FREE SUBGROUPS OF FREE PROFINITE GROUPS

by

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Introduction

In their paper [JL] Jarden and Lubotzky ask whether the following is true:

Twinning Principle: *Let m be an infinite cardinal. Given a statement $P(H,G)$ about a profinite group G and a closed subgroup H, the following are equivalent:*

(G) *If a closed subgroup H of $\hat{F}_m$ satisfies $P(H,\hat{F}_m)$, then $H \cong \hat{F}_m$.*

(F) *If a separable algebraic extension L of a Hilbertian field K satisfies $P(G(L),G(K))$, then L is Hilbertian.*

The Weak Twinning Principle [JL, p. 208], asserts that the following are equivalent:

(G_0) If a closed subgroup H of $\hat{F}_\omega$ satisfies $P(H,\hat{F}_\omega)$, then $H \cong \hat{F}_\omega$.

(F_0) If a separable algebraic extension L of a countable PAC Hilbertian field K satisfies $P(G(L),G(K))$, then L is Hilbertian.

(To be precise, [JL] uses the phrase "countable ω-free PAC field K, which is not perfect if $\mathrm{char}(K) > 0$" instead of "countable PAC Hilbertian field K", but these assertions are equivalent. This follows, apart from [FJ, Proposition 11.16] and [FJ, Corollary 24.38] from a result of Pop [Po, Theorem 1] which asserts that every PAC Hilbertian field K is ω-free. See also [HJ, Theorem A] for another proof.)

The authors list seven instances of a statement $P(H,G)$ for which the principle holds; we denote them (P1)–(P7) :

(P1) $(G : H) < \infty$.

(P2) H is normal in G and G/H is finitely generated.

(P3) H is a proper subgroup of finite index of a closed normal subgroup of G.

(P4) H is normal in G and G/H is abelian.

(P5) H is the intersection of two closed normal subgroups of G, neither of which is contained in the other.

(P6) H contains a closed normal subgroup N of G such that G/N is pronilpotent and $(G : H)$ is divisible by at least two primes.

(P7) $(G : H) = \prod_p p^{\alpha(p)}$, with all $\alpha(p)$ finite.

Following this list, Jarden and Lubotzky add: *"Although some of the group theoretical ingredients of the proofs of theorems (Gn) enter in the proofs of theorems (Fn),* *it is difficult to see a real analogy between the proofs of the group theoretical theorems and those of field theory.”*

In this paper we try to shed some light on this ‘mysterious’ principle.

The strategy is as follows:

(a) Give a general sufficient condition for an algebraic separable extension M of a Hilbertian field K to be Hilbertian.

(b) Show that this condition covers the extensions L/K that satisfy $P(G(L), G(K))$, where $P(H,G)$ is one of the statements(P1)–(P7).

(c) Prove that the group theoretic counterpart (via Galois theory) of this criterion is a condition for a closed subgroup of $\hat{F}_m$ to be isomorphic to $\hat{F}_m$.

(d) Show that the latter condition covers the pairs of groups (G, H) that satisfy $P(H, G)$, where $P(H,G)$ is one of the statements(P1)–(P7).

Parts (a) and (b) have been accomplished in [Ha]. The criterion [Ha, Theorem 3.2] roughly states that certain embedding problems over K should have no solution contained in some Galois extension of K containing M. It also yields [Ha, Theorem 4.1]:

Theorem (F8): *Let K be a Hilbertian field and let M_1, M_2 be two Galois extensions of K. Let M be an intermediate field of M_1M_2/K such that $M \not\subseteq M_1$ and $M \not\subseteq M_2$. Then M is Hilbertian.*

In this paper we present steps (c) and (d).

We obtain (Theorem 2.2), a technical criterion for a subgroup of $\hat{F}_m$ to be isomorphic to $\hat{F}_m$. It turns out that this criterion is responsible for essentially all known instances of the Twinning Principle. This, in our opinion, unveils the ‘mystery’ of the Twinning Principle.

