TEL AVIV UNIVERSITY
FACULTY OF MANAGEMENT
THE LEON RECANATI GRADUATE
SCHOOL OF BUISNESS ADMINISTRATION

אוניברסיטת תל-אביב
הפקולטה לניהול
בית הספר למוסמכים במינהל עסקים
ע"ש ליאון רקנאטי

אסטרטגיות שיווי-משקל של ספקי שרות
במערכת תורים:
תמחור במוצרים משלימים

תזה
לתואר מוסמך במדעי הניהול
חקר ביצועים והחלטות

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תקציר

אנו בוחנים מודל של תמחור, כאשר צרכנים חסרי סבלנות מחכים בתור כדי לקבל שרות.

כדי לקבל את השירות, הצרכנים צריכים לשלם גם לספק השרות (סכום אחיד) וגם לספק חנייה (סכום שנקבע לפי הזמן שלהם במערכת בפועל). ספקי השרות והחנייה הינם יזמים נפרדים – כל אחד מהם קובע את מחירו כדי למקסם את רווחיו.

מונח כי קיים קצב הגעה פוטנציאלי של לקוחות למערכת. הלקוחות אינם יכולים לצפות באורך התור, ולכן חלק מהלקוחות מצטרפים בפועל למערכת – חלק שנקבע בהתבסס על ניתוחים סטטיסטיים.

אנו מוצאים תוצאות אנאליטיות ברורות עבור ההתנהגות בשיווי משקל של שני היזמים ושל הצרכנים, ומגדירים תחום של שיווי-משקל שניתן להגיע אליהם בתנאים שונים (מותנה בגובה הביקוש הפוטנציאלי). אנו מראים באופן גרפי, שהמערכת מתכנסת לקבוצת שיווי המשקל שהגדרנו.

אנו משווים תוצאות אלו לתוצאות שיווי-משקל שיושגו ע"י מונופוליסט המספק גם את השירות וגם את החנייה. המונופוליסט יבחר סט נמוך יותר של מחירים, ויביא לאופטימום חברתי.

אנו מראים שחוסר-הסבלנות של הצרכנים הינו מאפיין הכרחי להצדקת המודל: במקרה שבו הצרכנים אינם חסרי-סבלנות, מחירו של ספק החנייה בשיווי משקל יאופס. אנו מראים, שעבור ערכים נמוכים של חוסר-סבלנות הלקוחות, מחיר החנייה הינו פונקציה עולה של חוסר-סבלנות הלקוחות.

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Equilibrium Strategies of Service Providers in a Queueing System: The Case of Complementary Pricing

A Thesis
For M.Sc Degree in The Management Sciences
Operations Research And Decision Sciences

Submitted by: Ari Veltman.

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Thesis Reader: Dr. Israel Brosh

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Abstract

We investigate a model of pricing when impatient customers queue to receive a service.

In order to receive the service, the customers have to pay both the service-provider (a bulk sum), and a parking-provider (a sum determined by the actual time they spend in the system). The service and parking providers are separate entrepreneurs, each choosing a price to maximize his own revenue.

A potential rate of arrival by customers is assumed. The customers are not able to observe the queue's length, and so a portion of the customers join the system – this portion is determined according to statistical arguments.

We find clear analytical results for the behavior of both entrepreneurs and the customers in equilibrium, and define a range of equilibria that may be reached in different situations (depending on the potential demand). We show graphically, that the system converges into this set of equilibria.

We compare the results to the results of equilibria that will be reached by a monopolist who supplies both the service and the parking. The monopolist will choose a lower set of prices, and achieve social optimality.

We show that customers' impatience is crucial for the justification of the model: in case the customers are not impatient, the parking-provider's price in equilibrium will be set to zero. Indeed, we show that for low values of customers' impatience, the parking-price is an increasing function of the customers' impatience.

We also find, that under our model, the parking-provider will generally prefer to change his pricing system, and set a bulk-sum for his service just like the service-provider.

When the potential demand in the market decreases, it either has no effect or it causes the set of prices to rise.

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Part 1 – Introduction

When potential customers decide to get a certain service, or even purchase a product, they consider the possibility that they will have to spend time waiting their turn in a queue and being serviced themselves.

When customers consider their time to be valuable, their decision whether to arrive at the facility and receive the service or not, is not only influenced by the charged price, but also by the time they expect to spend at that facility.

This interpretation of customers' demand, has been dealt with quite widely in recent years.

Consider the providers of the service. Since the customers spend time at their facility, they have to provide the customers the desired service and "accommodation" for the time they spend at the facility. This "accommodation" may be a parking service for their cars or a parallel for services provided via the internet.

In this work, we consider a separation of this "parking"-service from the basic service provided. We consider two separate entrepreneurs, one of which provides a "parking" service, and the other provides the regular service. The customers have to pay them both in order to receive the service.

We investigate such a model to see how this separation influences the price charged from the customers, how the prices are set in such a model, and to find whether there are any Nash-equilibria that may be expected in such a model.

