RELATIVELY PROJECTIVE GROUPS
AS ABSOLUTE GALOIS GROUPS*

by

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ABSTRACT

A group structure $G = (G, G_1, \ldots, G_n)$ is projective if and only if $G$ is isomorphic to a Galois group structure

$$\text{Gal}(K) = (\text{Gal}(K), \text{Gal}(K_1), \ldots, \text{Gal}(K_n))$$

of a field-valuation structure $K = (K, K_1, v_1, \ldots, K_n, v_n)$ where $(K_i, v_i)$ is the Henselian closure of $(K, v_i | K)$ and $K$ is pseudo closed with respect to $K_1, \ldots, K_n$.

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Introduction

A central problem in Galois theory and Field Arithmetic is the characterization of the absolute Galois groups among all profinite groups. To fix notation, let $K$ be a field. Denote its separable closure by $K_s$ and its absolute Galois group by $\text{Gal}(K) = \text{Gal}(K_s/K)$. Then $\text{Gal}(K)$ is a profinite group. An arbitrary profinite group $G$ is said to be an absolute Galois group if $G \cong \text{Gal}(K)$ for some field $K$.

A sufficient condition for a profinite group $G$ to be an absolute Galois group is that $G$ is projective. This means that each epimorphism $G' \to G$ of profinite groups has a section. Indeed, there is a Galois extension $L/K$ with $\text{Gal}(L/K) \cong G$ [Lep]. Each section of $\text{res} : \text{Gal}(K) \to \text{Gal}(L/K)$ gives a separable algebraic extension $F$ with $\text{Gal}(F) \cong G$. Lubotzky and v. d. Dries [FrJ, Cor. 20.16] improve on that by constructing $F$ with the PAC property. Conversely, the absolute Galois group of each PAC field is projective [FrJ, Thm. 10.17].

The goal of this work is to generalize this characterization of projective groups by proving Theorem A and Theorem B below:

**Theorem A:** Let $K$ be a field, $v_i$ a valuation of $K$, and $K_i$ a Henselian closure of $(K, v_i)$, $i = 1, \ldots, n$. Suppose $v_1, \ldots, v_n$ are independent and $K$ is pseudo closed with respect to $K_1, \ldots, K_n$. Then $\text{Gal}(K)$ is projective with respect to $\text{Gal}(K_1), \ldots, \text{Gal}(K_n)$.

Here $K$ is pseudo closed with respect to $K_1, \ldots, K_n$ if the following holds: Every absolutely irreducible variety $V$ over $K$ with a simple $K_i$-rational point, $i = 1, \ldots, n$, has a $K$-rational point.

A profinite group $G$ is projective with respect to $n$ closed subgroups $G_1, \ldots, G_n$ if the following holds: Suppose $G'$ is a profinite group, $G_1', \ldots, G_n'$ are closed subgroups and $\alpha : G' \to G$ is an epimorphism which maps $G_i'$ isomorphically onto $G_i$, $i = 1, \ldots, n$. Then there are an embedding $\alpha' : G \to G'$ with $\alpha \circ \alpha' = \text{id}_G$ and elements $a_1, \ldots, a_n \in G'$ with $\alpha'(G_i) = (G_i')^{a_i}$, $i = 1, \ldots, n$.

**Theorem B:** Let $G$ be a profinite group and $G_1, \ldots, G_n$ closed subgroups. Suppose each $G_i$ is an absolute Galois group and $G$ is projective with respect to $G_1, \ldots, G_n$. Then there are a field $K$, independent valuations $v_1, \ldots, v_n$ of $K$, and a Henselian
closure $K_i$ of $(K, v_i)$, $i = 1, \ldots, n$, with these properties: $K$ is pseudo closed with respect to $K_1, \ldots, K_n$ and has the approximation property with respect to $v, \ldots, v_n$, and there is an isomorphism $\text{Gal}(K) \to G$ that maps $\text{Gal}(K_i)$ onto $G_i$, $i = 1, \ldots, n$.

The approximation property is defined as follows: Let $V$ be an absolutely irreducible variety over $K$. Given a simple $K_i$-rational point $a_i$ of $V$ and $c_i \in K^\times$, $i = 1, \ldots, n$, there is an $a \in V(K)$ with $v_i(a - a_i) > v_i(c_i)$, $i = 1, \ldots, n$.

Special cases of Theorems A and B are consequences of the main result of [HaJ]. That paper characterizes a $p$-adically projective group as the absolute Galois group of a $\text{PpC}$ field. In particular, that result implies Theorems A and B when $G_1, \ldots, G_n$ are isomorphic to $\text{Gal}(\mathbb{Q}_p)$ for a fixed prime number $p$.

There is an overlapping between our results and those of [Pop]. An application of [Pop, Thm. 3.3] to the situation of Theorem A gives a weaker result than the projectivity in our sense: Let $\varphi: G \to A$ and $\psi: B \to A$ be epimorphisms with $B$ finite. Suppose $B_1, \ldots, B_n$ are subgroups of $B$ and $\psi$ maps $B_i$ isomorphically onto $\varphi(G_i)$, $i = 1, \ldots, n$. Then there is a homomorphism $\gamma: G \to B$ with $\psi \circ \gamma = \varphi$. However, no extra condition like $\gamma(G_i)$ is conjugate to $B_i$’ is proved. In other words, [Pop, Thm. 3.3] does not prove $G$ is, in his terminology, ‘strongly projective’.

Likewise, a somewhat weaker version of Theorem B can be derived from [Pop] and [HeP]. In the situation of Theorem B we may first use [HJK, Prop. 2.5] to construct fields $E, E_0, E_1, \ldots, E_n$ such that $\text{Gal}(E_0)$ is the free profinite group $\hat{F}$ of rank equal to $\text{rank}(G)$, $\text{Gal}(E_i) \cong G_i$, $i = 1, \ldots, n$, $E_i$ is a separable algebraic extension of $E$, and $\bigcap_{i=0}^n E_i = E$. Then there is an epimorphism $\psi: G^* = \hat{F} \ast \bigl( \prod_{i=1}^n \text{Gal}(E_i) \bigr) \to \text{Gal}(E)$ which maps $G_i$ isomorphically onto $\text{Gal}(E_i)$, $i = 1, \ldots, n$. This gives a ‘Galois approximation’ in the sense of [Pop, §2]. Using [Pop, Thm. 3.4], we can find a perfect field $K$, algebraic extensions $K_1, \ldots, K_n$, and an isomorphism $\lambda: G \to \text{Gal}(K)$ such that $\lambda(G_i) = \text{Gal}(K_i)$, $i = 1, \ldots, n$, and $K$ is pseudo closed with respect to $K_1, \ldots, K_n$. However, unlike Theorem B, [Pop, Thm. 3.4] does not equip the $K_i$’s with valuations. Furthermore, the approximation property of Theorem B allows $K_i$ to be algebraically closed, so it does not follow from [HeP, Thm. 1.9]. Thus, Theorem B is an improvement of what can be derived from [Pop] and [HeP].
The present work is a follow up of an earlier work [HJK] of the authors with Jochen Koenigsmann. Theorems A and B (except for the approximation property) appear also in [Koe]. While [Koe] uses model theoretic methods to prove Theorem A, our proof restricts to methods of algebraic geometry (Propositions 2.1 and 3.2) and is much shorter.