In addition, we add one more example of the Twinning Principle, the counterpart of (F8) above:

Theorem (Theorem 3.2): *Let $\hat{F}_m$ be a free profinite group of infinite rank m, and let M_1, M_2 be two normal subgroups of $\hat{F}_m$. Let M be a closed subgroup of $\hat{F}_m$ such that $M_1 \cap M_2 \subseteq M$ but $M_1 \not\subseteq M$ and $M_2 \not\subseteq M$. Then $M \cong \hat{F}_m$.*

On the other hand, the Twinning Principle, as stated in [JL], cannot hold in general. In Section 4 we discuss statements that can be considered counterexamples to the Twinning Principle.

1. Twisted wreath products

Let G and A be finite groups and let G' be a subgroup of G. Assume that G' acts on A (from the right). Let

$$\text{(1)} \qquad \mathrm{Ind}_{G'}^{G}(A) = \{f\colon G \to A \mid f(\sigma\rho) = f(\sigma)^{\rho}, \quad \text{for all } \sigma \in G,\ \rho \in G'\}$$

with the multiplication rule $(fg)(\sigma) = f(\sigma)g(\sigma)$. (We do not require that A be commutative.) Then G acts on $\mathrm{Ind}_{G'}^{G}(A)$ by the formula

$$\text{(2)} \qquad (f^{\tau})(\sigma) = f(\tau\sigma) \qquad \tau, \sigma \in G.$$

The semidirect product $G \ltimes \mathrm{Ind}_{G'}^{G}(A)$, together with the projection $G \ltimes \mathrm{Ind}_{G'}^{G}(A) \to G$, is called the **(twisted) wreath product** of A and G with respect to G'.

This construction is discussed in [Ha, Section 1]. We shall need the following property:

LEMMA 1.1 ([Ha, Lemma 1.4]): *Let $\pi\colon A \,\mathrm{wr}_{G'}\, G \to G$ be a twisted wreath product, where $A \neq 1$. Let $H_1 \triangleleft A \,\mathrm{wr}_{G'}\, G$ and $h_2 \in A \,\mathrm{wr}_{G'}\, G$. Let $G_1 = \pi(H_1)$.*

(a) *If $\pi(h_2) \notin G'$ and $(G_1G' : G') > 2$, then there is $h_1 \in H_1 \cap \mathrm{Ker}\pi$ such that $[h_1, h_2] \neq 1$.*

(b) *If $G_1 \not\subseteq G'$ and $\pi(h_2) \notin G_1G'$, there is $h_1 \in H_1 \cap \mathrm{Ker}\pi$ such that $[h_1, h_2] \neq 1$.*

We also include an easy consequence of the definitions:

LEMMA 1.2: *If $G \neq G'$ and $A \neq 1$, then $A \,\mathrm{wr}_{G'}\, G$ is not commutative.*

Proof: Choose $1 \neq a \in A$ and define $f\colon G \to A$ by $f(G \smallsetminus G') = 1$, $f(\rho) = a^{\rho}$, for all $\rho \in G'$. Then $f \in \mathrm{Ind}_{G'}^{G}(A)$. Choose $\sigma \in G \smallsetminus G'$. Then $f^{\sigma}(1) = f(\sigma) = 1 \neq a = f(1)$, and hence $f^{\sigma} \neq f$. Therefore $(\sigma, 1)(1, f) = (\sigma, f) \neq (\sigma, f^{\sigma}) = (1, f)(\sigma, 1)$. ∎

2. Subgroups of free groups

The aim of this section is to give a sufficient condition (Theorem 2.2) for a closed subgroup of a free profinite group $\hat{F}_m$ of an infinite rank to be isomorphic to $\hat{F}_m$. To this end we first need a workable definition of $\hat{F}_m$ (Lemma 2.1), in terms of the number of solutions of split embedding problems.

The following lemma extends the characterization of free profinite groups implicit in [FJ, Proposition 24.18].

LEMMA 2.1: *Let $\hat{F}_m$ be the free profinite group of infinite rank m, and let M be a closed subgroup of $\hat{F}_m$. The following three conditions are equivalent:*

(a) *Every finite embedding problem for M with a non trivial kernel has (at least) m solutions.*

(b) *Every finite split embedding problem for M with a non trivial kernel has (at least) m solutions.*

(c) $M \cong \hat{F}_m$.

