Quasi-formations*

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Abstract

We define quasi-formations, a generalization of formations of finite groups. For a quasi-formation $\mathcal{C}$ we construct an analogue of a free pro-$\mathcal{C}$ group.

Introduction

In the study of profinite groups (see [FrJ] or [RiZ]) one often considers pro-$\mathcal{C}$ groups for a class $\mathcal{C}$ of finite groups. In order to have a good theory, one has to impose several reasonable conditions on $\mathcal{C}$. The minimal requirement in the literature seems to be that $\mathcal{C}$ be a **formation**, that is, closed under quotients and fiber products. One then has free pro-$\mathcal{C}$ groups; these have nice properties (e.g. the embedding property) and play an important role in Galois theory: For instance, if K is a Hilbertian field with a projective absolute Galois group of rank at most $\aleph_0$, then $\mathrm{Gal}(K_{\mathrm{solv}}/K) \cong \hat{F}_\omega(\mathcal{C})$ where $\mathcal{C}$ is the formation of all solvable finite groups and K_{solv} is the maximal solvable extension of K ([FrJ, Corollary 24.8.4]). In particular, $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}/\mathbb{Q}_{\mathrm{ab}}) \cong \hat{F}_\omega(\mathcal{C})$ ([FrJ, Example 24.8.5(a)]).

Can one make a similar statement about $\mathrm{Gal}(K_{\mathrm{solv}})$ and, in particular, about $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$? In the latter case the problem is that the family $\mathcal{C}'$ of finite quotients of $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$ is not even a formation, since it is not closed under taking fiber products (Corollary 1.10). More generally, let $\mathcal{C}$ be a formation and let $\mathcal{C}'$ be the class of all finite groups which have no nontrivial quotients in $\mathcal{C}$. Then $\mathcal{C}'$ need not be closed under taking fiber products. Nevertheless (Lemma 3.2), $\mathcal{C}'$ satisfies the following weakening of the formation conditions:

Definition. A class of finite groups $\mathcal{C}'$ is a **quasi-formation** if it is closed under taking quotients and **compact** cartesian squares. The latter condition means that if

$$\begin{tikzcd} G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G_2 \arrow[d, "\varphi_2"] \\ G_1 \arrow[r, "\varphi_1"'] & A \end{tikzcd}$$

is a cartesian square of epimorphisms of finite groups with $G_1, G_2 \in \mathcal{C}'$, then $G \in \mathcal{C}'$, provided there is no subgroup $H \lneqq G$ with $\pi_i(H) = G_i, i = 1, 2$.

Another natural example of a quasi-formation is the class $\mathrm{Im}(G)$ of all finite quotients of a profinite group G with the embedding property (Proposition 4.3).

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For every formation $\mathcal{C}$ there exists a free pro-$\mathcal{C}$ group $F = \hat{F}_\omega(\mathcal{C})$. It is well known that F has the embedding property, $\mathrm{Im}(F) = \mathcal{C}$ and F with these properties is unique up to an isomorphism. The construction of F is based on the fact that $\mathcal{C}$ is a formation. We generalize it to quasi-formations: For every quasi-formation $\mathcal{C}'$ there exists a pro-$\mathcal{C}'$ group $E = \hat{E}(\mathcal{C}')$ with the embedding property and $\mathrm{Im}(E) = \mathcal{C}'$, and such E is unique up to an isomorphism (Theorem 4.4). Its rank is the supremum of the ranks of the groups in $\mathcal{C}'$ (in particular, E is of at most countable rank). We call it the **free pro-$\mathcal{C}'$ group**. We show (Theorem 5.5):

Theorem. *Let $\mathcal{C}'$ be the quasi-formation of finite groups which have no nontrivial quotients in a Melnikov formation $\mathcal{C}$. Then $\hat{E}(\mathcal{C}') \cong \hat{F}_\omega^{\mathcal{C}}$.*

In Section 6 we apply the theory of quasi-formations to countable fields for which a generalization of Shafarevich's conjecture is known to be true, namely, that the absolute Galois group of the maximal abelian extension of the field is free; e.g. the function fields of one variable over a countable algebraically closed field. For such a field K it immediately follows that the absolute Galois group of the maximal solvable extension of K is $\hat{E}(\mathcal{C}')$ (the free pro-$\mathcal{C}'$ group of rank $\aleph_0$), where $\mathcal{C}$ is the formation of all finite solvable groups.

Shafarevich's original conjecture asserts that the absolute Galois group of the maximal abelian extension of $\mathbb{Q}$ is free. In this case the conjecture is still open, but if it is true, then the absolute Galois group of the maximal solvable extension $\mathbb{Q}_{\mathrm{solv}}$ of $\mathbb{Q}$ is again $\hat{E}(\mathcal{C}')$.

It may be worthwhile to mention that the same conclusion regarding the absolute Galois group of $\mathbb{Q}_{\mathrm{solv}}$ follows also from a different conjecture, namely that $\mathbb{Q}_{\mathrm{solv}}$ is ample. This result appears in [Fri] and uses, among other ingredients, a generalization of the Hilbertianity notion.

Groups in this work are tacitly assumed to be profinite groups, their subgroups are assumed to be closed and all the homomorphisms between profinite groups are continuous.

1 Maximal pro-$\mathcal{C}$ quotients

We begin by recalling some well-known definitions and facts (cf. [FrJ] and [RiZ]).

Definition 1.1. Let $\mathcal{C}$ be a nonempty class of finite groups (this will always mean that $\mathcal{C}$ contains all the isomorphic images of the groups in $\mathcal{C}$).

(a) $\mathcal{C}$ is called a **formation** if it is closed under taking quotients and fiber products.

(b) Let $1 \to N \to G \to \bar{G} \to 1$ be a short exact sequence of finite groups. We call $\mathcal{C}$ **extension-closed** if from $N, \bar{G} \in \mathcal{C}$ it follows that $G \in \mathcal{C}$. If, in addition, the converse holds, i.e., from $G \in \mathcal{C}$ it follows that $N, \bar{G} \in \mathcal{C}$, we call $\mathcal{C}$ a **Melnikov formation**.

(c) A Melnikov formation $\mathcal{C}$ which is closed under taking subgroups is called a **full formation**.

(d) A **pro-$\mathcal{C}$ group** G is an inverse limit $G = \varprojlim G_i$, where $G_i \in \mathcal{C}$ for every i, such that the connecting homomorphisms $G_j \to G_i$ are epimorphisms.

Notice that a Melnikov formation is indeed a formation ([FrJ, p. 344]). We refer the reader to [FrJ, Example 17.3.3 and Remark 17.3.4] for examples of formations.

Definition 1.2. [FrJ, Definition 17.3.2] Let $\mathcal{C}$ be a formation and let G be a profinite group. Put

$$G^{\mathcal{C}} = \bigcap_{N \triangleleft G,\ G/N \in \mathcal{C}} N.$$

Lemma 1.3. *Let G, H be profinite groups and let $\mathcal{C}$ be a formation.*

(a) $G/G^{\mathcal{C}}$ is the largest pro-$\mathcal{C}$ quotient group of G, i.e., if $K \triangleleft G$ and G/K is a pro-$\mathcal{C}$ group, then $G^{\mathcal{C}} \leq K$.

(b) If $\varphi\colon G \to H$ is an epimorphism, then $\varphi(G^{\mathcal{C}}) = H^{\mathcal{C}}$.

(c) Suppose $\mathcal{C}$ is a full formation and let $\varphi\colon G \to H$ be a homomorphism. Then $\varphi(G^{\mathcal{C}}) \leq H^{\mathcal{C}}$.

(d) Assume $\mathcal{C}$ is a Melnikov formation. Then, if $G^{\mathcal{C}} \leq K \triangleleft G$, we have $G^{\mathcal{C}} = K^{\mathcal{C}}$.

Proof. See [RiZ, Lemma 3.4.1]. $\square$

Definition 1.4. Let $\mathcal{C}$ be a formation. Put

$$\mathcal{C}' = \{G \mid G \text{ is a finite group with } G^{\mathcal{C}} = G\}.$$

Clearly, a finite group is in $\mathcal{C}'$ if and only if it has no nontrivial quotients in $\mathcal{C}$.

Lemma 1.5. *Let $\mathcal{C}$ be a formation. Then*

(a) $\mathcal{C}'$ is closed under taking quotients.

(b) $\mathcal{C}'$ is extension-closed.

(c) If $\mathcal{C}$ is a Melnikov formation, then $\mathcal{C}' = \{G^{\mathcal{C}} \mid G \text{ is a finite group}\}$ and for every profinite group G we have that $G^{\mathcal{C}}$ is pro-$\mathcal{C}'$.

Proof. (a) If a group G has no nontrivial quotients in $\mathcal{C}$, then also a quotient of G has no nontrivial quotients in $\mathcal{C}$.