Finally, [HJP] gives a far reaching generalization of Theorems A and B. Instead of finitely many local objects (i.e. subgroups, algebraic extensions, and valuations), [HJP] deals with families of local objects subject to certain finiteness conditions. Unfortunately, [HJP] is a very long and complicated paper whose technical arguments may disguise the basic ideas lying underneath the proof. Some of these ideas, like “unirationally closed n-fold field structure” can be accessed much faster in this short note.
1. Relatively projective profinite groups

Consider a profinite group $G$ and closed subgroups $G_1, \ldots, G_n$ (with $n \geq 0$). Refer to $G = (G, G_1, \ldots, G_n)$ as a group structure (or as an $n$-fold group structure if $n$ is not clear from the context). An embedding problem for $G$ is a tuple

\begin{equation}
\mathcal{E} = (\varphi: G \to A, \psi: B \to A, B_1, \ldots, B_n)
\end{equation}

where $\varphi$ is a homomorphism and $\psi$ an epimorphism of profinite groups, $B_1, \ldots, B_n$ are subgroups of $B$, and $\psi$ maps $B_i$ isomorphically onto $\varphi(G_i)$, $i = 1, \ldots, n$. When $B$ is finite, we say $\mathcal{E}$ is finite. A weak solution of (1) is a homomorphism $\gamma: G \to B$ with $\psi \circ \gamma = \varphi$ and $\gamma(G_i) \leq B_i^{b_i}$ for some $b_i \in B$, $i = 1, \ldots, n$. Note that $\psi$ maps $B_i^{b_i}$ isomorphically onto $\varphi(G_i)^{\psi(b_i)}$. So, $\gamma(G_i) = B_i^{b_i}$.

We say $G$ is projective if each finite embedding problem $\mathcal{E}$ for $G$ where $\varphi$ is an epimorphism has a weak solution (cf. [Har, Def. 4.2]). Then every finite embedding problem $\mathcal{E}$ has a weak solution. Indeed, replace $A$ by $\varphi(G)$ and $B$ by $\psi^{-1}(\varphi(G))$ to obtain an embedding problem $\mathcal{E}'$ for $G$ with epimorphisms. By assumption, $\mathcal{E}'$ has a solution $\gamma$. This $\gamma$ is also a solution of $\mathcal{E}$.

Example 1.1: Let $G_0, G_1, \ldots, G_n$ be profinite groups with $G_0$ being free. Put $G = \prod_{k=0}^{n} G_k$. Then $(G, G_1, \ldots, G_n)$ is projective. ■

Lemma 1.2: Suppose $G = (G, G_1, \ldots, G_n)$ is a projective group structure. Then every embedding problem (1) for $G$ in which $A$ is finite and rank$(B) \leq \aleph_0$ is weakly solvable.

Proof: Assume without loss that $\varphi$ is an epimorphism. Then there is an inverse system of epimorphisms

\[ B \xrightarrow{\pi_j} B^{(j)} \xrightarrow{\psi_j} A, \quad B^{(j+1)} \xrightarrow{\psi_{j+1,j}} B^{(j)}, \quad j = 0, 1, 2, 3, \ldots \]

such that $B^{(0)} = A$, $\pi_0 = \psi$, the $B^{(j)}$ are finite groups, $\psi_{j+1} = \psi_j \circ \psi_{j+1,j}$, $\pi_j = \psi_{j+1,j} \circ \pi_{j+1}$, and $\psi: B \to A$ is the inverse limit of $\psi_j: B^{(j)} \to A$. For all $i$ and $j$ let $B_i^{(j)} = \pi_j(B_i)$.

Suppose by induction that $\gamma_j: G \to B^{(j)}$ is a homomorphism such that $\psi_j \circ \gamma_j = \varphi$ and $\gamma_j(G_i) = (B_i^{(j)})^{b_{ij}}$ with $b_{ij} \in B^{(j)}$, $i = 1, \ldots, n$. Choose $b'_i, j+1 \in B^{(j+1)}$ with
\(\psi_{j+1, j}(b_{i, j+1}^t) = b_{i, j}\). Then \(\psi_{j+1, j}\) maps \((B^j_{i, j+1})^{b_{i, j+1}}\) isomorphically onto \((B^j_i)^{b_{i, j}}\). So,

\[
(\gamma_j: G \to B^j, \psi_{j+1, j}: B^{j+1} \to B^j, (B^{j+1}_{i, j+1})^{b_{i, j+1}}, \ldots, (B^{j+1}_n)^{b_{n, j+1}})
\]

is a finite embedding problem for \(G\).

Since \(G\) is projective, there is a homomorphism \(\gamma_{j+1}: G \to B^{j+1}\) with \(\psi_{j+1, j} \circ \gamma_{j+1} = \gamma_j\) and \(\gamma_{j+1}(G_i) = (B^{j+1}_{i, j+1})^{b_{i, j+1}}\), for some \(b_{i, j+1} \in B^{j+1}\), \(i = 1, \ldots, n\). By assumption on \(\gamma_j\), we have \(\psi_{j+1} \circ \gamma_{j+1} = \varphi\).

The homomorphisms \(\gamma_j\) define a homomorphism \(\gamma: G \to B\) with \(\pi_j \circ \gamma = \gamma_j\), \(j = 0, 1, 2, \ldots\). So, \(\psi \circ \gamma = \varphi\).