The thesis is arranged as follows: a short discussion of relevant literature is given in sec. 1.1, followed by a short discussion of our goals in sec. 1.2. In Part 2, the model is presented. We give the main results – specific analytic results along with some other qualitative results. We solve the model to find Nash-equilibria and prove these results, and then give a graphical interpretation and show that system converges to these equilibria. In Part 3, we shortly investigate a few variations to this model, to get a feeling of its sensitivity to different kinds of changes. In part 4, we discuss the results, and summarize in part 5.

1.1 Overview of related literature

The subject of equilibrium strategies in queueing systems has been dealt with for about three decades now. The earliest works on this subject seem to be the one by Naor [15], and later on by Edelson and Hildebrand [5]. Since then, a quite extensive amount of articles dealt with this subject, beginning with different models concerning the relationships among customers and between the customers and a service provider (with the distinction between a public central planner and a private revenue maximizing entrepreneur), and continuing with models concerning competition between separate service providers (usually between competing revenue-maximizing servers).

A thorough mapping of existing literature on equilibrium strategies in queueing systems is given by Hassin & Haviv [6].

Following, is a brief presentation of a few of the models that had been dealt with in the past, that seem to be most relevant as a background for our model.

We begin with two early works that set the basis for our model.

1.1.1 Naor

Naor [15] considers whether the imposition of an "entrance fee" on arriving customers who wish to be serviced by a station is a rational measure.

He shows that self optimization of each customer does not optimize the general system (does not yield public optimization).

Naor describes a specific model, as follows (we give the assumptions in detail here and relate to them when we later present our model):

1. A stationary Poisson stream of customers – with parameter Λ - arrives at a single service station. Customers are identical except for their arrival time. Λ is the rate at which the need for the service arises, and not necessarily the actual rate of customers that decide to enter the station and be served.

2. The station renders service in such a way that service times are independently, identically, and exponentially distributed with intensity parameter μ.

3. On successful completion of service, the customer is endowed with a reward of R (expressible in monetary units). All customer rewards are equal.

4. The cost to a customer for staying in the system (either waiting or being served) is C monetary units per time unit. All customer costs are equal.

5. Customers are risk neutral, that is, they maximize the expected value of a linear utility function.

6. From the public (social) point of view, utilities of individuals (customers and servers) are identical and additive.

7. For the model to make sense, it is assumed that any customer will choose to join the queue if he finds the system empty (when he only incurs the costs of his own service-time), thus: $\frac{R\mu}{C} \geq 1$. Otherwise, There is no justification for operating the system at all.

8. An arriving customer has to choose between two alternatives:
   (a) He joins the queue, incurs the losses associated with spending some of his time in it, and finally obtains the reward.
   (b) He refuses to join the queue – an action which does not bring about any gain or loss.

   The choice is made by the customer on comparing the net gains associated with each of the alternatives, after observing the queue-length (option a is assumed on a tie).

Under these model assumptions, each customer chooses a "threshold" strategy as follows: on arriving at the system the customer observes the queue-size. If the queue is shorter than a predetermined threshold n, he joins the queue, otherwise he chooses not to join the queue (diverted).

The threshold chosen is the highest integer for which the expected utility gained by the service (R) exceeds or equals the expected costs of being served (consisting of the cost of time spent in the queue and being served, and of a paid toll if collected).

Clearly, the levying of tolls will increase the expected costs of being served, and so will result in a lower threshold (of course, this is done in "steps", as n is a discrete function of the costs).

Naor investigates the threshold strategy chosen "naturally" by customers without interference, and compares it to the Socially optimal threshold strategy (one which maximizes the total additive utilities of the all population). He further compares the socially optimal toll, with the toll chosen by a revenue-maximizer.

Naor finds the "natural"-threshold to be too high in comparison with the socially optimal one, which leads to the conclusion that levying a toll (which later may be redistributed) can be socially worthwhile. The range of the socially optimal toll is directly defined by the socially optimal threshold.

When a revenue-maximizer sets the toll, Naor finds that he tends to charge higher tolls than the socially optimal ones, resulting in an even lower threshold strategy than the socially optimal one.

In his concluding remarks, Naor emphasizes that his results are independent of the specifics of the model. This point is later criticized by Edelson & Hildebrand, as will be further discussed shortly. We mention now that by "specifics" Naor relates to the service-time distribution, possible heterogeneity of customers in their evaluation of R, and so forth, but does not relate to the possibility of changes in the basic model's structure (changing assumption 8 and changing of the toll's structure), or to changes in the individual and collective utility functions (assumptions 5 & 6).

1.1.2 Edelson and Hildebrand

Edelson and Hildebrand [5] further investigate the relationship between Pareto (Socially) optimal and revenue-maximizing tolls. They criticize Naor's conclusion that the revenue-maximizing toll is higher than the socially optimal one, and construct three models in which this conclusion does not hold.

They begin by giving Naor's R (individuals' benefit from service) a new interpretation, as the alternative cost of being served in a different facility, which in turn, is constructed by a constant toll - τ, and the cost of time spent there, where the expected time at the alternative facility is γ, which is assumed independent of the customer flow.