Proof: Clearly, (a) $\Rightarrow$ (b). Also, (c) $\Rightarrow$ (a) by [FJ, Lemma 24.14]. We prove (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

PART A: *We may assume that* $\mathrm{rank}(M) = m$. To justify this reduction, it suffices to show that (b) implies $\mathrm{rank}(M) = m$. So, assume (b). Let $\mathcal{ON}(M)$ be the family of open normal subgroups of M. Fix a non-trivial finite group G and consider the finite split embedding problem $(M \to 1,\ G \to 1)$. By (b) there are at least m epimorphisms $M \to G$. The cardinality of the family of their kernels is at least m, (since every open normal subgroup of G is the kernel of at most finitely many epimorphisms $M \to G$). Hence $|\mathcal{ON}(M)| \geq m$. By [FJ, Lemma 15.1], M is not finitely generated, hence [FJ, Supplement 15.12], $\mathrm{rank}(M) = |\mathcal{ON}(M)|$.

On the other hand, $\mathcal{ON}(M)$ has the same cardinality as the family $\mathcal{O}(M)$ of open subgroups of M, since $\mathcal{ON}(M) \subseteq \mathcal{O}(M)$ and each $U \in \mathcal{O}(M)$ is a union of finitely many cosets of some $N \in \mathcal{ON}(M)$. As $\mathcal{O}(M) = \{U \cap M \mid U \in \mathcal{O}(\hat{F}_m)\}$, we have $\mathrm{rank}(M) = |\mathcal{O}(M)| \leq |\mathcal{O}(\hat{F}_m)| = m$.

Part B: *(a) ⇒ (c).* As (c) ⇒ (a), both M and $\hat{F}_m$ satisfy (a). By [FJ, Proposition 24.18] with Part A we get $M \cong \hat{F}_m$.

Part C: *(b) ⇒ (a).* This is an elaboration on Jarden's Lemma [Ma, p. 231]. Let

(1)
$$\begin{array}{ccccccccc} & & & & & & M & & \\ & & & & & & \downarrow{\scriptstyle \varphi} & & \\ 1 & \longrightarrow & A & \longrightarrow & B & \xrightarrow[\alpha]{} & C & \longrightarrow & 1 \end{array}$$

be a finite embedding problem for M. As M is projective [FJ, Corollary 20.14], there is a homomorphism $\varphi_0\colon M \to B$ such that $\alpha \circ \varphi_0 = \varphi$. Let $B_0 = \varphi_0(M) \le B$. Then B_0 acts on A (via conjugation in B) and this gives rise to the semidirect product $B_0 \ltimes A$. Let $\pi\colon B_0 \ltimes A \to B_0$ be the canonical projection and let $\lambda\colon B_0 \ltimes A \to B$ be the unique epimorphism that is identity on A and on B_0. Finally, let α_0 be the restriction of α to B_0. Then we have a commutative diagram of epimorphisms

$$\begin{array}{ccccc} & & & & M \\ & & & {\scriptstyle \varphi_0}\swarrow & \\ B_0 \ltimes A & \xrightarrow[\pi]{} & B_0 & & \big\downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \alpha_0} & \swarrow & \\ B & \xrightarrow[\alpha]{} & C. & & \end{array}$$

By (b), there is a set $\{\psi_i\}_{i\in I}$ of epimorphisms $M \to B_0 \ltimes A$, of cardinality m, such that $\pi \circ \psi_i = \varphi_0$, for each $i \in I$. Then $\lambda \circ \psi_i$ is a solution of (1), for each $i \in I$.

We may assume that $\mathrm{Ker}\psi_i \neq \mathrm{Ker}\psi_j$ for $i \neq j$, otherwise replace $\{\psi_i\}_{i\in I}$ by a subset with this property. Thus, for $i \neq j$, there is $x \in M$ such that, say, $\psi_i(x) = 1$ but $\psi_j(x) \neq 1$. We have $\lambda \circ \psi_i(x) = 1$ and

$$\pi \circ \psi_j(x) = \varphi_0(x) = \pi \circ \psi_i(x) = 1,$$

and hence $\psi_j(x) \in \mathrm{Ker}\pi = A$. As λ is injective on A and $\psi_j(x) \neq 1$, we get $\lambda \circ \psi_j(x) \neq 1$. Therefore $\lambda \circ \psi_i \neq \lambda \circ \psi_j$. ∎