Fix \(i\) between 1 and \(n\). Let \(C_j = \{b \in B_i^j \mid \gamma_j(G_i) = (B^j_i)^b\}\). By construction, \(C_j\) is a nonempty finite subset of \(B_i^j\). Moreover, \(\psi_{j+1, j}(C_{j+1}) \subseteq C_j\). Hence, there is \(b_i \in B\) with \(\pi_j(b_i) \in C_j\) for \(j = 0, 1, 2, \ldots\). For each \(j\) we have \(\pi_j(\gamma(G_i)) = \pi_j(B_i^{b_i})\).

Hence, \(\gamma(G_i) = B_i^{b_i}\). Therefore, \(\gamma\) is a weak solution of (1).

**Lemma 1.3:** Let \(G = (G_1, \ldots, G_n)\) be a projective group structure. Suppose \(g \in G\) and \(G_i \cap G_j^g \neq 1\). Then \(i = j\) and \(g \in G_i\).

**Proof:** There is an epimorphism \(\varphi_0: G \to A_0\) with \(A_0\) finite and \(\varphi_0(G_i \cap G_j^g) \neq 1\). Consider an arbitrary epimorphism \(\varphi: G \to A\) with \(A\) finite and \(\text{Ker}(\varphi) \leq \text{Ker}(\varphi_0)\). Then \(\varphi(G_i \cap G_j^g) \neq 1\). Thus, there are \(g_i \in G_i\) and \(g_j \in G_j\) with \(g_i = g_j^g\) and \(\varphi(g_i) \neq 1\).

Let \(A_k = \varphi(G_k), k = 1, \ldots, n\). Put \(A_0 = A\). Consider the free profinite product \(A^* = \prod_{k=0}^{n} A_k\) together with the epimorphism \(\psi: A^* \to A\) whose restriction to \(A_k\) is the identity map, \(k = 0, 1, \ldots, n\).

The group \(A^*\) is infinite, but its rank is finite. Since \(G\) is projective, Lemma 1.2 gives a homomorphism \(\gamma: G \to A^*\) with \(\psi \circ \gamma = \varphi\) and \(\gamma(G_k) = A_k^\varphi\) for some \(a_k^\varphi \in A^*\); in particular, \(\psi(A_k^\varphi) = \varphi(G_k) = A_k, k = 1, \ldots, n\).

By the first paragraph, \(\gamma(g_i) = \gamma(g_j)^\gamma(g)\) and \(\psi(\gamma(g_i)) = \varphi(g_i) \neq 1\), which implies \(\gamma(g_i) \neq 1\). Hence, \(A_i^{\varphi} \cap A_j^{\varphi} \gamma(g) \neq 1\) in \(A^*\). Using the epimorphism \(A^* \to \prod_{k=0}^{n} A_k\) which is the identity map on each \(A_i\), we find that \(i = j\). By [HeR, Thm. B'], \(\gamma(g) \in A_i^{\varphi}\).

So, \(\varphi(g) \in \psi(A_i^{\varphi}) = \varphi(G_i)\). Since this holds for all \(\varphi\) as above, \(g \in G_i\).
Lemma 1.4: Suppose $G$ is a projective group structure. Then every finite embedding problem (1) has a solution $\gamma$ with $\gamma(G_i) = B_i^\gamma$ and $\psi(b_i) = 1$, $i = 1, \ldots, n$.

Proof: Without loss $\varphi$ is an epimorphism and $G_i \neq 1$, $i = 1, \ldots, n$. Let $i$ be between 1 and $n$. Consider $g \in G \setminus G_i \text{Ker}(\varphi)$. In particular, $g \notin G_i$. By Lemma 1.3, $G_i^g \neq G_i$. Hence, there is an open normal subgroup $N_{i,g} \leq \text{Ker}(\varphi)$ with $G_i^g N_{i,g} \neq G_i N_{i,g}$. The collection of all open sets $g N_{i,g}$ covers the compact set $G \setminus G_i \text{Ker}(\varphi)$. Hence, there are $g_1, \ldots, g_m$, depending on $i$, with

$$G \setminus G_i \text{Ker}(\varphi) = \bigcup_{j=1}^{m} g_j N_{i,g_j}.$$  \hfill (2)

Let $N = \bigcap_{i,j} N_{i,g_j}$. This is an open normal subgroup of $G$. Put $\hat{A} = G/N$ and let $\hat{\varphi}: G \to \hat{A}$ be the canonical homomorphism. Then there is an epimorphism $\alpha: \hat{A} \to A$ with $\alpha \circ \hat{\varphi} = \varphi$. Let $A_i = \varphi(G_i)$ and $\hat{A}_i = \hat{\varphi}(G_i)$.

Consider $a \in A \setminus A_i$. Choose $g \in G$ with $\varphi(g) = a$. Then $g \in G \setminus G_i \text{Ker}(\varphi)$. So, in the notation of (2), $g \in g_j N_{i,g_j}$ for some $j$. By definition, $G_i^g N_{i,g_j} \neq G_i N_{i,g_j}$. So, $G_i^g N_{i,g_j} \neq G_i N_{i,g_j}$. Hence, $G_i^g N \neq G_i N$ and therefore $\hat{A}_i^{\hat{\varphi}(g)} \neq \hat{A}_i$. Consequently, (3) if $\hat{a} \in \hat{A}$ and $\hat{A}_i^\hat{a} = \hat{A}_i$, then $\alpha(\hat{a}) \in A_i$.

Consider now the fiber product $\hat{B} = B \times_A \hat{A}$. Let $\beta: \hat{B} \to B$ and $\hat{\psi}: \hat{B} \to \hat{A}$ be the corresponding projections. For each $i$ let $\hat{B}_i = \{\hat{b} \in \hat{B} \mid \hat{\psi}(\hat{b}) \in \hat{A}_i \text{ and } \beta(\hat{b}) \in B_i\}$. Then $\hat{B}_i$ is a subgroup of $\hat{B}$ which $\hat{\psi}$ maps isomorphically onto $\hat{A}_i$. Also, $\beta(\hat{B}_i) = B_i$, $i = 1, \ldots, n$. So,

$$(\hat{\varphi}: G \to \hat{A}, \hat{\psi}: \hat{B} \to \hat{A}, \hat{B}_1, \ldots, \hat{B}_n)$$

is a finite embedding problem for $G$.