This interpretation, which may seem very appealing, actually makes some very fundamental changes in certain perspectives of analyzing the model. It does not influence the analyzation of the two first models presented, but it makes all the difference for the third one, as will be further discussed along with that model.

In the first model, the customers are not allowed to balk (Naor's 8^{th} assumption). This means, that when a need for service arises, a customer decides whether he wants to approach the analyzed facility or the alternative one, based on statistical information alone. The customer is not allowed to see the queue-size before making this decision. Once deciding, this decision is irrevocable. (The interpretation given here to Naor's R helps justify this decision-structure, but is not actually needed for this model. One could alternatively assume that the decision has to be made in advance before reaching the facility and being able to observe the queue-length).

The strategy to be followed by the customers in such a model, cannot be of the threshold type, and so the stream of customers is partitioned between joining the queue, and joining the alternative facility, in fractions $\lambda(\theta)$ and $[1-\lambda(\theta)]$ respectively (where θ is the toll collected) that ensure the expected benefits from joining the queue and joining the alternative facility are the same.

Following such a model, Edelson and Hildebrand find that the socially optimal and the revenue-maximizing tolls coincide.

The second model presented, allows the customers to balk (and adopt a threshold strategy), actually adopting all of Naor's assumptions. However, they allow the toll collector to impose a two-part tariff, selling rights to all the potential population to service with a specific toll if service is rendered.

Following this model, Edelson and Hildebrand show that again, the socially optimal and the revenue-maximizing tolls coincide.

This model, although very similar to Naor's, needs two assumptions to hold: the population is assumed finite, and more than that, it is assumed that the all potential population can be approached in advance to collect the pre-paid toll without further costs. These assumptions might be considered by some as a meaningful change of the model and it's underling assumptions, even though not positively expressed in Naor's paper. Still, minimal restructuring of this model will yield a model with the same result, in which these assumptions are not required.

The third model presented, allows balking and introduces heterogeneity among customers in the time-costs. Specifically, it divides the population (the stream of customers) into fractions with different evaluation for time spent – each fraction is homogenous. Edelson and Hildebrand give a simple example with two customer types. They find that in such a model, depending on the different parameters, the revenue-maximizing toll may be either higher, lower or the same as the socially optimal one. Their model enables them to explore some possibilities of price-discrimination.

It is in this model, that the alternative interpretation to Naor's R has it's effect, and it is important to emphasize the difference it makes, as it may raise the question whether the comparison between this model and Naor's results is significant.

When we increase the value of time for one type of customers under the alternative interpretation, we actually increase the value of R for these customers along with the value of time. Since the time spent in the other facility is supposedly fixed - γ, we increase a supposedly-fixed parameter, so that we order groups of customers in such a way that groups with a bigger evaluation for time, automatically have a bigger appreciation for the service. Still, their result shows that when costs and evaluation for service change among customers in a certain way, revenue-maximizing tolls may be lower than the socially optimal ones.

As opposed to Naor's interpretation of an objective appreciation of a given service (be it constant or random), we have an appreciation, which depends on the customers' evaluation of time. This may seem "economically right" at first, as it gives a feeling of competition. However, this other "facility" isn't actually analyzed under tools relevant to competition – we would expect the same rules to apply to a competitor, namely, dependence of the spent time in the system on the actual stream of customers to that system, and the ability to change the toll collected.

Further more, assuming that this model indeed represents a kind of degenerated competition (where the toll and time in the alternative system are assumed fixed, at least in the short future), a competitive market of one kind or another is a "whole different ball game" than a situation in which a Monopolist sets tolls to maximize his revenue as is the case in Naor's model (where naturally, one might expect higher tolls).

One last point on this alternative interpretation refers to the evaluation of the collective social utility. When dealing with two alternative facilities giving the service, one might wish to consider the utility achieved at both the facilities (is the transfer of one customer from one facility to the other results in socially losing all his utility ?).

Even if we do not consider the last model presented by Edelson and Hildebrand, there still remains the open question, what causes the basic difference between their results (in the first two models) and Naor's. Indeed, Naor's results show that the socially optimal toll cannot exceed the revenue-maximizing toll, while Edelson and Hildebrand show that small structural differences can impose equivalence of the tolls. One wonders if there is a consistency in this change, and if it can be expected under some rule-of-thumb. We discuss this point later on after presenting our model.

1.1.3 Other models

Levhari and Luski [8] and **Luski** [12] deal with the same model of competition between servers in a queueing model: two identical servers with exponential service rate, and a joint poisson arrival process. Customer's time value C is distributed by a continuous distribution function F(C), The value for service R is identical for all customers. The two servers compete with fixed prices they set for the service, and the customers decide which server to join, maximizing their expected utilities. The customers cannot observe the queues before making their decision.

The customers' reaction-functions to the prices, namely, the distribution of customers between the servers, are analytically described. However, the general solution for the servers' profit-maximization problem cannot be reached analytically, and the results are reached numerically.

