Real Hilbertianity and the field of totally real numbers

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Abstract: We use moduli spaces for covers of the Riemann sphere to solve regular embedding problems, with prescribed extendability of orderings, over PRC fields. As a corollary we show that the elementary theory of $\mathbb{Q}^{\mathrm{tr}}$ is decidable.

Introduction

The theory and use in [F] of moduli spaces of covers of the Riemann sphere with prescribed ramification data has been further developed in [FV1]. There the main theme is that K-rational points of the moduli spaces correspond to covers defined over K. Furthermore, [FV2] notes a correspondence between existence of K-rational points on certain related spaces and the solvability of regular embedding problems over K. Thus, using moduli spaces allows us to prove solvability of regular embedding problems over fields K suitably large for such varieties to have the requisite K-rational points.

This principle appears in [FV2] to show that the absolute Galois group of a countable Hilbertian PAC field of characteristic 0 is free. The natural extension of this to the (larger) class of Hilbertian PRC fields appears in [FV3].

Recall [FJ, p. 129] that K is **PAC (pseudo algebraically closed)** if every absolutely irreducible variety V defined over K has a K-rational point. Furthermore [P2], K is **PRC (pseudo real closed)** if every absolutely irreducible variety V defined over K has a K-rational point, provided that V has a non-singular point over each real closure of K. The latter condition on V is equivalent to the following one: the function field $K(V)$ of V is a **totally real** extension of K, that is, every ordering on K extends to $K(V)$.

Consider the field $\mathbb{Q}^{\mathrm{tr}}$ of all totally real algebraic numbers. It is the fixed field of all involutions in the absolute Galois group $G(\mathbb{Q})$ of $\mathbb{Q}$. By a recent result of Pop [P], $\mathbb{Q}^{\mathrm{tr}}$ is

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PRC. By Weissauer's theorem [FJ, Proposition 12.4] every proper finite extension of $\mathbb{Q}^{\rm tr}$ is Hilbertian. Also, by Prestel's extension theorem [P2, Theorem 3.1] every algebraic extension of $\mathbb{Q}^{\rm tr}$ is PRC. Hence, the absolute Galois group of a proper finite extension of $\mathbb{Q}^{\rm tr}$ is known [FV3]. The field $\mathbb{Q}^{\rm tr}$ is not Hilbertian, however. For example, $Z^2-(a^2+1)$ is reducible over $\mathbb{Q}^{\rm tr}$ for every $a \in \mathbb{Q}^{\rm tr}$.

In [FHV] we cover also the case of $\mathbb{Q}^{\rm tr}$. The key observation is that $\mathbb{Q}^{\rm tr}$ satisfies a certain weakening of the Hilbertian property. This allows specializing Galois extensions of $K(x)$ whose Galois group is generated by *real* involutions to obtain Galois extensions of K with the same Galois group. As a result we determine the absolute Galois group of $\mathbb{Q}^{\rm tr}$.

In the present paper we accomplish several tasks. We extend the methods and results of [FV3] to solve regular embedding problems over a PRC field K in such a way that the orderings of K extend to prescribed subfields (Theorem 5.2). This gives new information about the absolute Galois group of the field $K(x)$ of rational functions over K (Theorem 5.3).

The first 5 sections provide the prerequisites for the proof of Theorem 5.2. We need an approximation theorem for varieties over PRC fields (section 1), supplements about the moduli spaces (section 2), some group-theoretic lemmas (section 3), and the determination of the real involutions in Galois groups over $\mathbb{R}(x)$ (section 4).

In section 6 we define the concept of real Hilbertian, and show that $\mathbb{Q}^{\rm tr}$ has this property. In section 7 we combine all the previous results to determine the absolute Galois group of a countable real Hilbertian PRC field with no proper totally real algebraic extensions, whose space of orderings has no isolated points. This group is the free product of groups of order 2, indexed by the Cantor set X_ω (Theorem 7.6). In particular, $G(\mathbb{Q}^{\rm tr}) = {\rm Aut}(\tilde{\mathbb{Q}}/\mathbb{Q}^{\rm tr})$ is isomorphic to this group.

As a corollary we introduce the notion of *real Frobenius fields*: $\mathbb{Q}^{\rm tr}$ is an example (Corollary 8.3). Following the Galois stratification procedure of [FJ, Chap. 25] and [HL] we show that the elementary theory of real Frobenius fields allows elimination of quantifiers in the appropriate language. In particular, $\mathbb{Q}^{\rm tr}$ is primitive recursively decidable (Theorem 10.1). On the other hand (we thank A. Prestel for pointing this out to us) the ring of integers of $\mathbb{Q}^{\mathrm{tr}}$ is undecidable (Julia Robinson [R2]). Thus we obtain a natural example of an undecidable ring with a decidable quotient field (cf. Robinson [R1, p. 951]). Furthermore, we give (Corollary 10.5) a system of axioms for the theory of $\mathbb{Q}^{\mathrm{tr}}$.

F. Pop has told us that the characterization of $G(\mathbb{Q}^{\mathrm{tr}})$ also follows from his results, in particular his "$\frac{1}{2}$ Riemann existence theorem." His method is based on rigid analytic geometry, versus our use of the classical Riemann existence theorem. We look forward to seeing a written account of these results.

1. Ordered fields and an approximation theorem for PRC fields

Let K be a field of characteristic 0, and let $G(K)$ be its absolute Galois group. Recall [P1, §6] that the set of orderings $X(K)$ of K is a boolean topological space in its natural Harrison topology. This topology is given by a subbase consisting of sets of the form $H(c) = \{P \in X(K) | \ c \in P\}$, for $c \in K^\times$. Here P denotes the positive elements in an ordering.

By Artin-Schreier theory [L, XI,§2], the real closures of K (inside a fixed algebraic closure $\widetilde{K}$ of K) are the fixed fields of the involutions in $G(K)$. This identifies the set $\hat{\mathcal{X}}$ of real closures of K with a topological subspace of $G(K)$. It is a boolean space, since the set of involutions is closed in $G(K)$. Observe that $\overline{H}(z) = \{R \in \hat{\mathcal{X}} | \ z \in R\}$ is open in $\hat{\mathcal{X}}$, for each $z \in \widetilde{K}^\times$.

For each $R \in \hat{\mathcal{X}}$ let $\pi(R)$ be the restriction of the unique ordering of R to K. Then $\pi(R_1) = \pi(R_2)$ if and only if R_1 and R_2 are conjugate by an automorphism of $\widetilde{K}$ over K. The map $\pi\colon \hat{\mathcal{X}} \to X(K)$ is continuous: $\pi^{-1}(H(c)) = \overline{H}(\sqrt{c})$. Moreover, there exists a closed subset $\mathcal{X}$ of $\hat{\mathcal{X}}$ such that $\pi\colon \mathcal{X} \to X(K)$ is a homeomorphism [HJ1, Corollary 9.2]. The corresponding closed subset of involutions in $G(K)$ contains exactly one representative from each conjugacy class of involutions. Having fixed such $\mathcal{X}$, identify $X(K)$ with it.

Remark 1.1: (a) If K is PRC, then every clopen subset of $X(K)$ is of the form $H(c)$ for a suitable $c \in K^\times$ [P2, Proposition 1.3].

(b) Let R be a real closed field, and let $a \in R$, $c \in R^\times$. If either $c < 0$ or $a > 0$, then the system $\quad Y^2 + cZ^2 = a, \quad Y \neq 0 \quad$ has a solution in R.

(c) For $\mathbf{X} = (X_1, \ldots, X_n)$ put $\|\mathbf{X}\|^2 = \sum_{i=1}^n X_i^2$. Let K be an ordered field, and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in K^n$ and $\nu \in K^\times$. From the triangle inequality (over the real closure of K), if $\|\mathbf{a} - \mathbf{b}\|^2, \|\mathbf{b} - \mathbf{c}\|^2 < (\frac{\nu}{2})^2$ then $\|\mathbf{a} - \mathbf{c}\|^2 < \nu^2$.

PROPOSITION 1.2: *Let K be a PRC field, and let $V \subseteq \mathbb{A}^n$ be an absolutely irreducible affine variety defined over K. Let $\mathcal{X}$ be a closed set of real closures of K, one for each ordering of K. Let $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be disjoint clopen subsets of $\mathcal{X}$ that cover $\mathcal{X}$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ be nonsingular points on V such that $\mathbf{x}_j \in V(R)$ for every $R \in \mathcal{X}_j$, for $j = 1, \ldots, m$. Let $\nu_1, \ldots, \nu_m \in K^\times$. Then there is $\mathbf{x} \in V(K)$ such that for each $1 \leq j \leq m$*

$$\|\mathbf{x} - \mathbf{x}_j\|^2 < \nu_j^2 \text{ in } R, \qquad \text{for every } R \in \mathcal{X}_j. \tag{1}$$

Proof: Fix j and put $L = K(\mathbf{x}_j)$. Let $R \in \mathcal{X}_j$. Then $L \subseteq R$. As K is dense in R [P2, Proposition 1.4], there is $\mathbf{a}_j \in K^n$ such that

$$\|\mathbf{x}_j - \mathbf{a}_j\|^2 < \left(\frac{\nu_j}{2}\right)^2 \text{ in } R. \tag{2}$$

This $\mathbf{a}_j$ depends on R, but if $R' \in \mathcal{X}_j$ is sufficiently close to R, then (2) holds also with R' instead of R. Indeed, the restriction $\mathcal{X}_j \to X(L)$ is continuous, and (2) describes a basic open set in $X(L)$. Use compactness of $\mathcal{X}_j$ to partition $\mathcal{X}_j$ into smaller clopen subsets (and thereby increase m). Associate with each of them the original point $\mathbf{x}_j$ such that (2) holds with suitable $\mathbf{a}_j$ for all $R \in \mathcal{X}_j$.

By Remark 1.1(a), for each j there is $c_j \in K^\times$ such that $\mathcal{X}_j = H(c_j)$. Suppose that V is defined by $f_1(\mathbf{X}), \ldots, f_r(\mathbf{X}) \in K[X_1, \ldots, X_n]$. These together with the additional polynomials

$$f_{r+j}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \left(\frac{\nu_j}{2}\right)^2 - \|\mathbf{X} - \mathbf{a}_j\|^2 - Y_j^2 - c_j Z_j^2, \quad j = 1, \ldots, m$$

define an absolutely irreducible variety $W \subseteq \mathbb{A}^{n+2m}$ of dimension $\dim V + m$.

Indeed, by induction on m we may assume that $m = 1$. Let $\mathbf{x}$ be the generic point of V over $\widetilde{K}$, that is, the image of $\mathbf{X}$ in in the integral domain $\widetilde{K}[V] = \widetilde{K}[\mathbf{X}]/(f_1, \ldots, f_r)$.

Let $u = \left(\frac{\nu_1}{2}\right)^2 - \|\mathbf{x}-\mathbf{a}_1\|^2$, and let y_1 be transcendental over $\widetilde{K}(V)$. Observe that $u \neq 0$, since, by (2), $\left(\frac{\nu_1}{2}\right)^2 - \|\mathbf{x}_1 - \mathbf{a}_1\|^2 \neq 0$. Therefore $f_{r+1}(\mathbf{x}, y_1, Z_1) = u - y_1^2 - c_1 Z_1^2$ is irreducible over $\widetilde{K}(V)(y_1)$. Let z_1 be its root in the algebraic closure M of $\widetilde{K}(V)(y_1)$. Clearly

$$\widetilde{K}[W] = \widetilde{K}[\mathbf{X}, Y_1, Z_1]/(f_1, \ldots, f_r, f_{r+1}) \cong \widetilde{K}[V][y_1, z_1] \subseteq M.$$

It follows that $\widetilde{K}[W]$ is an integral domain, and $\mathrm{tr.deg.}(W) = \mathrm{tr.deg.}(V) + 1$. Thus W is absolutely irreducible and $\dim W = \dim V + 1$.

Let $R \in \mathcal{X}$. Without loss of generality $R \in \mathcal{X}_1$, and hence $R \notin \mathcal{X}_2, \ldots, \mathcal{X}_m$. Thus c_1 is positive, and $c_2, \ldots, c_m$ are negative in R. Apply (2) and Remark 1.1(b) to complete the $\mathbf{x}_1$ to a point $(\mathbf{x}_1, \mathbf{y}, \mathbf{z}) \in W(R)$ with $y_j \neq 0$ for each $1 \leq j \leq m$. In particular, $\frac{\partial f_{r+j}}{\partial Y_j}(\mathbf{x}_1, \mathbf{y}, \mathbf{z}) \neq 0$. Therefore $(\mathbf{x}_1, \mathbf{y}, \mathbf{z})$ is a nonsingular point on W.

By the PRC property of K there exists a point $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in W(K)$. Clearly $\mathbf{x} \in V(K)$, and for each j we have $\|\mathbf{x} - \mathbf{a}_j\|^2 \leq (\frac{\nu_j}{2})^2$ in R, for each $R \in \mathcal{X}_j$. This and (2) imply (1), by Remark 1.1(c). ▮

Applying Proposition 1.2 to $V = \mathbb{A}^1$ yields the Block Approximation Lemma of [P3, p. 354]:

COROLLARY 1.3: *Let K be a PRC field. Let $H_1, \ldots, H_m$ be disjoint clopen subsets of $X(K)$, and let $x_1, \ldots, x_m \in K$, and $\nu_1, \ldots, \nu_m \in K^\times$. Then there is $x \in K$ such that for every j*

$$(x - x_j)^2 \leq_P \nu_j^2 \qquad \text{for every } P \in H_j.$$

DEFINITION 1.4: *Let $\widetilde{K}$ be an algebraically closed field with $\iota \in \mathrm{Aut}(\widetilde{K})$ of order 2.*

(a) *For $c \in \widetilde{K}$ let $|c|_\iota^2 = c \cdot \iota(c)$.*

(b) *For $\mathbf{z} \in \widetilde{K}^n$ let $\|\mathbf{z}\|_\iota^2 = \sum_{i=1}^n |z_i|_\iota^2$.*

(c) *For a $\widetilde{K}$-linear morphism $f\colon \mathbb{A}^n \to \mathbb{A}^m$ given by a matrix $A = (a_{ij}) \in M_{m\times n}(\widetilde{K})$ let $\|f\|_\iota^2 = \sum_{i,j} |a_{ij}|_\iota^2$.*

In the above definition, the fixed field R of ι is real closed, and $||c|_\iota^2$, $\|\mathbf{z}\|_\iota^2$, $\|f\|_\iota^2$ are nonnegative elements of R. If $\mathbf{z} \in R^n$ then $\|\mathbf{z}\|_\iota^2 = \|\mathbf{z}\|^2$. Furthermore, in the unique ordering of R, the Schwartz inequality gives

$$\|f(\mathbf{z})\|_\iota^2 \leq \|f\|_\iota^2 \cdot \|\mathbf{z}\|_\iota^2, \tag{3}$$

for all $\mathbf{z} \in \widetilde{K}^n$.

REMARK 1.5: The space $X(\mathbb{Q}^{\mathrm{tr}})$ is homeomorphic to $X_\omega = \{0,1\}^{\aleph_0}$, the universal Boolean space of weight $\aleph_0$ (cf. the concluding Remark of [FV3]). In particular it has no isolated points.

LEMMA 1.6: *If K is a finitely generated field, then the set $X^a(K)$ of archimedian orderings on K is dense in $X(K)$.*

Proof: By induction on the number of generators of $K/\mathbb{Q}$ it suffices to show the following. Let K/K_0 be a simple extension of countable fields, let $P \in X(K)$, and let $P_0 = \mathrm{res}_{K_0} P \in X(K_0)$. If P_0 is in the closure of $X^a(K_0)$, then P is in the closure of $X^a(K)$.

The restriction $X(K) \to X(K_0)$ is open [ELW, 4.bis], hence we may assume that P_0 is archimedian. If K/K_0 is algebraic, then P is also archimedian. Otherwise K is the field of rational functions in one variable t over K_0. Replace (K_0, P_0) by its real closure (cf. [C, Lemma 8]) to assume that K_0 is real closed.

By [C, Corollary 9(c)], every neighborhood U of P in $X(K)$ contains a set of the form $\{Q \in X(K) |\ a < t < b \text{ in } Q\}$, where $a, b \in K_0$ and $a < b$ in P_0. As P_0 is archimedian, we can embed K_0 into $\mathbb{R}$. Since K_0 is countable, there is $c \in \mathbb{R} \smallsetminus K_0$ in the interval (a,b) in $\mathbb{R}$. This c is then transcendental over K_0. The K_0-embedding $K \to \mathbb{R}$ given by $t \mapsto c$ induces an archimedian ordering Q on K, and $a < t < b$ in Q. Thus $Q \in U$. ∎

For a subset I of a group G let $\mathrm{Con}_G(I) = \bigcup_{\sigma \in G} I^\sigma$. We say that I is a **conjugacy domain**, if I is closed under the conjugation, that is, $I = \mathrm{Con}_G(I)$.

DEFINITION 1.7: Let F/E be a Galois extension of fields with F not formally real, and let $\epsilon \in G(F/E)$ be an involution. We say that ϵ is **real** if its fixed field $F(\epsilon)$ in F is formally real. Equivalently, ϵ is the restriction of an involution in the absolute Galois group $G(E)$ of E. Denote the set of real involutions of $G(F/E)$ by $I(F/E)$.

Furthermore, assume that E is a totally real extension of a field K, and let $P \in X(K)$. Denote the involutions $\epsilon \in G(F/E)$ for which P extends to an ordering of $F(\epsilon)$

by $I_P(F/E)$. For $X \subseteq X(K)$, let $I_X(F/E) = \bigcup_{P\in X} I_P(F/E)$. If F is the algebraic closure of E, write $I_P(E)$ for $I_P(F/E)$, etc.

REMARK 1.8: (a) If $E = K$, then $I_P(F/E)$ is a conjugacy class in $G(F/E)$. In the general case $I_P(F/E)$ is a conjugacy domain in $G(F/E)$; in fact, $I_P(F/E) = \bigcup_{\substack{Q\in X(E)\\ Q\supseteq P}} I_Q(F/E)$.

(b) If M/N is a finitely generated extension of fields, then the restriction map of orderings $X(M) \to X(N)$ is closed and open [ELW, Theorem 4.1 and 4.bis]. In particular, let I be a set of involutions in $G(F/E)$, and assume that F/K is finitely generated. Then so is $F(\epsilon)/K$, for every involution $\epsilon \in G(F/E)$. Hence the set $\{P' \in X(K) |\ I_P(F/E) = I\}$ is closed and open in $X(K)$.

LEMMA 1.9: *Let $(K,P) \subseteq (K',P')$ be an extension of ordered fields. Let x be transcendental over K', and put $E = K(x)$ and $E' = K'(x)$. Furthermore, let F/E and F'/E' be Galois extensions with $F' = F \cdot E$. Assume that F, and hence also F', is not formally real. Then $I_P(F/E) = \mathrm{Con}_{G(F/E)}\mathrm{res}_F I_{P'}(F'/E')$*

Proof: We have $I_P(F/E) = \bigcup_{\substack{Q\in X(E)\\ Q\supseteq P}} I_Q(F/E)$ and $I_{P'}(F'/E') = \bigcup_{\substack{Q'\in X(E')\\ Q'\supseteq P'}} I_{Q'}(F'/E')$. As E and K' are linearly disjoint over K, each extension Q of P to E extends to an ordering Q' of E' that extends P'. Thus it suffices to show

$$I_Q(F/E) = \mathrm{Con}_{G(F/E)}\mathrm{res}_F I_{Q'}(F'/E')$$

for each ordering Q of E and for each extension Q' of Q to E'.