By assumption, there is a homomorphism $\hat{\gamma}: G \to \hat{B}$ such that $\hat{\psi} \circ \hat{\gamma} = \hat{\varphi}$ and $\hat{\gamma}(G_i) = \hat{B}_i^\gamma$ with $\hat{b}_i^\gamma \in \hat{B}$, $i = 1, \ldots, n$. Let $\gamma = \beta \circ \hat{\gamma}$, $b_i' = \beta(\hat{b}_i')$, $\hat{\alpha}_i' = \hat{\psi}(\hat{b}_i')$, and $\alpha_i' = \alpha(\hat{b}_i')$, $i = 1, \ldots, n$. Then $\psi \circ \gamma = \varphi$ and $\hat{A}_i^{\alpha_i'} = \hat{\psi}(\hat{B}_i^{b_i'}) = \hat{\varphi}(G_i) = \hat{A}_i$. By (3),
\( \alpha_i' \in A_i \).

There is (a unique) \( c_i \in B_i \) with \( \psi(c_i) = \alpha_i' \). Let \( b_i = c_i^{-1}b_i' \). Then \( \psi(b_i) = 1 \) and \( B_i^{b_i} = B_i^{b_i'} = \gamma(G_i) \), \( i = 1, \ldots, n \), as desired.

**Proposition 1.5:** Let \( G \) be a projective group structure. Then every embedding problem for \( G \) is solvable.

**Proof:** Let (1) be an embedding problem for \( G \). Assume without loss that \( \varphi \) and \( \psi \) are epimorphisms. Denote \( \text{Ker}(\psi) \) by \( K \).

**Part A:** Suppose \( K \) is finite. Then \( \hat{K} = K \setminus \{1\} \) is closed in \( B \). By assumption, \( B_i \cap \hat{K} = \emptyset \), \( i = 1, \ldots, n \). Hence, \( B \) has an open normal subgroup \( N \) with \( N \cap \hat{K} = \emptyset \) and \( B_iN \cap \hat{K}N = \emptyset \), \( i = 1, \ldots, n \). It follows that \( N \cap K = 1 \) and \( B_iN \cap KN = N \), \( i = 1, \ldots, n \). Let \( \bar{B} = B/N \), \( \bar{A} = A/\psi(N) \), \( \alpha: A \to \bar{A} \) and \( \beta: B \to \bar{B} \) be the quotient maps, and \( \bar{\psi}: \bar{B} \to \bar{A} \) the map induced by \( \psi \). Then \( \beta(K) = \text{Ker}(\bar{\psi}) \) and \( B = \bar{B} \times \bar{A} \).

Let \( \bar{\varphi} = \alpha \circ \varphi \). For each \( i \) let \( A_i = \varphi(G_i) \), \( \bar{A}_i = \alpha(A_i) \), and \( \bar{B}_i = \beta(B_i) \). From \( B_iN \cap KN = N \) it follows that \( \bar{B}_i \cap \text{Ker}(\bar{\psi}) = 1 \). So, \( \bar{\psi} \) maps \( \bar{B}_i \) isomorphically onto \( \bar{A}_i \), \( i = 1, \ldots, n \). This gives a finite embedding problem \( \bar{\mathcal{E}} = (\bar{\varphi}: G \to \bar{A}, \bar{\psi}: \bar{B} \to \bar{A}, \bar{B}_1, \ldots, \bar{B}_n) \) for \( G \).

Lemma 1.4 gives a homomorphism \( \bar{\gamma}: G \to \bar{B} \) such that \( \bar{\psi} \circ \bar{\gamma} = \bar{\varphi} \) and \( \bar{\gamma}(G_i) = \bar{B}_i^{\bar{b}_i} \) with \( \bar{b}_i \in \bar{B} \) and \( \bar{\psi}(\bar{b}_i) = 1 \). By the properties of fiber products, there is a homomorphism \( \gamma: G \to B \) with \( \psi \circ \gamma = \varphi \) and \( \beta \circ \gamma = \bar{\gamma} \).

(4)
Also, for each $i$ there is a $b_i \in B$ with $\beta(b_i) = \bar{b}_i$ and $\psi(b_i) = 1$. Let $g \in G_i$. Then $\varphi(g) \in A_i = \psi(B_i^{b_i})$. Hence, there is $b \in B_i^{b_i}$ with $\psi(b) = \varphi(g)$. It satisfies $\beta(b) \in \bar{B}_i^{\bar{b}_i}$ and $\tilde{\psi}(\beta(b)) = \alpha(\psi(b)) = \alpha(\varphi(g)) = \tilde{\psi}(\gamma(g))$. Since $\tilde{\psi}: \bar{B}_i^{\bar{b}_i} \rightarrow \bar{A}_i$ is injective, $\beta(b) = \bar{\gamma}(g)$. In addition, $\beta(\gamma(g)) = \bar{\gamma}(g)$ and $\psi(\gamma(g)) = \varphi(g)$. It follows that $\gamma(g) = b \in B_i^{b_i}$. So, $\gamma(G_i) \leq B_i^{b_i}$. Consequently, $\gamma$ is a solution to the embedding problem (1).

**PART B: Application of Zorn’s lemma.** Suppose (1) is an arbitrary embedding problem for $G$. For each normal subgroup $L$ of $B$ which is contained in $K$ let $\psi_L: B/L \rightarrow A$ be the epimorphism $\psi_L(bL) = \psi(b)$, $b \in B$. It maps $B_iL/L$ isomorphically onto $A_i = \varphi(G_i)$. This gives an embedding problem

$$\quad(5) \quad (\varphi: G \rightarrow A, \psi_L: B/L \rightarrow A, B_1L/L, \ldots, B_nL/L).$$

Let $\Lambda$ be the set of pairs $(L, \lambda)$, where $L$ is a closed normal subgroup of $B$ contained in $K$ and $\lambda$ is a solution of (5). The pair $(K, \psi_K^{-1} \circ \varphi)$ belongs to $\Lambda$. Partially order $\Lambda$ by $(L', \lambda') \leq (L, \lambda)$ if $L' \leq L$ and $\psi_{L',L} \circ \lambda' = \lambda$. Here $\psi_{L',L}: B/L' \rightarrow B/L$ is the epimorphism $\psi_{L',L}(bL') = bL$, $b \in B$.

