mathsf{F}) \in \mathrm{StrictClosure}(\mathrm{AlgExt}(\mathbb{Q}, \mathbb{F}))$. By the second paragraph of the proof, $\mathsf{F}_0 \in \mathrm{AlgExt}(\mathbb{Q}, \mathbb{F})$. Thus, $\mathsf{F}_0 \equiv \mathbb{F}$. Hence, by Proposition 8.2(m), $\mathrm{Gal}(\mathsf{F}_0) \cong \mathrm{Gal}(\mathbb{F})$, so $\mathrm{Gal}(\mathbb{F})$ is an image of $\mathrm{Gal}(\mathsf{F})$. In particular, every finite quotient of $\mathrm{Gal}(\mathbb{F})$ is a finite quotient of $\mathrm{Gal}(\mathsf{F})$. Combining with (1), we conclude that $\mathrm{Gal}(\mathsf{F})$ and $\mathrm{Gal}(\mathbb{F})$ have the same finite quotients. By Proposition 8.2(k), $\mathrm{Gal}(\mathbb{F})$ is finitely generated. Hence, by [7, Proposition 16.10.7(b)], $\mathrm{Gal}(\mathsf{F}) \cong \mathrm{Gal}(\mathbb{F})$. It follows that $\mathrm{Gal}(\mathsf{F})$ is finitely generated and isomorphic to $\mathrm{Gal}(\mathsf{F}_0)$. Since $\mathrm{res} : \mathrm{Gal}(\mathsf{F}) \to \mathrm{Gal}(\mathsf{F}_0)$ is surjective, it is bijective [7, Proposition 16.10.6(a)].

Next observe that the intersection of finitely many sets $\mathcal{W}_N$ contains a set of this form. Hence the intersection is nonempty. Therefore, there is an ultrafilter $\mathcal{D}$ of $\mathcal{X}$ which contains each $\mathcal{W}_N$. Put $\mathsf{F}^* = \prod_{E \in \mathcal{X}} E / \mathcal{D}$. By the fundamental property of ultraproducts, $\mathsf{F}^* \equiv \mathbb{F}$ [7, Corollary 7.7.1].

Embed F in F^* by mapping each $x \in \mathsf{F}$ onto the element $(x_E)/\mathcal{D}$, where x_E is x if $x \in E$ and $x_E = 0$ otherwise. Put $\mathsf{F}_0^* = \mathsf{F}^* \cap \tilde{\mathbb{Q}}$. Then, by Proposition 8.2(f), $\mathsf{F}_0 \equiv \mathbb{F} \equiv \mathsf{F}^* \equiv \mathsf{F}_0^*$. Also, $\mathsf{F}_0 \subseteq \mathsf{F}_0^*$. Let $x \in \mathsf{F}_0^*$. Put $f = \mathrm{irr}(x, \mathbb{Q})$. Since $\mathsf{F}_0 \equiv \mathsf{F}_0^*$, the number of roots of f in F_0 is equal to the number of roots of f in F_0^*. Hence, $x \in \mathsf{F}_0$. Therefore, $\mathsf{F}_0 = \mathsf{F}_0^*$. By Proposition 8.2(l),

res : $\mathrm{Gal}(\mathsf{F}^*) \to \mathrm{Gal}(\mathsf{F}_0)$ is an isomorphism. Hence, so is res : $\mathrm{Gal}(\mathsf{F}^* \cap \tilde{\mathsf{K}}) \to \mathrm{Gal}(\mathsf{F}_0)$. Since $\mathsf{F} \subseteq \mathsf{F}^* \cap \tilde{\mathsf{K}}$ and res : $\mathrm{Gal}(\mathsf{F}) \to \mathrm{Gal}(\mathsf{F}_0)$ is an isomorphism, we have $\mathsf{F} = \mathsf{F}^* \cap \tilde{\mathsf{K}}$. Again, by Proposition 8.2(f), $\mathsf{F} \equiv \mathsf{F}^*$. Consequently, $\mathsf{F} \in \mathfrak{X}$, as desired. $\blacksquare$

In order to formulate the first main result of this work we have to impose a certain restriction on $\mathcal{F}$.

Remark 10.2 (isomorphism of Galois groups of P-adic fields). We say $\mathcal{F}$ is *closed under Galois isomorphism* if for all classical local fields $\mathbb{F}$, $\mathbb{F}'$ the following holds:

(2) $\mathbb{F} \in \mathcal{F}$ and $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}')$ implies $\mathbb{F}' \in \mathcal{F}$.