Let $\epsilon' \in I_{Q'}(F'/E')$ and $\epsilon = \mathrm{res}_F\epsilon'$. There is an ordering R' of $F'(\epsilon')$ that extends Q'. Its restriction to $F(\epsilon)$ is an extension of Q, and hence $\epsilon \in I_Q(F/E)$. Since $I_Q(F/E)$ is a conjugacy class in $G(F/E)$, the assertion follows. ▮

2. Moduli spaces for covers of the Riemann sphere

In this section we recall notation and results from [FV1] in the form to be used later. We also add some remarks. Let G be a finite group, and $r \geq 3$ an integer.

(2.1) Let $\mathbb{P}^1 1 = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. We consider covers χ: $X \to \mathbb{P}^1 1$ of compact (connected) Riemann surfaces. Call two such covers χ: $X \to \mathbb{P}^1 1$ and χ': $X' \to \mathbb{P}^1 1$ **equivalent** if there exists an isomorphism α: $X \to X'$ with $\chi' \circ \alpha = \chi$. Let $\mathrm{Aut}(X/\mathbb{P}^1 1)$ denote the group of automorphisms α of X with $\chi \circ \alpha = \chi$. We say that χ is **Galois** if $\mathrm{Aut}(X/\mathbb{P}^1 1)$ is transitive on the fibers of χ. From now on χ will always denote a Galois cover. All but finitely many points of $\mathbb{P}^1 1$ have the same number of inverse images under χ. These finitely many exceptional points are called the **branch points** of χ.

(2.2) Let $a_1, \ldots, a_r \in \mathbb{P}^1 1$ be the branch points of (the Galois cover) χ: $X \to \mathbb{P}^1 1$, and set $\mathbf{a} = \{a_1, \ldots, a_r\}$. Then χ restricts to an (unramified) topological covering of the punctured sphere $\mathbb{P}^1 1 \smallsetminus \mathbf{a}$. Choose a base point a_0 on this punctured sphere, and consider the (topological) fundamental group $\Gamma = \Pi_1(\mathbb{P}^1 1 \smallsetminus \mathbf{a}, a_0)$, based at a_0 (with the composition law: $\gamma_1\gamma_2$ is the path γ_1 followed by γ_2).

Depending on the choice of a base point $p_0 \in \chi^{-1}(a_0)$, we get an epimorphism ι: $\Gamma \to \mathrm{Aut}(X/\mathbb{P}^1 1)$ as follows. For each path γ representing an element $[\gamma]$ of Γ, let p_1 be the endpoint of the unique lift of γ to $X \smallsetminus \chi^{-1}(\mathbf{a})$ with initial point p_0. Then, ι sends $[\gamma]$ to the unique element α of $\mathrm{Aut}(X/\mathbb{P}^1 1)$ with $\alpha(p_0) = p_1$.

(2.3) Let $\mathcal{H}^{\mathrm{ab}} = \mathcal{H}_r(G)^{\mathrm{ab}}$ be the set of equivalence classes $[\chi]$ of all Galois covers χ: $X \to \mathbb{P}^1 1$ with r branch points and with $\mathrm{Aut}(X/\mathbb{P}^1 1) \cong G$. Let $\mathcal{H}^{\mathrm{inn}} = \mathcal{H}_r(G)^{\mathrm{inn}}$ be the set of equivalence classes $[\chi, h]$ of pairs (χ, h) where χ: $X \to \mathbb{P}^1 1$ is a Galois cover with r branch points, and h: $\mathrm{Aut}(X/\mathbb{P}^1 1) \to G$ is an isomorphism. Here (χ, h) and $(\chi'$: $X' \to \mathbb{P}^1 1,\ h')$ are equivalent if there is an isomorphism δ: $X \to X'$ with $\chi' \circ \delta = \chi$ and $h' \circ \delta_* = h$, where δ_*: $\mathrm{Aut}(X/\mathbb{P}^1 1) \to \mathrm{Aut}(X'/\mathbb{P}^1 1)$ is the isomorphism $\alpha \mapsto \delta \circ \alpha \circ \delta^{-1}$. Let Λ: $\mathcal{H}^{\mathrm{inn}} \to \mathcal{H}^{\mathrm{ab}}$ be the map sending $[\chi, h]$ to $[\chi]$.

(2.4) Think of points of $\mathcal{H}^{\mathrm{inn}}$ as equivalence classes $[\mathbf{a}, a_0, f]$ of triples $(\mathbf{a}, a_0, f)$. Here $\mathbf{a} = \{a_1, \ldots, a_r\}$ is a set of r points of $\mathbb{P}^1 1$, and $a_0 \in \mathbb{P}^1 1 \smallsetminus \mathbf{a}$, and f: $\Gamma =$

$\Pi_1(\mathbb{P}^1 1\setminus\mathbf{a},a_0)\to G$ is an epimorphism that does not factor through the canonical map $\Gamma\to\Pi_1((\mathbb{P}^1 1\setminus\mathbf{a})\cup\{a_i\},a_0)$, for any i. (The latter condition means that the corresponding cover χ has **exactly** r branch points). Call two such triples $(\mathbf{a},a_0,f)$ and $(\tilde{\mathbf{a}},\tilde{a}_0,\tilde{f})$ equivalent if $\mathbf{a}=\tilde{\mathbf{a}}$ and there is a path ω from a_0 to $\tilde{a}_0$ in $\mathbb{P}^1 1\setminus\mathbf{a}$ such that $\tilde{f}\circ\omega^*=f$. Here $\omega^*\colon\Pi_1(\mathbb{P}^1 1\setminus\mathbf{a},a_0)\to\Pi_1(\mathbb{P}^1 1\setminus\mathbf{a},\tilde{a}_0)$ is the isomorphism $\gamma\mapsto\omega^{-1}\gamma\omega$.

(2.5) Here is the correspondence between the above pairs and triples [FV1, §1.2]. Given $[\chi,h]\in\mathcal{H}^{\mathrm{inn}}$, with $\chi\colon X\to\mathbb{P}^1 1$ as above, let $\mathbf{a}$ be the set of branch points of χ, and choose $a_0\in\mathbb{P}^1 1\setminus\mathbf{a}$ and $p_0\in\chi^{-1}(a_0)$. Set $\Gamma=\Pi_1(\mathbb{P}^1 1\setminus\mathbf{a},a_0)$ as above, and define $f\colon\Gamma\to G$ as $f=h\circ\iota$, where $\iota\colon\Gamma\to\mathrm{Aut}(X/\mathbb{P}^1 1)$ is the map from (2.2). Recall that ι is canonical up to composition with inner automorphisms of $\mathrm{Aut}(X/\mathbb{P}^1 1)$. Thus h and f determine each other up to inner automorphisms of G. This is compatible with the equivalence of pairs (resp., triples).

(2.6) Now define a topology on $\mathcal{H}^{\mathrm{inn}}$ as follows. To specify a neighborhood $\mathcal{N}$ of the point $[\mathbf{a},a_0,f]$ of $\mathcal{H}^{\mathrm{inn}}$, where $\mathbf{a}=\{a_1,\dots,a_r\}$, choose pairwise disjoint open discs $D_1,\dots,D_r$ around $a_1,\dots,a_r$, with $a_0\notin D_1\cup\cdots\cup D_r$. Then $\mathcal{N}$ consists of all points $[\tilde{\mathbf{a}},a_0,\tilde{f}]$ such that $\tilde{\mathbf{a}}$ has exactly one point in each D_i, and $\tilde{f}$ is the composition of the canonical isomorphisms

$$\Pi_1(\mathbb{P}^1 1\setminus\tilde{\mathbf{a}},a_0)\cong\Pi_1(\mathbb{P}^1 1\setminus(D_1\cup\cdots\cup D_r),a_0)\cong\Pi_1(\mathbb{P}^1 1\setminus\mathbf{a},a_0)$$

with f. These $\mathcal{N}$ form a basis for the topology. They are connected. The sets $\Lambda(\mathcal{N})$ form a basis for a topology on $\mathcal{H}^{\mathrm{ab}}$, such that $\Lambda\colon\mathcal{H}^{\mathrm{inn}}\to\mathcal{H}^{\mathrm{ab}}$ becomes an (unramified) covering.

(2.7) Let $\mathcal{U}_r$ denote the space of all subsets of cardinality r of the Riemann sphere $\mathbb{P}^1 1$. It has a natural structure of algebraic variety defined over $\mathbb{Q}$ [FV1, §1.1]; it is isomorphic to $\mathbb{P}^r\setminus D$, where D, the discriminant locus, is a hypersurface in $\mathbb{P}^r$. In particular, $\mathcal{U}_r$ is an affine variety. Furthermore, if K is a subfield of $\mathbb{C}$ and $\mathbf{a}=\{a_1,\dots,a_r\}\in\mathcal{U}_r$ with $a_1,\dots,a_r\neq\infty$, then $\mathbf{a}$ is K-rational if and only if $\prod_{i=1}^r(X-a_i)\in K[X]$. As a complex manifold, the topology of $\mathcal{U}_r$ has a basis consisting of sets $\mathcal{D}$ of the following form: Given pairwise disjoint open discs $D_1,\dots,D_r$ on $\mathbb{P}^1 1$, let $\mathcal{D}$ be the set of all $\mathbf{a}\in\mathcal{U}_r$ with $|\mathbf{a}\cap D_i|=1$ for $i=1,\dots,r$.

(2.8) Define $\Psi\colon \mathcal{H}^{\mathrm{inn}} \to \mathcal{U}_r$ and $\bar{\Psi}\colon \mathcal{H}^{\mathrm{ab}} \to \mathcal{U}_r$ by sending $[\chi, h]$ and $[\chi]$, respectively, to the set of branch points of χ. These maps are (unramified) coverings and $\bar{\Psi} \circ \Lambda = \Psi$. Through these coverings the spaces $\mathcal{H}^{\mathrm{ab}}$ and $\mathcal{H}^{\mathrm{inn}}$ inherit a structure of complex manifold from $\mathcal{U}_r$.

(2.9) Each cover $\chi\colon X \to \mathbb{P}^1 1$ as above is an algebraic morphism of algebraic varieties over $\mathbb{C}$, compatible with its analytic structure (Riemann's existence theorem). An automorphism β of $\mathbb{C}$ defines an automorphism β^* of $\mathbb{P}^1 1$ by $(x_0 : x_1) \mapsto (\beta^{-1}(x_0) : \beta^{-1}(x_1))$. Consider the cover $\beta(\chi)\colon \beta(X) \to \mathbb{P}^1 1$ obtained from $\chi\colon X \to \mathbb{P}^1 1$ through base change with β^*. Furthermore, for each $\alpha \in \mathrm{Aut}(X/\mathbb{P}^1 1)$ let $\beta_*(\alpha) = \beta(\alpha) \in \mathrm{Aut}(\beta(X)/\mathbb{P}^1 1)$ be the morphism obtained by the same base change.

(2.10) The spaces $\mathcal{H}^{\mathrm{ab}}$ and $\mathcal{H}^{\mathrm{inn}}$ have a unique structure of (the set of complex points of) a (reducible) algebraic variety defined over $\mathbb{Q}$ [FV1, Theorem 1]. This variety structure is compatible with the analytic structure of $\mathcal{H}^{\mathrm{ab}}$ and $\mathcal{H}^{\mathrm{inn}}$, and it makes the maps Ψ, $\bar{\Psi}$ and Λ into algebraic morphisms defined over $\mathbb{Q}$. Also, each automorphism β of $\mathbb{C}$—in its natural action $(x_1, \ldots, x_n) \mapsto (\beta(x_1), \ldots, \beta(x_n))$ on the complex points of (the affine pieces of) a variety defined over $\mathbb{Q}$—sends the point $[\chi] \in \mathcal{H}^{\mathrm{ab}}$ to $[\beta(\chi)]$ and the point $[\chi, h] \in \mathcal{H}^{\mathrm{inn}}$ to $[\beta(\chi), h \circ \beta_*^{-1}]$.

(2.11) We can describe the action of complex conjugation c on the triples of (2.4) that compose $\mathcal{H}^{\mathrm{inn}}$. Namely, c naturally acts on paths in $\mathbb{P}^1 1$. Thus, it induces a map $\Pi_1(\mathbb{P}^1 1 \smallsetminus \mathbf{a}, a_0) \to \Pi_1(\mathbb{P}^1 1 \smallsetminus c(\mathbf{a}), c(a_0))$. Denote this map by $\gamma \mapsto c\gamma$.

LEMMA: *If* $\mathbf{p} = [\mathbf{a}, a_0, f] \in \mathcal{H}^{\mathrm{inn}}$, *then* $c(\mathbf{p}) = [c(\mathbf{a}), c(a_0), cf]$, *where* $(cf)(c\gamma) = f(\gamma)$ *for each* $\gamma \in \Pi_1(\mathbb{P}^1 1 \smallsetminus \mathbf{a}, a_0)$.

Proof: Write $\mathbf{p}$ as $\mathbf{p} = [\chi, h]$. Then $c(\mathbf{p}) = [c(\chi), h \circ c_*^{-1}]$. It remains to show that this point is represented by the triple $(c(\mathbf{a}), c(a_0), cf)$. This is a straightforward consequence of the definitions (cf. [DF1, Lemma 2.1]). $\blacksquare$

(2.12) Let $\mathbf{b} = \{b_1, \ldots, b_r\} \in \mathcal{U}_r$ such that $0 \notin \mathbf{b}$. We can choose generators $\gamma_1, \ldots, \gamma_r$ for the fundamental group $\Pi_1(\mathbb{P}^1 1 \smallsetminus \mathbf{b}, 0)$ so that $\gamma_1 \cdots \gamma_r = 1$ is the only relation among them.

Indeed, we may assume that $b_j = \zeta^j$, for $j = 1, \ldots, r$, where $\zeta = e^{\frac{2\pi i}{r}}$. Otherwise apply a homeomorphism $\mathbb{P}^1 1 \to \mathbb{P}^1 1$ that maps 0 onto itself and b_j onto ζ_j. Let $\tilde{\gamma}_j$ be a path starting at 0, going up on a straight line to a neighborhood of b_j, traversing a small disk around b_j in the counterclockwise direction, and following the same straight line back to 0. Then $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ do not intersect except at 0. Let γ_j be the homotopy class of $\tilde{\gamma}_j$. Then $\gamma_1, \ldots, \gamma_r$ generate $\Pi_1(\mathbb{P}^1 1 \smallsetminus \mathbf{b}, 0)$ and $\gamma_1 \cdots \gamma_r = 1$.

Represent a point $\mathbf{p} \in \Psi^{-1}(\mathbf{b})$ by a triple $(\mathbf{b}, 0, f)$. The r-tuple $(\sigma_1, \ldots, \sigma_r) = (f(\gamma_1), \ldots, f(\gamma_r))$ determines the epimorphism $f \colon \Pi_1(\mathbb{P}^1 1 \smallsetminus \mathbf{b}, 0) \to G$. It has the following properties: $\sigma_1 \cdots \sigma_r = 1$, $\sigma_1, \ldots, \sigma_r$ generate G, and $\sigma_j \neq 1$ for all j [FV1, §1.3]. Let $\mathcal{E}_r$ denote the set of such r-tuples $(\sigma_1, \ldots, \sigma_r)$. Clearly, each $(\sigma_1, \ldots, \sigma_r) \in \mathcal{E}_r$ arises in the above way from some $\mathbf{p} \in \Psi^{-1}(\mathbf{b})$. Let $\mathcal{L}(G)$ be the collection of conjugacy classes $\neq \{1\}$ of G, and let $\mathcal{E}^{(r)}$ be all r-tuples $(\sigma_1, \ldots, \sigma_r) \in \mathcal{E}_r$ where each $\mathcal{C} \in \mathcal{L}(G)$ is represented exactly $r/|\mathcal{L}(G)|$ times among $\sigma_1, \ldots, \sigma_r$.

(2.13) For the rest of section 2 assume that r is a multiple of $|\mathcal{L}(G)|$ and suitably large [FV1, Appendix], and the Schur multiplier of G is generated by commutators. We explain the latter condition. Let R be a group of maximal order with the property that R has a subgroup $M \leq R' \cap Z(R)$ satisfying $R/M \cong G$. Then $M \cap \{g^{-1}h^{-1}gh |\, g, h \in R\}$ generates M, the Schur multiplier of G.

Fix $\mathbf{b} \in \mathcal{U}_r$ and $\gamma_1, \ldots, \gamma_r$ as above. By [FV1, §1.3] there is a (unique) connected component $\mathcal{H}$ of $\mathcal{H}^{\mathrm{inn}}$ containing $\{[\mathbf{b}, 0, f] |\, (f(\gamma_1), \ldots, f(\gamma_r)) \in \mathcal{E}^{(r)}\}$. Let $\bar{\mathcal{H}} = \Lambda(\mathcal{H})$ be its image in $\mathcal{H}^{\mathrm{ab}}$. We call $\mathcal{H}$ and $\bar{\mathcal{H}}$ **Hurwitz spaces**. By [FV1, Thm. 1] they are absolutely irreducible algebraic varieties defined over $\mathbb{Q}$. Moreover, since $\Psi \colon \mathcal{H} \to \mathcal{U}_r$ and $\bar{\Psi} \colon \bar{\mathcal{H}} \to \mathcal{U}_r$ are finite normal covers of an affine variety, $\mathcal{H}$ and $\bar{\mathcal{H}}$ are affine [H, Exc. III.4.1].

(2.14) For $A \in \mathrm{Aut}(G)$ (acting from the left on G), let $\delta_A \colon \mathcal{H} \to \mathcal{H}$ be the map sending the point $[\chi, h]$ to $[\chi, A \circ h]$. Then δ_A is an automorphism of the covering $\Lambda \colon \mathcal{H} \to \bar{\mathcal{H}}$. It depends only on the class of A modulo $\mathrm{Inn}(G)$. In fact, Λ is a Galois covering, and the map $A \mapsto \delta_A$ induces an isomorphism $\delta \colon \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G) \to \mathrm{Aut}(\mathcal{H}/\bar{\mathcal{H}})$ [FV1, §6.1]. Furthermore, δ_A is a morphism defined over $\mathbb{Q}$ [FV1, §6.2]. In the description of $\mathcal{H}^{\rm inn}$ in (2.4), δ_A sends the point $[\mathbf{a}, a_0, f]$ to $[\mathbf{a}, a_0, A \circ f]$. As $\Lambda \colon \mathcal{H}^{\rm inn} \to \mathcal{H}^{\rm ab}$ is an unramified covering (2.8), δ_A has no fixed points.

For the rest of this section we further assume that G has trivial center. Accordingly, identify G with the subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$ (acting from the left on G).

Let $\mathbf{p} \in \mathcal{H}$ and let $K \subseteq L$ be subfields of $\mathbb{C}$ such that $\Lambda(\mathbf{p}) \in \bar{\mathcal{H}}(K)$ and $L = K(\mathbf{p})$.