Suppose $\Lambda_0 = \{(L_j, \lambda_j) \mid j \in J\}$ is a descending chain in $\Lambda$. Then $\varprojlim B/L_j = B/L$ with $L = \bigcap_{j \in J} L_j$. The $\lambda_j$’s define a homomorphism $\lambda: G \rightarrow B/L$ with $\psi_{L,L_j} \circ \lambda = \lambda_j$ for each $j$. For each $i$ a compactness argument gives $b_i \in B$ with $\lambda(G_i) = B_i^{b_i}L/L$. Thus, $(L, \lambda)$ is a lower bound to $\Lambda_0$.

Zorn’s lemma gives a minimal element $(L, \lambda)$ for $\Lambda$. It suffices to prove that $L = 1$.

Assume $L \neq 1$. Then $B$ has an open normal subgroup $N$ with $L \not\leq N$. So, $L' = N \cap L$ is a proper open subgroup of $L$ which is normal in $B$. For each $i$ choose $b_i \in B$ with $\lambda(G_i) = B_i^{b_i}L/L$. Then $(\lambda: G \rightarrow B/L, \psi_{L',L}: B/L' \rightarrow B/L, B_1^{b_1}L'/L, \ldots, B_n^{b_n}L'/L)$ is an embedding problem for $G$. Its kernel $\text{Ker}(\psi_{L',L}) = L/L'$ is a finite group. By Part A, it has a solution $\lambda'$. The pair $(L', \lambda')$ is an element of $\Lambda$ which is strictly smaller than $(L, \lambda)$. This contradiction to the minimality of $(L, \lambda)$ proves that $L = 1$, as desired.

**Corollary 1.6:** Let $G = (G, G_1, \ldots, G_n)$ and $G' = (G', G'_1, \ldots, G'_n)$ be $n$-fold group structures with $G$ projective. Let $\psi: G' \rightarrow G$ be an epimorphism which maps $G'_i$
isomorphically onto $G_i$, $i = 1, \ldots, n$. Then there is a monomorphism $\psi': G \to G'$ with $\psi \circ \psi' = \text{id}_G$ and elements $a_1, \ldots, a_n \in G'$ with $\psi'(G_i) = (G_i')^a_i$ and $\psi(a_i) = 1$, $i = 1, \ldots, n$.

**Proof:** An application of Proposition 1.5 to the embedding problem

$$(\text{id}_G: G \to G, \, \psi: G' \to G, G'_1, \ldots, G'_n)$$

gives a section $\psi': G \to G'$ of $\psi$ and elements $a'_1, \ldots, a'_n \in G'$ with $\psi'(G_i) = (G_i')^a_i$. Thus, $G_i = G_i^\psi(a'_i)$. By Lemma 1.3, $\psi(a'_i) \in G_i$.

Choose $b_i \in G'_i$ with $\psi(b_i) = \psi(a'_i)$. Let $a_i = b_i^{-1}a'_i$. Then $(G_i')^a_i = (G_i')^a_i = \psi'(G_i)$ and $\psi(a_i) = 1$, $i = 1, \ldots, n$. ■
2. Unirationally closed \(n\)-fold field structures

Consider a field \(K\) and separable algebraic extensions \(K_1, \ldots, K_n\) (with \(n \geq 0\)). Refer to \(K = (K, K_1, \ldots, K_n)\) as a field structure (or as an \(n\)-fold field structure if \(n\) is not clear from the context). By an absolutely irreducible variety over \(K\) we mean a geometrically integral scheme of finite type over \(K\). (In the language of Weil’s Foundation, this is a variety defined over \(K\).) Let \(r\) be a positive integer and \(V\) an absolutely irreducible variety over \(K\). For each \(i\) let \(U_i\) be an absolutely irreducible variety over \(K_i\) birationally equivalent to \(\mathbb{A}^r_{K_i}\) and \(\varphi_i: U_i \to V \times_K K_i\) be a dominant separable morphism (of varieties over \(K_i\)). Refer to

\[
\Phi = (V, \varphi_1: U_1 \to V \times_K K_1, \ldots, \varphi_n: U_n \to V \times_K K_n)
\]

as a unirational arithmetical problem for \(K\). A solution to \(\Phi\) is a tuple \((a, b_1, \ldots, b_n)\) with \(a \in V(K)\), \(b_i \in U_i(K_i)\), and \(\varphi_i(b_i) = a\) for \(i = 1, \ldots, n\). Call \(K\) unirationally closed if each unirational arithmetical problem has a solution.

Associate to \(K\) its absolute Galois group structure

\[
\text{Gal}(K) = (\text{Gal}(K), \text{Gal}(K_1), \ldots, \text{Gal}(K_n)).
\]

**Proposition 2.1:** Let \(K = (K, K_1, \ldots, K_n)\) be a unirationally closed field structure. Then \(\text{Gal}(K)\) is a projective group structure.

**Proof:** By [HJK, Lemma 3.1] it suffices to weakly solve each embedding problem

\[
(\text{res}: \text{Gal}(K) \to \text{Gal}(L/K), \text{res}: \text{Gal}(F/E) \to \text{Gal}(L/K), \text{Gal}(F_{i}/F_{1}), \ldots, \text{Gal}(F_{i}/F_{n}))
\]

satisfying the following conditions: \(L/K\) is a finite Galois extension, \(E\) is a finitely generated regular extension of \(K\), \(F\) is a finite Galois extension of \(E\) which contains \(L\), \(F_i\) is a finite subextension of \(F/E\) that contains \(L_i = K_i \cap L\), \(F_i/L_i\) is a purely transcendental extension of transcendence degree \(r = [F : E]\), and \(\text{res}: \text{Gal}(F_{i}/F_{i}) \to \text{Gal}(L/L_i)\) is an isomorphism, \(i = 1, \ldots, n\).

It is possible to choose \(x_1, \ldots, x_k \in E, y_i \in F_i, i = 1, \ldots, n,\) and \(z \in F\) with this:

\[
(2a) \quad E = K(x) \quad \text{and} \quad V = \text{Spec}(K[x]) \quad \text{is a smooth subvariety of} \quad \mathbb{A}^k_K \quad \text{with generic point} \quad x.
\]

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(2b) For each \( i \), \( F_i = L_i(\mathbf{x}, y_i) \) and \( U_i = \text{Spec}(L_i[\mathbf{x}, y_i]) \) is a smooth subvariety of \( \mathbb{A}^{k+1}_{L_i} \) with generic point \((\mathbf{x}, y_i)\).

(2c) \( y_i \) is integral over \( L_i[\mathbf{x}] \) and the discriminant of \( \text{irr}(y_i, L_i(\mathbf{x})) \) is a unit of \( L_i[\mathbf{x}] \).