They reach the following results:

- The equilibrium prices may either be equal or differ. The socially-optimal prices are never reached.
- In case of a price-differentiation, the profits differ as well.
- Usually, equilibrium competitive-prices will be above monopoly prices and below socially-optimal prices.

Loch [11] considers a competition between two identical M/G/1 servers with unobservable queues. Customers are inhomogeneous with respect to their evaluation of the service – an aggregate decreasing demand function is applied.

He concludes that there has to be a symmetric equilibrium in which both servers set the same price and serve the same rates of arriving customers.

Loch also reaches the conclusion that the competitive result for the total arrival rate is bigger than the result for a monopolist and smaller than the socially optimal result.

Chen and Frank [1] investigate the pricing-strategy of a monopolist that faces a potential stream of customers, and serves them via a queue which they cannot observe.

The basic model is very similar to the one we present in this work (following Edelson and Hildebrand's no-balking model), and indeed they reach similar conclusions for the case of a monopolist. They also show the significance of the assumption of linear preferences: they investigate a model where the preferences of the customers are not necessarily linear and show that the conclusions of the basic model do not hold.

An interesting observation is that of the long-run maximization problem: Chen and Frank present a model where in the long run, the monopolist can choose his service rate. Under the assumption that the costs of maintaining the service rate are linear (per unit of the intensity parameter), they show that a monopolist will either choose not to operate at all or will choose such a service rate so that he will serve all the potential customers. This observation, however, does not hold for the kind of competition we present in our model.

Li and Lee [9] construct the following model: two servers with different exponential service (different parameters - μ_i); Identical customers with time value C; Joint arrival Process; Customers observe the queues continuously, and can move from one queue to the (end of the) other; The case of indifference is resolved by a randomization with equal probabilities. The servers compete by choosing fixed prices to be charged from customers that receive (complete) the service. The customers only pay the server in which they actually complete being served.

The servers' strategies are supposed to be such that if, for an empty system, the general price (combined of the fixed price and the expected time cost) at one of the servers is lower than the general price at the other (so that an arriving customer will prefer to join the first), this difference in the general price can be expressed as an integer multiple of the first server's service-time cost (C/μ_1). This integer, m, will serve the first customers as a threshold strategy, namely, if there are up to (m) customers in the system, they will all prefer waiting in the queue of the first server. The (m+1)th customer will be indifferent between the servers and randomize. The choices of the next customers arriving, are not important for Li & Lee as they consider the fact that in any state of the system where there are more than (m+1) customers in the system, both servers will be busy. Hence, they consider the system as one-dimensional.

Maximizing the servers' revenues, they arrive at the result that m=0, namely, under equilibrium a customer who arrives at an empty system is indifferent between the two servers. The faster server determines a higher price and still gets a higher share of the market.

Stidham [16], on investigating a single-server queueing system, uses an interpretation of demand and service curves to the customer's problem of choosing an actual arriving rate. He uses the term "demand"-curve for the total benefit to the customers for any given arrival rate, and the term "supply"-curve for the total cost they will endure (including the costs of spending time in the system) for any given arrival rate given the (predetermined) price for service.

He also uses the graphical interpretation to check for the convergence and stability of the reached equilibrium using "cobweb" diagrams.

We later use these techniques while analyzing our model to gain further insight to the solutions, and check for its stability.

1.2 Goals

Our goal may be generally described as reaching a better understanding of the kinds of equilibria that may be reached under different structures of congestion models, and further our ability to analyze (and make rational decisions in) real-life applications.

This is of course, quite a general goal, and we have no pretensions of giving a complete all-around manual of what will happen in any such model. Rather, we focus our attention on a specific model, and aim at two main goals, which can generally be described as achieving explicit analytic ready-to-use results and gaining some rules-of-thumb.

1.2.1 ready-to-use

As mentioned in the previous section, a lot of work has been done in the field of analyzing the kind of equilibria that may be reached under different models of queueing systems. Most of the models developed produce important qualitative results and intuitive thumb-rules for the application and expected results of such systems.

However, most of the models do not produce specific analytic results. Even the most robust results available, while holding under different assumptions are, a lot of times, difficult for use in relevant applications.

The ability to produce specific analytic "ready-for-use" results, even under very specific assumptions, may be considered an important characteristic of a model. This is especially true in models dealing with Nash-equilibrium. The concept of such a model is based on the assumption that all the decision-makers in the system may be able to understand it and find their rationally-optimal behavior. The harder it is to find this optimal behavior in specific situations, when one needs to use complicated computer-programs in order to reach specific results, the harder it is to accept the applicability of such a model.

The main model that we present involves competition between service suppliers in a queueing system. The competition examined is not of the type usually considered in previous works – between two or more servers, but rather between two separate entrepreneurs that divide the service between them. Specifically, the service is separated into a parking service, and the usual service. This interpretation helps us analyze different situations, and develop a different price structure: one which consists of a constant, along with a cost differentiated by time spent in the system.