(2.15) Write $\mathbf{p}$ as $\mathbf{p} = [\chi, h]$. Then, the cover $\chi \colon X \to \mathbb{P}^11$ can be defined over L (in a unique way) such that all automorphisms of χ are defined over L [FV1, Cor. 1]. Thus, there is a unique cover $\chi_L \colon X_L \to \mathbb{P}^1_L$ such that base change with the embedding $L \to \mathbb{C}$ gives χ from χ_L and the automorphisms of χ from the automorphisms of χ_L.

(2.16) We recall some facts from [FV1, §6.3]. The function field $F = L(X_L)$ is regular over L, and the extension $F/L(x)$ induced by χ is Galois. Here, x is the identity function on $\mathbb{P}^11$. The group $G(F/L(x))$ (acting from the left on F) is canonically isomorphic to $\mathrm{Aut}(X/\mathbb{P}^11)$, via the map that sends $\alpha \in \mathrm{Aut}(X/\mathbb{P}^11)$ to the element $g \mapsto g \circ \alpha^{-1}$ of $G(F/L(x))$. Let $h_0 \colon G(F/L(x)) \to G$ be the composition of this isomorphism with $h \colon \mathrm{Aut}(X/\mathbb{P}^11) \to G$.

(2.17) Furthermore, both L/K and $F/K(x)$ are Galois extensions, and the centralizer of $G(F/L(x))$ in $G(F/K(x))$ is trivial. This implies that h_0 extends to a unique embedding $h_1 \colon G(F/K(x)) \to \mathrm{Aut}(G)$. [FV1, Proposition 3] says:

$$H := h_1(G(F/K(x))) = \{A \in \mathrm{Aut}(G) |\ \delta_A(\mathbf{p}) \text{ is conjugate to } \mathbf{p} \text{ under } G(L/K)\}.$$

(2.18) Let β be an automorphism of $\mathbb{C}$, and let K and K' be two subfields of $\mathbb{C}$ such that $\beta(K) \subseteq K'$. Put $\mathbf{p}' = \beta(\mathbf{p})$ and $L' = K'(\mathbf{p}')$. Then $\beta(L) \subseteq L'$, and $\Lambda(\mathbf{p}') \in \bar{\mathcal{H}}(K')$. Let $F'/L'(x)$ be the Galois extension associated to K' and the point $\mathbf{p}'$ of $\mathcal{H}$, and let $h_1' \colon G(F'/K'(x)) \to \mathrm{Aut}(G)$ be the associated embedding. Then the following holds: Let $\beta \colon L(x) \to L'(x)$ be the extension of β (fixing x). This map extends further to $\beta \colon F \to F'$ such that canonically

$$F' \cong \beta(F) \otimes_{\beta(L)} L' \cong F \otimes_L L'. \tag{1}$$

The 'restriction' map β^*: $G(F'/K'(x)) \to G(F/K(x))$ that sends $\sigma \in G(F'/K'(x))$ to $\beta^{-1}\sigma|_{\beta(F)}\beta$ is injective, and yields an isomorphism $G(F'/L'(x)) \to G(F/L(x))$. Further, it makes the following diagram commutative:

(2)
$$\begin{tikzcd}
G(F'/K'(x)) \arrow[rr, "\beta^*"] \arrow[dr, "h_1'"'] & & G(F/K(x)) \arrow[dl, "h_1"] \\
& \mathrm{Aut}(G) &
\end{tikzcd}$$

Proof: We have $\mathbf{p}' = [\beta(\chi), h \circ \beta_*^{-1}]$ by (2.10). The natural action of $\beta \in \mathrm{Aut}(\mathbb{C})$ on functions defined over L extends β to a map from $F = L(X)$ to $F' = L(\beta(X))$. Then (1) follows from the fact that F is regular over L, and $[F' \colon L'(x)] = [F : L(x)]$ $(= \deg(\chi))$. The proof of (2) is straightforward from the definitions. ∎

(2.19) In the situation of (2.18) and Lemma 1.9 we have

$$h_1'(I_{P'}(F'/E')) \subseteq h_1(I_P(F/E)) \quad \text{and} \quad \mathrm{Con}_H h_1'(I_{P'}(F'/E')) = h_1(I_P(F/E)),$$

where H is the image of h_1 in $\mathrm{Aut}(G)$. If the 'restriction' map β^*: $G(F'/K'(x)) \to G(F/K(x))$ is an isomorphism, then $h_1'(I_{P'}(F'/E')) = h_1(I_P(F/E))$.

Proof: Without loss of generality assume that the map β: $K \to K'$ is an inclusion of fields. Hence $\beta^* = \mathrm{res}_F$. In the commutative diagram (2) we may replace $\mathrm{Aut}(G)$ by H, so that h_1 is an isomorphism. The assertions follow from the commutativity of that diagram. ∎

3. Group-theoretic lemmas

Fix the following notation. For a group G and an integer $r \geq 0$ put $\dot{G} = G \smallsetminus \{1\}$. Let $\mathcal{L}(G)$ be the collection of nontrivial conjugacy classes of G, and put $l = |\mathcal{L}(G)|$. For an r-tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \dot{G}^r$ and $\mathcal{C} \in \mathcal{L}(G)$, let $n_{\mathcal{C}}(\boldsymbol{\sigma})$ be the number of indices, $1 \leq i \leq r$, with $\sigma_i \in \mathcal{C}$. Then $\sum_{\mathcal{C} \in \mathcal{L}(G)} n_{\mathcal{C}}(\boldsymbol{\sigma}) = r$.

Lemma 3.1: *Let G be a finite group. Every sufficiently large multiple r of $4l$ satisfies the following conditions. Let $\epsilon \in \mathrm{Aut}(G)$ be of order 2, and let I be a set of involutions in $(G \rtimes \langle \epsilon \rangle) \smallsetminus G$ such that $\epsilon \in I$. Let $e = 8 \cdot |G|!$ if $|I| \geq 2$, and $e = 0$ if $|I| = 1$. Put $m = r - e$. Then there are sequences $\boldsymbol{\sigma} \in \dot{G}^e$, $\boldsymbol{\tau} \in \dot{G}^m$ with the following properties:*

(a1) $\sigma_i^\epsilon = \sigma_1 \cdots \sigma_{i-1} \sigma_i^{-1} \sigma_{i-1}^{-1} \cdots \sigma_1^{-1}$, *for each* $1 \leq i \leq e$;

(a2) $\tau_j^\epsilon = \tau_{m+1-j}^{-1}$ *for each* $1 \leq j \leq m$;

(b) $I = \{\epsilon, \epsilon\sigma_1, \epsilon\sigma_1\sigma_2, \ldots, \epsilon\sigma_1\sigma_2 \cdots \sigma_e\}$;

(c) $(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \mathcal{E}^{(r)}(G)$, *that is,* $\langle \sigma_1, \ldots, \sigma_e, \tau_1, \ldots, \tau_m \rangle = G$, $\sigma_1 \cdots \sigma_e \tau_1 \cdots \tau_m = 1$, *and* $n_{\mathcal{C}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = r/l$ *for each* $\mathcal{C} \in \mathcal{L}(G)$.

Proof: It suffices to consider only one $\epsilon \in \mathrm{Aut}(G)$ of order 2, and only one set I of involutions.

Part A: *Separation of $\boldsymbol{\sigma}$ from $\boldsymbol{\tau}$.* Define an equivalence relation on $\mathcal{L}(G)$ where the class $[\mathcal{C}]$ of $\mathcal{C}$ is $\{\mathcal{C}, \mathcal{C}^{-1}, \mathcal{C}^\epsilon, \mathcal{C}^{-\epsilon}\}$. Part B constructs $\boldsymbol{\sigma} \in \dot{G}^e$ satisfying (a1), (b), and two further conditions

(c1) $\sigma_1 \cdots \sigma_e = 1$; and

(d1) for each $[\mathcal{C}]$ there is $\mu_{[\mathcal{C}]} \geq 0$ such that $n_{\mathcal{C}}(\boldsymbol{\sigma}) = 4\mu_{[\mathcal{C}]}$.

Observe that $e = \sum_{\mathcal{C} \in \mathcal{L}(G)} n_{\mathcal{C}}(\boldsymbol{\sigma}) = 4 \sum_{\mathcal{C} \in \mathcal{L}(G)} \mu_{[\mathcal{C}]}$.

Furthermore, for each $[\mathcal{C}]$ let $\nu_{[\mathcal{C}]}$ be a positive integer, and put $m = 4 \sum_{\mathcal{C} \in \mathcal{L}(G)} \nu_{[\mathcal{C}]}$. Part C shows that there is an m-tuple $\boldsymbol{\tau} \in \dot{G}^m$ satisfying (a2), and the following two conditions:

(c2) $\langle \tau_1, \ldots, \tau_m \rangle = G$ and $\tau_1 \cdots \tau_m = 1$;

and

(d2) $n_{\mathcal{C}}(\boldsymbol{\tau}) = 4\nu_{[\mathcal{C}]}$, for each $\mathcal{C}$.

Now, let $n = \frac{r}{4l}$, and assume that $n > \mu_{[\mathcal{C}]}$ for each $[\mathcal{C}]$. Do the last step with $\nu_{[\mathcal{C}]} = n - \mu_{[\mathcal{C}]}$. Then

$$e + m = 4 \sum_{\mathcal{C}\in\mathcal{L}(G)} \mu_{[\mathcal{C}]} + 4 \sum_{\mathcal{C}\in\mathcal{L}(G)} \nu_{[\mathcal{C}]} = 4 \sum_{\mathcal{C}\in\mathcal{L}(G)} n = 4ln = r.$$

Clearly, (c1), (c2), (d1), and (d2) imply (c). In fact, $n_{\mathcal{C}}(\boldsymbol{\sigma},\boldsymbol{\tau}) = 4n = \frac{r}{l}$ for each $\mathcal{C} \in \mathcal{L}(G)$.

PART B: *Construction of* $\boldsymbol{\sigma}$. If $I = \{\epsilon\}$, let $e = 0$ and $\boldsymbol{\sigma} = ()$. Otherwise put

$$\boldsymbol{\sigma} = (\epsilon_1\epsilon_2, \epsilon_2\epsilon_3, \ldots, \epsilon_e\epsilon_1),$$

where $\epsilon_1 = \epsilon, \epsilon_2, \ldots, \epsilon_e \in I$, not necessarily distinct, but $\epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_e \neq \epsilon_1$. Then $\boldsymbol{\sigma}$ satisfies (a1) and (c1). Furthermore, if $I = \{\epsilon_1, \ldots, \epsilon_e\}$, then $\boldsymbol{\sigma}$ also satisfies (b). To construct such $\epsilon_1, \ldots, \epsilon_e$, let $n' = \frac{e}{2|I\smallsetminus\{\epsilon\}|}$. Observe that n' is an integer divisible by 4, because $|I\smallsetminus\{\epsilon\}| < |G|$. Let $\epsilon_i = \epsilon$ for odd i, and choose $\epsilon_2, \epsilon_4, \ldots, \epsilon_e$ so that each element of $I\smallsetminus\{\epsilon\}$ occurs in this sequence exactly n' times.

Let $g \in G$, and let $n_g(\boldsymbol{\sigma})$ be the number of indices $1 \leq i \leq e$ for which $\sigma_i = g$. From the above, $4|n_g(\boldsymbol{\sigma})$. Moreover, $\sigma_{2i-1}^{\epsilon} = \epsilon(\epsilon\epsilon_{2i})\epsilon = \sigma_{2i-1}^{-1} = \sigma_{2i}$, for each $1 \leq i \leq e/2$. Hence, $n_g(\boldsymbol{\sigma}) = n_{g^{-1}}(\boldsymbol{\sigma}) = n_{g^{\epsilon}}(\boldsymbol{\sigma}) = n_{g^{-\epsilon}}(\boldsymbol{\sigma})$. This yields (d1).

PART C: *Construction of* $\boldsymbol{\tau}$. For each $[\mathcal{C}]$ let $\nu_{[\mathcal{C}]}$ be a positive integer. Choose k and $\mathbf{g} = (g_1, \ldots, g_k) \in \dot{G}^k$ such that $n_{\mathcal{C}}(\mathbf{g}) = \nu_{[\mathcal{C}]}$, for each $\mathcal{C}$. In particular, $\mathbf{g}$ contains an entry from each $\mathcal{C} \in \mathcal{L}(G)$. A proper subgroup of G misses some conjugacy classes of G [FJ, Lemma 12.4]. Therefore, $G = \langle g_1, \ldots, g_k \rangle$. Furthermore, $k = \sum_{\mathcal{C}\in\mathcal{L}(G)} \nu_{[\mathcal{C}]} = \frac{m}{4}$. Put

$$\boldsymbol{\tau} = (g_1, g_1^{-1}, \ldots,\; g_k, g_k^{-1}, g_k^{\epsilon}, g_k^{-\epsilon}, \ldots, g_1^{\epsilon}, g_1^{-\epsilon}).$$

This choice satisfies (a2) and (c2), and, for each $\mathcal{C}$,

$$n_{\mathcal{C}}(\boldsymbol{\tau}) = n_{\mathcal{C}}(\mathbf{g}) + n_{\mathcal{C}^{-1}}(\mathbf{g}) + n_{\mathcal{C}^{\epsilon}}(\mathbf{g}) + n_{\mathcal{C}^{-\epsilon}}(\mathbf{g}) = 4\nu_{[\mathcal{C}]}. \qquad ■$$

LEMMA 3.2: *Let* $1 \to G \to H \xrightarrow{\pi} C \to 1$ *be an exact sequence of finite groups, and let*

I be a set of involutions in $H \smallsetminus G$. There exists a commutative diagram of finite groups

(1)
$$\begin{tikzcd}
1 \arrow[r] & \Gamma \arrow[r] \arrow[d] & \Gamma \arrow[r, "\tilde{\pi}"'] \arrow[d] & C \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G \arrow[r] & H \arrow[r, "\pi"'] & C \arrow[r] & 1
\end{tikzcd}$$

with exact rows and surjective vertical maps such that the Schur multiplier of Γ is generated by commutators and $C_?(\Gamma) = 1$. Finally, every involution in I lifts to at least two involutions in Γ.

Proof: Choose a presentation $1 \to \mathcal{R} \to \mathcal{F} \to H \to 1$, where $\mathcal{F}$ is the free product of a free group of finite rank with finitely many groups of order 2, say $\langle\delta_1\rangle, \ldots, \langle\delta_e\rangle$, such that $\{\delta_1, \ldots, \delta_e\}$ maps onto I. The inverse image $\mathcal{F}_1$ of G in $\mathcal{F}$ contains no conjugates of $\delta_1, \ldots, \delta_e$, hence by the Kurosh Subgroup Theorem [M, Theorem VII.5.1 and Proposition VII.5.3] it is a free group of finite rank. Let $\mathcal{N} = [\mathcal{F}_1, \mathcal{R}]$ be the group generated by commutators $[f, r]$ with $f \in \mathcal{F}_1, r \in \mathcal{R}$. Set $F = \mathcal{F}/\mathcal{N}$, $F_1 = \mathcal{F}_1/\mathcal{N}$, and $R = \mathcal{R}/\mathcal{N}$. Then $1 \to R \to F_1 \to G \to 1$ is a central extension.

By the general theory of the Schur multiplier [Hu, Kap.5, §23], R is the direct product of the Schur multiplier $M(G) = R \cap (F_1)'$ and a free abelian group A. Let A_0 be the intersection of all the F-conjugates of A. Then $A_0 \triangleleft F$. Since $(R : A) = |M(G)| < \infty$, also $(F : A_0) < \infty$. Set $\Gamma = F/A_0$, $\Gamma = F_1/A_0$, and $S = R/A_0$, to get diagram (1). The image $\widetilde{I}$ of $\{\delta_1, \ldots, \delta_e\}$ in Γ maps onto I. Notice that S is the direct product of $S \cap (\Gamma)' \cong M(G)$ and A/A_0. As in the proof of [FV3, Lemma 2] the Schur multiplier of Γ is generated by commutators.

Replace G, H, and I by Γ, Γ, and $\widetilde{I}$, to assume that the Schur multiplier of G is generated by commutators.

Let T be a non-abelian finite simple group with trivial Schur multiplier. For example, take $T = \mathrm{SL}_2(8)$ [Hu, Satz 25.7]. Form the regular wreath product Γ of H with T (e.g., [Hu, Def. 15.6]). Thus $\Gamma = T^j \rtimes H$, with $j = |H|$, and H acts on T^j by permuting the factors in its regular representation. Let Γ be $T^j \rtimes G \leq \Gamma$. Clearly, $C_?(T^j) = 1$, and hence $C_?(\Gamma) = 1$. If $\epsilon \in I$ and $\tau \in T^j$, then $\tau^{-1}\epsilon\tau$ is an involution in Γ that maps to ϵ. This proves the last assertion of the lemma.

Since $M(T) = 1$, every central extension of T splits. This implies that every representation group of Γhas a normal subgroup isomorphic to T^j such that the quotient by this subgroup is a representation group of G. Therefore, $M(\Gamma) \cong M(G)$ is generated by commutators. ▮

LEMMA 3.3: *Let $\pi\colon H \to \overline{H}$ be an epimorphism of finite groups, and let $I_1, \dots, I_m \subseteq H$ and $\overline{I}_1, \dots, \overline{I}_m \subseteq \overline{H}$ be sets of involutions such that $\pi(I_j) = \overline{I}_j$. Then there exists a finite group $\widetilde{H}$, a surjection $\rho\colon \widetilde{H} \to H$, and sets of involutions $\widetilde{I}_1, \dots, \widetilde{I}_m \subseteq \widetilde{H}$ such that $\rho(\widetilde{I}_j) = I_j$ for every j, and every automorphism $\bar\alpha$ of $\overline{H}$ that satisfies $\bar\alpha(\overline{I}_j) = \overline{I}_j$ for all j, lifts to an automorphism $\tilde\alpha$ of $\widetilde{H}$ (that is, $(\pi \circ \rho) \circ \tilde\alpha = \bar\alpha \circ (\pi \circ \rho)$) that satisfies $\tilde\alpha(\widetilde{I}_j) = \widetilde{I}_j$ for all j. Moreover, if the I_j are conjugacy domains in H, then the $\widetilde{I}_j$ can be taken conjugacy domains in $\widetilde{H}$.*

Proof: Let K be a set of cardinality $\mathrm{Ker}(\pi)$. Let $\widetilde{H}$ be the free product of cyclic groups

$$\widetilde{H} = \Big(\mathop{\ast\!\!\!\prod}_{\substack{\bar h \in \overline{H} \\ k \in K}} \langle x_{\bar h, k}\rangle\Big) * \Big(\mathop{\ast\!\!\!\prod}_{\substack{\bar\epsilon \in \overline{I} \\ k \in K}} \langle \tilde\epsilon_{\bar\epsilon, k, 1}\rangle\Big) * \Big(\mathop{\ast\!\!\!\prod}_{\substack{\bar\epsilon \in \overline{I} \\ k \in K}} \langle \tilde\epsilon_{\bar\epsilon, k, 2}\rangle\Big) * \cdots * \Big(\mathop{\ast\!\!\!\prod}_{\substack{\bar\epsilon \in \overline{I} \\ k \in K}} \langle \tilde\epsilon_{\bar\epsilon, k, m}\rangle\Big),$$

where $\langle x_{\bar h, k}\rangle \cong \mathbb{Z}$ and $\langle \tilde\epsilon_{\bar\epsilon, k, j}\rangle \cong \mathbb{Z}/2\mathbb{Z}$, and let $\widetilde{I}_j = \{\tilde\epsilon_{\bar\epsilon, k, j} \mid \bar\epsilon \in \overline{I}_j, k \in K\}$. (Of course, $\widetilde{H}$ is not yet finite.) Define a surjection $\rho\colon \widetilde{H} \to H$ by mapping $\{x_{\bar h, k} \mid k \in K\}$ onto $\{h \in H \mid \pi(h) = \bar h\}$ and $\{\tilde\epsilon_{\bar\epsilon, k, j} \mid \bar\epsilon \in \overline{I}_j\}$ onto $\{\epsilon \in I_j \mid \pi(\epsilon) = \bar\epsilon\}$. Then $\rho(\widetilde{I}_j) = I_j$. Every automorphism $\bar\alpha$ of $\overline{H}$ that satisfies $\bar\alpha(\overline{I}_j) = \overline{I}_j$ for all j, lifts to an automorphism $\tilde\alpha$ of $\widetilde{H}$ defined by $x_{\bar h, k} \mapsto x_{\bar\alpha(\bar h), k}$ and $\tilde\epsilon_{\bar\epsilon, k, j} \mapsto \tilde\epsilon_{\bar\alpha(\bar\epsilon), k, j}$. Clearly $\tilde\alpha(\widetilde{I}_j) = \widetilde{I}_j$. If I_j and $\overline{I}_j$ are conjugacy domains, we can replace $\widetilde{I}_j$ by the conjugacy domain that it generates in $\widetilde{H}$.