Thus, \( L_i[\mathbf{x}, y_i]/L_i[\mathbf{x}] \) is, in the terminology of [FrJ, Definition 5.4], a ring cover.

So, the projection on the first \( k \) coordinates is an étale morphism \( \pi_i: U_i \to V \times_K L_i \).

(2d) \( F = K(\mathbf{x}, z) \) and \( L[\mathbf{x}, z]/L[\mathbf{x}] \) is a ring cover.

By assumption, there are points \( \mathbf{a} \in V(K) \) and \( (\mathbf{a}, b_i) \in U_i(K_i), i = 1, \ldots, n \).

Since \( \mathbf{a} \) is simple on \( V \), there is a \( K \)-place \( \rho_0: E \to K \cup \{\infty\} \) with \( \rho_0(\mathbf{x}) = \mathbf{a} \) [JaR, Cor. A2]. Extend \( \rho_0 \) to an \( L \)-place \( \rho: F \to \bar{K} \cup \{\infty\} \). Let \( \bar{F} \cup \{\infty\} \) be the residue field of \( \rho \). By (2d) and [FrJ, Lemma 5.5], \( \bar{F} \) is a finite Galois extension of \( K \) which contains \( L \).

Moreover, there is an embedding \( \rho^*: \text{Gal}(\bar{F}/K) \to \text{Gal}(F/E) \) with \( \rho(\rho^*(\sigma)u) = \sigma(\rho(u)) \) for each \( \sigma \in \text{Gal}(\bar{F}/K) \) and \( u \in F \) with \( \rho(u) \neq \infty \). Let \( \gamma = \rho^* \circ \text{res}_{K_i/\bar{F}} \). This is a homomorphism from \( \text{Gal}(K) \) to \( \text{Gal}(F/E) \) with \( \text{res}_{F/L} \circ \gamma = \text{res}_{K_i/L} \).

For each \( i \), (2c) gives an \( L_i \)-place \( \rho_i: F_i \to \bar{K_i} \cup \{\infty\} \) which extends \( \rho_0 \) such that \( \rho_i(\mathbf{x}, y_i) = (\mathbf{a}, b_i) \). Extend \( \rho_i \) to an \( L \)-place \( \rho_i: F \to \bar{K} \cup \{\infty\} \). Since \( \rho_i|_{EL} = \rho|_{EL} \), there is \( \sigma_i \in \text{Gal}(F/EL) \) with \( \rho_i = \rho \circ \sigma_i^{-1} \).

Thus, \( \rho(F_i^{\sigma_i}) = \rho \circ \sigma_i^{-1}(F_i) = \rho_i(F_i) \subseteq L_i(b_i) \cup \{\infty\} \).

This implies \( \gamma(\text{Gal}(K_i)) \leq \gamma(\text{Gal}(L_i(b_i)) = \rho^*(\bar{F}/L_i(b_i)) \leq \text{Gal}(F/F_i)^{\sigma_i} \).

Consequently, \( \gamma \) is a solution of the embedding problem.  ✷
3. Pseudo closed fields

Let $n \geq 0$. A field structure $K = (K, K_1, \ldots, K_n)$ is **pseudo closed** if every absolutely irreducible variety $V$ over $K$ with $K_i$-rational simple points has a $K$-rational point. In this case we also say $K$ is **pseudo closed** with respect to $K_1, \ldots, K_n$.

**Lemma 3.1:** Suppose $0 \leq m \leq n$.

(a) Let $(G, G_1, \ldots, G_m)$ be a projective group structure. Then $(G, G_1, \ldots, G_m, \overbrace{1, \ldots, 1}^{(n-m)})$ is projective.

(b) Let $(K, K_1, \ldots, K_n)$ be a pseudo closed field structure. Suppose for each $m < i \leq n$ either $K_i = K_s$ or there is $1 \leq j \leq m$ with $K_j \subseteq K_i$. Then $(K, K_1, \ldots, K_m)$ is pseudo closed.

**Proof of (a):** Standard checking.

**Proof of (b):** Use that $V_{\text{simp}}(K_s) \neq \emptyset$ for every absolutely irreducible variety $V$ over $K_s$. □

A **field-valuation structure** is a tuple $K = (K, K_1, v_1, \ldots, K_n, v_n)$ such that $(K, K_1, \ldots, K_n)$ is a field structure and $v_i$ is a valuation of $K_i$, $i = 1, \ldots, n$. If $(K_i, v_i)$ is Henselian, then $v_i$ has a unique extension to $K_s$ which we also denote by $v_i$. We say $v_1, \ldots, v_n$ are **independent** if for all $1 \leq i \neq j \leq n$ the ring generated by the valuation rings of the restrictions of $v_i$ and $v_j$ to $K$ is $K$. This is equivalent to the weak approximation theorem [Jar, Prop. 4.2 and 4.4]. The **absolute Galois structure** of $K$ is the one associated with $(K, K_1, \ldots, K_n)$, namely, $\text{Gal}(K) = (\text{Gal}(K), \text{Gal}(K_1), \ldots, \text{Gal}(K_n))$.

**Proposition 3.2:** Let $K = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure. Suppose $(K_i, v_i)$ is a Henselian closure of $K$ at $v_i$, $i = 1, \ldots, n$, the valuations $v_1, \ldots, v_n$ are independent, and $K$ is pseudo closed with respect to $K_1, \ldots, K_n$. Then $\text{Gal}(K)$ is projective.

**Proof:** By Lemma 3.1 we may assume $K_i \neq K_s$, $i = 1, \ldots, n$. By [Jar, Lemma 13.2], $K_i \not\subseteq K_j$ for $i \neq j$. By Proposition 2.1 it suffices to show that $(K, K_1, \ldots, K_n)$ is unirationally closed.
Consider a unirational arithmetical problem $\Phi$ for $K$ as in (1) of Section 2. Let $V_i = V \times_K K_i, i = 1, \ldots, n$. Since $U_i$ is a rational variety, it is smooth and there is a point $b'_i \in U_i(K_i)$. Let $a_i = \varphi_i(b'_i)$. By [GPR, Cor. 9.5], $b'_i$ has a $v_i$-open neighborhood $\mathcal{U}_i$ in $U(K_i)$ which $\varphi_i$ maps $v_i$-homeomorphically onto a $v_i$-open neighborhood $\mathcal{V}_i$ of $a_i$ in $V_i(K_i)$.