We reach results of analytic form. Specifically, we give results for the behavior chosen by the customers and by each of the profit maximizing entrepreneurs. These results are compared to the socially-optimal results, and to the results expected when the all system belongs to one monopolistic entity.

1.2.2 rules-of-thumb

However important one might consider specific analytic results, we want to achieve some more general conclusions about the behavior of such systems.

We seek rules-of-thumb that can help us predict in which model a monopolist will produce the socially optimal result and in which he will divert from this policy ; predict what will happen to the prices (and the congestion in the system) when we add competition to the system.

Our results help us develop some intuitive rules-of thumb that may be applied to other models of congestion. To further our understanding, we analyze a few variations to our main model, so we can get a feeling og what will happen when small changes in the model take place.

These rules-of-thumb we develop, help us, for example, in the comparison of the papers of Naor [13] and of Edelson & Hildebrand [3]. These papers seem to produce a contradiction, although based on very similar assumptions. Naor reaches the conclusion that a monopolist will always charge more than the socially optimal price. Edelson & Hildebrand show that the monopolist's price may also be equal to or lower than the socially optimal price. Using the intuition developed, we can easily understand why this happens.

Part 2 – The Model

2.1 General outline

Consider a given service, which is given in a "facility" in which customers have to line up in queue and be served in their turn. The queue's properties (the distributions of queueing times, service times, queue's length, idle time, etc.) are determined by its basic parameters which may vary widely (arrival and service rates, service-discipline, number of servers and places in queue, etc.).

Generally, Our model may be considered an extension to the models first analyzed by Naor and by Edelson & Hildebrand, in which the question is raised of an optimal price to be charged from each of the customers who receive the service.

We focus our discussion on a profit-maximizing **service-supplier** and **parking-supplier** as follows: when a customer arrives to receive service he needs to use a "parking lot" for the duration of his stay in the system. The "parking" service is considered as a separate service run by a separate entity. The customer has to pay both for the parking and for the service itself.

According to the "rules-of-behavior" of the customers, and the other parameters involved, the service-supplier and parking-supplier act in their pricing-policy to maximize their profits.

For comparison, we also check the optimal pricing-policy of a monopolist (owning both the service and the parking facilities), and of a social planner (maximizing the collective social utility).

2.2 Specific structure of the main Model and definitions

We adopt Naor's assumptions 1 through 7 (see 1.1.1) as they are. We change the 8^{th} assumption in the spirit of Edelson & Hildebrand and state the alternative assumption as follows:

8. At the time a customer's need for service arises, he does not know the queue size, but he is well informed about its statistical distribution, on which he is

basing his decision whether to join the queue or not. The decision to join or balk is irrevocable.

We are dealing with a queueing system of the type M/M/1 with infinite population, infinite queueing positions and a First-Come-First-Served discipline.

Before we proceed, we give some definitions to the symbols that will be used throughout this paper (2.2.1) and give some thought to the justification and problematic character of some of the assumptions (2.2.2).

2.2.1 parameter & symbol definitions

$\Re$: Rational numbers.

U: Expected utility for a single customer.

Λ: The basic potential Poisson rate of arrival of customers to the system. This parameter is considered fixed and exogenous to the decisions of the participants. We consider Λ as the rate in which the need for the service arises among the customer population (which we consider infinite).

λ: The actual rate of arrival to the system. This parameter is the decision variable of the customers as follows: when the need for service arises for service a customer employs a mixed strategy to the decision whether to join the system or not. The probability every customer gives to joining the system in equilibrium is λ/Λ. This interpretation to the customers' strategy allows us to consider the actual arrival rate λ as Poisson (This is due to the nature of the Poisson process).

μ: The service facility's intensity parameter of the exponentially distributed service times (assumption 2)

R: The reward (in monetary units) endowed to the customer on service completion (assumption 3)

C: The monetary cost of time to a customer per time unit (assumption 4) w: The expected time in the system (including queueing and service times) for a customer.

π_S : The profit of the Service-supplier. Constitutes the objective function of the revenue-maximizing service supplier.

π_N : The profit of the parking-supplier. Constitutes the objective function of the revenue-maximizing parking supplier. We use N here rather than P to avoid a mix up between the parking supplier and different Probabilities; N is borrowed from queueing literature as it usually refers to treatment of the queue (as opposed to treatment of the service itself – S).

π_M : The profit of a Monopolist. Constitutes the objective function of a revenue-maximizing monopolist who owns both the service and the parking facilities.

π_W : The collective social Welfare. Constitutes the objective function of a social planner.

P_S, P_S^M, P_S^W : The price/toll collected for the Service. Serves as the decision variable of the revenue-maximizing service facility owner / revenue-maximizing monopolist / social (Welfare) planner, respectively.

P_N, P_N^M, P_N^W : The price/toll collected for the parking. Serves as the decision variable of the revenue-maximizing parking facility owner / revenue-maximizing monopolist / social (Welfare) planner, respectively. In this model we assume that this toll is collected per time unit spent in the facility (queueing and being served).