Theorem 2.2: *Let $K = \hat{F}_m$ and let M be a closed subgroup of K. Suppose that for all open subgroups K_α and K_β of K such that $M \le K_\alpha$ there exist:*

(i) *an open subgroup K' of K_α such that $M \le K'$,*

(ii) *an open normal subgroup L of K such that $L \le K' \cap K_\beta$,*

(iii) *a closed normal subgroup N of K such that $N \le M \cap L$,*

such that for every finite nontrivial group A_0 and every action of the subgroup $G' = K'/L$ of $G = K/L$ on A_0 the following embedding problem has **no** *(strong) solution*

(2)
$$\begin{array}{ccc} & & K/N \\ & & \big\downarrow{\scriptstyle \bar{\varphi}} \\ A_0 \,\mathrm{wr}_{G'}\, G & \xrightarrow[\alpha_1]{} & G. \end{array}$$

Then $M \cong \hat{F}_m$.

Proof:

Part A: *Preliminaries.* Let a finite group G'' act on a nontrivial finite group A, let $p''\colon G'' \ltimes A \to G''$ be the projection of the semidirect product, and let $\varphi''\colon M \to G''$ be an epimorphism. By Lemma 2.1 we have to show that the split embedding problem

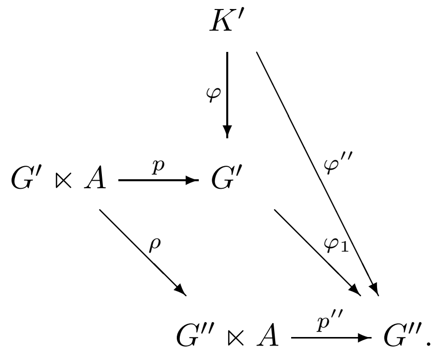
(3)
$$\begin{array}{ccc} & & M \\ & & \big\downarrow{\scriptstyle \varphi''} \\ G'' \ltimes A & \xrightarrow{p''} & G''. \end{array}$$

has m solutions.

There is an open $K_\alpha \le K$ such that $M \le K_\alpha$ and φ'' extends to a continuous epimorphism $\varphi''\colon K_\alpha \to G''$. Let $K_\beta = \mathrm{Ker}\varphi''$. Then K_β is an open normal subgroup of K_α such that $MK_\beta = K_\alpha$.

For these K_α, K_β let K', L, and N be as in (i) - (iii). Put $G = K/L$ and $G' = K'/L \le G$ and denote by $\varphi\colon K \to G$ the quotient map, as well as its restriction $K' \to G'$ to K'. Furthermore, from now on restrict φ'' to K'. As $L \le K' \cap K_\beta = \mathrm{Ker}\varphi''$, $\varphi''\colon K' \to G''$ factors through $\varphi\colon K' \to G'$. Thus we have the following commutative diagram:

(4)



in which p is the canonical projection of the semidirect product, G' acts on A through $\varphi_1\colon G' \to G''$ and the action of G'' on A, and ρ is the map induced from φ_1 and the identity of A.

Part B: *Epimorphisms into the wreath product.* Let $\alpha\colon A\,\mathrm{wr}_{G'}\,G \to G$ be the wreath product. Fix a set I of cardinality m. For each $i \in I$ we now construct an epimorphism $\psi_i\colon K \to A\,\mathrm{wr}_{G'}\,G$ such that $\alpha \circ \psi_i = \varphi\colon K \to G$.

As $K \cong \hat{F}_m$, it has a basis X of cardinality m converging to 1. Write X as $X_0 \cup X_1$, where $X_0 = X \smallsetminus \mathrm{Ker}\varphi$ and $X_1 = X \cap \mathrm{Ker}\varphi$. Then X_0 is finite and $|X_1| = m$. Therefore there exists a bijection $\mathrm{Ind}_{G'}^{G}(A) \times I \to X_1$; we write it as $(f,i) \mapsto x_{f,i}$. Thus $X = X_0 \cup \{x_{f,i} \mid f \in \mathrm{Ind}_{G'}^{G}(A),\ i \in I\}$.