Actually, by Remark 8.4, it suffices to impose condition (2) only for a finite extension $\mathbb{F}$ of $\mathbb{Q}_p$ and a finite extension $\mathbb{F}'$ of $\mathbb{Q}_{p'}$. By Proposition 9.2, $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}')$ implies $p = p'$ and $[\mathbb{F}' : \mathbb{Q}_p] = [\mathbb{F} : \mathbb{Q}_p]$. So, for each $\mathbb{F} \in \mathcal{F}$ there are only finitely many fields $\mathbb{F}'$ with $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(\mathbb{F})$.

Section 2 of [**17**] gives for each p examples of nonisomorphic extensions F and F' of $\mathbb{Q}_p$ with $\mathrm{Gal}(\mathsf{F}) \cong \mathrm{Gal}(\mathsf{F}')$. Indeed, [**17**, page 2, Theorem] and [**31**, page 281, Theorem] prove for arbitrary finite extensions F, F' of $\mathbb{Q}_p$ (if $p = 2$, the theorem assumes $\sqrt{-1} \in \mathsf{F}$) that $\mathrm{Gal}(\mathsf{F}) \cong \mathrm{Gal}(\mathsf{F}')$ if and only if $[\mathsf{F} : \mathbb{Q}_p] = [\mathsf{F}' : \mathbb{Q}_p]$ and $\mathsf{F} \cap \mathbb{Q}_{p,ab} = \mathsf{F}' \cap \mathbb{Q}_{p,ab}$. Here $\mathbb{Q}_{p,ab}$ is the maximal abelian extension of $\mathbb{Q}_p$.

Finally consider classical local fields F and F' of characteristic 0. Suppose F is elementarily equivalent to F'. Then F is isomorphic to F'. Indeed, we may assume F is a finite extension of $\mathbb{Q}_p$ and F' is a finite extension of $\mathbb{Q}_{p'}$. By Proposition 8.2(h), $p = p'$. Let $\mathsf{F}_0 = \mathsf{F} \cap \tilde{\mathbb{Q}}$ and $\mathsf{F}_0' = \mathsf{F}' \cap \tilde{\mathbb{Q}}$. By Proposition 8.2(f), $\mathsf{F}_0 \equiv \mathsf{F}$ and $\mathsf{F}_0' \equiv \mathsf{F}'$. Hence, by [**7**, Lemma 20.6.3], $\mathsf{F}_0 \cong \mathsf{F}_0'$. We may therefore assume $\mathsf{F}_0 = \mathsf{F}_0'$ and F_0 is a finite extension of $\mathbb{Q}_{p,alg}$. But then the isomorphism res : $\mathrm{Gal}(\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_{p,alg})$ (see Proposition 8.2(l)) maps both $\mathrm{Gal}(\mathsf{F})$ and $\mathrm{Gal}(\mathsf{F}')$ onto $\mathrm{Gal}(\mathsf{F}_0)$. Consequently, $\mathsf{F} = \mathsf{F}'$.

Lemma 10.3. Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Suppose $\mathcal{F}$ is closed under Galois isomorphism. Then for every field K,

(3) $\bigcup_{\mathbb{F} \in \mathcal{F}}\{\mathsf{F} \in \mathrm{AlgExt}(\mathsf{K}) \mid \mathsf{F} \equiv \mathbb{F}\} = \bigcup_{\mathbb{F} \in \mathcal{F}}\{\mathsf{F} \in \mathrm{AlgExt}(\mathsf{K}) \mid \mathrm{Gal}(\mathsf{F}) \cong \mathrm{Gal}(\mathbb{F})\}.$ $\square$

Proof. By Proposition 8.2(m), the left-hand side of (3) is contained in its right-hand side. Conversely, let $\mathsf{F} \in \mathrm{AlgExt}(\mathsf{K})$ and $\mathbb{F} \in \mathcal{F}$ be fields with $\mathrm{Gal}(\mathsf{F}) \cong \mathrm{Gal}(\mathbb{F})$. If $\mathbb{F}$ is real closed, then so is F and $\mathsf{F} \equiv \mathbb{F}$ (Remark 8.4). Otherwise, $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$ for some p. By Proposition 8.2(n), F is elementarily equivalent to a finite extension $\mathbb{F}'$ of $\mathbb{Q}_p$. Hence, by Proposition 8.2(m), $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(\mathsf{F}) \cong \mathrm{Gal}(\mathbb{F})$. Since $\mathcal{F}$ is closed under Galois isomorphism, $\mathbb{F}' \in \mathcal{F}$. Consequently, F belongs to the left-hand side of (3).