Thus $\widetilde{H}$ satisfies the requirements of the lemma, except that it is not finite. To make $\widetilde{H}$ finite, replace it by its quotient $\widetilde{H}/N$, and ρ by the induced quotient map, where N is a characteristic subgroup of finite index in $\widetilde{H}$, contained in $\mathrm{Ker}(\rho)$. For example, take N to be the intersection of all normal subgroups M of $\widetilde{H}$ with $\widetilde{H}/M \cong H$. ▮

They are connected.

4. Points over ordered fields

Let G be a finite group with a trivial center such that the Schur multiplier of G is generated by commutators. Identify G with the subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$. Fix a sufficiently large integer r that satisfies (2.13) and the assertions of Lemma 3.1. Associate with G and r the moduli spaces $\mathcal{H}^{\mathrm{inn}}$ and $\mathcal{H}^{\mathrm{ab}}$ (2.3).

Our aim is to choose Hurwitz spaces $\mathcal{H}$ and $\bar{\mathcal{H}}$ and some points $\mathbf{q} = [\mathbf{b}, 0, f_0]$ on $\mathcal{H}$ (2.4). First, let

$$e = 8 \cdot |G|! \qquad \text{and} \qquad m = \frac{r-e}{2},$$

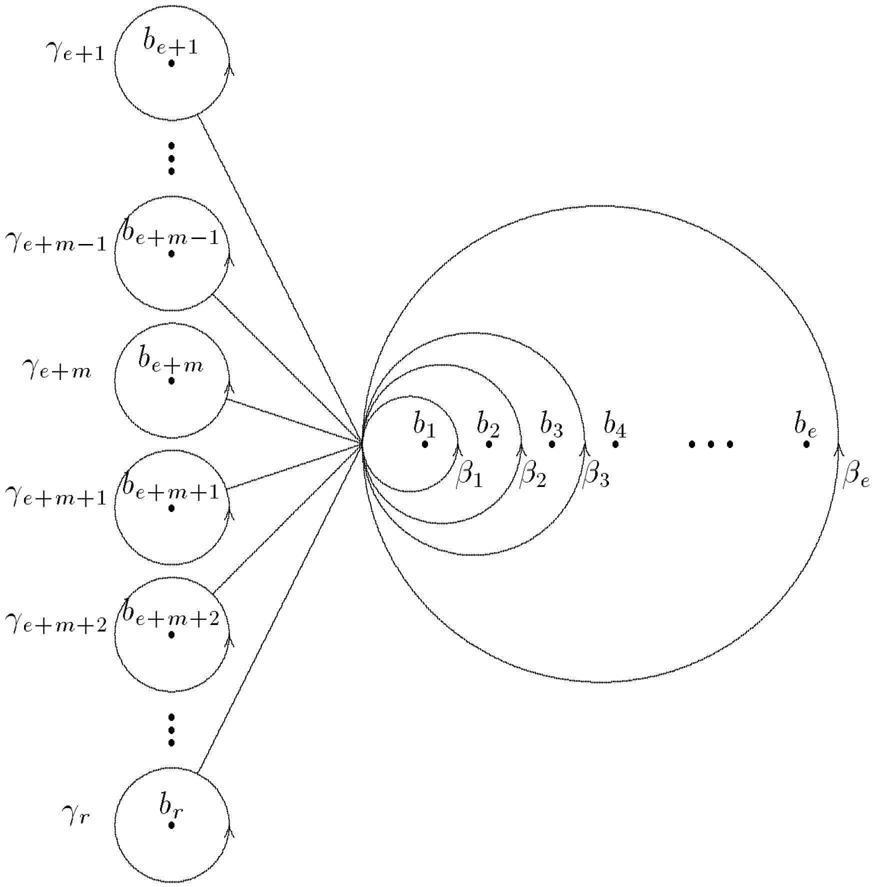
so that $r = e + 2m$, and define the base point $\mathbf{b} = \{b_1, \dots, b_r\}$ in $\mathcal{U}_r$ by

$$b_1 = 1, \dots, b_e = e, \text{ and } b_{e+j} = -3 + (2m + 1 - 2j)\sqrt{-1}, \text{ for } j = e, \dots, 2m.$$

Next, fix generators of $\Pi_1(\mathbb{P}^1 \smallsetminus \mathbf{b}, 0)$. For each $1 \le j \le r$ let D_j be the disc of diameter $\frac{1}{2}$ around b_j (so that $D_1, \dots, D_r$ are disjoint). Define loops $\gamma_1, \dots, \gamma_r$ in the complex plane with the initial and final point 0 in the following way:

(1) $\gamma_1 = \beta_1$, $\gamma_2 = \beta_1^{-1}\beta_2$, $\dots$, $\gamma_e = \beta_{e-1}^{-1}\beta_e$, where β_j is the circle in the counterclockwise direction with diameter $[0, b_j + \frac{1}{2}]$ on the real axis;

(2) for $e < j \le r$ the path γ_j goes on a straight line from 0 towards b_j, then travels on a circle of diameter $\frac{1}{2} < \rho < 1$ in the counterclockwise direction around b_j, and returns on a straight line to 0.



These loops are homotopic to the loops constructed in (2.12). Therefore they represent generators of the fundamental group $\Pi_1(\mathbb{P}^1 \setminus \bigcup D_j, 0)$, subject only to the relation $\gamma_1 \cdots \gamma_r = 1$. If $\mathbf{a}$ is an r-tuple with $|\mathbf{a} \cap D_j| = 1$ for $j = 1, \dots, r$, then $\gamma_1, \dots, \gamma_r$ also represent generators of $\Gamma = \Pi_1(\mathbb{P}^1 \setminus \mathbf{a}, 0)$. Indeed, $\Pi_1(\mathbb{P}^1 \setminus \bigcup D_j, 0) \cong \Gamma$ via the inclusion $\mathbb{P}^1 \setminus \bigcup D_j \to \mathbb{P}^1 \setminus \mathbf{a}$. Furthermore, for such $\mathbf{a}$, we may use $\gamma_1, \dots, \gamma_r$ also to represent free generators of $\tilde{\Gamma} = \Pi_1(\mathbb{P}^1 \setminus (\mathbf{a} \cup \{\infty\}), 0)$. The canonical epimorphism $\lambda_*\colon \tilde{\Gamma} \to \Gamma$ induced by the inclusion $\lambda\colon \mathbb{P}^1 \setminus (\mathbf{a} \cup \{\infty\}) \to \mathbb{P}^1 \setminus \mathbf{a}$ maps the class of γ_j in $\tilde{\Gamma}$ onto the class of γ_j in Γ.

Using $\mathbf{b}$ and $\gamma_1, \dots, \gamma_r$, define the Hurwitz spaces $\mathcal{H}$ and $\bar{\mathcal{H}}$ and the maps Λ, Ψ, and $\bar{\Psi}$ as in (2.13).

Finally, assume that G has non-inner automorphisms of order 2, and let ϵ be such an automorphism. Let $G \rtimes \langle \epsilon \rangle$ be the subgroup of $\mathrm{Aut}(G)$ generated by G and ϵ. In particular, the centralizer of G in $G \rtimes \langle \epsilon \rangle$ is trivial. Let $I \subseteq G \rtimes \langle \epsilon \rangle \setminus G$ be a set of involutions, with $\epsilon \in I$ and $|I| \geq 2$. Lemma 3.1 (with m replaced by $2m$) produces an r-tuple $(\sigma_1, \ldots, \sigma_r) \in \mathcal{E}^{(r)}(G)$ (see (2.12)) with the following properties:

(3) $\sigma_j^\epsilon = \sigma_1 \cdots \sigma_{j-1}\sigma_j^{-1}\sigma_{j-1}^{-1} \cdots \sigma_1^{-1}$, for each $1 \leq j \leq e$;

(4) $\sigma_{e+j}^\epsilon = \sigma_{e+(2m+1-j)}^{-1}$ for $j = 1, \ldots, 2m$;

(5) $I = \{\epsilon, \epsilon\sigma_1, \epsilon\sigma_1\sigma_2, \ldots, \epsilon\sigma_1\sigma_2\cdots\sigma_e\}$.

Fix (for each ϵ and each I) such an r-tuple $(\sigma_1, \ldots, \sigma_r)$. As $\sigma_1 \cdots \sigma_r = 1$, there is a unique epimorphism $f_0\colon \Pi_1(\mathbb{P}^1 \smallsetminus \mathbf{b}, 0) \to G$ with $f_0(\gamma_j) = \sigma_j$, for $j = 1, \ldots, r$.

Definition 4.1: The point $\mathbf{q} = [\mathbf{b}, 0, f_0] \in \mathcal{H}$ is called **the basic point associated with** G, ϵ, and I. The neighborhood

$$\mathcal{N} = \{\mathbf{p} = [\mathbf{a}, 0, f] \in \mathcal{H}^{in} |\ \mathbf{a} \cap D_j = 1, f(\gamma_j) = f_0(\gamma_j) = \sigma_j,\ \text{for } j = 1, \ldots, r\}$$

of $\mathbf{q}$ in $\mathcal{H}$ is called **the basic neighborhood** of $\mathbf{q}$.

Remark 4.2: (a) A priori, $\mathcal{N}$ is a neighborhood of $\mathbf{q}$ in $\mathcal{H}^{\text{inn}}$ (see (2.6)). Yet, $\mathcal{N}$ is connected. Hence, $\mathcal{N} \subseteq \mathcal{H}$.

(b) The point $\mathbf{b}$ is $\mathbb{Q}$-rational (2.7). Hence $\mathbf{q}$ is algebraic over $\mathbb{Q}$.

(c) Let $\mathbf{p} \in \mathcal{N}$, and let $\mathbf{a} = \{a_1, \ldots, a_r\} = \Psi(\mathbf{p})$. Then without loss of generality $a_j \in D_j$, for $j = 1, \ldots, r$. If $\mathbf{a}$ is $\mathbb{R}$-rational (i.e., $(X - a_1)\cdots(X - a_r) \in \mathbb{R}[X]$), then $a_1 < \cdots < a_e$ are real, and $a_{e+(2m+1-j)}$ is the complex conjugate of a_{e+j}, for $j = 1, \ldots, m$.

(d) Let c be the complex conjugation. As $\mathcal{H}$ is an affine variety, we may embed it in a fixed affine space $\mathbb{A}^n$. Then the complex topology on it is given by the norm $|| - ||_c$ defined in Definition 1.4. There are only finitely many choices of ϵ and I. Hence there are only finitely many basic points associated with G. Thus there is a positive rational number ν (that depends only on G) such that if $\mathbf{q}$ is a basic point, $\mathbf{p} \in \mathcal{H}$, and $||\mathbf{p} - \mathbf{q}||_c{}^2 < \nu^2$, then $\mathbf{p}$ is in the basic neighborhood $\mathcal{N}$ of $\mathbf{q}$.

Lemma 4.3: *Let* $\mathbf{p} \in \mathcal{N}$ *such that* $\Psi(\mathbf{p})$ *is* $\mathbb{R}$*-rational. Then* $\delta_\epsilon(\mathbf{p}) = c(\mathbf{p})$, *where* c *is complex conjugation.*

Proof: Write $\mathbf{p}$ as $[\mathbf{a}, 0, f]$. Then $\mathbf{a} = \Psi(\mathbf{p})$. We have $\delta_\epsilon(\mathbf{p}) = [\mathbf{a}, 0, \epsilon \circ f]$ by (2.14) and $c(\mathbf{p}) = [c(\mathbf{a}), 0, cf] = [\mathbf{a}, 0, cf]$ by (2.11). It remains to show that $cf = \epsilon \circ f$.

Observe that $c\beta_j = \beta_j^{-1}$, for $j = 1, \ldots, e$. Recursively:

$$(6) \qquad c\gamma_1 = \gamma_1^{-1},\ c\gamma_2 = \gamma_1\gamma_2^{-1}\gamma_1^{-1}, \ldots, c\gamma_e = \gamma_1 \cdots \gamma_{e-1}\gamma_e^{-1}\gamma_{e-1}^{-1} \cdots \gamma_1^{-1}.$$

Furthermore,

$$(7) \qquad c\gamma_{e+j} = \gamma_{e+(2m+1-j)}^{-1}, \text{ for } j = 1, \ldots, 2m.$$

Recall that $(cf)(c\gamma) = f(\gamma)$. Combine (6) and (7) with (3) and (4) to get $cf(\gamma_j) = f(c\gamma_j) = \sigma_j^{\epsilon} = (\epsilon \circ f)(\gamma_j)$, for each $1 \le j \le r$. ▮

Proposition 4.4: *Let (K, P) be an ordered field, and let ι be an involution in $G(K)$ inducing P on K. Assume that $\widetilde{K} \subseteq \mathbb{C}$. Let $\lambda \in G(\mathbb{Q})$, and let $\mathbf{p} \in \mathcal{H}(\widetilde{K})$ such that*

$$(8) \qquad ||\mathbf{p} - \lambda(\mathbf{q})||_{\iota}{}^2 < \nu^2 \textit{ in } \widetilde{K}(\iota),$$

and $\bar{\mathbf{p}} = \Lambda(\mathbf{p})$ is K-rational. Let $F/K(x)$ be the Galois extension and $h_1\colon G(F/K(x)) \to \mathrm{Aut}(G)$ the embedding corresponding to $\mathbf{p}$ over K (2.17). Put $L = K(\mathbf{p})$ and let H be the image of h_1. The following hold:

(a) *$\delta_\epsilon(\mathbf{p}) = \iota(\mathbf{p})$;*

(b) *P does not extend to L; in particular, P does not extend to F;*

(c) *$G \rtimes \langle \epsilon \rangle \le H$, and therefore $I \subseteq H$;*

(d) *$h_1(I_P(F/K(x))) = \mathit{Con}_H(I)$.*

Proof: By (2.14), $\delta_\epsilon(\mathbf{p}) \neq \mathbf{p}$. Therefore (a) implies $\iota(\mathbf{p}) \neq \mathbf{p}$. Hence $L \not\subseteq \widetilde{K}(\iota)$, and this implies (b). Furthermore, the criterion of (2.17) implies that $\epsilon \in H$. Since $G \le H$, $G \rtimes \langle \epsilon \rangle \le H$. So it suffices to prove (a) and (d).

Part I: *Reduction to K with archimedean orderings dense in $X(K)$.* Let K_0 be a finitely generated subfield of K, containing the finitely generated subfield $\mathbb{Q}(\bar{\mathbf{p}})$ of K. Let P_0 be the restriction of P to K_0. This ordering is induced from the restriction $\iota_0 \in G(K_0)$ of ι. Let $F_0/K_0(x)$ be the Galois extension and $(h_1)_0\colon G(F_0/K_0(x)) \to \mathrm{Aut}(G)$ the embedding corresponding to $\mathbf{p}$ over K_0. We may assume that $F = F_0 \cdot K$ (2.18). If K_0 is sufficiently large, then the restriction map $\mathrm{res}_{F_0}\colon G(F/K(x)) \to G(F_0/K_0(x))$ is an isomorphism. If we can show that the assertions hold for K_0, P_0, ι_0, $(h_1)_0$, then, by (2.19) and since $\iota_0(\mathbf{p}) = \iota(\mathbf{p})$, they also hold for K, P, ι, h_1. By Lemma 1.6(a), the set of archimedian orderings on K_0 is dense in $X(K_0)$. So we may assume that K enjoys this property.

Part II: *Reduction to P archimedian.* By Remark 1.8, if P' is an (archimedian) ordering of K sufficiently near to P, then $I_P(F/K(x)) = I_{P'}(F/K(x))$. We may assume that P' is induced by an involution ι' in $G(K)$, which is so near to ι that

$$||\mathbf{p} - \lambda(\mathbf{q})||_\iota{}^2 = ||\mathbf{p} - \lambda(\mathbf{q})||_{\iota'}{}^2 \quad \text{and} \quad \iota(\mathbf{p}) = \iota'(\mathbf{p}).$$

Thus we may replace P by P' and ι by ι'.

Part III: *Reduction to $K = \mathbb{R}$ and $\lambda = 1$.* Assume that P is archimedian. Extend λ^{-1} to an automorphism β of $\mathbb{C}$, and let $\iota' = \beta\iota\beta^{-1}$. Then $\beta(\widetilde{K}(\iota)) = (\beta(\widetilde{K}))(\iota')$ is a real closure of $(\beta(K), \beta(P))$. Hence it is also archimedian. Thus we may assume that $(\beta(\widetilde{K}))(\iota') \subseteq \mathbb{R}$. Hence $\beta\iota\beta^{-1} = \iota' = \mathrm{res}_{\beta(\widetilde{K})} c$, where c is complex conjugation on $\mathbb{C}$.

Since $\mathbf{q}$ is algebraic over $\mathbb{Q}$, we have $\beta\lambda(\mathbf{q}) = \mathbf{q}$. Therefore an application of β to (8) yields

$$||\beta(\mathbf{p}) - \mathbf{q}||_c{}^2 < \nu^2 \text{ in } \mathbb{R}. \tag{$8'$}$$

As $\bar{\mathbf{p}}$ is K-rational, $\iota(\bar{\mathbf{p}}) = \bar{\mathbf{p}}$. Thus $c(\beta(\bar{\mathbf{p}})) = \beta(\bar{\mathbf{p}})$, and hence $\beta(\bar{\mathbf{p}})$ is $\mathbb{R}$-rational. Also, since δ_ϵ is defined over $\mathbb{Q}$, it commutes with β. Therefore (a) is equivalent to

(a$'$) $\delta_\epsilon(\beta(\mathbf{p})) = c(\beta(\mathbf{p}))$.