Define $\psi_i\colon X \to A\,\mathrm{wr}_{G'}\,G$ by

$$\psi_i(x_{f,k}) = \begin{cases} f & k = i \\ 1 & k \neq i \end{cases}$$

and $\psi_i(x_0) = \varphi(x_0)$ for each $x_0 \in X_0$ (here we identify G with a subgroup of $A\,\mathrm{wr}_{G'}\,G$ via α). Clearly $\alpha \circ \psi_i(x) = \varphi(x)$ for every $x \in X$ and $\psi_i(X_1) = \mathrm{Ind}_{G'}^{G}(A) = \mathrm{Ker}\alpha$. Therefore ψ_i extends to an epimorphism $\psi_i\colon K \to A\,\mathrm{wr}_{G'}\,G$ such that

(5) $$\alpha \circ \psi_i = \varphi.$$

Let $\pi\colon \mathrm{Ind}_{G'}^{G}(A) \to A$ be the map given by $f \mapsto f(1)$. It is compatible with the action of G'. Let π also denote its extension $\pi\colon G' \ltimes \mathrm{Ind}_{G'}^{G}(A) \to G' \ltimes A$ by the identity of G'.

Part C: *If* $\psi\colon K \to A\,\mathrm{wr}_{G'}\, G$ *is an epimorphism and* $\alpha \circ \psi = \varphi$*, then* $\pi \circ \psi(N) = A$. Indeed, since $\varphi(N) = 1$ and $N \triangleleft K$, we get that $\psi(N)$ is a normal subgroup of $A\,\mathrm{wr}_{G'}\, G$ contained in $\mathrm{Ind}_{G'}^{G}(A)$. Hence $\psi(N)$ is a normal G-invariant subgroup of $\mathrm{Ind}_{G'}^{G}(A)$. Therefore $A_1 = \pi \circ \psi(N)$ is a normal G'-invariant subgroup of A. We have the following three commutative diagrams

(6)

$$\begin{array}{ccc} & & K \\ & \swarrow^{\psi} & \downarrow^{\varphi} \\ A\,\mathrm{wr}_{G'}\, G & \xrightarrow{\alpha} & G \\ \downarrow^{\lambda} & & \| \\ (A/A_1)\,\mathrm{wr}_{G'}\, G & \longrightarrow & G \end{array} \qquad \begin{array}{ccc} & & K' \\ & \swarrow^{\mathrm{res}_{K'}\psi} & \downarrow^{\mathrm{res}_{K'}\varphi} \\ G' \ltimes \mathrm{Ind}_{G'}^{G}(A) & \xrightarrow{\alpha} & G' \\ \downarrow^{\pi} & & \| \\ G' \ltimes A & \xrightarrow[p]{} & G' \end{array} \qquad \begin{array}{ccc} & & N \\ & \swarrow^{\mathrm{res}_N\psi} & \downarrow^{\mathrm{res}_N\varphi} \\ \mathrm{Ind}_{G'}^{G}(A) & \longrightarrow & 1 \\ \downarrow^{\pi} & & \| \\ A & \longrightarrow & 1 \end{array}$$

in which λ is the epimorphism induced from the quotient map $A \to A/A_1$.

Now, $\psi(N) \le \pi^{-1}(A_1) = \{f \in \mathrm{Ind}_{G'}^{G}(A) \mid f(1) \in A_1\}$ and $\psi(N)$ is a G-invariant subgroup of $\mathrm{Ind}_{G'}^{G}(A)$, hence

$$\psi(N) \le \bigcap_{\sigma \in G} \{f \in \mathrm{Ind}_{G'}^{G}(A) \mid f(1) \in A_1\}^{\sigma} = \bigcap_{\sigma \in G} \{f \in \mathrm{Ind}_{G'}^{G}(A) \mid f(\sigma) \in A_1\} = \mathrm{Ker}\lambda.$$

It follows that $\lambda \circ \psi$ induces an epimorphism $\bar{\psi}\colon K/N \to (A/A_1)\,\mathrm{wr}_{G'}\, G$ that solves (2) with $A_0 = A/A_1$. By assumption, this cannot happen unless $A_1 = A$, as claimed.