(b) Let $1 \to N \to G \to \bar{G} \to 1$ be a short exact sequence of finite groups and assume $N, \bar{G} \in \mathcal{C}'$. By Lemma 1.3(b), it induces the short exact sequence $1 \to NG^{\mathcal{C}}/G^{\mathcal{C}} \to G/G^{\mathcal{C}} \to \bar{G}/\bar{G}^{\mathcal{C}} \to 1$. As $\bar{G}/\bar{G}^{\mathcal{C}} = 1$, we have

$$N/(N \cap G^{\mathcal{C}}) \cong NG^{\mathcal{C}}/G^{\mathcal{C}} \cong G/G^{\mathcal{C}} \in \mathcal{C}.$$

But N has no nontrivial quotients in $\mathcal{C}$. Thus, $N/(N \cap G^{\mathcal{C}}) = 1$. Conclude that $G/G^{\mathcal{C}} = 1$.

(c) By Lemma 1.3(d), $(G^{\mathcal{C}})^{\mathcal{C}} = G^{\mathcal{C}}$ for every profinite group G. Now let $\varphi\colon G^{\mathcal{C}} \to A$ be an epimorphism with A a finite group. By Lemma 1.3(b) and (d), $A = \varphi(G^{\mathcal{C}}) = \varphi((G^{\mathcal{C}})^{\mathcal{C}}) = A^{\mathcal{C}}$. Hence, $A \in \mathcal{C}'$. It follows that $G^{\mathcal{C}}$ is pro-$\mathcal{C}'$. $\square$

Remark 1.6. Let $\mathcal{C}$ be a formation. It may well happen that $\mathcal{C}'$ is **trivial**, i.e., $\mathcal{C}' = \{1\}$. Obviously, this happens if and only if $\mathcal{C}$ contains all finite simple groups. If $\mathcal{C}$ is a Melnikov formation, then by [FrJ, Remark 17.3.4], this happens if and only if $\mathcal{C}$ is the class of all finite groups.

In general, if $\mathcal{C}$ is a formation, $\mathcal{C}'$ need not be a formation, for it need not be closed under taking fiber products. This is demonstrated in Example 1.9 below.

Proposition 1.7. *Let $1 \to P \to H \xrightarrow{\pi} S \to 1$ be a central extension of groups. Put $\hat{H} = H \times_S H$. Then*

(a) $\hat{H} \cong P \times H$.

(b) Let $\mathcal{C}$ be a formation such that $1 \neq P \in \mathcal{C}$ and $H^{\mathcal{C}} = H$. Then $\hat{H}^{\mathcal{C}} \neq \hat{H}$. In particular, $\mathcal{C}'$ is not a formation.

Proof. (a) Let Λ be the image of the diagonal map $H \to \hat{H}$ given by $x \mapsto (x, x)$. Clearly, $(1 \times P) \cap \Lambda = 1$. Moreover, $\hat{H} = (1 \times P)\Lambda$. Indeed, let $(h, h') \in \hat{H}$. Then $\pi(h) = \pi(h')$ and therefore $h' = gh$ for some $g \in P$. Thus, $(h, h') = (1, g)(h, h) \in (1 \times P)\Lambda$. As $1 \times P$ is central in $\hat{H}$, it follows that $\hat{H} = (1 \times P) \times \Lambda \cong P \times H$.

(b) By (a), $\hat{H}$ has a nontrivial quotient in $\mathcal{C}$. $\square$

Lemma 1.8. *Let p be a prime number and let $P = \mathbb{Z}/p\mathbb{Z}$. Then there exists a central extension $1 \to P \to H \to S \to 1$ of a finite simple group S by P such that H is perfect.*

Proof. By Dirichlet's Theorem ([Ser, p. 61]), there exists a prime number $q \geq 5$ such that $p|q-1$. Let $H = \mathrm{SL}_p(\mathbb{F}_q)$ and $S = \mathrm{PSL}_p(\mathbb{F}_q)$. By [Rot, Theorems 8.13 and 8.23], S is simple. By [Rot, Theorems 8.9 and 8.10], H is a central extension of S by P. By [Lan, Chapter XIII, Theorems 8.3 and 9.2], H is perfect. □

Example 1.9. If $\mathcal{C}$ is the class of all finite p-groups for a fixed prime p, or abelian or nilpotent or solvable groups, then $\mathcal{C}'$ is not a formation. Indeed, in each of these four cases there is a prime p such that $P = \mathbb{Z}/p\mathbb{Z} \in \mathcal{C}$. Consider a central extension $1 \to P \to H \to S \to 1$ as in Lemma 1.8. Since a nontrivial group in $\mathcal{C}$ has a nontrivial abelian quotient, the perfect group H is in $\mathcal{C}'$, that is, $H^{\mathcal{C}} = H$. By Proposition 1.7, $\mathcal{C}'$ is not a formation. (Notice that the last three cases produce the same class $\mathcal{C}'$.)

Corollary 1.10. *The class $\mathcal{C}$ of all finite images of $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$ is not a formation.*

Proof. By [Son, Theorem 3], $H = \mathrm{SL}_2(\mathbb{F}_5) \in \mathcal{C}$. Let $S = \mathrm{PSL}_2(\mathbb{F}_5)$ and $P = \mathbb{Z}/2\mathbb{Z}$. By Lemma 1.8, $1 \to P \to H \to S \to 1$ is a central extension. By Proposition 1.7(a), $H \times_S H$ has P as a quotient. But $\mathbb{Q}_{\mathrm{solv}}$ has no nontrivial solvable extensions. Thus, $H \times_S H$ is not realizable over $\mathbb{Q}_{\mathrm{solv}}$ and therefore $\mathcal{C}$ is not a formation. □

On the other hand, we show in Proposition 1.13 below that if $\mathcal{C}$ is a Melnikov formation generated by non-abelian finite simple groups ([FrJ, Remark 17.3.4]), then $\mathcal{C}'$ is a formation. To that end we need the following two lemmas:

Lemma 1.11. *Let G be a group and let K_1, K_2 and N be normal subgroups of G. Assume $K_1 \cap K_2 = 1$ and $NK_1 = NK_2 = G$. Then G/N is abelian.*

Proof. Let $\bar{x}, \bar{y} \in G/N$. By assumption, there exist $x \in K_1, y \in K_2$ that lift $\bar{x}, \bar{y}$. As K_1 and K_2 are normal, $[x,y] \in [K_1, K_2] \subseteq K_1 \cap K_2 = 1$. Hence, $[\bar{x}, \bar{y}] = 1$. □

Lemma 1.12. *Let $\pi_i \colon G_i \to A, i = 1,2$ be two epimorphisms of finite groups and let $G = G_1 \times_A G_2$. Let $\mathcal{C}$ be a Melnikov formation generated by finite non-abelian simple groups. Then $G^{\mathcal{C}} = G_1^{\mathcal{C}} \times_{A^{\mathcal{C}}} G_2^{\mathcal{C}}$.*

Proof. By Lemma 1.3(b), $\pi_i(G_i^{\mathcal{C}}) = A^{\mathcal{C}}$ for $i = 1,2$. Thus, $G' = G_1^{\mathcal{C}} \times_{A^{\mathcal{C}}} G_2^{\mathcal{C}}$ and $\bar{G} = (G_1/G_1^{\mathcal{C}}) \times_{A/A^{\mathcal{C}}} (G_2/G_2^{\mathcal{C}})$ are well defined. Moreover, the quotient maps $G_i \to G_i/G_i^{\mathcal{C}}$ for $i = 1,2$ induce an epimorphism $G \to \bar{G}$ with kernel G'. As $\bar{G} \in \mathcal{C}$, it follows from Lemma 1.3(a) that $G^{\mathcal{C}} \leq G'$. Since G' is a normal subgroup of G, by Lemma 1.3(d), $G^{\mathcal{C}} = (G')^{\mathcal{C}}$. Thus, it suffices to show that $G' = (G')^{\mathcal{C}}$. Replacing G, G_1, G_2 and A by $G', G_1^{\mathcal{C}}, G_2^{\mathcal{C}}$ and $A^{\mathcal{C}}$, respectively, we may therefore assume that $G_1 = G_1^{\mathcal{C}}$, $G_2 = G_2^{\mathcal{C}}$ and $A = A^{\mathcal{C}}$. Thus, $G = G'$. Suppose $G \neq G^{\mathcal{C}}$.

For $i = 1,2$ let K_i be the kernel of the canonical projection $G \to G_i$. Then $G^{\mathcal{C}} K_i = G$. Indeed, as a homomorphic image of $G/G^{\mathcal{C}} \in \mathcal{C}$, $G/G^{\mathcal{C}}K_i \in \mathcal{C}$. On the other hand, $G/G^{\mathcal{C}}K_i$ is a homomorphic image of $G/K_i \cong G_i$ which, by Definition 1.4, has no nontrivial quotients in $\mathcal{C}$. Hence, $G/G^{\mathcal{C}}K_i = 1$ and therefore $G^{\mathcal{C}}K_i = G$.