Finally, let $F'/\mathbb{R}(x)$ be the Galois extension and h_1': $G(F'/\mathbb{R}(x)) \to \mathrm{Aut}(G)$ the embedding corresponding to $\beta(\mathbf{p})$ over $\mathbb{R}$, and let H' be the image of h_1'. Then by (2.19), condition (d) follows from

(d$'$) $h_1'(I_{P'}(F'/\mathbb{R}(x))) = \mathrm{Con}_{H'}(I)$,

where P' is the unique ordering of $\mathbb{R}$.

Thus, replacing K by $\mathbb{R}$ and $\mathbf{p}$ by $\beta(\mathbf{p})$, we may assume that $K = \mathbb{R}$ and $\lambda = 1$.

Part IV: *$K = \mathbb{R}$ and $\lambda = 1$.* By Remark 4.2(d) we have $\mathbf{p} \in \mathcal{N}$. Write $\mathbf{p}$ as $[\mathbf{a}, 0, f]$. Then $\mathbf{a} = \Psi(\mathbf{p}) = \bar{\Psi}(\bar{\mathbf{p}})$ is $\mathbb{R}$-rational. So Lemma 4.3 gives assertion (a).

Proof of (d): By (c) – that follows from (a) – we have $G \rtimes \langle \epsilon \rangle \leq H$. Check:

$$|G \rtimes \langle \epsilon \rangle| = 2 \cdot |G| = 2 \cdot [F : \mathbb{C}(x)] = [F : \mathbb{R}(x)] = |H|,$$

so $H = G \rtimes \langle \epsilon \rangle$.

Write $\mathbf{p}$ in the form $\mathbf{p} = [\chi, h]$, with $\chi \colon X \to \mathbb{P}^1$ (2.1). Fix a point $y \in \chi^{-1}(0)$. Let $Y_0 = \mathbb{P}^1 \smallsetminus (\mathbf{a} \cup \{\infty\})$, and let $\psi \colon \widehat{Y}_0 \to Y_0$ be the universal unramified covering of Y_0. Fix a point $\hat{y} \in \psi^{-1}(0)$. Put $Y = \chi^{-1}(Y_0) \subseteq X$. As $\chi \colon Y \to Y_0$ is unramified, there exists a unique covering $\varphi \colon \widehat{Y}_0 \to Y$ such that $\chi \circ \varphi = \psi$ and $\varphi(\hat{y}) = y$. Let $\widehat{F}$ be the field of algebraic meromorphic functions on $\widehat{Y}_0$ (in the sense of [KN, p. 199]). Then the field extension $\widehat{F}/\mathbb{C}(x)$ induced by ψ is the maximal extension of $\mathbb{C}(x)$ unramified in Y_0.

Let $F = \mathbb{C}(X) = \mathbb{C}(Y)$. We shall identify $G(F/\mathbb{C}(x))$ with G via h_0, and $G(F/\mathbb{R}(x))$ with H via h_1 (see (2.16) and (2.17)), so that $h \colon \mathrm{Aut}(X/\mathbb{P}^1) \to G$ is the canonical isomorphism $\mathrm{Aut}(X/\mathbb{P}^1) \to G(F/\mathbb{C}(x))$ that sends $\alpha \in \mathrm{Aut}(X/\mathbb{P}^1)$ to the element $f \mapsto f \circ \alpha^{-1}$ of $G(F/\mathbb{C}(x))$. Similarly, let $\widehat{G} = G(\widehat{F}/\mathbb{C}(x))$, and let $\hat{h} \colon \mathrm{Aut}(\widehat{Y}_0/Y_0) \to \widehat{G}$ be the canonical map that sends $\hat{\alpha}$ to the element $\hat{f} \mapsto \hat{f} \circ \hat{\alpha}^{-1}$.

Let $\imath \colon \Pi_1(\mathbb{P}^1 \smallsetminus \mathbf{a}, 0) \to \mathrm{Aut}(X/\mathbb{P}^1)$ be the epimorphism associated to the point $y \in \chi^{-1}(0)$ (see (2.2)). Similarly define $\hat{\imath} \colon \Pi_1(Y_0, 0) \to \mathrm{Aut}(\widehat{Y}_0/Y_0)$, associated to the point $\hat{y} \in \psi^{-1}(0)$. Then there is a commutative diagram

$$\begin{tikzcd}
\Pi_1(Y_0, 0) \arrow[r, "\hat{\imath}"'] \arrow[d, "\lambda_*"] & \mathrm{Aut}(\widehat{Y}_0/Y_0) \arrow[r, "\hat{h}"'] \arrow[d, "\varphi_*"] & \widehat{G} \arrow[d, "\mathrm{res}_F"] \\
\Pi_1(\mathbb{P}^1 \smallsetminus \mathbf{a}, 0) \arrow[r, "\imath"'] & \mathrm{Aut}(X/\mathbb{P}^1) \arrow[r, "h"'] & G
\end{tikzcd}$$

where λ_* is induced from the inclusion $\lambda \colon Y_0 \to \mathbb{P}^1 \smallsetminus \mathbf{a}$.

Put $\hat{\sigma}_j = \hat{h} \circ \hat{\imath}(\gamma_j)$, for $j = 1, \ldots, r$. Then

$$\text{(9)} \qquad \mathrm{res}_F \hat{\sigma}_j = h \circ \imath \circ \lambda_*(\gamma_j) = f(\gamma_j) = \sigma_j, \qquad \text{for } j = 1, \ldots, r.$$

Let $n = (r + e)/2 = e + m$. By Remark 4.2(c) we may assume $a_1 < \cdots < a_e$ are real, and $a_{e+(2m+1-j)}$ is the complex conjugate of a_{e+j}, for $j = 1, \ldots, m$. Observe that

$\widehat{F}$ is the maximal extension of $\mathbb{R}(x)$ unramified outside the primes of $\mathbb{R}(x)$ induced by $a_1,\dots,a_r,\infty$. In this situation the proof of [KN, Satz 2] shows that there is $\hat\epsilon\in\widehat{H}=G(\widehat{F}/\mathbb{R}(x))$ such that $\hat\epsilon,\hat\sigma_1,\dots,\hat\sigma_n$ form a system of generators for $\widehat{H}$ with the defining relations

$$(10)\qquad \hat\epsilon^2=1 \quad\text{and}\quad \hat\sigma_j^{\hat\epsilon}=\hat\sigma_1^{-1}\cdots\hat\sigma_{j-1}^{-1}\hat\sigma_j^{-1}\hat\sigma_{j-1}\cdots\hat\sigma_1, \qquad \text{for } 1\le j\le e.$$

Further, $\hat\sigma_{e+j}^{\hat\epsilon}=\hat\sigma_{e+(2m+1-j)}^{-1}$ for $j=1,\dots,2m$. By (3) and (4) this implies that $\mathrm{res}_F\hat\epsilon$ and ϵ act on G in the same way. Since H is a subgroup of $\mathrm{Aut}(G)$, this implies that $\mathrm{res}_F\hat\epsilon=\epsilon$.

Further, each involution of $\widehat{H}$ is conjugate to one of $\hat\epsilon$, $\hat\sigma_1\hat\epsilon$, $\hat\sigma_2\hat\sigma_1\hat\epsilon,\dots,\hat\sigma_e\cdots\hat\sigma_2\hat\sigma_1\hat\epsilon$. Indeed, by [HJ2, Lemma 4.2 (Part E)] it follows that $\widehat{H}$ is the free profinite product of the free profinite group $\langle\hat\sigma_{e+1},\dots,\hat\sigma_n\rangle$ of rank $n-e=m$ with $e+1$ groups

$$\langle\hat\epsilon\rangle,\ \langle\hat\sigma_1\hat\epsilon\rangle,\ \langle\hat\sigma_2\hat\sigma_1\hat\epsilon\rangle,\ \dots\ ,\langle\hat\sigma_e\cdots\hat\sigma_2\hat\sigma_1\hat\epsilon\rangle$$

that are of order two. Thus by [HR, Theorem A$'$] the elements of finite order in $\widehat{H}$ are the conjugates of the elements of these $e+1$ subgroups.

By Lemma 4.5 below, all involutions of $\widehat{H}$ are real. Using (9) and (5) we finally get

$$I(F/\mathbb{R}(x))=\mathrm{res}_F I(\widehat{F}/\mathbb{R}(x))=\mathrm{res}_F\mathrm{Con}_{\widehat{H}}(\{\hat\epsilon,\hat\sigma_1\hat\epsilon,\dots,\hat\sigma_e\cdots\hat\sigma_1\hat\epsilon\})=$$

$$=\mathrm{Con}_H(\{\epsilon,\sigma_1\epsilon,\dots,\sigma_e\cdots\sigma_1\epsilon\})=\mathrm{Con}_H(I). \qquad ∎$$

Lemma 4.5: *Let S be a finite set of finite prime divisors of the field $\mathbb{R}(x)$. Let $\mathbb{R}(x)^S$ be the maximal extension of $\mathbb{R}(x)$ unramified outside $S\cup\{\infty\}$, and set $G^S=G(\mathbb{R}(x)^S/\mathbb{R}(x))$. Then all involutions of G^S are real.*

Proof: By [KN, Satz 3] the absolute Galois group $\mathcal{G}$ of $\mathbb{R}(x)$ possesses a system of generators
$\{\delta,\ \tau_{\mathfrak p}\mid \mathfrak p$ a finite prime of $\mathbb{R}(x)/\mathbb{R}\}$ with the defining profinite relations

$$\delta^2=1 \quad\text{and}\quad \tau_{\mathfrak p}^{\delta}=\Big(\prod_{\mathfrak p'<\mathfrak p}\tau_{\mathfrak p'}^{-1}\Big)\,\tau_{\mathfrak p}^{-1}\Big(\prod_{\mathfrak p'<\mathfrak p}\tau_{\mathfrak p'}^{-1}\Big)^{-1} \quad \text{for all real } \mathfrak p.$$

Here $\prod_{\mathfrak{p}'<\mathfrak{p}} \tau_{\mathfrak{p}'}^{-1}$ is the unique accumulation point of the set of products $\tau_{\mathfrak{p}_1}^{-1}\cdots\tau_{\mathfrak{p}_r}^{-1} \in \mathcal{G}$ for real primes $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ with $\mathfrak{p}_1 < \cdots < \mathfrak{p}_r < \mathfrak{p}$. Furthermore, [KN] constructs this system of generators in such a way that for every finite set S of finite primes and every finite prime $\mathfrak{p} \notin S$ the natural restriction map $\mathcal{G} \to G^S$ maps $\tau_{\mathfrak{p}}$ onto 1 [KN, p. 207].

Let $\mathfrak{p}_1 < \cdots < \mathfrak{p}_e$ be the real, and $\mathfrak{p}_{e+1},\dots,\mathfrak{p}_n$ the complex primes of S. Let $\hat\sigma_j$ be the image of $\tau_{\mathfrak{p}_j}$, for $j = 1,\dots,n$, and let $\hat\epsilon$ be the image of δ in G^S. Then $\hat\epsilon, \hat\sigma_1,\dots,\hat\sigma_n$ generate G^S and satisfy (10). These are in fact defining relations for G^S by [KN, Satz 2]. As in the last part of the proof of Proposition 4.4, each involution of G^S is conjugate to some $\hat\sigma_j\cdots\hat\sigma_1\hat\epsilon$, where $0 \le j \le e$. Thus it suffices to show that each $\hat\sigma_j\cdots\hat\sigma_1\hat\epsilon$ lifts to an involution of $\mathcal{G}$. To this end put

$$\delta_0 = \delta \quad \text{and} \quad \delta_j = \Big(\prod_{\mathfrak{p}'\le\mathfrak{p}_j} \tau_{\mathfrak{p}'}^{-1}\Big)^{-1}\delta = \tau_{\mathfrak{p}_j}\Big(\prod_{\mathfrak{p}'<\mathfrak{p}_j} \tau_{\mathfrak{p}'}^{-1}\Big)^{-1}\delta$$

for $1 \le j \le e$. Then δ_j maps onto the involution $\hat\sigma_j\cdots\hat\sigma_1\hat\epsilon$ in G^S. In fact, given another finite set S' of finite primes of $\mathbb{R}(x)$ that contains S, the same argument shows that δ_j restricts to an involution in $G^{S'}$. As $\mathcal{G} = \varprojlim_{S'} G^{S'}$, we get that δ_j is an involution in $\mathcal{G}$. ∎

5. The regular real embedding problem over a PRC field

Let K be a PRC field of characteristic 0, and let $\nu > 0$ be a rational number. We proceed similarly as in the case of PAC fields [FV2, Section 1]. We take some extra care at places.

Let $X_1, \ldots, X_m$ be a partition of $X(K)$ into disjoint clopen subsets. Fix a closed system $\mathcal{X}$ of representatives of the conjugacy classes of involutions in $G(K)$; then $\{\widetilde{K}(\iota) |\ \iota \in \mathcal{X}\}$ is a closed subset of real closures of K, one for each ordering of K (see Section 1). Put $\mathcal{X}_j = \mathcal{X} \cap I_{X_j}(\widetilde{K}/K)$. Then $\mathcal{X}_1, \ldots, \mathcal{X}_m$ is a partition of $\mathcal{X}$ into disjoint clopen subsets.

Lemma 5.1: *Let $\Lambda\colon \mathcal{H} \to \bar{\mathcal{H}}$ be an unramified Galois cover of absolutely irreducible, non-singular varieties defined over K. Assume that all the automorphisms of $\mathcal{H}/\bar{\mathcal{H}}$ are defined over K. Let $\beta\colon G(K) \to \mathrm{Aut}(\mathcal{H}/\bar{\mathcal{H}})$ be a homomorphism, and let L be the fixed field of* $\ker(\beta)$. *Assume that L is not formally real. Let $\mathbf{q}_1, \ldots, \mathbf{q}_m \in \mathcal{H}(\widetilde{K})$ satisfy*

(1) $\iota\mathbf{q}_j = \beta(\iota)(\mathbf{q}_j)$, *for each $\iota \in \mathcal{X}_j$, for $j = 1, \ldots, m$.*

Then there exists $\mathbf{p} \in \mathcal{H}(\widetilde{K})$ such that

(2) $\sigma\mathbf{p} = \beta(\sigma)(\mathbf{p})$ *for each $\sigma \in G(K)$;*

(3) $||\mathbf{p} - \mathbf{q}_j||_\iota^2 \le \nu^2$ *in $\widetilde{K}(\iota)$, for each $\iota \in \mathcal{X}_j$, for $j = 1, \ldots, m$;*

(4) *the point $\Lambda(\mathbf{p})$ of $\bar{\mathcal{H}}$ is K-rational and $K(\mathbf{p}) = L$.*

Proof: First notice that (4) follows from (2). Indeed, an automorphism of the unramified cover $\mathcal{H} \to \bar{\mathcal{H}}$ has no fixed points. If (2) holds, then for each $\sigma \in G(K)$ we have $\sigma(\Lambda(\mathbf{p})) = \Lambda(\sigma(\mathbf{p})) = \Lambda(\beta(\sigma)(\mathbf{p})) = \Lambda(\mathbf{p})$ and

$$\sigma(\mathbf{p}) = \mathbf{p} \iff \beta(\sigma)(\mathbf{p}) = \mathbf{p} \iff \beta(\sigma) = 1 \iff \sigma \in G(L).$$

The rest is a straightforward modification of the proof of [FV2, Lemma 1]. Apply Weil's descent [W, Theorem 3] to the maps $f_{\tau,\rho} = \beta(\tau) \circ \beta(\rho)^{-1}$ to get a variety $\mathcal{H}'$ defined over K, and a linear isomorphism $f\colon \mathcal{H}' \to \mathcal{H}$ defined over L with these properties. The map $\Lambda \circ f\colon \mathcal{H}' \to \bar{\mathcal{H}}$ is defined over K and $\sigma f = \beta(\sigma) \circ f$, for each $\sigma \in G(K)$. In particular, suppose that $\mathbf{q}' \in \mathcal{H}'(\widetilde{K})$ and $\mathbf{q} = f(\mathbf{q}') \in \mathcal{H}(\widetilde{K})$. Then, for every $\sigma \in G(K)$

$$\beta(\sigma)(\mathbf{q}) = (\beta(\sigma) \circ f)(\mathbf{q}') = (\sigma f)(\mathbf{q}') = \sigma(f(\sigma^{-1}\mathbf{q}')).$$

Conclude that

$$(5) \qquad \sigma\mathbf{q} = \beta(\sigma)(\mathbf{q}) \quad \Longleftrightarrow \quad \mathbf{q}' = \sigma\mathbf{q}' \quad \Longleftrightarrow \quad \mathbf{q}' \in \mathcal{H}'(\widetilde{K}(\sigma)).$$

Use (5) and equation (3) of Section 1 to translate (1)–(3) via f from $\mathcal{H}$ to $\mathcal{H}'$. Let $\mathbf{q}_j' = f^{-1}(\mathbf{q}_j) \in \mathcal{H}'(\widetilde{K})$, for $j = 1, \dots, m$. Then

(1′) $\mathbf{q}_j' \in \mathcal{H}'(\widetilde{K}(\iota))$, for each $\iota \in \mathcal{X}_j$, for $j = 1, \dots, m$.

We must find $\mathbf{p}' \in \mathcal{H}'(\widetilde{K})$ such that

(2′) $\mathbf{p}' \in \mathcal{H}'(\widetilde{K}(\sigma))$ for each $\sigma \in G(K)$, that is, $\mathbf{p}' \in \mathcal{H}'(K)$;

(3′) $||\mathbf{p}' - \mathbf{q}_j'||_\iota^2 \le \nu^2/||f||_\iota^2$ in $\widetilde{K}(\iota)$, for each $\iota \in \mathcal{X}_j$, for $j = 1, \dots, m$

Suppose that $\iota \in \mathcal{X}_j$ for some $1 \le j \le m$. As $||f||_\iota^2$ is algebraic over K, there is $a_j \in K$ such that $a_j^2 > ||f||_\iota^2$ in $\widetilde{K}(\iota)$. Replace $\mathcal{X}_1, \dots, \mathcal{X}_m$ by a finer partition to assume that this is true for each $\iota \in \mathcal{X}_j$. Thus (3′) will follow from a stronger statement:

(3″) $||\mathbf{p}' - \mathbf{q}_j'||_\iota^2 \le \nu^2/a_j^2$ in $\widetilde{K}(\iota)$, for each $\iota \in \mathcal{X}_j$, for $j = 1, \dots, m$.

By Proposition 1.2 there is $\mathbf{p}' \in \mathcal{H}'(K)$ such that (3″) holds. ▮

THE VALUE OF THEOREM 5.2: Let K be a PRC field with just two orderings, and $L = K(\sqrt{-1})$. Let H be a direct product of groups of order 2, with an epimorphism $\pi\colon H \to G(L/K)$. Then we can realize H over $K(x)$ such that π coincides with the restriction to $L(x)$, and: all involutions in $H \setminus \ker(\pi)$ are real. Previously you would only get that at least one involution in $H \setminus \ker(\pi)$ is real. Using tricks (replace L by a larger field, replace H by a cover of it), you could achieve that at least two involutions in $H \setminus \ker(\pi)$ are real. This is the improvement, and this is all the improvement. However, suppose you want to use it to say something about $G(K)$, where K has some kind of (real) hilbertian property. Thus you want to specialize the above realization of H over $K(x)$ to a realization of H over K. Presumably you want to keep the real involutions still real (otherwise why bother to make them real in the first placeΓ). But $G(K)$ has only two conjugacy classes of real involutions, so you can do it only if there are only two involutions in $H \setminus \ker(\pi)$. So the improvement is not used.