$\blacksquare$

Theorem 10.4. Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 not containing $\mathbb{C}$ which is closed under Galois isomorphism. Let K be a P$\mathcal{F}$C field. Put

$$\mathcal{G} = \bigcup_{\mathbb{F} \in \mathcal{F}} \big\{ \operatorname{Gal}(F) \mid F \in \operatorname{AlgExt}(K),\ \operatorname{Gal}(F) \cong \operatorname{Gal}(\mathbb{F}) \big\}. \tag{10.1}$$

Then $\operatorname{Gal}(K)$ is strongly $\mathcal{F}$-projective and $(\operatorname{Gal}(K), \mathcal{G}_{\max})$ is a proper projective group structure. □

Proof. Let $\mathcal{C} = \{\operatorname{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$. For each $\Gamma \in \mathcal{C}$ let

$$\mathcal{G}_\Gamma = \bigcup_{\substack{\mathbb{F} \in \mathcal{F} \\ \operatorname{Gal}(\mathbb{F}) \cong \Gamma}} \big\{ \operatorname{Gal}(F) \mid F \in \operatorname{AlgExt}(K, \mathbb{F}) \big\}. \tag{10.2}$$

By **Lemma 10.1**, $\mathcal{G}_\Gamma$ is strictly closed in $\operatorname{Subgr}(\operatorname{Gal}(K))$. By **Lemma 10.3**, $\mathcal{G} = \bigcup_{\Gamma \in \mathcal{C}} \mathcal{G}_\Gamma$, so $\mathcal{G}$ is strictly closed in $\operatorname{Subgr}(\operatorname{Gal}(K))$. Hence, by [**12**, Remark 1.2], $\mathcal{G}$ is étale compact. By **Proposition 4.1**, $\operatorname{Gal}(K)$ is $\mathcal{G}$-projective. By **Lemma 9.4**, $\mathcal{C}$ has a system of big quotients. It follows from **Proposition 7.5** that $\operatorname{Gal}(K)$ is strongly $\mathcal{G}$-projective and $(\operatorname{Gal}(K), \mathcal{G}_{\max})$ is a proper projective group structure. ∎

11   Realization of strongly projective groups as absolute Galois groups

The second main result of this work is a converse to **Theorem 10.4**. We consider again a finite set $\mathcal{F}$ of classical local fields of characteristic 0 not containing $\mathbb{C}$. We prove that each $\mathcal{F}$-projective group G which satisfies the group-theoretic analog of **Lemma 10.1** is isomorphic to $\operatorname{Gal}(K)$ for some P$\mathcal{F}$C field K. Moreover, we construct K equipped with a "field-valuation structure" satisfying the "block approximation condition." We recall the definitions of these concepts from [**12**].

A *field structure* is data $\mathbf{K} = (K, X, K_x)_{x \in X}$, where K is a field, X is a profinite space with a continuous action of $\operatorname{Gal}(K)$ on X, and for each $x \in X$, K_x is a separable algebraic extension of K satisfying the following conditions:

(1a) for each finite separable extension L of K the set $X_L = \{x \in X \mid L \subseteq K_x\}$ is open;

(1b) $K_{x^\sigma} = K_x^\sigma$ for all $x \in X$ and $\sigma \in \operatorname{Gal}(K)$;

(1c) $\{\sigma \in \operatorname{Gal}(K) \mid x^\sigma = x\} \subseteq \operatorname{Gal}(K_x)$.

Thus, $\operatorname{Gal}(\mathbf{K}) = (\operatorname{Gal}(K), X, \operatorname{Gal}(K_x))_{x \in X}$ is a group structure called the *absolute Galois structure* associated with $\mathbf{K}$ [**12**, Section 6].

Denote the set of all valuations, including the trivial one, of a field L by $\mathrm{Val}(L)$. A subbasis for the *patch topology* of $\mathrm{Val}(L)$ consists of all sets

$$
\begin{aligned}
\mathrm{Val}_a(K) &= \{v \in \mathrm{Val}(K) \mid v(a) > 0\}, \\
\mathrm{Val}'_a(K) &= \{v \in \mathrm{Val}(K) \mid v(a) \geq 0\}
\end{aligned}
\tag{11.1}
$$

with $a \in K$.