It follows from Lemma 1.11 that $G/G^{\mathcal{C}}$ is abelian, in contradiction to the assumption. □

Proposition 1.13. *Let $\mathcal{C}$ be the Melnikov formation generated by finite non-abelian simple groups. Then $\mathcal{C}'$ is a formation.*

Proof. By Lemma 1.5(a), $\mathcal{C}'$ is closed under taking quotients. It remains to show that if $G = G_1 \times_A G_2$ where $G_1, G_2 \in \mathcal{C}'$, then $G \in \mathcal{C}'$. For $i = 1,2$, $G_i^{\mathcal{C}} = G_i$. Furthermore, $A^{\mathcal{C}} = A$ since A is a quotient of G_1. By Lemma 1.12, $G^{\mathcal{C}} = G$, that is, $G \in \mathcal{C}'$. □

2 Compact cartesian squares

We recall the definition and basic properties of a cartesian square ([FrJ, Proposition 22.2.1 and Definition 22.2.2]):

Definition 2.1. A commutative diagram of epimorphisms of profinite groups

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "\alpha"'] & G_2 \arrow[d, "\pi_2"] \\ G_1 \arrow[r, "\pi_1"'] & A \end{tikzcd} \tag{1}$$

(abbreviated as (G, G_1, G_2, A)) is called a **cartesian square** if $G \cong G_1 \times_A G_2$, that is, whenever H is a profinite group and $\varphi\colon H \to G_1$, $\psi\colon H \to G_2$ are homomorphisms such that $\pi_1 \circ \varphi = \pi_2 \circ \psi$, there exists a unique homomorphism $\pi\colon H \to G$ such that $\alpha \circ \pi = \varphi$ and $\beta \circ \pi = \psi$.

The following is taken from [FrJ, p. 570]: Let G_1 and G_2 be profinite groups. Consider the collection $\mathcal{P}$ of all triples (π_1, π_2, A) with $\pi_i\colon G_i \to A$ an epimorphism, $i = 1, 2$. Partially order $\mathcal{P}$ by $(\pi_1, \pi_2, A) \le (\pi_1', \pi_2', A')$ if there exists an epimorphism $\pi\colon A' \to A$ which makes the diagram

$$\begin{tikzcd} G_1 \arrow[dr, "\pi_1'"] \arrow[ddr, "\pi_1"'] & & G_2 \arrow[dl, "\pi_2'"'] \arrow[ddl, "\pi_2"] \\ & A' \arrow[d, "\pi"] & \\ & A & \end{tikzcd} \tag{2}$$

commutative. Write $(\pi_1, \pi_2, A) \sim (\pi_1', \pi_2', A')$ if π is an isomorphism. Then $\le$ induces a partial ordering on the quotient set $\mathcal{P}(G_1, G_2) := \mathcal{P}/\sim$.

Let $\mathrm{pr}_i\colon G_1 \times G_2 \to G_i$ be the projection onto the ith coordinate, $i = 1, 2$. Define $\mathcal{H} = \mathcal{H}(G_1, G_2)$ to be the collection of all closed subgroups H of $G_1 \times G_2$ with $\mathrm{pr}_i(H) = G_i$, $i = 1, 2$. Partially order $\mathcal{H}$ by inclusion.

The definition of quasi-formations in the next section is based on the following definition:

Definition 2.2. A cartesian square (1) is called **compact** if one of the following equivalent ([FrJ, Lemma 24.4.1]) conditions is satisfied:

(a) $G_1 \times_A G_2$ is minimal in $\mathcal{H}(G_1, G_2)$.

(b) (π_1, π_2, A) is maximal in $\mathcal{P}(G_1, G_2)$.

Notice that if π_1 or π_2 is an isomorphism, then (b) holds, so (1) is compact.

Lemma 2.3. *Let* $\mathrm{I} = (G, G_1, B, B_1)$ *and* $\mathrm{II} = (B, B_1, G_2, A)$ *be two commutative diagrams of epimorphisms of profinite groups. Let* $\mathrm{III} = (G, G_1, G_2, A)$ *be the induced commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[r, "\xi"] \arrow[d, "\gamma"] & G_2 \arrow[d, "\eta"] \\ G_1 \arrow[r, "\delta"'] & B_1 \arrow[r, "\zeta"'] & A \end{tikzcd} \tag{3}$$

Then:

(a) *If two of the commutative squares are cartesian, then so is the third.*

(b) *Assume* I, II *and* III *are cartesian. Then* III *is compact if and only if both* I *and* II *are compact.*

Proof. (a) Assume both I and II are cartesian. We show that III is cartesian. Let H be a profinite group and let $\varphi\colon H \to G_1$ and $\psi\colon H \to G_2$ be homomorphisms such that $\zeta \circ \delta \circ \varphi = \eta \circ \psi$. As II is cartesian, there exists a unique homomorphism $\pi_2\colon H \to B$ such that $\xi \circ \pi_2 = \psi$ and $\gamma \circ \pi_2 = \delta \circ \varphi$. As I is cartesian, there exists a unique homomorphism $\pi\colon H \to G$ such that $\beta \circ \pi = \pi_2$ and $\alpha \circ \pi = \varphi$. It follows that $\alpha \circ \pi = \varphi$ and $\xi \circ \beta \circ \pi = \xi \circ \pi_2 = \psi$.

In order to see that π is unique, assume there exists $\pi'\colon H \to G$ such that $\xi \circ \beta \circ \pi' = \psi$ and $\alpha \circ \pi' = \varphi$. Define $\pi_2' = \beta \circ \pi'\colon H \to B$. Then $\xi \circ \pi_2' = \xi \circ \beta \circ \pi' = \psi$ and $\gamma \circ \pi_2' = \gamma \circ \beta \circ \pi' = \delta \circ \alpha \circ \pi' = \delta \circ \varphi$. As π_2 is unique with this property, $\pi_2' = \pi_2$. Thus, $\pi_2 = \beta \circ \pi'$. Since $\alpha \circ \pi' = \varphi$, it follows from the uniqueness of π applied to I that $\pi' = \pi$.

Assume both I and III are cartesian. By [FrJ, Lemma 22.2.4] we may assume that (3) is of the form

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/K_2 \arrow[r] \arrow[d] & G/L \arrow[d] \\ G/K_1 \arrow[r] & G/K_1K_2 \arrow[r] & G/K_1L \end{tikzcd}$$

with $K_1, K_2, L \triangleleft G$ such that $K_2 \le L$ and $K_1 \cap K_2 = K_1 \cap L = 1$. It suffices to show that $K_1K_2L = K_1L$ and $K_1K_2 \cap L = K_2$. The former equality is obvious. Let $g \in K_1$ and $h \in K_2$ such that $gh \in L$. As $K_2 \le L$, we have $g \in K_1 \cap L = 1$. It follows that $gh = h \in K_2$.

Assume both II and III are cartesian. Again, we may assume that (3) is of the form

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/K_2 \arrow[r] \arrow[d] & G/L \arrow[d] \\ G/K_1 \arrow[r] & G/M \arrow[r] & G/K_1L \end{tikzcd}$$

with $K_1, K_2, L, M \triangleleft G$ such that $K_2 \le L, K_1, K_2 \le M, K_1 \cap L = 1, M \cap L = K_2$ and $ML = K_1L$. It suffices to show that $K_1K_2 = M$ and $K_1 \cap K_2 = 1$. The latter equality is obvious. Let $g \in M$. Then $g = hl$ for some $h \in K_1, l \in L$. As $h \in M$, also $l \in M$ and therefore $l \in M \cap L = K_2$. It follows that $g \in K_1K_2$.

(b) Assume both I and II are compact. Let H be a subgroup of G such that $\alpha(H) = G_1$ and $\xi \circ \beta(H) = G_2$. Put $H' = \beta(H)$. Then $\xi(H') = \xi \circ \beta(H) = G_2$ and $\gamma(H') = \gamma \circ \beta(H) = \delta \circ \alpha(H) = B_1$. By the compactness of II, $H' = B$. By the compactness of I, $H = G$. Thus, III is compact.

Assume III is compact. Let H be a subgroup of G such that $\alpha(H) = G_1$ and $\beta(H) = B$. Then $\xi \circ \beta(H) = G_2$. By the compactness of III, $H = G$ and therefore I is compact. Now let H' be a subgroup of B such that $\xi(H') = G_2$ and $\gamma(H') = B_1$. Let $H = \beta^{-1}(H')$. Then $\xi \circ \beta(H) = G_2$ and $B_1 = \gamma \circ \beta(H)$. Denote $K_1 = \ker(\alpha)$ and $K_2 = \ker(\beta)$. By [FrJ, Lemma 22.2.4], $\ker(\gamma \circ \beta) = K_1K_2$. As $(\gamma \circ \beta)(H) = B_1$, we have $HK_1K_2 = G$. But $K_2 \le H$ and therefore $HK_1 = G$. Thus, $\alpha(H) = G_1$. By the compactness of III, $H = G$. Hence, $H' = \beta(H) = \beta(G) = B$. Conclude that II is compact. $\square$

The following notion helps to characterize compact cartesian squares:

Definition 2.4. An epimorphism $\varphi\colon A \to B$ of profinite groups is called **indecomposable** if whenever $\psi\colon A \to C$ and $\rho\colon C \to B$ are epimorphisms such that $\varphi = \rho \circ \psi$, then either ψ or ρ is an isomorphism. In particular, an isomorphism is indecomposable.

Lemma 2.5. *Consider a cartesian square* (1).

(a) If $\pi_2\colon G_2 \to A$ is indecomposable, then (1) is compact if and only if either π_2 is an isomorphism or there exists no epimorphism $\theta\colon G_1 \to G_2$ such that $\pi_1 = \pi_2 \circ \theta$.