THEOREM 5.2: *Let L/K be a finite Galois extension with L not formally real and let $\pi\colon H \to G(L/K)$ be an epimorphism of finite groups. For each $1 \le j \le m$ let $I_j \subseteq H$*

be a conjugacy domain of involutions such that $\pi(I_j) = I_{X_j}(L/K)$. Then there exists a regular extension F of L, Galois over $K(x)$, and an isomorphism $h_1\colon G(F/K(x)) \to H$ that maps $I_{X_j}(F/K(x))$ onto I_j. In addition, the following diagram commutes.

$$\begin{tikzcd}
G(F/K(x)) \arrow[rr, "h_1"] \arrow[dr, "\mathrm{res}_L"'] & & H \arrow[dl, "\pi"] \\
& G(L/K) &
\end{tikzcd}$$

In particular, h_1 maps $I(F/K(x))$ onto $\bigcup_j I_j$.

Proof: By Skolem-Löwenheim Principle [FJ, Proposition 6.4] we may assume that $K \subseteq \mathbb{C}$. We divide the proof into five parts.

PART 1: *Weakening of the commutativity.* Let $G = \mathrm{Ker}(\pi)$. Instead of the commutativity of the diagram it suffices to show that h_1 maps $G(F/L(x))$ onto G. Indeed, apply Lemma 3.3. This gives an epimorphism of finite groups $\rho\colon \widetilde{H} \to H$ and conjugacy domains of involutions $\widetilde{I}_1, \ldots, \widetilde{I}_m \subseteq \widetilde{H}$ such that $\rho(\widetilde{I}_j) = I_j$. In addition, every automorphism of $G(L/K)$ that preserves the $I_{X_j}(L/K)$ lifts (under the map $\rho \circ \pi$) to an automorphism of $\widetilde{H}$ that preserves the $\widetilde{I}_j$. Let $\widetilde{G} = \mathrm{Ker}(\rho \circ \pi)$.

Assume that we can find a regular extension $\widehat{F}$ of L, Galois over $K(x)$, and an isomorphism $\hat{h}_1\colon G(\widehat{F}/K(x)) \to \widehat{H}$ that maps $G(\widehat{F}/L(x))$ onto $\widehat{G}$ and the $I_{X_j}(\widehat{F}/K(x))$ onto the $\widehat{I}_j$. In particular, $\mathrm{Ker}(\pi \circ \rho \circ \hat{h}_1) = G(\widehat{F}/L(x)) = \mathrm{Ker}(\mathrm{res}_L)$. Hence there exists an automorphism α of $G(L/K)$ such that $\alpha \circ \pi \circ \rho \circ \hat{h}_1 = \mathrm{res}_L$ and α preserves the $I_{X_j}(L/K)$. We can lift α to an automorphism $\hat{\alpha}$ of $\widehat{H}$ that preserves the $\widehat{I}_j$. Thus, by composing h_1 with $\hat{\alpha}$ we may assume that $(\pi \circ \rho) \circ \hat{h}_1 = \mathrm{res}_L$.

Now let F be the fixed field of $\mathrm{Ker}(\rho)$ in $\widehat{F}$. Then h_1 induces an isomorphism $h_1\colon G(F/K(x)) \to H$ with the required properties.

PART 2: *Reduction to commutators generate $M(G)$, $H \subseteq \mathrm{Aut}(G)$, and $|I_j| \geq 2$.* As L is not formally real, $1 \notin I_{X_j}(L/K) = \pi(I_j)$, for each $1 \leq j \leq m$. Thus $I_j \subseteq H \smallsetminus G$. Let $\widetilde{H}$ and $\widetilde{G}$ be as in Lemma 3.2, and let $\widetilde{I}_j$ be the inverse image of I_j in the set of involutions of $\widetilde{H}$. Suppose that there is $\widehat{F}$ regular over L, such that $\widehat{F}/K(x)$ is Galois, and an isomorphism $\tilde{h}\colon G(\widehat{F}/K(x)) \to \widetilde{H}$ that maps $G(\widehat{F}/L(x))$ onto $\widetilde{G}$ and

$I_{X_j}(\widehat{F}/K(x))$ onto $\widetilde{I}_j$. As in part 1, the subfield of $\widehat{F}$ corresponding to the kernel of the map $\widetilde{H} \to H$ (sending $\widetilde{G}$ to G) is the desired F.

Thus, assume that commutators generate $M(G)$, $C_H(G) = 1$, and $|I_j| \geq 2$ for each j. In particular, the conjugation action of H on G induces a monomorphism $H \to \mathrm{Aut}(G)$. Identify H with its image in $\mathrm{Aut}(G)$ (and G with $\mathrm{Inn}(G)$). Then $G(L/K)$ is a subgroup of $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, and π: $H \to G(L/K)$ is the restriction of the quotient map π: $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ to H.

Part 3: *Construction.* Let $\Lambda : \mathcal{H} \to \bar{\mathcal{H}}$ be the cover of Hurwitz spaces, associated with G, defined in Section 4. Let β: $G(K) \to \mathrm{Aut}(\mathcal{H}/\bar{\mathcal{H}})$ be the composition of the restriction $G(K) \to G(L/K) \leq \mathrm{Out}(G)$ with the isomorphism δ: $\mathrm{Out}(G) \to \mathrm{Aut}(\mathcal{H}/\bar{\mathcal{H}})$ (2.14). Furthermore, let ν be as in Remark 4.2(d).

Let M be the field generated over $\mathbb{Q}$ by $\sqrt{-1}$ and the conjugates of basic points associated with G, ϵ, and I as in Definition 4.1, for all possible ϵ and I. This is a finite extension of $\mathbb{Q}$ (Remark 4.2(b)). Refine the partition $X_1, \ldots, X_m$ of $X(K)$, and hence also the corresponding partition $\mathcal{X}_1, \ldots, \mathcal{X}_m$ of $\mathcal{X}$, so that for each $1 \leq j \leq m$ there are unique $\bar{\epsilon}_j \in G(L/K)$ and $\bar{\iota}_j \in G(M/\mathbb{Q})$ such that $\mathrm{res}_L \mathcal{X}_j = \{\bar{\epsilon}_j\}$ and $\mathrm{res}_M \mathcal{X}_j = \{\bar{\iota}_j\}$.

Fix $1 \leq j \leq m$. Put $I_j' = \{\epsilon \in I_j | \ \pi(\epsilon) = \bar{\epsilon}_j\}$ and choose $\epsilon_j \in I_j'$. Then $I_j' \subseteq G \rtimes \langle \epsilon_j \rangle$. Let $\mathbf{q}$ be the basic point associated with G, ϵ_j, and I_j'. Then $\mathbb{Q}(\mathbf{q}) \subseteq M$. As the real involutions in $G(M/\mathbb{Q})$ are conjugate, there is $\lambda_j \in G(M/\mathbb{Q})$ such that $\lambda_j^{-1} \bar{\iota}_j \lambda_j = \mathrm{res}_M c$, where c is complex conjugation. Set $\mathbf{q}_j = \lambda_j(\mathbf{q})$. By Lemma 4.3, $\delta_\epsilon(\mathbf{q}) = c(\mathbf{q})$. Therefore, $\delta_\epsilon(\mathbf{q}_j) = \delta_\epsilon \lambda_j(\mathbf{q}) \lambda_j \delta_\epsilon(\mathbf{q}) = (\lambda_j c \lambda_j^{-1})(\mathbf{q}_j) = \bar{\iota}_j(\mathbf{q}_j)$.

Let $\iota \in \mathcal{X}_j$. Then $\delta_{\epsilon_j} = \delta(\pi(\epsilon_j)) = \delta(\bar{\epsilon}_j) = (\delta \circ \mathrm{res}_L)(\iota) = \beta(\iota)$. So $\iota(\mathbf{q}_j) = \bar{\iota}_j(\mathbf{q}_j) = \delta_{\epsilon_j}(\mathbf{q}_j) = \beta(\iota)(\mathbf{q}_j)$. Thus $\mathbf{q}_1, \ldots, \mathbf{q}_m$ satisfy (1). Therefore there exists $\mathbf{p} \in \mathcal{H}(\widetilde{K})$ that satisfies (2)-(4). Let $F/K(x)$ be the Galois extension and h_1: $G(F/K(x)) \to \mathrm{Aut}(G)$ the embedding associated with $\mathbf{p}$ over K (2.17).

Part 4: *The image of h_1.* Let $\tau \in H$. There is $\sigma \in G(K)$ such that $\mathrm{res}_L \sigma = \pi(\tau)$. By (2), $\sigma(\mathbf{p}) = \delta(\mathrm{res}_L \sigma)(\mathbf{p}) = \delta_\tau(\mathbf{p})$. Hence by the criterion of (2.17), τ is in the image of h_1. Thus $H \leq \mathrm{im}(h_1)$. But

$$|H| = |G| \cdot |G(L/K)| = [F : L(x)] \cdot [L : K] = [F : K(x)] = |\mathrm{im}(h_1)|,$$

and hence $H = \mathrm{im}(h_1)$.

Part 5: $h_1(I_{X_j}(F/K(x))) = I_j$. Let $1 \leq j \leq m$ and let $P \in X_j$. By Proposition 4.4(d), $h_1(I_P(F/K(x))) = \mathrm{Con}_H(I'_j)$. As $\mathrm{Con}_{G(K)}(\mathcal{X}_j) = I_{X_j}(\widetilde{K}/K)$, $\mathrm{Con}_{G(L/K)}(\bar{\epsilon}_j) = I_{X_j}(L/K)$. Conclude that $\mathrm{Con}_H(I'_j) = I_j$. Thus $h_1(I_P(F/K(x))) = I_j$. ∎

Theorem 5.2 says more about the structure of the absolute Galois group of $K(x)$. For instance, every finite group is realizable over $K(x)$. This is not new [DF2, Theorem 5.7]. Still, the precise information about real involutions gives more.

Theorem 5.3: *Let K be a formally real PRC field. Let G be a finite group, and let G_0 be a normal subgroup of G generated by involutions. There is a Galois extension $N/K(x)$ with Galois group G such that the fixed field of G_0 in N is the maximal totally real extension of $K(x)$ in N.*

Proof: Let $\langle \epsilon \rangle$ be a group of order 2. Put $H = G \times \langle \epsilon \rangle$ and $H_0 = G_0 \times \langle \epsilon \rangle$, and let $\pi \colon H \to G(K(\sqrt{-1})/K)$ be the epimorphism with kernel G. The set I_0 of involutions in G_0 generates G_0. Therefore $I_1 = (I_0 \cup \{1\}) \times \{\epsilon\}$ generates H_0. It is a conjugacy domain in H. Theorem 5.2 (with $m = 1$) gives a Galois extension F of $E = K(x)$ that contains $\sqrt{-1}$ such that $G(F/E) = H$, $G(F/E(\sqrt{-1})) = G$, and I_1 is the set of real involutions in $G(F/E)$. Let N be the fixed field of ϵ and N' the fixed field of $H_0 = \langle I_1 \rangle$. The last condition means that N' is the maximal totally real extension of E in F, and, thererefore, also in N. Clearly $G(N/E) \cong G$ and N' is the fixed field of G_0 in N. ∎

6. Real Hilbertian fields

As in the preceding sections, all fields are of characteristic 0.

Let S/R be a Galois cover of rings [FJ, Definition 5.4], and let F/E be the corresponding Galois extension of the quotient fields. Thus R is an integrally closed domain and there is $z \in S$ integral over R such that $S = R[z]$ and the discriminant $d_E(z)$ of z over E is a unit of R. We call such z a **primitive element** for S/R. Assume that S/R is **real** [HL, Definition 4.2], that is, R is a regular ring and F is not formally real.

LEMMA 6.1: *The integral closure S' of R in each intermediate extension F' of F/E is also a regular ring.*

Proof: Observe that S/S' is also a Galois cover. By [R, p. 75] it suffices to show that S'/R is étale. i.e., flat and unramified. We have $S = \oplus_{i=0}^{d-1} Rz^i$, where $d = [F : E]$, and so S/R is faithfully flat. Similarly S/S' is faithfully flat. By the descent property [Ma, (4.B)], S'/R is (faithfully) flat.

To show that S'/R is unramified, let $\mathfrak{q}$ be a prime of S, and let $\mathfrak{p} = \mathfrak{q} \cap S'$ and $\mathfrak{m} = \mathfrak{q} \cap R$. Replace R, S', and S by their localizations at these primes to assume that they are local rings. Then S/S' and S'/R are still faithfully flat. As S/S' is unramified, the field extension $(S/\mathfrak{q})/(R/\mathfrak{m})$ is separable and finite. Hence so is its subextension $(S'/\mathfrak{p})/(R/\mathfrak{m})$. As S/R and S/S' are unramified, $\mathfrak{m}S = \mathfrak{q}$ and $\mathfrak{p}S = \mathfrak{q}$. Thus $(\mathfrak{m}S')S \cap S' = \mathfrak{p}S \cap S'$. But S/S' is faithfully flat, hence [Ma, (4.C)], $\mathfrak{m}S' = \mathfrak{p}$.

∎

Let M be a field. Every homomorphism $\varphi\colon R \to M$ extends to a homomorphism $\psi\colon S \to \widetilde{M}$, and ψ induces a group homomorphism $\psi^*\colon G(M) \to G(F/E)$, given by

$$\psi\big(\psi^*(\sigma)(x)\big) = \sigma(x) \qquad \text{for } x \in S. \tag{1}$$

[FJ, Lemma 5.5]. If ψ' is another extension of φ, then ψ'^* and ψ^* differ by an inner automorphism of $G(F/E)$ [L, Corollary 1 on p. 247]. In particular, for $\sigma \in G(M)$ and a conjugacy domain $I \subseteq G(F/E)$ we have $\psi'^*(\sigma) \in I$ if and only if $\psi^*(\sigma) \in I$. This allows us to abuse the notation and write $\varphi^*(\sigma) \in I$ instead of $\psi^*(\sigma) \in I$. (Cf. also [HL, Remark 4.1].)

REMARK 6.2: (a) Let S_0/R_0 be another real Galois cover of rings, with L/K the corresponding extension of the quotient fields. Assume that R is finitely generated over R_0, the field K is algebraically closed in E, and L is the algebraic closure of K in F. Furthermore, $R_0 \subseteq M$ and $\varphi\colon R \to M$ is an R_0-homomorphism.

We may choose the extension ψ of φ to be an S_0-homomorphism. It then follows from (1) that the composition of ψ^* with $\mathrm{res}_L\colon G(F/E) \to G(L/K)$ is the restriction map $\mathrm{res}_L\colon G(M) \to G(L/K)$.

(b) For each $P \in X(K)$ we have $\varphi^*(I_P(M)) \subseteq I_P(F/E)$. In particular, $\varphi^*(I(M)) \subseteq I(F/E)$.

Indeed, let $\epsilon \in I_P(M)$, and let $\psi\colon S \to \widetilde{M}$ be an extension of φ. It follows from (1) that ψ maps the integral closure S' of R in $F(\psi^*(\epsilon))$ into $\widetilde{M}(\epsilon)$. The latter field is real closed. Thus, by Knebusch' Theorem [HL, Proposition 1.2], P extends to an ordering on $F(\psi^*(\epsilon))$.

For the rest of this section and for an ambient field K consider the following setup:

(2) Let x be transcendental over K, let $E = K(x)$ and $R = K[x, h(x)^{-1}]$. Let S be a real Galois cover of R and let F/E be the corresponding extension of the quotient fields. Let $X_1, \ldots, X_m$ be a partition of $X(K)$ into disjoint clopen subsets, and for each $1 \le j \le m$ let $Q_j \in X(E)$ such that $\mathrm{res}_K Q_j \in X_j$.

In this setup, each $a \in K$ with $h(a) \neq 0$ defines the homomorphism $\varphi_a\colon R \to K$ by $x \mapsto a$.

LEMMA 6.3: *In (2) let Q be an ordering on E and P its restriction to K. Let $\overline{K}$ be the real closure of (K, P). There exist branch points $x_1 < x_2$ in $\overline{K}$ of the extension F/E with no other $\overline{K}$ branch points between them, having the following property. For each $a \in K$ in the interval (x_1, x_2), $\varphi_a^*(I_P(K)) \subseteq I_Q(F/E)$*

Proof: Let $\epsilon \in I_Q(F/E)$. Choose a primitive element y for $F(\epsilon)/E$, integral over $K[x]$, and let $f_\epsilon = \mathrm{irr}(y, E) \in K[x, Y]$. Let S' be the integral closure of R in $F(\epsilon)$.

The sentence $(\exists X, Y)[f_\epsilon(X, Y) = 0 \wedge \frac{\partial f}{\partial Y}(X, Y) \neq 0]$ holds in $F(\epsilon)$, and therefore also in the real closure of (E, Q). By Tarski's principle it is valid in $\overline{K}$. Thus there is $\bar{a} \in \overline{K}$ such that $f_\epsilon(\bar{a}, Y)$ has a simple root in $\overline{K}$. This certainly remains true if $\bar{a}$ is replaced by a in the neighborhood $\overline{U}$ of $\bar{a}$ in $\overline{K}$ determined by nearest branch points of F/E in $\overline{K}$.

Let ι be the generator of $G(\overline{K})$, and let S' be the integral closure of R in $F(\epsilon)$. Let $a \in \overline{U} \cap K$. Then $\varphi_a\colon R \to K \subseteq \overline{K}$ extends to a homomorphism $\psi\colon S' \to \overline{K}$. It follows that its extension $\psi\colon S \to \widetilde{K}$ satisfies $\psi^*(\iota) = \epsilon$. As $I_P(K)$ and $I_Q(F/E)$ are the conjugacy classes of ι and ϵ in the respective groups, we have $\varphi_a^*(I_P(K)) \subseteq I_Q(F/E)$.

∎

Definition 6.4: A formally real field K is **real Hilbertian**, if in each setup (2) the following holds. Assume that $G(F/E) = \langle \bigcup_{j=1}^m I_{Q_j}(F/E)\rangle$. Then there exists a K-homomorphism $\varphi\colon R \to K$ such that $\varphi^*(G(K)) = G(F/E)$ and $\varphi^*(I_{X_j}(K)) = I_{Q_j}(F/E)$, for each $1 \le j \le m$. In particular, $\varphi^*(I(K)) = I(F/E)$.