2.2.2 Definition of objective functions

We specify here the objectives of the different participants in this model. The Leading principal in later finding and checking for Nash-equilibria is that each of the participants acts to maximize is objective function, given the strategies he expects the other participants to follow.

For each of the customers we have:

$U=R-P_S-P_N \cdot w-C \cdot w$

As we expect the customers to generate an aggregate arrival rate (following together the same mixed strategy) we write:

$U(\lambda)=R-P_S-P_N \cdot w(\lambda)-C \cdot w(\lambda)$

For the revenue-maximizing service supplier, we have:

$\pi_S(P_S) = \lambda \cdot P_S$

For the revenue-maximizing parking supplier, we have:

$\pi_N(P_N) = \lambda \cdot P_N \cdot w(\lambda)$

For a monopolist, we have:

$\pi_M(P_S,P_N) = \lambda \cdot P_S + \lambda \cdot P_N \cdot w(\lambda)$

And for the social planner, we have:

$\pi_W(P_S,P_N) = \lambda \cdot (R-C \cdot w(\lambda))$

2.2.3 Focus on some assumptions

The following assumptions are somewhat "side"-assumptions, as they are not the ones that give the real-life picture the model represents but rather they serve as mathematical simplifications for our convenience. However these assumption are not to be taken as trivial – their adequacy to real life situations is far from obvious, and a change in these assumptions may change the results considerably.

Still, as we seek analytic results that will help us understand what is going on, we stick to them, while pointing out the issue here.

Assumption 5 - This assumption of linear preferences might be considered problematic when applying the model to realistic situations. Chen and Frank [1] show that when this assumption does not hold, the comparative results change.

To justify the use of this assumption, we point out that while considering customers for whom the relative expenses are of small consequence, we can relax the assumption to risk neutral in the relevant range, which will be quite realistic.

Another problem with this assumption, is the principal that customers maximize their expected utilities. This widely used assumption requires that each of the customers is aware of his own utility function and is able to analyze the system properly in order to derive his utility-maximizing behavior. We can only suggest that in a case where a Nash-equilibrium exists and is applied in effect, a stable system will justify itself in the sense that even if we don't expect the customers to be acquainted with the mathematics used here, they will statistically find out the relative information through some kind of a learning process.

Assumption 6 - is needed for the comparative calculation of the socially optimal solution. The utilities of the entrepreneurs are considered in the same way as the customers' utilities, and all are considered linearly additive for the purpose of computing the collective utility. This implies that the distribution of funds among the population is of no importance, a point that might be criticized.

We use this assumption in the context of our background of other existing models that regularly use this assumption in comparing the strategies of social planners to those of private entrepreneurs.

2.3 Main Results

The following use of the concept Nash-equilibria relates to sets of strategies by all the relevant "players" in our model, namely solutions of the kind: (P_S, P_N, λ).

Theorem 2.1: Under assumptions 1-8, in the case of separate revenue-maximizing service provider and parking provider, there always exists at least one Nash-equilibrium. There is a unique Nash-equilibrium when $\Lambda \geq \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}$, and a convex set of Nash-equilibria when $0 < \Lambda < \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}$.

Specifically, the strategies and results are as follows:

$$\lambda = Min \quad \left\{ \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}, \Lambda \right\}$$

$$(P_N, P_S) = \begin{cases} \left(\sqrt[3]{\mu \cdot C^2 \cdot R} - C, R - \sqrt[3]{\frac{C \cdot R^2}{\mu}} \right) & \Lambda \geq \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \\ \left\{ \left(x, R - \frac{x + C}{\mu - \Lambda} \right) \middle| x \in \left[\frac{C \cdot \mu}{\mu - \Lambda} - C, \frac{(\mu - \Lambda)^2 \cdot R}{\mu} - C \right] \right\} & 0 < \Lambda < \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \end{cases}$$

$$(\pi_N, \pi_S) = \begin{cases} \left(\left(\sqrt[6]{C \cdot R^2 \cdot \mu^2} - \sqrt{C} \right)^2, \left(\sqrt{R \cdot \mu} - \sqrt[6]{C^2 \cdot R \cdot \mu} \right)^2 \right) & \Lambda \geq \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \\ \left\{ \left(x, \Lambda \cdot R - \frac{\Lambda \cdot C}{\mu - \Lambda} - x \right) \middle| x \in \left[\frac{C \cdot \Lambda^2}{(\mu - \Lambda)^2}, \frac{(\mu - \Lambda) \cdot \Lambda \cdot R}{\mu} - \frac{C \cdot \Lambda}{\mu - \Lambda} \right] \right\} & 0 < \Lambda < \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \end{cases}$$

$$w = Min \quad \left\{ \sqrt[3]{\frac{R}{C \cdot \mu^2}}, \frac{1}{\mu - \Lambda} \right\}$$

Theorem 2.2: When the solution is not limited by Λ (i.e. $\Lambda \geq \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}$), the service provider is better off than the parking provider: $\pi_S \geq \pi_N$.