(b) Assume both $\pi_1\colon G_1 \to A$ and $\pi_2\colon G_2 \to A$ are indecomposable and not isomorphisms. Then (1) is compact if and only if there exists no isomorphism $\theta\colon G_1 \to G_2$ such that $\pi_1 = \pi_2 \circ \theta$.

Proof. (a) Let $\theta\colon G_1 \to G_2$ be an epimorphism such that $\pi_1 = \pi_2 \circ \theta$ and assume that π_2 is not an isomorphism. Then $(\pi_1, \pi_2, A) < (\theta, \mathrm{id}, G_2)$ in $\mathcal{P}(G_1, G_2)$:

$$
\begin{tikzcd}
G_1 \arrow[dr, "\theta"] \arrow[ddr, "\pi_1"'] & & G_2 \arrow[dl, "\mathrm{id}"'] \arrow[ddl, "\pi_2"] \\
& G_2 \arrow[d, "\pi_2"] & \\
& A &
\end{tikzcd}
$$

Thus (1) is not compact.

Conversely, assume (1) is not compact. Then there is a commutative diagram (2) of epimorphisms with π not an isomorphism. As π_2 is indecomposable, π_2' is an isomorphism. Thus, π_2 is not an isomorphism. Put $\theta = (\pi_2')^{-1} \circ \pi_1'$. Then $\theta\colon G_1 \to G_2$ is an epimorphism such that $\pi_1 = \pi_2 \circ \theta$.

(b) If $\theta\colon G_1 \to G_2$ is an epimorphism such that $\pi_1 = \pi_2 \circ \theta$, then θ is an isomorphism, since π_1 is indecomposable and π_2 is not an isomorphism. Hence, the assertion follows from (a). $\square$

Lemma 2.7 below is used in the next section to characterize quasi-formations. The following technical lemma is needed in its proof:

Lemma 2.6. *Consider a cartesian square (1). If there exist either*

(a) epimorphisms $\delta_1\colon G_1 \to B$ and $\delta_2\colon B \to A$ such that $\delta_2 \circ \delta_1 = \pi_1$; or

(b) epimorphisms $\beta_1\colon G \to D$ and $\beta_2\colon D \to G_2$ such that $\beta_2 \circ \beta_1 = \beta$,

then there exists a commutative diagram with three cartesian squares

$$
\begin{tikzcd}
G \arrow[r, "\beta_1"'] \arrow[rr, bend left, "\beta"] \arrow[d, "\alpha"'] & D \arrow[r, "\beta_2"'] \arrow[d, "\pi"'] & G_2 \arrow[d, "\pi_2"] \\
G_1 \arrow[r, "\delta_1"] \arrow[rr, bend right, "\pi_1"'] & B \arrow[r, "\delta_2"] & A
\end{tikzcd}
\tag{4}
$$

Proof. (a) Let $D = B \times_A G_2$ and let $\beta_2\colon D \to G_2, \pi\colon D \to B$ be the canonical projections. Then (D, B, G_2, A) is a cartesian square. By Definition 2.1, there exists a unique homomorphism $\beta_1\colon G \to D$ such that $\beta_2 \circ \beta_1 = \beta$ and $\pi \circ \beta_1 = \delta_1 \circ \alpha$. As (G, G_1, G_2, A) is cartesian, by [FrJ, Lemma 22.2.4], $\ker(\pi_1 \circ \alpha) = \ker(\alpha)\ker(\beta)$. Thus, $\ker(\pi_1 \circ \alpha) \leq \ker(\delta_1 \circ \alpha)\ker(\beta)$. By [FrJ, Lemma 22.2.6(b)], β_1 is surjective. By Lemma 2.3(a), (G, G_1, D, B) is a cartesian square.

(b) Let $B = G_1/\alpha(\ker(\beta_1))$ and let $\delta_1\colon G_1 \to B$ be the canonical projection. We have $\ker(\delta_1 \circ \alpha) = \alpha^{-1}(\ker(\delta_1)) = \alpha^{-1}(\alpha(\ker(\beta_1))) = \ker(\alpha)\ker(\beta_1)$. Thus, $\ker(\beta_1) \subseteq \ker(\delta_1 \circ \alpha)$. It follows that there exists a unique epimorphism $\pi\colon D \to B$ such that $\delta_1 \circ \alpha = \pi \circ \beta_1$. Moreover, $\ker(\alpha) \cap \ker(\beta_1) \subseteq \ker(\alpha) \cap \ker(\beta) = 1$. Hence, $\ker(\delta_1 \circ \alpha) = \ker(\alpha) \times \ker(\beta_1)$.

By [FrJ, Proposition 22.2.4], (G, G_1, D, B) is a cartesian square. Furthermore, by [FrJ, Lemma 22.2.5], $\alpha(\ker(\beta)) = \ker(\pi_1)$. It follows that $\ker(\delta_1) = \alpha(\ker(\beta_1)) \subseteq \alpha(\ker(\beta)) = \ker(\pi_1)$. Hence, there exists a unique epimorphism $\delta_2 \colon B \to A$ such that $\delta_2 \circ \delta_1 = \pi_1$. We have $\delta_2 \circ \pi \circ \beta_1 = \delta_2 \circ \delta_1 \circ \alpha = \pi_1 \circ \alpha = \pi_2 \circ \beta = \pi_2 \circ \beta_2 \circ \beta_1$. Hence, $\delta_2 \circ \pi = \pi_2 \circ \beta_2$. By Lemma 2.3(a), (D, B, G_2, A) is a cartesian square. $\square$

Lemma 2.7. *Consider a cartesian square (1). Then*

(a) α (resp. β) is an isomorphism if and only if π_2 (resp. π_1) is an isomorphism.

(b) α (resp. β) is indecomposable if and only if π_2 (resp. π_1) is indecomposable.

Proof. (a) By [FrJ, Lemma 22.2.5], $\ker(\pi_1) \cong \ker(\beta)$. By symmetry, $\ker(\pi_2) \cong \ker(\alpha)$.

(b) We prove the claim in the parentheses, since diagram (4) is better suited to this proof. The other case follows by symmetry.

Since isomorphisms are, by definition, indecomposable, by (a), we may assume that both π_1 and β are not isomorphisms.

Suppose β is indecomposable and let $\delta_1 \colon G_1 \to B, \delta_2 \colon B \to A$ be epimorphisms such that $\delta_2 \circ \delta_1 = \pi_1$. By Lemma 2.6(a), there exists a commutative diagram (4) with cartesian squares. As β is indecomposable, either β_1 or β_2 is an isomorphism. By (a), either δ_1 or δ_2 is an isomorphism. Hence, π_1 is indecomposable.

Conversely, suppose π_1 is indecomposable and let $\beta_1 \colon G \to D, \beta_2 \colon D \to G_2$ be epimorphisms such that $\beta_2 \circ \beta_1 = \beta$. By Lemma 2.6(b), there exists a commutative diagram (4) with cartesian squares. As π_1 is indecomposable, either δ_1 or δ_2 is an isomorphism. By (a), either β_1 or β_2 is an isomorphism. Hence, β is indecomposable. $\square$

3 Quasi-formations

Definition 3.1. Let $\mathcal{C}$ be a class of finite groups. We call $\mathcal{C}$ a **quasi-formation** if it is closed under taking quotients and compact cartesian squares (i.e., if (G, G_1, G_2, A) is a compact cartesian square with $G_1, G_2 \in \mathcal{C}$, then $G \in \mathcal{C}$).

We have seen in Proposition 1.13 that if $\mathcal{C}$ is a Melnikov formation generated by finite non-abelian simple groups, then $\mathcal{C}'$ is a formation. If we take out the non-abelianity assumption, we get a quasi-formation:

Lemma 3.2. *Let $\mathcal{C}$ be a formation. Then $\mathcal{C}'$ is a quasi-formation.*

Proof. By Lemma 1.5(a), $\mathcal{C}'$ is closed under taking quotients. Consider a compact cartesian square (G, G_1, G_2, A) with $G_1, G_2 \in \mathcal{C}'$. By Lemma 1.3(b), for $i = 1, 2$, the image of $G^{\mathcal{C}}$ in G_i is $G_i^{\mathcal{C}} = G_i$. Thus, $G^{\mathcal{C}} \in \mathcal{H}(G_1, G_2)$. By the compactness, $G^{\mathcal{C}} = G$, that is, $G \in \mathcal{C}'$. $\square$

Example 3.3. Let $\mathcal{C}$ be the class of all finite p-groups for a fixed prime p, or abelian, or nilpotent, or solvable groups. Then $\mathcal{C}'$ is a quasi-formation but not a formation. Indeed, by Example 1.9, $\mathcal{C}'$ is not a formation. By Lemma 3.2, $\mathcal{C}'$ is a quasi-formation.

Definition 3.4. Let $\mathcal{C}$ be a class of finite groups. Define the **rank of** $\mathcal{C}$ as

$$\mathrm{rank}(\mathcal{C}) = \sup\{\mathrm{rank}(G) \mid G \in \mathcal{C}\}.$$

For a cardinal number d define $\mathcal{C}_d = \{G \in \mathcal{C} \mid \mathrm{rank}(G) \leq d\}$.