Since $K$ is pseudo closed and $K_i \not\subseteq K_j$ for $i \neq j$, [HeP, Thm. 1.9] gives a point $a \in V(K)$ which belongs to $\mathcal{V}_i$, $i = 1, \ldots, n$. Hence, there is a $b_i \in U_i(K_i)$ with $\varphi_i(b_i) = a$, $i = 1, \ldots, n$. Note that [HeP, p. 298] makes the assumption $\text{char}(K) = 0$. Nevertheless, the proof of [HeP, Thm. 1.9] is also valid in positive characteristic. See also [Sch, Thm. 4.9] which generalizes [HeP, Thm. 1.9]. Therefore, $K$ is unirationally closed.

An isomorphism $\alpha: (G,G_1,\ldots,G_n) \to (G',G'_1,\ldots,G'_n)$ of group structures is an isomorphism $\alpha: G \to G'$ of groups with $\alpha(G_i) = G'_i$, $i = 1, \ldots, n$.

**Lemma 3.3:** Let $G = (G,G_1,\ldots,G_n)$ be a projective group structure. Suppose each $G_i$ is an absolute Galois group. Then there is a field structure $K$ of characteristic 0 with $\text{Gal}(K) \cong G$. If each $G_i$ is an absolute Galois group of a field of characteristic $p$ independent of $i$, then $K$ may be chosen to be of characteristic $p$.

**Proof:** Let $\hat{F}_m$ be the free profinite group of rank $m \geq \text{rank}(G)$. Since $\hat{F}_m$ is projective [FrJ, Example 20.13], it is an absolute Galois group in each characteristic [FrJ, Cor. 20.16]. Put $G^* = \hat{F}_m \ast \prod_{i=1}^n G_i$. By [HJK, Thm. 3.4], $G^* \cong \text{Gal}(F)$ for a field $F$ of characteristic 0. If there is $p$ such that each $G_i$ with $i \geq 1$ is a Galois group in characteristic $p$, then we may choose $F$ to be of characteristic $p$.

By [FrJ, Cor. 15.20] there is an epimorphism $\psi_0: \hat{F}_m \to G$. Let $\psi: G^* \to G$ be the unique epimorphism that extends $\psi_0$ and the identity maps of $G_1,\ldots,G_n$. Corollary 1.6 gives an embedding of $G$ into $G^*$. Let $K$ be the fixed field of $G$ in $F_\psi$. For each $i \geq 1$ let $K_i$ be the fixed field of $G_i$ in $F_\psi$. Then $\text{Gal}(K,K_1,\ldots,K_n) \cong G$.  

Let $K = (K,K_1,\ldots,K_n,v_n)$ and $K' = (K',K'_1,\ldots,K'_n,v'_n)$ be field-valuation structures. We say $K'$ is an extension of $K$ if $K \subseteq K'$, $K_i = K'_i \cap K$, and $v_i$ is the restriction of $v'_i$ to $K_i$, $i = 1, \ldots, n$. In this case $K$ is a substructure of $K'$. 

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Let \((K, v)\) be a valued field. For \(a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r) \in K^r\) we write \(v(a - b) = \min_j v(a_j - b_j)\).

**Lemma 3.4:** Let \(K = (K, K_1, v_1, \ldots, K_n, v_n)\) be a field-valuation structure and let \(\bar{K} = (\bar{K}, \bar{K}_1, \bar{v}_1, \ldots, \bar{K}_n, \bar{v}_n)\) be a substructure of \(K\). Assume:

1a) \(\bar{K}_i\) is perfect and \(\bar{v}_i\) is trivial, \(i = 1, \ldots, n\).

1b) \(\text{Gal}(\bar{K})\) is projective.

1c) \((K, v_i)\) is a Henselian field with residue field \(K_i, i = 1, \ldots, n\).

1d) \(\text{res}: \text{Gal}(K) \rightarrow \text{Gal}(\bar{K})\) is an isomorphism.

Suppose \(V \subseteq \mathbb{A}^r\) is an affine variety over \(K\) and \(b_i \in V_{\text{simp}}(K_i), i = 1, \ldots, n\). Then \(K\) has an extension \(K' = (K', K'_1, v'_1, \ldots, K'_n, v'_n)\) with these properties:

2a) \((K', v'_i)\) is a Henselian field with residue field \(K_i, i = 1, \ldots, n\).

2b) \(\text{res}: \text{Gal}(K') \rightarrow \text{Gal}(K)\) is an isomorphism.

2c) There is \(x \in V(K')\) with \(v'_i(x - b_i) > \gamma\) for each \(\gamma \in v_i(K_i^\times), i = 1, \ldots, n\).

**Proof:** Let \(x\) be a generic point of \(V\) over \(K\) and let \(F = K(x)\). For each \(i\) put \(M_i = K_i(x)\). Then [JaR, p. 456, Cor. 2] gives a \(K_i\)-place \(\varphi_i: M_i \rightarrow K_i \cup \{\infty\}\) with \(\varphi_i(x) = b_i\). Now let \(\rho_i: K_i \rightarrow \bar{K}_i \cup \{\infty\}\) be the \(\bar{K}_i\)-place associated with \(v_i\). The compositum \(\varphi'_i = \rho_i \circ \varphi_i: M_i \rightarrow \bar{K}_i \cup \{\infty\}\) is a \(\bar{K}_i\)-place of \(M_i\) that extends \(\rho_i\). Denote the corresponding valuation of \(M_i\) by \(w_i\). Then \(w_i\) extends \(v_i\), \(\bar{K}_i\) is the residue field of \(w_i\), and for every \(c \in K_i^\times\) and every coordinate \(1 \leq j \leq r, \varphi'_i(x - b_j) = \rho_i\left(\frac{c}{\infty}\right) = 0\).

Thus, \(w_i(x - b_i) > w(c), i = 1, \ldots, n\).

Extend \(w_i\) to a Henselization \(M'_i\) of \((M_i, w_i)\). By [HJK, Prop. 2.4], \((M'_i, w_i)\) has a separable algebraic extension \((N_i, w_i)\) such that the map \(\text{res}: \text{Gal}(N_i) \rightarrow \text{Gal}(K_i)\) is an isomorphism and \(\bar{K}_i\) is the residue field of \(N_i\). In particular, \(N_i\) is Henselian.