Corollary 6.5: *If K is a number field, then K is real Hilbertian.*

Proof: Consider (2). For each j put $P_j = \mathrm{res}_K Q_j$. Since $|X(K)| < \infty$, by refining $X_1, \ldots, X_m$ we may assume that $X_j = \{P_j\}$. As K is dense in each of its real closures, Lemma 6.3 gives a nonempty open subset U_j of K (with respect to P_j) such that $\varphi_a^*(I_{P_j}(K)) \subseteq I_{Q_j}(F/E)$ for each $a \in U_j$. Consider the Hilbert set

$$H_K = \{a \in K|\ h(a) \neq 0 \text{ and } \varphi_a^*(G(K)) = G(F/E)\}$$

[FJ, Lemma 12.12]. By [G, Lemma 3.4], H_K is dense in K in the product topology induced by $P_1, \ldots, P_n$, that is, there is $a \in H_K \cap U_1 \cap \cdots \cap U_m$. Observe that $I_{Q_j}(F/E)$ is a conjugacy class in $G(F/E)$. The surjectivity of φ_a^* implies that $\varphi_a^*(I_{P_j}(K)) = I_{Q_j}(F/E)$. ∎

Proposition 6.6: *Let $K = \mathbb{Q}^{\mathrm{tr}}$. Then K is real Hilbertian.*

Proof: Assume (2). Let z be a primitive element for the cover S/R. Let $K' \subseteq K$ be a number field such that $h(x) \in K'[x]$. Put $R' = K'[x, h(x)^{-1}]$ and $S' = R'[z]$, and let E' and F' be their quotient fields. For each j let Q_j' be the restriction of Q_j to E', and let X_j' be the restriction of X_j to K'. Take K' sufficiently large to assume that

(i) S'/R' is a real Galois cover;

(ii) $[F' : E'] = [F : E]$, and therefore K and F' are linearly disjoint over K'; and

(iii) the sets $X'_1, \dots, X'_m$ are distinct.

Then $X'_1, \dots, X'_m$ is a partition of $X(K')$ and $G(F'/E') \cong G(F/E)$.

By Corollary 6.4 there exists $a \in K'$ such that an extension $\psi'\colon S' \to \widetilde{K}$ of $\varphi_a\colon R' \to K'$ satisfies $\psi'^*(I_{X'_j}(K)) = I_{Q'_j}(F'/E')$. Extend φ_a to the K-homomorphism $\varphi_a\colon R \to K$. As K and S' are linearly disjoint over K', it is possible to extend this φ_a and ψ' to the same K-homomorphism $\psi\colon S \to \widetilde{K}$. By (1), the following diagram commutes.

$$\begin{tikzcd}
G(F/E) \arrow[d, "\mathrm{res}_{F'}"] & G(K) \arrow[l, "\psi^*"'] \arrow[d] & I(K) \arrow[l] \arrow[d] \\
G(F'/E') & G(K') \arrow[l, "\psi'^*"'] & I(K') \arrow[l]
\end{tikzcd}$$

From (ii), the left vertical map is an isomorphism. As $K = \mathbb{Q}^{\mathrm{tr}}$, $I(K) = I(\mathbb{Q}) = I(K')$. Thus, the right vertical inclusion is surjective and maps $I_{X_j}(K)$ onto $I_{X'_j}(K')$. Diagram chasing yields $\psi^*(I_{X_j}(K)) = I_{Q_j}(F/E)$. But $G(F/E) = \langle \bigcup_{j=1}^{m} I_{Q_j}(F/E) \rangle$, and hence $\psi^*(G(K)) = G(F/E)$. ▮

7. The absolute Galois group of the field of totally real algebraic numbers

In this section we consider the following category. An **involutory structure** is a pair $(G, I_G) = \mathbf{G}$ for short, where G is a profinite group and I_G is a closed set of involutions in G. A **morphism** of involutory structures $\varphi\colon \mathbf{G} \to \mathbf{H}$ is a continuous homomorphism of groups $\varphi\colon G \to H$ such that $\varphi(I_G) \subseteq I_H$. We say that $\varphi\colon \mathbf{G} \to \mathbf{H}$ is an **epimorphism** if $\varphi(G) = H$ and $\varphi(I_G) = I_H$.

Example 7.1: (a) Let L/K be a Galois extension with L not formally real. Then $\mathbf{G}(L/K) = (G(L/K), I(L/K))$ is an involutory structure. Let E be an extension of K, and let F/E be a Galois extension such that $L \subseteq F$. Then the restriction $\mathrm{res}_L\colon \mathbf{G}(F/E) \to \mathbf{G}(L/K)$ is a morphism. Moreover, suppose that E/K is regular and totally real: every ordering on K extends to E. Then res_L is an epimorphism (cf. [HJ1, Lemma 3.5]).

(b) Let S/R be a real Galois cover with F/E the corresponding Galois extension of fields. Let M be a field and let $\varphi\colon R \to M$ be a homomorphism. Extend φ to a homomorphism $\psi\colon S \to \widetilde{M}$. Then the group homomorphism $\psi^*\colon G(M) \to G(F/E)$ is a morphism of involutory structures $\psi^*\colon \mathbf{G}(M) \to \mathbf{G}(F/E)$ (Remark 6.2(b)).

A **finite image** of $\mathbf{G}$ is a finite involutory structure $\mathbf{H}$ for which there exists an epimorphism $\varphi\colon \mathbf{G} \to \mathbf{H}$. Clearly, up to an isomorphism, it is of the form $(G/N, \{\epsilon N/N |\ \epsilon \in I_G\})$, where N is an open normal subgroup of G not meeting I_G. Let $\mathrm{Im}\,\mathbf{G}$ be the class of all finite images of $\mathbf{G}$.

A **finite embedding problem** for $\mathbf{G}$ consists of an epimorphism $\pi\colon \mathbf{H} \to \mathbf{A}$ of finite involutory structures, together with an epimorphism $\varphi\colon \mathbf{G} \to \mathbf{A}$. A **solution** is an epimorphism $\psi\colon \mathbf{G} \to \mathbf{H}$ such that $\pi \circ \psi = \varphi$. We say that $\mathbf{G}$ has the **embedding property** if every finite embedding problem $(\pi\colon \mathbf{H} \to \mathbf{A},\ \varphi\colon \mathbf{G} \to \mathbf{A})$ for $\mathbf{G}$, in which H is a finite image of $\mathbf{G}$, has a solution.

Example 7.2: Let D be the free profinite product $\coprod_{x \in X_\omega} \langle \epsilon_x \rangle$ of groups of order 2 over X_ω (Remark 1.5), and let $I_D^0 = \{\epsilon_x |\ x \in X_\omega\}$. A finite involutory structure (A, I_A) is a finite image of (D, I_D^0) if and only if A is generated by I_A. Furthermore, (D, I_D^0) has the embedding property.

With D and I_D^0 as above, put $\mathbf{D} = (D, I_D)$, where I_D is the conjugacy domain $\mathrm{Con}_D(I_D^0)$ of D generated by I_D^0.

LEMMA 7.3: (a) *I_D is the set of all involutions in D, and D is of rank $\aleph_0$.*

(b) *A finite involutory structure $\mathbf{A}$ is in $\mathrm{Im}\,\mathbf{D}$ if and only if $I_A \neq \emptyset$ is a conjugacy domain in A and $A = \langle I_A \rangle$.*

(c) *$\mathbf{D}$ has the embedding property.*

Proof of (a): See [HJ2, Corollary 3.2 and Lemma 2.2]

Proof of (b): Immediate from Example 7.2.

Proof of (c): Let $\pi\colon \mathbf{H} \to \mathbf{A}$, $\varphi\colon \mathbf{D} \to \mathbf{A}$ be a finite embedding problem for $\mathbf{D}$. Then $I_A \subseteq A$ and $I_H \subseteq H$ are conjugacy domains. Let $I_A^0 = \varphi(I_D^0)$ and let $I_H^0 = \{\epsilon \in I_H |\ \pi(\epsilon) \in I_A^0\}$. As $\mathrm{Con}_D(I_D^0) = I_D$, we have $\mathrm{Con}_A(I_A^0) = I_A$; it follows that $\mathrm{Con}_H(I_H^0) = I_H$.

By Example 7.2, I_A^0 generates A. But $\mathbf{H} \in \mathrm{Im}\,\mathbf{D}$ implies that I_H generates H. We have $\pi(I_H^0) = I_A^0$ and $\mathrm{Con}_H(I_H^0) = I_H$. By an analogue of Gaschütz' lemma [HL, Lemma 3.3 with $n = 0$], I_H^0 generates H.

By Example 7.2 there is an epimorphism $\psi\colon D \to H$ such that $\pi \circ \psi = \varphi$ and $\psi(I_D^0) = I_H^0$. Clearly $\psi(I_D) = I_H$. ∎

THEOREM 7.4: *Let K be a real Hilbertian PRC field. Assume that K has no proper totally real algebraic extensions and $X(K)$ has no isolated points. Put $\mathbf{G} = (G, I_G)$, where G is the absolute Galois group of K and I_G is the conjugacy domain of all involutions in G. Then*

(a) *A finite embedding problem $(\pi\colon \mathbf{H} \to \mathbf{A},\ \varphi\colon \mathbf{G} \to \mathbf{A})$ for $\mathbf{G}$ has a solution, if*

(*) *$I_H \neq \emptyset$ is a conjugacy domain in H and $H = \langle I_H \rangle$.*

(b) *A finite involutory structure $\mathbf{H}$ is in $\mathrm{Im}\,\mathbf{G}$ if and only if (*) holds.*

(c) *$\mathbf{G}$ has the embedding property.*

Proof: The fixed field of I_G in G is totally real over K. Thus $G = \langle I_G \rangle$.

Proof of (a): As $1 \notin I_A = \varphi(I_G)$, $\mathrm{Ker}(\varphi) \cap I_G = \emptyset$. Therefore the fixed field L of $\mathrm{Ker}(\varphi)$ is not formally real. Without loss of generality $\mathbf{A} = \mathbf{G}(L/K)$ and φ is the restriction map.

Theorem 5.2 (with $m = 1$, $X_1 = X(K)$, and $I_1 = I_H$) identifies $\pi\colon \mathbf{H} \to \mathbf{A}$ with the restriction map $\mathrm{res}_L\colon \mathbf{G}(F/E) \to \mathbf{G}(L/K)$, where E is a simple transcendental extension of K, and F is a Galois extension of E that contains L and is regular over L.

In particular, $G(F/E) = \langle I(F/E)\rangle$. Choose $Q_1, \ldots, Q_m \in X(E)$ with $I(F/E) = \bigcup_{j=1}^m I_{Q_j}(F/E)$. We may assume that their restrictions $P_1, \ldots, P_m \in X(K)$ to K are distinct. Indeed, each P_j is not isolated in $X(K)$, and hence there is $P \in X(K)$ distinct from $P_1, \ldots, P_m$ and arbitrarily close to P_j. By Remark 1.8(b) we may assume that $I_P(F/E) = I_{P_j}(F/E)$. As $I_P(F/E) = \bigcup_{\substack{Q \in X(E) \\ Q \supseteq P}} I_Q(F/E)$, there is $Q \in X(E)$ above P such that $I_Q(F/E) = I_{Q_j}(F/E)$. We replace Q_j by Q.

Let $X_1, \ldots, X_m$ be a partition of $X(K)$ into disjoint clopen sets such that $P_j \in X_j$. This gives the setup (2) of Section 6. As K is real Hilbertian, there is $a \in K$ and an epimorphism $\varphi_a^*\colon \mathbf{G}(K) \to \mathbf{G}(F/E)$. By Remark 6.2(a), φ_a^* is a solution to our embedding problem.

Proof of (b): Condition (*) is necessary, since $I_G \neq \emptyset$ is a conjugacy domain in G and $G = \langle I_G\rangle$. Conversely, assume (*). Let $A = \langle a\rangle = G(K(\sqrt{-1})/K)$ and $\mathbf{A} = (A, \{a\})$, where a is the generator of A, and let $\varphi\colon \mathbf{G} \to \mathbf{A}$ be the restriction map. We construct below a finite involutory structure $\hat{\mathbf{H}}$ that satisfies (*), with epimorphisms $\hat{\mathbf{H}} \to \mathbf{H}$ and $\pi\colon \hat{\mathbf{H}} \to \mathbf{A}$. By (a) there is an epimorphism $\psi\colon \mathbf{G} \to \hat{\mathbf{H}}$, and hence $\mathbf{H} \in \mathrm{Im}\,\mathbf{G}$.

If there is an epimorphism $\pi\colon \mathbf{H} \to \mathbf{A}$, let $\hat{\mathbf{H}} = \mathbf{H}$. If not, let $\hat{\mathbf{H}} = (H \times A, I_H \times \{a\})$. Both $\mathbf{A}$ and $\mathbf{H}$ are quotients of $\hat{\mathbf{H}}$. Observe that (*) holds for $\hat{\mathbf{H}}$. Otherwise $I_H \times \{a\}$ generates a proper subgroup Γ of $H \times A$ such that the projection $H \times A \to H$ maps Γ onto $H = \langle I_H\rangle$. Thus $\Gamma = \{(h, \pi(h)) | \; h \in H\}$, where $\pi\colon H \to A$ is an epimorphism. As $I_H \times \{a\} \subseteq \Gamma$, we have $\pi(I_H) = a$. Thus π induces an epimorphism $\mathbf{H} \to \mathbf{A}$, a contradiction.

Proof of (c): Clear from (a) and (b). ▮

If, in addition to the assumptions of the theorem, K is countable, then G is of rank at most $\aleph_0$. Thus the involutory structures $\mathbf{G}$ and $\mathbf{D}$ are very similar, by Lemma 7.3 and Theorem 7.4.

In fact, we have the following straightforward modification of [FJ, Lemma 24.1]:

LEMMA 7.5: *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be involutory structures with embedding property, such that* G *and* H *are of rank at most* $\aleph_0$. *If* $\operatorname{Im}\mathbf{G} = \operatorname{Im}\mathbf{H}$, *then* $\mathbf{G} \cong \mathbf{H}$.

THEOREM 7.6: *Let* K *be a countable real Hilbertian PRC field. Assume that* K *has no proper totally real algebraic extensions and* $X(K)$ *has no isolated points. Then* $\mathbf{G}(K) \cong \mathbf{D}$, *and hence* $G(K) \cong D$.

The field $\mathbb{Q}^{\mathrm{tr}}$ of totally real algebraic numbers is PRC by [P]. (We remark that although Pop [P] states this result, he only gives the proof for an analog. Therefore in all our results about $\mathbb{Q}^{\mathrm{tr}}$ the reference [P] should be replaced by a subsequent version, where this omission will be remedied.) It is clearly countable. By Proposition 6.6 it is real Hilbertian, and by Remark 1.5, $X(\mathbb{Q}^{\mathrm{tr}})$ has no isolated points. Therefore:

COROLLARY 7.7: *The absolute Galois group of the field* $\mathbb{Q}^{\mathrm{tr}}$ *of totally real algebraic numbers is the free profinite product* D *of groups of order 2 over the universal Boolean space* $X_\omega = \{0,1\}^{\aleph_0}$ *of weight* $\aleph_0$.

8. Real Frobenius fields

Let S/R be a real Galois ring cover, and let F/E be the corresponding field extension. Let K be a subfield of R and L the algebraic closure of K in F.

We say [HL, Definition 4.2] that

(a) S/R is **regular over** K, if the extension E/K is regular. In that case L/K is a finite Galois extension.

(b) S/R is **finitely generated** over K, if R and S are finitely generated rings over K.

(c) F/E is **amply real** over K if E/K is a regular extension, the algebraic closure L of K in F is not formally real, and the extension $F(\epsilon)/L(\epsilon)$ is totally real for every real involution $\epsilon \in G(F/E)$.

DEFINITION 8.1: A field M is said to be **real Frobenius** if it satisfies the following condition: Let S/R be a real Galois ring cover, finitely generated and regular over M, with F/E the corresponding field extension amply real over M. Let N be the algebraic closure of M in F. Let $\mathbf{H} \leq \mathbf{G}(F/E)$ such that $\mathbf{H} \in \operatorname{Im}\mathbf{G}(M)$ and $\operatorname{res}_N \mathbf{H} =$

$\mathbf{G}(N/M)$. Then there exists an M-homomorphism $\psi\colon S \to \widetilde{M}$ such that $\psi(R) = M$ and $\psi^*(\mathbf{G}(M)) = \mathbf{H}$.

PROPOSITION 8.2: *Let M be a PRC field. If $\mathbf{G}(M)$ has the embedding property, then M is real Frobenius.*

Proof: (Cf. [HL, Proposition 5.6].) Let S/R, F/E, N, and $\mathbf{H}$ be as in Definition 8.1. The embedding property gives an epimorphism of involutory structures $h\colon \mathbf{G}(M) \to \mathbf{H}$ with $\mathrm{res}_N \circ h = \mathrm{res}_N$. Put $L = \widetilde{M}F$. Then

$$G(L/E) = G(\widetilde{M}E/E) \times_{G(NE/E)} G(F/E) = G(M) \times_{G(N/M)} G(F/E).$$

Let D be the fixed field of $\Delta = \{(\delta, h(\delta))|\ \delta \in G(M)\}$ in L. Then D/M is regular, $DF = D\widetilde{M} = L$, and $D \cap F = E$ [FJ, p. 354]. We show that D/M is totally real. Let P be an ordering on M. There is $\epsilon \in I(M)$ such that P is the restriction of P_ϵ from $\widetilde{M}(\epsilon)$. Then $h(\epsilon) \in I_H \subseteq I(F/E)$. Observe that $\widetilde{M}(\epsilon)$ and $F(h(\epsilon))$ are linearly disjoint over $N(\epsilon)$ and $L(\epsilon, h(\epsilon)) = D(\epsilon)F(h(\epsilon))$ contains D. By assumption there is an ordering Q of $F(h(\epsilon))$ such that $\mathrm{res}_{N(\epsilon)}Q = \mathrm{res}_{N(\epsilon)}P_\epsilon$. Therefore P_ϵ and Q extend to an ordering of $L(\epsilon, h(\epsilon))$ [J, p. 241]. The restriction of this ordering to D extends P.

The integral closure U of R in D is finitely generated over M [FJ, p. 354] and hence U is the coordinate ring of an absolutely irreducible variety V defined over M. Since M is PRC, there exists an M-homomorphism $\psi_0\colon U \to M$. Extend ψ_0 to an $\widetilde{M}$-epimorphism $\widetilde{M}U \to \widetilde{M}$, and let $\psi\colon S \to \widetilde{M}$ be its restriction to S. Then $\psi(R) = M$, and, by [FHJ, Remark on p. 9], we may arrange it so that $\psi^*\colon G(M) \to G(F/E)$ coincides with h. Therefore $\psi^*(\mathbf{G}(M)) = \mathbf{H}$. ▮

By Corollary 7.7 and Lemma 7.3(c), $\mathbf{G}(\mathbb{Q}^{\mathrm{tr}})$ has the embedding property. By [P], $\mathbb{Q}^{\mathrm{tr}}$ is PRC. Therefore:

COROLLARY 8.3: *$\mathbb{Q}^{\mathrm{tr}}$ is real Frobenius.*

9. Real Galois Stratification

This section gives a quantifier elimination procedure for the theory of real Frobenius fields in the language below. The procedure is similar to that in [FJ, Chapter 25] and almost the same as in [HL]. Therefore we only comment on the differences.

A **Galois ring/set cover** C/A over a field K [FJ, p. 403] is **real** if A is nonsingular, $\mathrm{char}(K) = 0$, and $K(C)$ is not formally real. Put $\mathbf{G}(C/A) = \mathbf{G}(K(C)/K(A))$ (Example 7.1(a)) and let $\mathrm{Sub}[C/A]$ be the set of involutory substructures of $\mathbf{G}(C/A)$.