(Here $\mathrm{rank}(G)$ denotes the minimal number of elements in a generating set of G, cf. [FrJ, p. 328].)

This definition gives rise to new quasi-formations:

Lemma 3.5. *Let $\mathcal{C}$ be a quasi-formation and d a cardinal number. Then $\mathcal{C}_d$ is a quasi-formation.*

Proof. As all groups in $\mathcal{C}$ are finite, if $d \geq \aleph_0$, then $\mathcal{C}_d = \mathcal{C}$ and the assertion follows. Assume $d < \aleph_0$. Obviously, $\mathcal{C}_d$ is closed under taking quotients.

Let (G, G_1, G_2, A) be a compact cartesian square with $G_1, G_2 \in \mathcal{C}_d$. Then $G \cong G_1 \times_A G_2 \in \mathcal{C}$. By assumption, there exist $g_1, \ldots, g_d \in G_1$ that generate G_1. By Gaschütz Lemma ([FrJ, Lemma 17.7.2]), the images of $g_1, \ldots, g_d$ in A can be lifted to generators $h_1, \ldots, h_d$ of G_2. Let $G' = \langle (g_1, h_1), \ldots, (g_d, h_d) \rangle \leq G$. For $i = 1, 2$ the projection $G \to G_i$ maps G' onto G_i. Thus, by the compactness of the cartesian square, $G' = G$ and therefore G is generated by d elements. It follows that $G \in \mathcal{C}_d$. $\square$

In fact, a quasi-formation (modulo isomorphisms) can even be a finite set:

Example 3.6. Let p be a prime number and d a nonnegative integer. Let $\mathcal{C}$ be the class of all finite elementary abelian p-groups. Clearly, $\mathcal{C}$ is a formation. By Lemma 3.5, $\mathcal{C}_d = \{(\mathbb{Z}/p\mathbb{Z})^n \mid 0 \leq n \leq d\}$ is a quasi-formation. Thus, modulo isomorphisms, $\mathcal{C}_d$ is a finite set.

In contrast, a formation is always an infinite set (modulo isomorphisms):

Lemma 3.7. *Let $\mathcal{C}$ be a class of finite groups, closed under taking direct products that contains a nontrivial group. Then* $\mathrm{rank}(\mathcal{C}) = \aleph_0$. *In particular, a formation that contains a nontrivial group is infinite (modulo isomorphisms).*

Proof. Let $G \in \mathcal{C}$ be a nontrivial group. By assumption, $G^n \in \mathcal{C}$ for every $n \in \mathbb{N}$ and by [Dey, Theorem 2], $\mathrm{rank}(G^n)$ tends to infinity with n. As direct products are a particular case of fiber products, the last assertion follows. $\square$

The following characterization of quasi-formations will be used in the next section.

Lemma 3.8. *Let $\mathcal{C}$ be a class of finite groups, closed under taking quotients. The following conditions are equivalent:*

(a) $\mathcal{C}$ is a quasi-formation.

(b) For every pair of indecomposable epimorphisms $(\beta\colon B \to A, \gamma\colon C \to A)$ with $B, C \in \mathcal{C}$ there exist $G \in \mathcal{C}$ and indecomposable epimorphisms $\pi_B\colon G \to B$, $\pi_C\colon G \to C$ such that $\beta \circ \pi_B = \gamma \circ \pi_C$.

(c) For every pair of epimorphisms $(\beta\colon B \to A, \gamma\colon C \to A)$ with $B, C \in \mathcal{C}$ there exist $G \in \mathcal{C}$ and epimorphisms $\pi_B\colon G \to B, \pi_C\colon G \to C$, such that $\beta \circ \pi_B = \gamma \circ \pi_C$.

Proof. "(a) $\Rightarrow$(b)": Let $G = B \times_A C$ and let $\pi_B\colon G \to B$, $\pi_C\colon G \to C$ be the canonical epimorphisms. If β (or γ) is an isomorphism, by Lemma 2.7(a), so is π_C (or π_B), hence $G \in \mathcal{C}$. So we may assume that β, γ are not isomorphisms. If there is no isomorphism $\theta\colon C \to B$ such that $\beta \circ \theta = \gamma$, then (G, B, C, A) is compact by Lemma 2.5(b), hence $G \in \mathcal{C}$. If there is such θ, put $G = C \in \mathcal{C}$, $\pi_B = \theta$ and $\pi_C = \mathrm{id}$.

"(b) $\Rightarrow$(c)": First assume that one of the maps β, γ, say γ, is indecomposable. There is a sequence

$$B = B_m \xrightarrow{\beta_m} B_{m-1} \xrightarrow{\beta_{m-1}} \cdots \xrightarrow{\beta_2} B_1 \xrightarrow{\beta_1} B_0 = A \tag{5}$$

of indecomposable epimorphisms such that $\beta_1 \circ \cdots \circ \beta_m = \beta$. We complete it to a commutative diagram of epimorphisms of groups in $\mathcal{C}$

$$\begin{tikzcd}
G \arrow[r, equal] \arrow[d, "\pi_B"] & G_m \arrow[r, "\pi_m"] \arrow[d, "\gamma_m"'] & G_{m-1} \arrow[r, "\pi_{m-1}"] \arrow[d, "\gamma_{m-1}"] & \cdots \arrow[r, "\pi_2"] & G_1 \arrow[r, "\pi_1"] \arrow[d, "\gamma_1"] & G_0 \arrow[r, equal] \arrow[d, "\gamma_0"] & C \arrow[d, "\gamma"] \\
B \arrow[r, equal] & B_m \arrow[r, "\beta_m"'] & B_{m-1} \arrow[r, "\beta_{m-1}"'] & \cdots \arrow[r, "\beta_2"'] & B_1 \arrow[r, "\beta_1"'] & B_0 \arrow[r, equal] & A
\end{tikzcd} \tag{6}$$

with $\gamma_0, \gamma_1, \ldots, \gamma_m$ indecomposable: Suppose $i \geq 1$ and G_j, γ_j, π_j have already been constructed for $j < i$. By (b), there exist $G_i \in \mathcal{C}$ and indecomposable epimorphisms

$\gamma_i\colon G_i \to B_i$, $\pi_i\colon G_i \to G_{i-1}$ such that $\gamma_{i-1} \circ \pi_i = \beta_i \circ \gamma_i$. Then $\pi_B = \gamma_m$ and $\pi_C = \pi_1 \circ \cdots \circ \pi_m$ satisfy $\beta \circ \pi_B = \gamma \circ \pi_C$.

In the general case in which both β and γ are decomposable we proceed similarly. Let (5) be a decomposition of β into indecomposable epimorphisms. We complete it to a commutative diagram (6) of epimorphisms of groups in $\mathcal{C}$: Suppose $i \geq 1$ and G_j, γ_j, π_j have already been constructed for $j < i$. By the preceding paragraph, applied to the pair $(\gamma\colon G_{i-1} \to B_{i-1}, \beta_i\colon B_i \to B_{i-1})$ instead of the pair $(\beta\colon B \to A, \gamma\colon C \to A)$, there exist $G_i \in \mathcal{C}$ and epimorphisms $\gamma_i\colon G_i \to B_i$, $\pi_i\colon G_i \to G_{i-1}$ such that $\gamma_{i-1} \circ \pi_i = \beta_i \circ \gamma_i$. Then $\pi_B = \gamma_m$ and $\pi_C = \pi_1 \circ \cdots \circ \pi_m$ satisfy $\beta \circ \pi_B = \gamma \circ \pi_C$.

"(c) $\Rightarrow$(a)": Let $(\beta\colon B \to A, \gamma\colon C \to A)$ be two epimorphisms such that $B, C \in \mathcal{C}$. Let $p_B\colon B \times_A C \to B$ and $p_C\colon B \times_A C \to C$ be the canonical projections and assume $(B \times_A C, B, C, A)$ is compact. By assumption, there exist $G \in \mathcal{C}$ and epimorphisms $\pi_B\colon G \to B, \pi_C\colon G \to C$, such that $\beta \circ \pi_B = \gamma \circ \pi_C$. By Definition 2.1, there exists a homomorphism $\pi\colon G \to B \times_A C$ such that $p_B \circ \pi = \pi_B$ and $p_C \circ \pi = \pi_C$. Let $H = \pi(G)$. Then $p_B(H) = \pi_B(G) = B$ and $p_C(H) = \pi_C(G) = C$. Since $(B \times_A C, B, C, A)$ is compact, $B \times_A C = H \in \mathcal{C}$. □

Remark 3.9. For every class of finite groups $\mathcal{C}$ there exists the smallest quasi-formation that contains it, namely, the intersection of all quasi-formations containing $\mathcal{C}$. Indeed, the class of all finite groups is a quasi-formation containing $\mathcal{C}$ and any intersection of quasi-formations is also a quasi-formation.

4 Free groups over quasi-formations

Let $\mathcal{C}$ be a formation and let $m \leq \aleph_0$ be a cardinal number. In [FrJ, Lemma 17.4.2] a pro-$\mathcal{C}$ group $\hat{F}$ of rank m is constructed. It follows from its definition that $\mathrm{Im}(\hat{F}) = \mathcal{C}_m$. In [FrJ, Lemma 24.3.3] it is shown that $\hat{F}$ has the embedding property. Furthermore, [FrJ, Theorem 24.8.1] shows that a pro-$\mathcal{C}$ group G of at most countable rank with the embedding property and such that $\mathrm{Im}(G) = \mathcal{C}$ is unique up to an isomorphism. These results remain true if we replace $\mathcal{C}$ by a quasi-formation. This is the content of Theorem 4.4.