By (1b) and (1d), \(\text{Gal}(K)\) is projective. So, we may apply Corollary 1.6 to the map \(\text{res}: \text{Gal}(F) \rightarrow \text{Gal}(K)\) with the isomorphisms \(\text{res}: \text{Gal}(N_i) \rightarrow \text{Gal}(K_i), i = 1, \ldots, n\). This gives elements \(\sigma_1, \ldots, \sigma_n \in \text{Gal}(F)\) such that \(\sigma_i|_{K_i} = \text{id}, i = 1, \ldots, n\), and an \(n\)-fold field structure \((K', (N_1)^{\sigma_1}, \ldots, (N_n)^{\sigma_n})\) such that

\[
\text{res}: \text{Gal}(K', (N_1)^{\sigma_1}, \ldots, (N_n)^{\sigma_n}) \rightarrow \text{Gal}(K, K_1, \ldots, K_n)
\]
is an isomorphism.

Finally let $K'_i = N^\sigma_i$ and $v'_i = w_i \circ \sigma_i^{-1}$, $i = 1, \ldots, n$. Note that $\sigma_i$ fixes $x$ as well as each element of $K_s$. So, (2) holds.

Let $K = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure. We say $K$ is pseudo-closed with the approximation property if it has this property:

(3) Suppose $V \subseteq A^r$ is an affine absolutely irreducible variety over $K$, $a_i \in V_{\text{simp}}(K_i)$, and $\gamma_i \in v_i(K_i^\times)$, $i = 1, \ldots, n$. Then there is $a \in V(K)$ with $v_i(a - a_i) > \gamma_i$, $i = 1, \ldots, n$.

**Proposition 3.5:** Let $K = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure and $K = (\bar{K}, \bar{K}_1, \bar{v}_1, \ldots, \bar{K}_n, \bar{v}_n)$ a substructure of $K$ satisfying conditions (1).

Then $K$ has an extension $K' = (K', K_1', v_1', \ldots, K_n', v_n')$ with these properties:

(4a) $(K'_i, v'_i)$ is a Henselian field with residue field $\bar{K}_i$, $i = 1, \ldots, n$.

(4b) The map res: $\text{Gal}(K') \to \text{Gal}(K)$ is an isomorphism.

(4c) $K'$ is pseudo closed with the approximation property.

**Proof:** Well-order all tuples $(V, b_1, \ldots, b_n)$ where $V$ is an affine absolutely irreducible variety over $K$ and $b_i \in V_{\text{simp}}(K_i)$, $i = 1, \ldots, n$. Use transfinite induction and Lemma 3.4 to construct a transfinite tower of field-valuation structures whose union is a field-valuation structure $L_1 = (L_1, L_{1,1}, v_{1,1}, \ldots, L_{1,n}, v_{1,n})$ with these properties:

(5a) $(L_{1,i}, v_{1,i})$ is a Henselian field with residue field $\bar{K}_i$, $i = 1, \ldots, n$.

(5b) The map res: $\text{Gal}(L_1) \to \text{Gal}(K)$ is an isomorphism.

(5c) Suppose $V$ is an absolutely irreducible affine variety over $K$ and $b_i \in V_{\text{simp}}(K_i)$, $i = 1, \ldots, n$. Then there is $x \in V(L_1)$ with $v_{1,i}(x - b_i) > \gamma_i$ for all $\gamma_i \in v_i(K_i^\times)$, $i = 1, \ldots, n$.

Use ordinary induction to construct an ascending sequence of $n$-fold field-valuation structures $L_j$, $j = 1, 2, 3, \ldots$ with $L_{j+1}$ relating to $L_j$ as $L_1$ relates to $K$, $j = 1, 2, 3, \ldots$. Then $K' = \bigcup_{j=1}^{\infty} L_j$ satisfies (4).

**Lemma 3.6:** Let $(K, v)$ be a Henselian field and $L$ a separable algebraic extension of $K$. Suppose $K$ is $v$-dense in $L$. Then $K = L$.
Proof: Consider $x \in L$ and let $x_1, \ldots, x_n$ be the conjugates of $x$ over $K$. By assumption, there is $y \in K$ with $v(y - x) > \max_{i \neq j} v(x_i - x_j)$. By Krasner's Lemma [Jar, Lemma 12.1], $K(x) \subseteq K(y) = K$. Therefore, $x \in K$. ■

Theorem 3.7: Let $G = (G, G_1, \ldots, G_n)$ be a projective group structure. Suppose each $G_i$ is an absolute Galois group. Then $G$ is the group structure of a field structure $K = (K, K_1, \ldots, K_n)$ with these properties: $\text{char}(K) = 0$, $K_i$ is the Henselian closure of $K$ at a valuation $v_i$, $i = 1, \ldots, n$, and $(K, K_1, v_1, \ldots, K_n, v_n)$ is pseudo closed with the approximation property. If all $G_i$ are absolute Galois groups of fields of the same characteristic $p$, then $K$ can be chosen to have characteristic $p$.

Proof: Lemma 3.3 gives a field structure $(E, E_1, \ldots, E_n)$ with $G \cong \text{Gal}(E, E_1, \ldots, E_n)$. Let $\tilde{v}_i$ be the trivial valuation of $\tilde{E}_i$. Put $\tilde{E} = (\tilde{E}, \tilde{E}_1, \tilde{v}_1, \ldots, \tilde{E}_n, \tilde{v}_n)$.

The pair $(\tilde{E}, \tilde{E})$ has all properties that $(\tilde{K}, K)$ of Proposition 3.5 has. So, Proposition 3.5 gives an extension $K = (K, K_1, v_1, \ldots, K_n, v_n)$ of $E$ with these properties:

(6a) $(K_i, v_i)$ is a Henselian field, $i = 1, \ldots, n$.

(6b) The map $\text{res}: \text{Gal}(K) \rightarrow \text{Gal}(E)$ is an isomorphism.

(6c) $K$ is pseudo closed with the approximation property.

By (6b), $\text{Gal}(K) \cong G$. By (6a), $(K, v_i)$ has a Henselian closure $(H_i, v_i)$ which is contained in $(K_i, v_i)$. By (6c) applied to $A^1_K$, $K$ is $v_i$-dense in $K_i$. Hence, $H_i$ is $v_i$-dense in $K_i$. Therefore, by Lemma 3.6, $(K_i, v_i)$ is the Henselian closure of $K$ at $v_i$. ■

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