When the solution is limited by Λ ($\Lambda < \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}$), the relation between the revenues of the service-provider and the parking-provider depends on the specific values of the parameters, and the parking-provider's profit may be greater equal or less than that of the service-provider.

Theorem 2.3: Under assumptions 1-8, in the case of a revenue maximizing monopolist that owns both service and parking facilities, there always exist Nash-equilibria. The result for the customers' strategy is unique.

Specifically, the results are:

$$\lambda_M = Min \quad \left\{ \mu - \sqrt{\frac{C \cdot \mu}{R}}, \Lambda \right\}$$

$$\left(P_N^M, P_S^M\right) = \begin{cases} \left\{ \left(x, R - \sqrt{\frac{R \cdot (x+C)^2}{\mu \cdot C}} \right) \middle| x \in \Re \right\} & \Lambda \geq \mu - \sqrt{\frac{C \cdot \mu}{R}} \\ \left\{ \left(x, R - \frac{x+C}{\mu - \Lambda} \right) \middle| x \in \Re \right\} & 0 < \Lambda < \mu - \sqrt{\frac{C \cdot \mu}{R}} \end{cases}$$

(results limited to positive prices are given in sec. 2.4, see 2.21a and 2.22a)

$$\pi_M = \begin{cases} \left(\sqrt{\mu \cdot R} - \sqrt{C}\right)^2 & \Lambda \geq \mu - \sqrt{\frac{C \cdot \mu}{R}} \\ \Lambda \cdot R - \frac{\Lambda \cdot C}{\mu - \Lambda} & 0 < \Lambda < \mu - \sqrt{\frac{C \cdot \mu}{R}} \end{cases}$$

$$w = Min \quad \left\{ \sqrt{\frac{R}{C \cdot \mu}}, \frac{1}{\mu - \Lambda} \right\}$$

Theorem 2.4: Under assumptions 1-8, the Nash-equilibrium in the case of a monopolist is socially optimal. Specifically, the possible sets of prices chosen by the monopolist are either the same chosen by a social planner or serve as an upper bound of a convex set of sets of prices that may be reached by a social planner. The set of results for the prices when $\Lambda < \mu - \sqrt{\frac{C \cdot \mu}{R}}$ is explicitly given in equation 2.23.

Theorem 2.5: The result (for λ) reached in the case of a competition between revenue maximizing service and parking providers is lower than the socially optimal one. The set of prices set in equilibrium is socially too high.

Theorem 2.6: The prices are monotone decreasing as a function of the potential rate of arrival (Λ).

Theorem 2.7: In the unbounded case:

For low values of C (up to $C = \frac{8}{27}\mu R$), the parking provider's price is an increasing function of C, and for higher values of C, it is a decreasing function of C.

For low values of C (up to $C = \frac{1}{27}\mu R$), the parking provider's profit is an increasing function of C, and for higher values of C, it is a decreasing function of C.

2.4 Mathematical Proofs

Consider a single customer's utility from receiving the service:

$$U = R - P_S - P_N \cdot w - C \cdot w$$

From the no-balking rule (assumption 8) we know that the actual time in the system is not relevant to the customer's decision whether to join the queue. Furthermore, we know that the customers are risk-neutral and therefore we can consider w as the expected time spent in the system by a single customer, and U the expected utility gained by joining.

In case the above expression is negative, the customer will choose to give up his need for service and avoid joining the system.

In case the above expression is positive, the customer will choose to join the queue, incur the costs and get the reward.

We know from basic queueing theory that w is influenced from the actual rate of arrival to the system:

$$w(\lambda) = \frac{1}{\mu - \lambda}$$

This means, that if the expected utility is positive and all potential customers will decide to join the queue, the costs may increase as to turn the expected utility to negative.

This means that as long as the potential stream of customers is big enough (we shortly discuss what that means), we can expect the expected utility of each customer to be zero, where he will be indifferent between joining the queue or giving up service.

In these circumstances, we can expect a mixed strategy to be taken by the customers such that a probability is chosen by which a customer actually joins the system when the need arises.

We get an actual rate of arrival to the system: λ, which is a Poisson rate of arrival due to the use of probability on the potential rate of arrival Λ.

We keep in mind that $0 \le \lambda \le \Lambda$. We also assume $\lambda<\mu$ - otherwise, there will be an endless queue and no point coming to receive service. This enables us to use the existing simple results of queueing-theory. We will assume for now that Λ is "big enough" so that we will get an internal solution to the problem (unbounded solution). This could be either defined as Λ or we can settle for $\Lambda \ge \mu$. Later, we show what happens when this does not hold.

2.4.1 Unbounded Solution (Non-limiting Λ) - Competition

Under these definitions (U=0), we can expect the following:

$$R = P_S + P_N \cdot w(\lambda) + C \cdot w(\lambda)$$

Or:

$$R = P_S + (P_N + C) \cdot \tfrac{1}{\mu - \lambda}$$

Hence, given a set of prices, we can expect the customers to act so that we get an actual rate of arrival:

$$\boxed{\lambda = \mu - \frac{P_N + C}{R - P_S}} \qquad (2.1)$$

We now turn our attention to the service and parking providers.