The following definition is taken from [FrJ, Definitions 22.3.1 and 24.1.2] and [FrJ, p. 506].

Definition 4.1. Let G be a profinite group and let $\mathcal{C}$ be a class of finite groups.

(a) An **embedding problem for** G is a pair

$$(\varphi\colon G \to A, \alpha\colon B \to A) \tag{7}$$

in which φ and α are epimorphisms. We call (7) a **$\mathcal{C}$-embedding problem** if G, A, B are pro-$\mathcal{C}$ groups, **finite** if B is finite, and **split** if there exists a homomorphism $\alpha'\colon A \to B$ with $\alpha \circ \alpha' = \mathrm{id}_A$. A **weak solution** to (7) is a homomorphism $\gamma\colon G \to B$ such that $\alpha \circ \gamma = \varphi$. A weak solution γ to (7) is a **solution** if γ is surjective.

(b) We say that G has the **embedding property** if every finite embedding problem (7) for G such that $B \in \mathrm{Im}(G)$ has a solution.

(c) A profinite group G is **projective** if every embedding problem (7) for G has a weak solution.

The following easy lemma seems to be nowhere explicitly stated:

Lemma 4.2. *Let $\mathcal{C}$ be a class of finite groups, closed under taking quotients. If every finite $\mathcal{C}$-embedding problem (7) with α indecomposable has a solution, then every finite $\mathcal{C}$-embedding problem has a solution.*

Proof. Let $(\varphi_0\colon G \to B_0, \alpha\colon B \to B_0)$ be a finite $\mathcal{C}$-embedding problem. There exists a sequence

$$B = B_m \xrightarrow{\alpha_m} \cdots \xrightarrow{\alpha_2} B_1 \xrightarrow{\alpha_1} B_0$$

of indecomposable epimorphisms such that $\alpha_1 \circ \cdots \circ \alpha_m = \alpha$. By induction on the assumption for each $1 \le i \le m$ there exists an epimorphism $\varphi_i\colon G \to B_i$ such that $\alpha_i \circ \varphi_i = \varphi_{i-1}$. Then $\alpha \circ \varphi_m = \varphi_0$. $\square$

Profinite groups with the embedding property give rise to quasi-formations:

Proposition 4.3. *Let G be a profinite group with the embedding property. Then* $\mathrm{Im}(G)$ *is a quasi-formation.*

Proof. Clearly, $\mathrm{Im}(G)$ is closed under taking quotients. That $\mathrm{Im}(G)$ is closed under taking compact cartesian squares follows from [FrJ, Lemma 24.5.1(a)]. $\square$

Conversely, quasi-formations give rise to profinite groups with the embedding property:

Theorem 4.4. *Let $\mathcal{C}$ be a quasi-formation of finite groups. Then there exists a pro-$\mathcal{C}$ group $\hat{E}(\mathcal{C})$ of at most countable rank, unique up to an isomorphism, with the embedding property and* $\mathrm{Im}(\hat{E}(\mathcal{C})) = \mathcal{C}$.

Proof. The cardinality of $\mathcal{C}$, modulo isomorphisms, is at most countable. Given $B, A \in \mathcal{C}$ there is only a finite number of epimorphisms $\alpha\colon B \to A$. Thus, the cardinality of all epimorphisms $\alpha\colon B \to A$, up to composition with isomorphisms, is at most countable. We may therefore construct a sequence $\mathcal{E} = (\alpha_1, \alpha_2, \ldots)$ of all indecomposable epimorphisms $\alpha_j\colon B_j \to A_j$ with $A_j, B_j \in \mathcal{C}$, that contains each element, up to composition with isomorphisms, countably many times.
Part A: *Construction of the group:* Inductively construct a sequence (possibly finite)

$$\cdots \xrightarrow{\varphi_{3,2}} G_2 \xrightarrow{\varphi_{2,1}} G_1 \xrightarrow{\varphi_{1,0}} G_0 = 1$$

of indecomposable epimorphisms (denoting $\varphi_{k,i} = \varphi_{i+1,i} \circ \cdots \circ \varphi_{k,k-1}\colon G_k \to G_i$ for $i \le k$) of groups in $\mathcal{C}$ and a sequence

$$\ldots > j_2 > j_1 > j_0 = 0$$

of integers in the following way: Assume G_i, $\varphi_{i,i-1}$, and j_i have already been constructed for each $0 \le i \le k$. If there exists no $j \in \mathbb{N}$ such that

(a) $j > j_k$, and

(b) there exists $i \in \{0, 1, \ldots, k\}$ such that $G_i = A_j$, and the finite $\mathcal{C}$-embedding problem $(\varphi_{k,i}\colon G_k \to G_i, \alpha_j\colon B_j \to G_i = A_j)$ has no solution,

end the construction and let $G = G_k$; thus, $G = \varprojlim G_i$. Otherwise let j_{k+1} be the minimal j that satisfies (a) and (b), set $G_{k+1} = B_j \times_{G_i} G_k$, and let $\varphi_{k+1,k}$ be the canonical projection $G_{k+1} \to G_k$. By (b) and by Lemma 2.5(a), the cartesian square (G_{k+1}, B_j, G_k, G_i) is compact. As $\mathcal{C}$ is a quasi-formation, $G_{k+1} \in \mathcal{C}$.

If the sequence $(G_0, G_1, \ldots)$ is infinite, let $G = \varprojlim G_i$. For every G_i in the sequence let $\varphi_i\colon G \to G_i$ be the canonical map of the inverse limit.

In any case, by [FrJ, Example 17.1.7(a)], $\mathrm{rank}(G) \le \aleph_0$.
Part B: *Every finite $\mathcal{C}$-embedding problem $(\varphi_i\colon G \to G_i, \alpha\colon B \to G_i)$ with α indecomposable has a solution:* Assume no solution exists. Then for every $k > i$ (such that G_k is defined) the finite $\mathcal{C}$-embedding problem $(\varphi_{k,i}\colon G_k \to G_i, \alpha\colon B \to G_i)$ has no solution. As α appears infinitely many times in $\mathcal{E}$, there exists $j > j_{i+1}$ such that $\alpha = \alpha_j$. Let $k > i$ be maximal such that $j > j_k$. Then j satisfies conditions (a) and (b). Moreover, j

is minimal with that property. Indeed, if $j' < j$ is minimal, then $j' = j_{k+1}$, so $j > j_{k+1}$, a contradiction to the choice of k. By Part A, we have the following cartesian square:

$$\begin{tikzcd} G_{k+1} \arrow[r, "\varphi_{k+1,k}"] \arrow[d, "\pi_B"'] & G_k \arrow[d, "\varphi_{k,i}"] \\ B \arrow[r, "\alpha"] & G_i \end{tikzcd}$$

Thus, π_B solves the finite $\mathcal{C}$-embedding problem $(\varphi_{k+1,i}\colon G_{k+1} \to G_i, \alpha\colon B \to G_i)$, contrary to the assumption.

Part C: *Every finite $\mathcal{C}$-embedding problem (7) has a solution:* By Lemma 4.2, we may assume that α is indecomposable. There exist $i \in \mathbb{N}$ and an epimorphism $\bar{\varphi}\colon G_i \to A$ such that $\varphi = \bar{\varphi} \circ \varphi_i$. If there exists an epimorphism $\theta\colon G_i \to B$ such that $\alpha \circ \theta = \bar{\varphi}$, then $\theta \circ \varphi_i$ solves (7). If no such epimorphism exists, by Lemma 2.5(a), the cartesian square $(B \times_A G_i, B, G_i, A)$ is compact. As $\mathcal{C}$ is a quasi-formation, $B \times_A G_i \in \mathcal{C}$. Let $\pi_{G_i}\colon B \times_A G_i \to G_i$ (resp. $\pi_B\colon B \times_A G_i \to B$) be the canonical projection onto G_i (resp. B). By Lemma 2.7, π_{G_i} is indecomposable. It follows from Part B that the finite $\mathcal{C}$-embedding problem $(\varphi_i\colon G \to G_i, \pi_{G_i}\colon B \times_A G_i \to G_i)$ has a solution. Thus, there exists an epimorphism $\gamma\colon G \to B \times_A G_i$ such that $\pi_{G_i} \circ \gamma = \varphi_i$. Then $\pi_B \circ \gamma$ is a solution of (7).

Part D: *Every $B \in \mathcal{C}$ is a quotient of G:* Put $A = 1$ in Part C.

Part E: *Uniqueness:* Since G is pro-$\mathcal{C}$, it follows from Part D that $\mathrm{Im}(G) = \mathcal{C}$. By Part C, G has the embedding property. In Part A we have seen that $\mathrm{rank}(G) \le \aleph_0$. By [FrJ, Lemma 24.4.7], G with these properties is unique up to an isomorphism. □

Definition 4.5. We call the group $\hat{E}(\mathcal{C})$ of Theorem 4.4 the **free pro-$\mathcal{C}$ group**.