Part D: *Solutions of (3).* Fix $i \in I$. We have $N \le M \le K'$. By (5) and the middle diagram of (6), $p \circ \pi \circ \mathrm{res}_{K'}\psi_i = \mathrm{res}_{K'}\varphi$. From (4) we deduce that $p'' \circ \rho \circ \pi \circ \mathrm{res}_{K'}\psi_i = \varphi''$. In particular, $p'' \circ (\rho \circ \pi \circ \mathrm{res}_M\psi_i) = \mathrm{res}_M\varphi''$.

By Part C, $\pi \circ \psi_i(N) = A$. Therefore,

$$\rho \circ \pi \circ \psi_i(M) \supseteq \rho \circ \pi \circ \psi_i(N) = \rho(A) = A = \mathrm{Ker}p''.$$

Thus $\rho \circ \pi \circ \mathrm{res}_M\psi_i$ is onto $G'' \ltimes A$, and hence solves (3).

Part E: *The solutions are distinct.* Let $i \ne j$. We have to show that $\rho \circ \pi \circ \mathrm{res}_M\psi_i \ne \rho \circ \pi \circ \mathrm{res}_M\psi_j$. Let $\hat{A} = A \times A$ and let $p_1\colon \hat{A} \to A$ and $p_2\colon \hat{A} \to A$ be the coordinate projections. Let G' act on $\hat{A}$ coordinatewise; this defines the wreath product

$\hat{\alpha}\colon \hat{A} \operatorname{wr}_{G'} G \to G$, and we get the following commutative diagram

$$(7) \qquad \begin{array}{ccc} \hat{A} \operatorname{wr}_{G'} G & \xrightarrow{p_2^*} & A \operatorname{wr}_{G'} G \\ \downarrow{\scriptstyle p_1^*} & \searrow^{\hat{\alpha}} & \downarrow{\scriptstyle \alpha} \\ A \operatorname{wr}_{G'} G & \xrightarrow{\alpha} & G \end{array}$$

in which p_i^* is the identity on G and $p_i^*(f) = p_i \circ f$, for every $f \in \operatorname{Ind}_{G'}^G(\hat{A})$, for $i = 1, 2$.

Use the basis X of K from Part B to define $\hat{\psi}\colon X \to \hat{A} \operatorname{wr}_{G'} G$ by

$$\hat{\psi}(x_{f,k}) = \begin{cases} (f, 1) & k = i \\ (1, f) & k = j \\ 1 & k \neq i, j \end{cases}$$

and $\hat{\psi}(x_0) = \varphi(x_0)$ for each $x_0 \in X_0$ (again, we identify G with a subgroup of $\hat{A} \operatorname{wr}_{G'} G$ via $\hat{\alpha}$). Then $\hat{\psi}$ extends to an epimorphism $\hat{\psi}\colon K \to \hat{A} \operatorname{wr}_{G'} G$ such that $\hat{\alpha} \circ \hat{\psi} = \varphi$. Furthermore, $p_1^* \circ \hat{\psi} = \psi_i$ and $p_2^* \circ \hat{\psi} = \psi_j$.

By Part C (with $\hat{A}$ instead of A), $\hat{\pi} \circ \hat{\psi}(N) = \hat{A}$, where $\hat{\pi}\colon \operatorname{Ind}_{G'}^G \hat{A} \to \hat{A}$ is the map given by $f \mapsto f(1)$. Thus there is $x \in N$ such that $\hat{\pi} \circ \hat{\psi}(x) = (a, 1)$, where $1 \neq a \in A$. Clearly $p_1 \circ \hat{\pi} = \pi \circ p_1^*$ and $p_2 \circ \hat{\pi} = \pi \circ p_2^*$. It follows that $\pi \circ \psi_i(x) = a \neq 1$ and $\pi \circ \psi_j(x) = 1$. But ρ is identity on A, and hence $\rho \circ \pi \circ \psi_i(x) \neq \rho \circ \pi \circ \psi_j(x)$. ∎

3. The Diamond Theorem

If we take either one of the equivalent properties (a) or (b) of Lemma 2.1 as the definition of the free profinite group $\hat{F}_m$, then Theorem 2.2 gives a new proof of the following result, devoid of combinatorial constructions for discrete free groups:

PROPOSITION 3.1 ([FJ, Proposition 15.27]): *Let m be an infinite cardinal. Then every open subgroup M of $\hat{F}_m$ is isomorphic to $\hat{F}_m$.*

Proof: Let $K = \hat{F}_m$. Given open subgroups K_α and K_β of K such that $M \le K_\alpha$, choose an open subgroup $L \triangleleft K$ such that $L \le M \cap K_\beta$. Put $N = L$ and $K' = M$. Then $\bar{\varphi}$ in embedding problem (2) of Section 2 is an isomorphism of finite groups, while α_1 is not. Therefore (2) has no strong solution. ∎

Our main application of Theorem 2.2 it the following result.