A *field-valuation structure* is a structure $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ satisfying the following conditions:

(2a) $(K, X, K_x)_{x \in x}$ is a field structure;

(2b) v_x is a valuation of K_x satisfying $v_{x^\sigma} = v_x^\sigma$ for all $x \in X$ and $\sigma \in \mathrm{Gal}(K)$; here $v_x^\sigma(u^\sigma) = v_x(u)$ for each $u \in K_x$;

(2c) for each finite separable extension L of K define a map $v_L : X_L \to \mathrm{Val}(L)$ by $v_L(x) = v_x|_L$; then v_L is continuous.

We call $\mathbf{K}$ *Henselian* if (K_x, v_x) is Henselian for each $x \in X$.

The *absolute Galois structure* associated with $\mathbf{K}$ is the same one associated with the underlying field structure, namely $\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), X, \mathrm{Gal}(K_x))_{x \in X}$. We call $\mathbf{K}$ *proper* if $\mathrm{Gal}(\mathbf{K})$ is proper.

Definition 11.1 (block approximation condition). A *block approximation problem* for a field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ is data $(V, X_i, L_i, a_i, c_i)_{i \in I_0}$ satisfying the following:

(3a) $(\mathrm{Gal}(L_i), X_i)_{i \in I_0}$ is a *special partition* of $\mathrm{Gal}(\mathbf{K})$: for each $i \in I_0$ the set X_i is open-closed in X, and for all $x \in X_i$, $L_i \subseteq K_x$, $\mathrm{Gal}(L_i) = \{\sigma \in \mathrm{Gal}(K) \mid X_i^\sigma = X_i\}$, and $X = \bigcup_{i \in I_0} \bigcup_{\rho \in R_i} X_i^\rho$, where R_i is any subset of $\mathrm{Gal}(K)$ satisfying $\mathrm{Gal}(K) = \bigcup_{\rho \in R_i} \mathrm{Gal}(L_i)\rho_i$;

(3b) V is a smooth affine variety over K;

(3c) $a_i \in V(L_i)$;

(3d) $c_i \in K^\times$.

A *solution* of the problem is a point $a \in V(K)$ with $v_x(a - a_i) > v_x(c_i)$ for all $i \in I_0$ and $x \in X_i$. We say $\mathbf{K}$ satisfies the *block approximation condition* if each block approximation problem for $\mathbf{K}$ is solvable.

The block approximation condition has several interesting consequences.

Proposition 11.2 [12, Proposition 12.3]**.** Let $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ be a Henselian field-valuation structure satisfying the block approximation condition.

(a) Put $\mathcal{K} = \{K_x \mid x \in X\}$. Then K is $P\mathcal{K}C$.

(b) Suppose $x_1, \ldots, x_n \in X$ lie in distinct $\mathrm{Gal}(K)$-orbits. Then $v_{x_1}|_K, \ldots, v_{x_n}|_K$ satisfy the weak approximation theorem.

(c) Suppose $x, y \in X$ lie in distinct $\mathrm{Gal}(K)$-orbits. Then $v_x|_K$ and $v_y|_K$ are independent.

(d) Suppose X has more than one $\mathrm{Gal}(K)$-orbit. Then the trivial valuation is not in $\nu_K(X)$.

(e) For each $x \in X$, K is v_x-dense in K_x.

(f) (K_x, v_x) is a Henselian closure of $(K, v_x|_K)$. □

Theorem 11.3. Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 and G an $\mathcal{F}$-projective group. Let

(4) $\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$, $\mathcal{G} = \mathrm{Subgr}(G, \mathcal{C}) = \bigcup_{\Gamma \in \mathcal{C}} \mathrm{Subgr}(G, \Gamma)$.

Suppose

(5a) $\mathbb{C} \notin \mathcal{F}$,

(5b) $\mathcal{F}$ is closed under Galois isomorphism,

(5c) $\mathrm{Subgr}(G, \Gamma)$ is strictly closed in $\mathrm{Subgr}(G)$ for each $\Gamma \in \mathcal{C}$.