Remark 4.6. As $\mathrm{Im}(\hat{E}(\mathcal{C})) = \mathcal{C}$, we have $\mathrm{rank}(\hat{E}(\mathcal{C})) = \mathrm{rank}(\mathcal{C})$.

Lemma 4.7. *If $\mathcal{C}$ is a formation, then $\hat{E}(\mathcal{C}) \cong \hat{F}_\omega(\mathcal{C})$ and $\hat{E}(\mathcal{C}_d) \cong \hat{F}_d(\mathcal{C})$, for every $d \in \mathbb{N}$.*

Proof. By [FrJ, Lemma 24.3.3], $\hat{F}_\omega(\mathcal{C})$ has the embedding property. Clearly $\mathrm{Im}(\hat{F}_\omega(\mathcal{C})) = \mathcal{C}$. Hence, the first assertion follows from the uniqueness in Theorem 4.4. The second assertion follows from [FrJ, Lemma 17.7.1] (we tacitly assume that there is $G \in \mathcal{C}$ such that $1 \le \mathrm{rank}(G) \le d$, otherwise $\hat{F}_d(\mathcal{C})$ is not defined). □

5 Some properties of $\hat{E}(\mathcal{C})$

We examine projectivity and connection to smallest embedding covers.

Recall ([FrJ, Definition 22.5.1]) that an epimorphism $\varphi\colon G \to H$ of profinite groups is called a **Frattini cover** if for every closed subgroup G_0 of G satisfying $\varphi(G_0) = H$ it follows that $G_0 = G$.

By [FrJ, Lemma 22.5.6], if $\varphi\colon G \to H$ is an epimorphism of profinite groups, then G has a closed subgroup G_0 such that $\varphi|_{G_0}\colon G_0 \to H$ is a Frattini cover.

Definition 5.1. We call a class $\mathcal{C}$ of finite groups **Frattini-closed** if the following holds: Let $\varphi\colon G \to H$ be a Frattini cover of finite groups such that $H \in \mathcal{C}$. Then $G \in \mathcal{C}$.

Notice that in [FrJ, Exercise 11 on p. 541] the term "admissible" is mentioned in a similar situation.

Lemma 5.2. *Let $\mathcal{C}$ be a formation. Then $\mathcal{C}'$ is Frattini-closed.*

Proof. Let $\varphi\colon G \to H$ be a Frattini cover of finite groups such that $H \in \mathcal{C}'$. By Lemma 1.3(b), $\varphi(G^{\mathcal{C}}) = H^{\mathcal{C}} = H$. Since φ is a Frattini cover, $G^{\mathcal{C}} = G$, that is, $G \in \mathcal{C}'$. □

Lemma 5.3. *Let G be a profinite group with the embedding property and let $\mathcal{C} = \mathrm{Im}(G)$. Then G is projective if and only if $\mathcal{C}$ is Frattini-closed.*

Proof. Suppose G is projective and let $\alpha\colon B \to A$ be a Frattini cover of finite groups with $A \in \mathcal{C}$. As $\mathcal{C} = \mathrm{Im}(G)$, there exists an epimorphism $\varphi\colon G \to A$. By assumption there exists a homomorphism $\psi\colon G \to B$ such that $\alpha \circ \psi = \varphi$. As α is a Frattini cover, ψ is surjective. Hence, $B \in \mathrm{Im}(G) = \mathcal{C}$.

Conversely, assume $\mathcal{C}$ is Frattini-closed and let $(\varphi\colon G \to A, \alpha\colon B \to A)$ be a finite embedding problem for G. Then $A \in \mathrm{Im}(G) = \mathcal{C}$. Let B_0 be a subgroup of B such that $\alpha_0 = \alpha|_{B_0}\colon B_0 \to A$ is a Frattini cover. By assumption $B_0 \in \mathcal{C}$. As G has the embedding property, there exists an epimorphism $\psi\colon G \to B_0$ such that $\alpha_0 \circ \psi = \varphi$. We may regard ψ as a homomorphism into B. It follows that every finite embedding problem for G has a weak solution. Thus, G is projective ([FrJ, Lemma 22.3.2]). $\square$

Recall ([FrJ, p. 565]) that a profinite group is called **superprojective** if it is both projective and has the embedding property.

Proposition 5.4. *Let $\mathcal{C}$ be a formation such that $\mathcal{C}'$ is nontrivial. Then $\hat{E}(\mathcal{C}')$ is superprojective of rank $\aleph_0$.*

Proof. By Lemma 3.2, $\mathcal{C}'$ is a quasi-formation. By Lemma 5.2, $\mathcal{C}'$ is Frattini-closed and by Lemma 1.5(b), it is extension-closed. In particular, $\mathcal{C}'$ is closed under taking direct products. Therefore, by Lemma 3.7, $\mathrm{rank}(\mathcal{C}') = \aleph_0$. By Theorem 4.4, $\hat{E}(\mathcal{C}')$ has the embedding property and $\mathrm{Im}(\hat{E}(\mathcal{C}')) = \mathcal{C}'$. By Remark 4.6, $\mathrm{rank}(\hat{E}(\mathcal{C}')) = \aleph_0$. By Lemma 5.3, $\hat{E}(\mathcal{C}')$ is projective. $\square$

For Melnikov formations $\mathcal{C}$ there is a deep connection between maximal pro-$\mathcal{C}$ kernels of free profinite groups and free pro-$\mathcal{C}'$ groups:

Theorem 5.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be two Melnikov formations such that $\mathcal{C} \cup \mathcal{C}' \subseteq \mathcal{D}$. Suppose $\mathcal{C}$ is different from the class of all finite groups. Then $\hat{E}(\mathcal{C}') \cong \hat{F}_d(\mathcal{D})^{\mathcal{C}}$ for every cardinal $2 \le d \le \aleph_0$ such that there exists $G \in \mathcal{C}$ with* $\mathrm{rank}(G) \le d$.

Proof. It suffices to prove the assertion only for $d = \aleph_0$. Indeed, let $F = \hat{F}_d(\mathcal{D})$ and $N = F^{\mathcal{C}}$. By [FrJ, Lemma 17.4.10], $F/N \cong \hat{F}_d(\mathcal{C})$. By [FrJ, Corollary 17.6.5], $\hat{F}_d(\mathcal{C})$ is infinite. Thus, $(F : N) = \infty$. By a result of Melnikov [FrJ, Proposition 25.8.3], there exists a normal subgroup K of F which contains N and is isomorphic to $\hat{F}_\omega(\mathcal{D})$. By Lemma 1.3(d), $K^{\mathcal{C}} = N$. Thus, $F^{\mathcal{C}} \cong (\hat{F}_\omega(\mathcal{D}))^{\mathcal{C}}$.

So assume $F = \hat{F}_\omega(\mathcal{D})$. By Lemma 1.5(c), N is pro-$\mathcal{C}'$. By Lemma 3.2, $\mathcal{C}'$ is a quasi-formation. Thus, by Theorem 4.4, it suffices to show that $\mathcal{C}' \subseteq \mathrm{Im}(N)$, that N has the embedding property and that $\mathrm{rank}(N) = \aleph_0$.

Let $A \in \mathcal{C}' \subseteq \mathcal{D}$. By [FrJ, Theorem 24.8.1], there exists an epimorphism $\varphi\colon F \to A$. By Lemma 1.3(b), $\varphi(N) = A^{\mathcal{C}} = A$. Thus, $\mathcal{C}' \subseteq \mathrm{Im}(N)$. By Remark 1.6, $\mathcal{C}'$ is nontrivial. It follows that N is nontrivial.

Let $(\varphi\colon N \to A, \alpha\colon B \to A)$ be a finite $\mathcal{C}'$-embedding problem for N. By [FrJ, Lemma 1.2.5(c)], φ extends to an epimorphism $\varphi'\colon N' \to A$ where N' is an open normal subgroup of F containing N. By [FrJ, Proposition 25.2.2], $N' \cong F$. By [FrJ, Lemma 24.3.3], N' has the embedding property and by [FrJ, Theorem 24.8.1], $B \in \mathrm{Im}(N')$. Thus, there exists an epimorphism $\gamma'\colon N' \to B$ such that $\alpha \circ \gamma' = \varphi'$. By Lemma 1.3(d), $(N')^{\mathcal{C}} = N$. Therefore, by Lemma 1.3(b), $\gamma'(N) = \gamma'((N')^{\mathcal{C}}) = B^{\mathcal{C}} = B$. It follows that $\gamma = \gamma'|_N\colon N \to B$ is an epimorphism with $\alpha \circ \gamma = \varphi$.

Finally, since N is a nontrivial closed subgroup of F of infinite index, $\mathrm{rank}(N) = \aleph_0$ ([FrJ, Theorem 25.4.7(b)]). $\square$

Remark 5.6. (a) The previous theorem gives an alternative proof of Proposition 5.4 in the case of a Melnikov formation $\mathcal{C}$: Free profinite groups are projective ([FrJ, Corollary

22.4.5]) and every closed subgroup of a projective profinite group is projective ([FrJ, Proposition 22.4.7]). Thus, $\hat{E}(\mathcal{C}') \cong \hat{F}_\omega^{\mathcal{C}}$ is projective.