THEOREM 3.2: *Let m be an infinite cardinal, and let M_1, M_2 be two closed normal subgroups of $\hat{F}_m$. Let M be a closed subgroup of $\hat{F}_m$ that contains $M_1 \cap M_2$ but $M_1 \not\subseteq M$ and $M_2 \not\subseteq M$. Then $M \cong \hat{F}_m$.*

Proof: By Proposition 3.1 we may assume that $(\hat{F}_m : M) = \infty$.

PART A: *We may assume that*

(a) *either $M_1M_2 = \hat{F}_m$ or $(M_1M : M) > 2$.*

Indeed, we cannot have $(M_1M : M) = 1$, since $M_1 \not\subseteq M$. Suppose that $(M_1M : M) = 2$. There is an open subgroup K_2 of $\hat{F}_m$ containing M but not M_1M. Thus $K_2 \cap M_1M = M$. Put $K = K_2M_1M$. Then $(K : K_2) = (M_1M : M_1) = 2$, hence K_2 is a normal subgroup of K. Observe that $M_1K_2 = K$ and $K_2 \cap M_1 \subseteq K_2 \cap M_1M = M \subseteq K$. Furthermore, $K_2 \not\subseteq M$, since $(K_2 : M)$ is infinite.

Again, using Proposition 3.1, replace $\hat{F}_m$ by its open subgroup K and M_2 by K_2 to achieve $M_1M_2 = \hat{F}_m$.

PART B: *Construction of L and N.* We apply the criterion of Theorem 2.2. Let $K = \hat{F}_m$. Let K_α and K_β be two open subgroups of K such that $M \le K_\alpha$.

Choose an open normal subgroup L of K contained in $K_\alpha \cap K_\beta$. Let $G = K/L$, and let $\varphi\colon K \to G$ be the quotient map. Let G_1, G_2, and G' be the images in G of M_1, M_2, and M, respectively, under φ. Put $K' = ML$; then $K' \subseteq K_\alpha$ and $G' = K'/L$. Then

(b) $G_1, G_2 \triangleleft G$.

Condition $M_1, M_2 \not\subseteq M$ implies that if L is sufficiently small then

(c) $G_1, G_2 \not\subseteq G'$.

Similarly, $(K : M) = \infty$ implies, with L sufficiently small, that

(d) $(G : G') > 2$.

Finally, (a) implies, with L sufficiently small, that

(e) either $G_1 G_2 = G$ or $(G_1 G' : G') > 2$.

In particular,

(e$'$) either $G_2 \not\subseteq G_1 G'$ or $(G_1 G' : G') > 2$.

Indeed, if both $G_2 \subseteq G_1 G'$ and $G_1 G_2 = G$, then $G = G_1 G'$. By (d), $(G_1 G' : G') > 2$.

Let $N = L \cap M_1 \cap M_2$.

Part C: *An embedding problem.* Let $A_0 \neq 1$ be a finite group on which G' acts, and let $H = A_0 \,\mathrm{wr}_{G'}\, G$. By Theorem 2.2 it suffices to show that

(1)
$$\begin{array}{ccc} & & K \\ & & \downarrow{\scriptstyle\varphi} \\ H & \xrightarrow{\pi} & G \end{array}$$

has no (strong) solution $\psi\colon K \to H$ that factors through $K \to K/N$.

Let $\psi\colon K \to H$ be a solution such that $\psi(N) = 1$. For $i = 1,2$ let $H_i = \psi(M_i)$. Then $H_i \triangleleft H$ and $\pi(H_i) = \varphi(M_i) = G_i$.