Then there is a proper field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ such that

(6a) $\mathbf{K}$ satisfies the block approximation condition,

(6b) there is an isomorphism $\varphi : (G, \mathcal{G}_{\max}) \to \mathrm{Gal}(\mathbf{K})$; in particular, $G \cong \mathrm{Gal}(K)$,

(6c) $\{K_x \mid x \in X\} = \mathrm{AlgExt}(K, \mathcal{F})_{\min}$,

(6d) K is $\mathrm{P}\mathcal{F}\mathrm{C}$. □

Proof. By **Lemma 9.4**, $\mathcal{C}$ has a system of finite big quotients. By **Proposition 8.2**(k), each $\Gamma \in \mathcal{C}$ is finitely generated and prosolvable. Finally, by assumption, G is $\mathcal{G}$-projective. Hence, by **Proposition 7.5**, $\mathbf{G} = (G, \mathcal{G}_{\max})$ is a proper projective group structure. Moreover, **Proposition 7.5** gives a homomorphism $\delta : G \to \circledast_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ injectively into a conjugate of some $\Gamma \in \mathcal{C}$. By assumption, each $\Gamma \in \mathcal{C}$ is the absolute Galois group of a Henselian algebraic extension of $\mathbb{Q}$ or a real closure of $\mathbb{Q}$. Therefore, by [**8**, Theorem 10.1], we may identify $\circledast_{\Gamma \in \mathcal{C}} \Gamma$ with $\mathrm{Gal}(D)$ for some algebraic extension field D of $\mathbb{Q}$. Let E be the fixed field of $\delta(G)$ in $\tilde{\mathbb{Q}}$. Then $\delta : G \to \mathrm{Gal}(E)$ is an epimorphism of profinite groups which extends to a cover $\delta : \mathbf{G} \to \mathrm{Gal}(\mathbf{E})$ of group structures, with $\mathbf{E}$ being a field structure whose underlying field is E. Indeed, $\mathbf{E}$ is the field structure associated to the quotient structure $(G, \mathcal{G}_{\min})/\mathrm{Ker}(\delta)$ [**12**, Example 2.5]. Note that $\mathrm{Gal}(\mathbf{E})$ need not be proper.

Put $X = \mathcal{G}_{\max}$. By [**12**, Theorem 15.4], there is a proper Henselian field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ which satisfies the block approximation condition and there is an isomorphism $\varphi : \mathbf{G} \to \mathrm{Gal}(\mathbf{K})$ such that

(7a) $E = K \bigcap \tilde{\mathbb{Q}}$, $E_{\delta(x)} = K_x \cap \tilde{\mathbb{Q}}$, and ν_x is trivial on $E_{\delta(x)}$ for all $x \in X$,

(7b) $\mathrm{res}_{\tilde{K}/\tilde{E}} \circ \varphi = \delta$.

By Lemma 10.3, $F \in \mathrm{AlgExt}(K, \mathcal{F})$ if and only if $\mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})$ for some $\mathbb{F} \in \mathcal{F}$. Therefore the isomorphism $\varphi : \mathbf{G} \to \mathrm{Gal}(\mathbf{K})$ establishes, via Galois correspondence, a bijection of $\mathrm{Subgr}(G, \mathcal{C})$ onto $\mathrm{AlgExt}(K, \mathcal{F})$ which maps $\mathrm{Subgr}(G, \mathcal{C})_{\max}$ onto $\mathrm{AlgExt}(K, \mathcal{F})_{\min}$. This proves (6c).

By Proposition 11.2(a), K is pseudo-$\{K_x \mid x \in X\}$-closed. Therefore, by (6c), K is P$\mathcal{F}$C. ∎

Problem 11.4. Is it possible to remove the condition "$\mathcal{F}$ is closed under Galois isomorphism" from Theorem 11.3? □

Acknowledgment

References

[1]   J. Ax, *A metamathematical approach to some problems in number theory*, 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, NY, 1969), American Mathematical Society, Rhode Island, 1971, pp. 161–190.

[2]   G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York, 1972.

[3]   J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, London, 1967, proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union.

[4]   I. Efrat, *A Galois-theoretic characterization of $\mathfrak{p}$-adically closed fields*, Israel J. Math. **91** (1995), no. 1–3, 273–284.

[5]   Y. L. Ershov, *Projectivity of absolute Galois groups of* RC_ζ^**-fields*, Algebra (Krasnoyarsk, 1993) (Y. L. Ershov et al., eds.), de Gruyter, Berlin, 1996, pp. 63–80.