(b) By [FrJ, Corollary 24.9.4], every closed normal subgroup N of a free profinite group F has the embedding property. We give a different proof of this fact in the special case $N = (\hat{F}_\omega(\mathcal{D}))^{\mathcal{C}}$ that uses the properties of maximal pro-$\mathcal{C}$ quotients (Lemma 1.3).

We have seen (Theorem 5.5) that —for Melnikov formations $\mathcal{C}$ different from the class of all finite groups— free pro-$\mathcal{C}'$ groups arise as maximal pro-$\mathcal{C}$ kernels of free profinite groups. It turns out that for quasi-formations $\mathcal{C}$ the free pro-$\mathcal{C}$ groups are related to smallest embedding covers as well. Recall ([FrJ, Definition 24.4.3]) that an epimorphism $\varphi\colon H \to G$ of profinite groups is called an **embedding cover** if H has the embedding property. An epimorphism $\varepsilon\colon E \to G$ is called a **smallest embedding cover** if ε is an embedding cover and if for every embedding cover $\varphi\colon H \to G$ there exists an epimorphism $\gamma\colon H \to E$ such that $\varepsilon \circ \gamma = \varphi$.

By [FrJ, Proposition 24.4.5], every profinite group G has a smallest embedding cover E. By [FrJ, Lemma 24.4.6(b)], $\mathrm{rank}(E) = \mathrm{rank}(G)$. By [FrJ, Corollary 24.4.8], if G is of at most countable rank, then E is unique up to an isomorphism. It turns out that in this case E is the free group constructed in this section:

Proposition 5.7. *Let G be a profinite group of at most countable rank. Let $\mathcal{C}$ be the smallest quasi-formation containing* $\mathrm{Im}(G)$*. Let $\varepsilon\colon E \to G$ be a smallest embedding cover of G. Then $E \cong \hat{E}(\mathcal{C})$.*

Proof. For the existence of $\mathcal{C}$ see Remark 3.9.

We first show that there exists an epimorphism $\psi\colon \hat{E}(\mathcal{C}) \to G$. By [FrJ, Example 17.1.7(a)], there exists a sequence of epimorphisms of finite groups in $\mathcal{C}$ $\cdots \xrightarrow{\pi_2} G_2 \xrightarrow{\pi_1} G_1 \xrightarrow{\pi_0} G_0 = 1$ such that $G = \varprojlim G_i$. We construct inductively a family of epimorphisms $\psi_i\colon \hat{E}(\mathcal{C}) \to G_i$ with $\psi_{i-1} = \pi_{i-1} \circ \psi_i$ for each i: Let $\psi_0\colon \hat{E}(\mathcal{C}) \to G_0 = 1$ be the trivial map. Assume we have already constructed $\psi_0, \ldots, \psi_i$. By Theorem 4.4, $\hat{E}(\mathcal{C})$ has the embedding property, hence there exists an epimorphism $\psi_{i+1}\colon \hat{E}(\mathcal{C}) \to G_{i+1}$ such that $\psi_i = \pi_i \circ \psi_{i+1}$. By [FrJ, Corollary 1.1.6], there exists an epimorphism $\psi\colon \hat{E}(\mathcal{C}) \to G$.

By assumption, E is a smallest embedding cover. Thus, there exists an epimorphism $\gamma\colon \hat{E}(\mathcal{C}) \to E$ such that $\psi = \varepsilon \circ \gamma$. By Proposition 4.3, $\mathrm{Im}(E)$ is a quasi-formation. As G is a quotient of E, $\mathrm{Im}(G) \subseteq \mathrm{Im}(E)$. Therefore, $\mathcal{C} \subseteq \mathrm{Im}(E)$. It follows that $\mathrm{Im}(E) \subseteq \mathrm{Im}(\hat{E}(\mathcal{C})) = \mathcal{C} \subseteq \mathrm{Im}(E)$. Hence, $\mathrm{Im}(E) = \mathrm{Im}(\hat{E}(\mathcal{C}))$. By [FrJ, Lemma 24.4.6(b)], $\mathrm{rank}(E) = \mathrm{rank}(G)$. As both E and $\hat{E}(\mathcal{C})$ are of at most countable rank and have the embedding property, by [FrJ, Lemma 24.4.7], $E \cong \hat{E}(\mathcal{C})$. □

Corollary 5.8. *Let G be a profinite group of rank $\aleph_0$. Let E be a smallest embedding cover of G.*

(a) Suppose $\mathcal{C}$ is a Melnikov formation such that $\mathrm{Im}(G) = \mathcal{C}'$*. Then $E \cong (\hat{F}_\omega)^{\mathcal{C}}$.*

(b) Suppose $\mathcal{C} = \mathrm{Im}(G)$ is a formation. Then $E \cong \hat{F}_\omega(\mathcal{C})$.

Proof. (a) By Lemma 3.2 and Proposition 5.7, $E \cong \hat{E}(\mathcal{C}')$. By Theorem 5.5, $\hat{E}(\mathcal{C}') \cong (\hat{F}_\omega)^{\mathcal{C}}$.

(b) By Proposition 5.7, $E \cong \hat{E}(\mathcal{C})$. By Lemma 4.7, $\hat{E}(\mathcal{C}) \cong \hat{F}_\omega(\mathcal{C})$. □

6 Connection to Shafarevich's conjecture

During a series of talks in 1964 I. R. Shafarevich posed the following assertion, now called the Shafarevich's conjecture: The absolute Galois group $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})$ of the maximal abelian extension of $\mathbb{Q}$ is free. As of 2017, the conjecture remains open.

If the conjecture is true, we immediately get $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) \cong \hat{E}(\mathcal{C}')$, where $\mathcal{C}$ is the Melnikov formation of all finite solvable groups. Indeed, $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) \le \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}) \triangleleft \mathrm{Gal}(\mathbb{Q})$; by Lemma 1.3(d) and Theorem 5.5,

$$\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) = \mathrm{Gal}(\mathbb{Q})^{\mathcal{C}} = \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})^{\mathcal{C}} \cong (\hat{F}_\omega)^{\mathcal{C}} = \hat{E}(\mathcal{C}').$$

Shafarevich's conjecture has a natural generalization to global fields which asserts that the absolute Galois group of the maximal abelian extension of any global field is free. It was proven in the function field case by D. Harbater [Ha2, Theorem 4.1]. In this case the conjecture follows from the fact that the absolute Galois group of the function field of every curve over the algebraic closure $\widetilde{\mathbb{F}_p}$ of $\mathbb{F}_p$ is free.

In [Ha1], [Pop], and [HaJ] even more is shown: The absolute Galois group of the function field of every curve over any algebraically closed field is free. This fact naturally leads to the question what happens over fields K which are "almost" algebraically closed, i.e, $[\tilde{K} : K]$ is finite, where $\tilde{K}$ is an algebraic closure of K. By Artin-Schreier theory such fields are precisely the real closed fields. In [Ha2, Theorem 4.2] it is proven that the absolute Galois group of a function field of a curve X over a real closed field K is free if and only if X has no K-points.

While the above generalizations of Shafarevich's conjecture concern one-dimensional function fields, it is possible to consider fields of higher dimension: In [Ha2, Theorem 4.5] it is proven that if K is a separably closed field, then the absolute Galois group of the maximal abelian extension of $K((x,y))$ is free.

The theory of quasi-formations developed in the previous chapters and, in particular, Theorem 5.5 together with the above mentioned proven cases of Shafarevich's conjecture immediately give the following:

Theorem 6.1. *Let $\mathcal{C}$ be a Melnikov formation different from the class of all finite groups and let F be a function field in one variable over a countable algebraically closed field K. Then the absolute Galois group of the maximal $\mathcal{C}$-extension $F_{\mathcal{C}}$ of F is isomorphic to $\hat{E}(\mathcal{C}')$.*

Proof. By definition, $F_{\mathcal{C}}$ is the compositum of all finite Galois extensions E of F with $\mathrm{Gal}(E/F) \in \mathcal{C}$. By [Ha1, Theorem 4.4] or [HaV, Theorem 4.6], $\mathrm{Gal}(F) \cong \hat{F}_\omega$. Hence, by Theorem 5.5, $\mathrm{Gal}(F_{\mathcal{C}}) \cong (\hat{F}_\omega)^{\mathcal{C}} \cong \hat{E}(\mathcal{C}')$. □

Remark 6.2. (a) Theorem 5.5 does not apply to the result concerning $K((x,y))$ since $K((x,y))$ is uncountable for every field K.

(b) Suppose $\mathcal{C}$ is the Melnikov formation of all finite solvable groups. It is then possible in Theorem 6.1 to replace K with any subfield of $\widetilde{\mathbb{F}_p}$: Let F be a function field in one variable over K. Then $F' = F\widetilde{\mathbb{F}_p}$ is a function field in one variable over the algebraically closed field $\widetilde{\mathbb{F}_p}$ and F' is a Galois extension of F, contained in $F_{\mathcal{C}}$. By Lemma 1.3(d), $F_{\mathcal{C}} = F'_{\mathcal{C}}$. Thus, we may apply Theorem 6.1 to the function field F' over $\widetilde{\mathbb{F}_p}$.

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