There exist $h_1 \in H_1$ and $h_2 \in H_2$ such that $\pi(h_1) = 1$ and $[h_1, h_2] \neq 1$. Indeed, if the first assertion of (e$'$) holds, there is $h_2 \in H_2$ such that $\pi(h_2) \notin G_1 G'$. Condition (c) and Lemma 1.1(b) provide the required $h_1 \in H_1$. If the second assertion of (e$'$) holds, by (c) there is $h_2 \in H_2$ such that $\pi(h_2) \notin G'$. Lemma 1.1(a) gives the required $h_1 \in H_1$.

For $i = 1,2$ there is $\gamma_i \in M_i$ such that such that $\psi(\gamma_i) = h_i$. As $h_1 \in \mathrm{Ker}\pi$, we have $\gamma_1 \in \mathrm{Ker}\varphi = L$. But then

$$[\gamma_1, \gamma_2] \in [L, M_2] \cap [M_1, M_2] \subseteq L \cap (M_1 \cap M_2) = N, \tag{2}$$

hence $[h_1, h_2] = [\psi(\gamma_1), \psi(\gamma_2)] \in \psi(N) = 1$, a contradiction. ∎

4. About the Twinning Principle

Let $G = \hat{F}_m$ and let H be a closed subgroup of G. We now show how to deduce from Theorem 2.2 that if one of the conditions (P1)–(P6) from the introduction holds, then $H \cong \hat{F}_m$.

Case (P1) is Proposition 3.1. Case (P2) is a straightforward Galois theoretic translation of [Ha, Proposition 4.5]. Cases (P3) and (P5) immediately follow from Theorem 3.2. So does (P6): Since $(G/N : H/N)$ is divisible by two primes and the Sylow subgroups of G/N are normal in G/N, there are two (Sylow) normal subgroups P_1, P_2 of G/N such that $P_1 \cap P_2 = 1$ and $P_1, P_2 \not\leq H/N$. The preimages M_1, M_2 of P_1, P_2 are normal in G, satisfy $M_1 \cap M_2 = N \leq H$, but $M_1, M_2 \not\leq H$.

Case (P4) can be easily deduced from Theorem 2.2 by Lemma 1.2.

The somewhat bizarre case (F7) is not covered by Theorem 2.2. However, the original proofs for the group theoretical statement and the field theoretical statement are analogous to each other.

Nevertheless, the principle cannot hold in full generality:

Example 4.1: Let $P(H,G)$ mean "the cohomological dimension of H is 2". Since $\hat{F}_\omega$ has cohomological dimension 1, every subgroup has cohomological dimension≤ 1. Therefore condition (G) of the Twinning Principle holds (vacuously). However, $K = \mathbb{Q}$ is Hilbertian, and the field $L = \mathbb{Q}_3$ of algebraic 3-adic integers is Henselian and hence not Hilbertian [FJ, Section 14, Exercise 8], although its cohomological dimension is 2 [Ri, Corollary V.6.2]. ∎

A more interesting counterexample is the following (found together with Moshe Jarden):

Example 4.2: Let $P(H,G)$ be "$H \cong \hat{F}_\omega$". Then condition (G) of the Twinning Principle for $m = \omega$ trivially holds. Let F be a field of characteristic 0 with absolute Galois group $G(F) \cong \hat{F}_\omega$ (cf. [FJ, Corollary 20.16]). The field of formal power series in one variable $F((t))$ is a regular extension of F, and hence the restriction map $G(F((t))) \to G(F)$ is surjective. As $G(F)$ is projective, this map splits. Hence there is a separable extension L of $F((t))$ such that $G(L) \cong G(F) \cong \hat{F}_\omega$. Furthermore, as

$F((t))$ is a complete valued field and hence Henselian, L is Henselian, and hence not Hilbertian [FJ, Section 14, Exercise 8]. Choose a transcendence base B for L/F and let $K = F(B)$. Then K is Hilbertian [FJ, Theorem 12.9]. Thus (F) does not hold. ∎

Of course, the principle fails because the notion of 'statement' (applied to $P(H, G)$) is somewhat vague. We could now reformulate the principle to hold only for those extensions (resp., subgroups) that satisfy the conditions of Theorem 2.2. However, this seems to be too restrictive: one can hope to replace the construction of twisted wreath product by something more general. Until such generalization has been found, we leave the question of proper formulation of the principle open.

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