Let $K \subseteq M$ be a field. Each $\mathbf{a} \in A(M)$ determines a K-homomorphism $\varphi \colon K[A] \to M$, and therefore (see Section 6) a homomorphism $\varphi^* \colon G(M) \to G(C/A)$ (unique up to an inner automorphism of $G(C/A)$). Example 7.1(b) says that $\varphi^*(\mathbf{G}(M) \le \mathbf{G}(C/A)$. Omitting the reference to C and M, define the **Artin symbol** $\mathbf{Ar}(A, \mathbf{a})$ as the set $\{\varphi^*(\mathbf{G}(M)^\sigma \mid \sigma \in G(C/A)\}$. This is a conjugacy class in $\mathrm{Sub}[C/A]$. For properties of the Artin symbol see [HL, Section 6].

For $n \ge 0$ let $\pi \colon \mathbb{A}^{n+1} \to \mathbb{A}^n$ be the projection on the first n coordinates. Let $A \subseteq \mathbb{A}^{n+1}$ and $B \subseteq \mathbb{A}^n$ be two non-singular basic sets [FJ, p. 244] such that $\pi(A) = B$. Then $K[B] \subseteq K[A]$. Let $\mathbf{x}$ and $(\mathbf{x}, y)$ be generic points of B and A, respectively. Then $K(A) = K(B)(y)$. Furthermore, let C/A and D/B be real Galois covers such that $K(D)$ contains the algebraic closure of $K(B)$ in $K(C)$.

Definition 9.1 ([HL, Definition 7.1]): Let M be a field extension of K. An M-**specialization** of the pair $(C/A, D/B)$ is a K-homomorphism φ from C into an overfield of M with these properties: $\varphi(K[B]) \subseteq M$; and if y is transcendental over $K(B)$, then $\varphi(y)$ is transcendental over M.

For such a specialization put $y' = \varphi(y)$, $N = M[\varphi(D)]$, $R = M[\varphi(K[A])]$, $E = M(y')$ (the quotient field of R), $S = M[\varphi(C)]$, and $F = E[\varphi(C)]$ (the quotient field of S). Then φ induces an embedding $\varphi^* \colon G(F/E) \to G(C/A)$.

Assume that $\dim A = \dim B + 1$. The pair $(C/A, D/B)$ is **specialization compatible** if the following properties hold for every M and each M-specialization φ as above.

(i) $K(D)$ is the algebraic closure of $K(B)$ in $K(C)$.

and for every M and each M-specialization φ as above

(ii) $[K(C) : K(D)(y)] = [F : N(y')]$.

(iii) The cover $K(C)/K(A)$ is amply real over $K(B)$.

(iv) For each involution $\epsilon \in G(F/E)$ with $\varphi^*(\epsilon)$ real the extension $F(\epsilon)/N(\epsilon)$ is totally real.

Assume that $\dim A = \dim B$. The pair $(C/A, D/B)$ is said to be **specialization compatible** if $K[A]$ is integral over $K[B]$ and $C = D$.

Lemma 9.2: *Assume that* $\dim A = \dim B + 1$ *and that* $(C/A, D/B)$ *is specialization compatible. Let* $\mathrm{Con}(A)$ *be a conjugacy domain in* $\mathrm{Sub}[C/A]$*, and let* $\mathcal{S}$ *be a a set of (isomorphism types of) involutory structures. Define*

$$\mathrm{Con}(B) = \begin{cases} \mathrm{res}_{K(D)}(\mathcal{S} \cap \mathrm{Con}(A)) & \textit{if } \dim A = \dim B + 1; \\ \{\mathbf{G}^\sigma |\ \mathbf{G} \in \mathrm{Con}(A),\ \sigma \in G(C/B)\} & \textit{if } \dim A = \dim B. \end{cases}$$

Let M *be a real Frobenius field that contains* K*, and let* $\mathbf{b} \in B(M)$*. Assume* $\mathrm{Im}\,\mathbf{G}(M) \cap \mathrm{Sub}[C/A] = \mathcal{S}$*. Then* $\mathbf{Ar}(B, \mathbf{b}) \subseteq \mathrm{Con}(B)$ *if and only if there is* $\mathbf{a} \in A(M)$ *such that* $\pi(\mathbf{a}) = \mathbf{b}$ *and* $\mathbf{Ar}(A, \mathbf{a}) \subseteq \mathrm{Con}(A)$.

Proof: See [HL, Lemma 7.2] in case $\dim A = \dim B + 1$ and [HL, Lemma 7.3] in case $\dim A = \dim B$. (Replace everywhere the e-structures of [HL] by our involutory structures.) ▮

Lemma 9.3 (= [HL, Lemma 7.5]): *Let* K_1 *be a finite extension of* $K(D)$*. There are Zariski open subsets* $A' \subseteq A$*,* $B' \subseteq B$ *and a specialization compatible pair of real Galois covers* $(C'/A'\ , D'/B')$ *such that* $K(C) \subseteq K(C')$ *and* $K_1 \subseteq K(D')$.

From now on we can proceed exactly as in [FJ, Chapter 25]. Replace Galois covers with real Galois covers, and conjugacy classes of subgroups of $G(C_i/A_i)$ with conjugacy classes of involutory substructures of $\mathbf{G}(C_i/A_i)$ (cf. [HL, Sections 8 and 9]).

This includes the definition of Galois stratification [FJ, p. 410], and Galois formulas [FJ, p. 410]. Thus, a **real** Galois stratification

$$\mathcal{A} = \langle \mathbb{A}^n, C_i/A_i, \mathrm{Con}(A_i) |\ i \in I \rangle,$$

is a partition of the affine space $\mathbb{A}^n$ over K as a finite disjoint union $\mathbb{A}^n = \bigcup_{i\in I} A_i$ of nonsingular K-basic sets, each of them equipped with a real Galois cover C_i/A_i and a conjugacy domain $\mathrm{Con}(A_i)$ in $\mathrm{Sub}[C_i/A_i, \mathbf{P}_0]$. The corresponding **real** Galois formula is a formal expression $\mathbf{Ar}(\mathcal{A}, \mathbf{X}) \subseteq \mathrm{Con}(\mathcal{A})$ with the following interpretation. For be an extension M of K and $\mathbf{a} \in M^n$ write $M \models \mathbf{Ar}(\mathcal{A}, \mathbf{a}) \subseteq \mathrm{Con}(\mathcal{A})$ if $\mathbf{Ar}(A_i, \mathbf{a}) \subseteq \mathrm{Con}(A_i)$ for the unique i such that $\mathbf{a} \in A_i(M)$.

If K is a presented field with elimination theory [FJ, Definition 17.9], we get an effective elimination of quantifiers for the theory of real Frobenius fields in this language.

Moreover, every formula in the language $\mathcal{L}(K)$ of rings with parameters from K is equivalent to a real Galois formula (cf. [FJ, Remark 25.8]): The corresponding real Galois stratification may satisfy $C_i = K[A_i][\sqrt{-1}]$, for each $i \in I$. Thus we get:

PROPOSITION 9.4 (cf. [HL, Theorem 9.2(a)): *Let K be a presented field with elimination theory, and let ϑ be a sentence in $\mathcal{L}(K)$. We can effectively find a finite Galois extension L of K with $\sqrt{-1} \in L$, a finite family $\mathcal{H} \supseteq \mathrm{Sub}[L/K]$ of (isomorphism types of) finite involutory structures, and for each $\mathcal{S} \subseteq \mathcal{H}$ a conjugacy domain $\mathrm{Con}(\mathcal{S})$ in $\mathrm{Sub}[L/K]$ contained in $\mathcal{S}$ with the following property. For every real Frobenius field M that contains K and satisfies $\mathrm{Im}\,\mathbf{G}(M) \cap \mathcal{H} = \mathcal{S}$ we have $M \models \vartheta$ if and only if $\mathrm{res}_L \mathbf{G}(M) \in \mathrm{Con}(\mathcal{S})$.*

In particular, Proposition 9.4 holds for $K = \mathbb{Q}$.

10. Model theoretic results.

Let K' be a given finite extension of $\mathbb{Q}$, say as $K' = \mathbb{Q}[X]/(f)$, where $f \in \mathbb{Z}[X]$ is a given monic irreducible polynomial. Then K' is formally real (resp. $K' \subseteq \mathbb{Q}^{\mathrm{tr}}$) if and only if f has a root in $\mathbb{R}$ (resp. f splits over $\mathbb{R}$). We can effectively decide whether this condition holds [L, p. 276].

In particular, given a finite Galois extension L of $\mathbb{Q}$, we can effectively find the field $L \cap \mathbb{Q}^{\mathrm{tr}}$ and the involutory structure $\mathbf{G}(L/L \cap \mathbb{Q}^{\mathrm{tr}})$.

Let $\mathcal{L}(K)$ denote the elementary language of fields with parameters from K.

THEOREM 10.1: *The elementary theory of $\mathbb{Q}^{\mathrm{tr}}$ is effectively decidable.*

Proof: Apply Proposition 9.4. The field $\mathbb{Q}^{\mathrm{tr}}$ is real Frobenius (Corollary 8.3) and $\mathrm{Im}\,\mathbf{G}(\mathbb{Q}^{\mathrm{tr}}) = \mathrm{Im}\,\mathbf{D}$ (Corollary 7.7) is the family of finite involutory structures $\mathbf{H}$ in which $I_H \neq \emptyset$ is a conjugacy domain in H and $H = \langle I_H \rangle$ (Lemma 7.3(b)). Furthermore, if $L/\mathbb{Q}$ is a finite Galois extension and $K' = L \cap \mathbb{Q}^{\mathrm{tr}}$, then $\mathbb{Q}^{\mathrm{tr}}/K'$ is totally real, and hence $\mathrm{res}_{K'} X(\mathbb{Q}^{\mathrm{tr}}) = X(K')$. Therefore $\mathrm{res}_L \mathbf{G}(\mathbb{Q}^{\mathrm{tr}}) = (G(L/K'), I(L/K')) = \mathbf{G}(L/K')$.

Let ϑ be a sentence in $\mathcal{L}(\mathbb{Q})$. Proposition 9.4 effectively gives a finite Galois extension L of $\mathbb{Q}$ with $\sqrt{-1} \in L$, a finite family $\mathcal{H}$ of (isomorphism types of) finite involutory structures, and, for

$$\mathcal{S} = \{\mathbf{H} \in \mathcal{H} |\ I_H \neq \emptyset \text{ is a conjugacy domain in } H \text{ and } H = \langle I_H \rangle\},$$

a conjugacy domain $\mathrm{Con} = \mathrm{Con}(\mathcal{S})$ in $\mathrm{Sub}[L/\mathbb{Q}]$ contained in $\mathcal{S}$. For these, $\mathbb{Q}^{\mathrm{tr}} \models \vartheta$ if and only if $\mathbf{G}(L/L \cap \mathbb{Q}^{\mathrm{tr}}) \in \mathrm{Con}$. This condition is checkable, by the remarks preceding this corollary. ▮

LEMMA 10.2: *There is a formula $\theta(X_1, \ldots, X_n) \in \mathcal{L}(\mathbb{Q})$ with the following property. Let M be a PRC field, let $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$, and put $f = Z^n + a_1 Z^{n-1} + \cdots + a_n \in M[Z]$. Then $M \models \theta(\mathbf{a})$ if and only if*

(∗) *f has a root α in $\widetilde{M}$ such that $M(\alpha)$ is formally real.*

Proof: Condition (∗) is equivalent to this: There is an ordering P on M such that f has a root in the real closure of (M, P). By Tarski's principle [HL, Proposition

1.4] this statement is equivalent to a finite disjunction of statements of the form: There is an ordering P on M such that $\bigwedge_{i=1}^{r} f_i(\mathbf{a}) = 0 \ \wedge \ \bigwedge_{j=1}^{m} g_j(\mathbf{a}) \in P$, where $f_1, \ldots, f_r, g_1, \ldots, g_m \in \mathbb{Z}[X_1, \ldots, X_n]$ do not depend on M and $\mathbf{a}$.

Put $g_0 = 1$, and let Δ be the set of finite sums of squares in M. By [P1, Corollary 1.6], $\sum_{i,j=0}^{m} g_i(\mathbf{a})g_j(\mathbf{a})\Delta$ is the intersection of all orderings on M that contain $g_1(\mathbf{a}), \ldots, g_m(\mathbf{a})$. Therefore the last statement is equivalent to: $\bigwedge_{i=1}^{r} f_i(\mathbf{a}) = 0 \ \wedge \ -1 \notin \sum_{i,j=0}^{m} g_i(\mathbf{a})g_j(\mathbf{a})\Delta$. As Δ is the set of sums of two squares in the PRC field M [P2, Proposition 1.5], a formula in $\mathcal{L}(\mathbb{Q})$ expresses this statement. ∎

PROPOSITION 10.3: *Every real Galois formula θ over a field K is equivalent to a formula in $\mathcal{L}(K)$, modulo the theory of PRC fields M containing K.*

Proof: It suffices to express in $\mathcal{L}(K)$ the statement $\mathbf{Ar}(A, \mathbf{X}) \in \mathrm{Con}$, with C/A a real Galois ring/set cover over K and $\mathrm{Con} = \{\mathbf{H}^\sigma | \ \sigma \in G(C/A)\}$, where $\mathbf{H} = (H, I_H)$ is a involutory substructure of $\mathbf{G}(C/A)$. Let $E = K(A)$ and $F = K(C)$ be the quotient fields. For each $G \leq G(C/A) = G(F/E)$ let $F(G)$ be the fixed field of G in F, and let z_G be a primitive element for $F(G)/E$. Replacing A by an open subset A' (that is, replacing the given Galois stratification by its refinement) we may assume that $K[A][z_G]/K[A]$ is a ring cover [FJ, Definition 5.4] and z_G is a primitive element for it.

Write $K[A]$ as $K[\mathbf{x}, g(\mathbf{x})^{-1}]$, where $\mathbf{x}$ is a generic point of A over K. Let f_G be a polynomial over K such that $f_G(\mathbf{x}, g(\mathbf{x})^{-1}, Z) = \mathrm{irr}(z_G, E)$. Furthermore, for every $\epsilon \in H$ let h_ϵ be a polynomial over K such that $h_\epsilon(\mathbf{x}, g(\mathbf{x})^{-1}, z_H, Z') = \mathrm{irr}(z_{\langle\epsilon\rangle}, F(H))$.

Then $M \models \mathbf{Ar}(A, \mathbf{a}) \in \mathrm{Con}$ means:

(a) $\mathbf{a} \in A$, that is, there is a specialization $\mathbf{x} \to \mathbf{a}$ such that $g(\mathbf{a}) \neq 0$; and

(b) $\mathbf{x} \to \mathbf{a}$ extends to a homomorphism ψ: $C \to \widetilde{M}$ such that $\psi^*(G(M)) = H$ and

(c) $\psi^*(I(M)) = I_H$, that is, for every involution $\epsilon \in H$ we have $\epsilon \in I_H$ if and only if

(c$_\epsilon$) $\epsilon \in \psi^*(I(M))$.

Assume (a). Then (b) means the conjunction of the following two statements:

(b1) $f_H(\mathbf{a}, g(\mathbf{a})^{-1}, Z)$ has a root $c \in M$;

(b2) if $G < H$, then $f_G(\mathbf{a}, g(\mathbf{a})^{-1}, Z)$ has no root in M.

(Cf. [FJ, Remark 25.14].) Furthermore, assume (a) and (b), and let $\epsilon \in H$ be an involution. Condition (c_ϵ) says

(c'_ϵ) $h_\epsilon(\mathbf{a}, g(\mathbf{a})^{-1}, c, Z')$ has a root $\alpha \in \widetilde{M}$ such that $M(\alpha)$ is formally real. Therefore the assertion follows by Lemma 10.2. ▮

Remark 10.4: *The following collection of conditions on a field M is equivalent to a primitive recursive set of elementary sentences in $\mathcal{L}(\mathbb{Q})$:*

(1) *M is PRC.*

(2) $M \cap \tilde{\mathbb{Q}} = \mathbb{Q}^{\mathrm{tr}}$.

(3) $\operatorname{Im} \mathbf{G}(M) = \{\mathbf{H} |\ H = \langle I_H \rangle\}$.

(4) *$\mathbf{G}(M)$ has the embedding property.*

(5) *$M/\mathbb{Q}^{\mathrm{tr}}$ is totally real.*

Proof: For (1) see [P2, Theorem 4.1].

Condition (2) says, for each irreducible polynomial $f \in \mathbb{Q}[X]$, that f has a root in M if an only if f has a root in $\mathbb{Q}^{\mathrm{tr}}$, that is, f splits over the real closure of $\mathbb{Q}$. The latter clause is expressible in $\mathcal{L}(\mathbb{Q})$ either by Tarski's principle [HL, Proposition 1.4] or by Sturm's Theorem [L, Chapter XI, §2].

Conditions (3) and (4) easily follow from Lemma 10.2.

Assume (1) and (2). By Remark 1.8(b), the image X of the restriction map $X(M) \to X(\mathbb{Q}^{\mathrm{tr}})$ is closed in $X(\mathbb{Q}^{\mathrm{tr}})$. Thus (5) is equivalent to 'X is dense in $X(\mathbb{Q}^{\mathrm{tr}})$'. Now, $X(\mathbb{Q}^{\mathrm{tr}})$ has a basis consisting of sets $\{P \in X(\mathbb{Q}^{\mathrm{tr}}) |\ P$ extends to $\mathbb{Q}^{\mathrm{tr}}(\alpha)\}$, where α runs through the elements of $\tilde{\mathbb{Q}}$. (Indeed, by Remark 1.8(b) these sets are clopen. By [P1, Corollary 9.2], $\mathbb{Q}^{\mathrm{tr}}$ is SAP, that is, the sets

$$H(c) = \{P \in X(\mathbb{Q}^{\mathrm{tr}}) |\ c \in P\} = \{P \in X(\mathbb{Q}^{\mathrm{tr}}) |\ P \text{ extends to } \mathbb{Q}^{\mathrm{tr}}(\sqrt{c})\}$$

form a basis for the Harrison topology on $X(\mathbb{Q}^{\mathrm{tr}})$, as c varies on $\mathbb{Q}^{\mathrm{tr}}$.) It suffices to consider only those $\alpha \in \tilde{\mathbb{Q}}$ with $\mathbb{Q}(\alpha)$ formally real, otherwise the corresponding set of orderings is empty. Thus (5) is equivalent to 'for every finite formally real extension $Q(\alpha)$ of $\mathbb{Q}$ the field $M(\alpha)$ is formally real'. Now use Lemma 10.2. ▮

Corollary 10.5: *A field M is a model of* $\mathrm{Th}(\mathbb{Q}^{\mathrm{tr}})$ *if and only if it satisfies conditions (1)–(5).*

Proof: The conditions hold for $M = \mathbb{Q}^{\mathrm{tr}}$. Hence by Remark 10.4 they also hold for each model M of $\mathrm{Th}(\mathbb{Q}^{\mathrm{tr}})$. Conversely, assume (1)–(5). Then M is a real Frobenius field (Proposition 8.3), and $\mathrm{Im}\,\mathbf{G}(M) = \mathrm{Im}\,\mathbf{G}(\mathbb{Q}^{\mathrm{tr}})$, by (3). Furthermore, (2) and (5) imply that $\mathrm{res}_{\tilde{\mathbb{Q}}}\mathbf{G}(M) = \mathbf{G}(\mathbb{Q}^{\mathrm{tr}})$. Therefore by Proposition 9.4 (with $K = \mathbb{Q}$) the fields M and $\mathbb{Q}^{\mathrm{tr}}$ satisfy the same sentences in $\mathcal{L}(\mathbb{Q})$. ▮

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