[6]   ———, *Projectivity of the absolute Galois groups of multi-valued fields*, Siberian Adv. Math. **14** (2004), no. 1, 1–6, translation from Mat. Tr. **6** (2003), no. 1, 28–33.

[7]   M. D. Fried and M. Jarden, *Field Arithmetic*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 11, Springer, Berlin, 2005, revised and enlarged by Moshe Jarden.

[8]   W-D. Geyer, *Galois groups of intersections of local fields*, Israel J. Math. **30** (1978), no. 4, 382–396.

[9]   D. Haran, *On closed subgroups of free products of profinite groups*, Proc. London Math. Soc. (3) **55** (1987), no. 2, 266–298.

[10]   D. Haran and M. Jarden, *The absolute Galois group of a pseudo real closed field*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 3, 449–489.

[11]   ———, *The absolute Galois group of a pseudo p-adically closed field*, J. reine angew. Math. **383** (1988), 147–206.

[12]   D. Haran, M. Jarden, and F. Pop, *Projective group structures as absolute Galois structures with block approximation*, to appear in Mem. Amer. Math. Soc.

[13]   W. Herfort and L. Ribes, *Torsion elements and centralizers in free products of profinite groups*, J. reine angew. Math. **358** (1985), 155–161.

[14]   U. Jannsen, *Über Galoisgruppen lokaler Körper* [*On Galois groups of local fields*], Invent. Math. **70** (1982), no. 1, 53–69 (German).

[15]   M. Jarden, *Algebraic realization of p-adically projective groups*, Compositio Math. **79** (1991), no. 1, 21–62.

[16]   ———, *Intersections of local algebraic extensions of a Hilbertian field*, Generators and Relations in Groups and Geometries (Lucca, 1990) (A. Barlotti et al., eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 333, Kluwer Acadmic Publishers, Dordrecht, 1991, pp. 343–405.

[17]   M. Jarden and J. Ritter, *On the characterization of local fields by their absolute Galois groups*, J. Number Theory **11** (1979), no. 1, 1–13.

[18]   ———, *On the Frattini subgroup of the absolute Galois group of a local field*, Israel J. Math. **74** (1991), no. 1, 81–90.

[19]   M. Jarden and P. Roquette, *The Nullstellensatz over p-adically closed fields*, J. Math. Soc. Japan **32** (1980), no. 3, 425–460.

[20]   H. Koch, *Galoissche Theorie der P-Erweiterungen*, Springer, Berlin, 1970.

[21]   J. Koenigsmann, *From p-rigid elements to valuations (with a Galois-characterization of p-adic fields)*, J. reine angew. Math. **465** (1995), 165–182.

[22]   ———, *Relatively projective groups as absolute Galois groups*, Israel J. Math. **127** (2002), 93–129.

[23]   J. Neukirch, *Kennzeichnung der p-adischen und der endlichen algebraischen Zahlkörper*, Invent. Math. **6** (1969), 296–314 (German).

[24]   F. Pop, *Galoissche Kennzeichnung p-adisch abgeschlossener Körper* [*Galois characterization of p-adically closed fields*], J. reine angew. Math. **392** (1988), 145–175 (German).

[25]   ———, *On prosolvable subgroups of profinite free products and some applications*, Manuscripta Math. **86** (1995), no. 1, 125–135.

[26]   ———, *Classically projective groups and pseudo classically closed fields*, Valuation Theory and Its Applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Commun., vol. 33, American Mathematical Society, Rhode Island, 2003, pp. 251–283.

[27]   A. Prestel, *Lectures on Formally Real Fields*, Lecture Notes in Mathematics, vol. 1093, Springer, Berlin, 1984.

[28]   A. Prestel and P. Roquette, *Formally P-adic Fields*, Lecture Notes in Mathematics, vol. 1050, Springer, Berlin, 1984.

[29]   L. Ribes, *Introduction to Profinite Groups and Galois Cohomology*, Queen′s Papers in Pure and Applied Mathematics, no. 24, Queen′s University, Ontario, 1970.

[30]   L. Ribes and P. Zalesskii, *Profinite Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 40, Springer, Berlin, 2000.

[31]   J. Ritter, $\mathfrak{P}$-*adic fields having the same type of algebraic extensions*, Math. Ann. **238** (1978), no. 3, 281–288.

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