We have:

$\pi_S = \lambda \cdot P_S$

$\pi_N = \lambda \cdot P_N \cdot w(\lambda) = \frac{\lambda \cdot P_N}{\mu - \lambda}$

Using our available result for λ, we get:

$$\boxed{\pi_S = P_S \cdot \mu - P_S \cdot \frac{P_N + C}{R - P_S}} \qquad (2.2)$$

and:

$$\boxed{\pi_N = \frac{P_N \cdot \mu \cdot R - P_N \cdot \mu \cdot P_S - P_N^2 - P_N \cdot C}{P_N + C}} \qquad (2.3)$$

Each of the suppliers acts to maximize his profit, using the price he controls, given the behavior of his counterpart.

We have:

$$\frac{\partial \pi_S}{\partial P_S} = \mu - \frac{(P_N + C) \cdot (R - P_S) + P_S \cdot (P_N + C)}{(R - P_S)^2} \qquad \boxed{\frac{\partial \pi_S}{\partial P_S} = \mu - \frac{(P_N + C) \cdot R}{(R - P_S)^2}} \qquad (2.4)$$

And to check that we will get a Max-point:

$\frac{\partial^2 \pi_S}{\partial^2 P_S} = -\frac{2(P_N + C) \cdot R}{(R - P_S)^3}$ We need $R > P_S$, which is a trivial demand. We can expect unreasonable outcomes if this will not be the case. We will check later to verify that our outcome will, in fact, satisfy this.

For the parking-supplier, we have:

$$\frac{\partial \pi_N}{\partial P_N} = \frac{(\mu \cdot R - \mu \cdot P_S - 2P_N - C)\cdot(P_N + C) - (P_N \cdot \mu \cdot R - P_N \cdot \mu \cdot P_S - P_N^2 - P_N \cdot C)}{(P_N + C)^2},$$ which can be reduced to:

$$\boxed{\frac{\partial \pi_N}{\partial P_N} = \frac{C \cdot \mu \cdot (R - P_S)}{(P_N + C)^2} - 1} \qquad (2.5)$$

And again: $\frac{\partial^2 \pi_N}{\partial^2 P_N} = -\frac{2C \cdot \mu \cdot (R - P_S)}{(P_N + C)^3}$, we need $R > P_S$.

We are now ready to try and find equilibrium points. We do this by Equating equations 2.4 and 2.5 to zero.

We first extract the reaction functions of the service provider and parking provider from equating 2.4 and 2.5 respectively to zero, for later use:

$$\boxed{P_S(P_N) = R - \sqrt{\frac{(P_N + C)\cdot R}{\mu}}} \qquad (2.4a)$$

$$\boxed{P_N(P_S) = \sqrt{C \cdot \mu \cdot (R - P_S)} - C} \qquad (2.5a)$$

From 2.4, we can easily extract: $$P_N + C = \frac{\mu (R - P_S)^2}{R} \qquad (2.4b)$$

We use this in 2.5, to get: $$\frac{C \cdot R^2}{\mu \cdot (R - P_S)^3} - 1 = 0$$

Which means: $$(R - P_S)^3 = \frac{C \cdot R^2}{\mu}$$

So we get: $$\boxed{P_S = R - \sqrt[3]{\frac{C \cdot R^2}{\mu}}} \qquad (2.6)$$

We have a single solution, which reassures us that $R > P_S$, so that we get maximization for both suppliers.

Using 2.6 in 2.4b, we get: $$P_N + C = \sqrt[3]{\frac{\mu^3 \cdot C^2 \cdot R^4}{R^3 \cdot \mu^2}}$$

So that: $$\boxed{P_N = \sqrt[3]{\mu \cdot C^2 \cdot R} - C} \qquad (2.7)$$

Using the results in equations 2.1,2.2,2.3 we get:

$$\lambda = \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \qquad (2.8)$$

$$w = \sqrt[3]{\frac{R}{C \cdot \mu^2}} \qquad (2.9)$$

$$\pi_S = \left(\sqrt{R \cdot \mu} - \sqrt[6]{R \cdot C^2 \cdot \mu} \right)^2 \qquad (2.10)$$

$$\pi_N = \left(\sqrt[6]{C \cdot R^2 \cdot \mu^2} - \sqrt{C} \right)^2 \qquad (2.11)$$

2.4.1.1 Theorem 2.3

We want to show that $\pi_S \geq \pi_N$ - (2.10) ≥ (2.11)

We begin with assumption 7: $R\mu \geq C$.

$$\sqrt[3]{R\mu} \geq \sqrt[3]{C}$$

$$\sqrt[6]{R^2\mu^2}\left(\sqrt[6]{R\mu} - \sqrt[6]{C}\right) \geq \sqrt[6]{C^2}\left(\sqrt[6]{R\mu} - \sqrt[6]{C}\right)$$

$$\sqrt{R\mu} - \sqrt[6]{R^2\mu^2 C} \geq \sqrt[6]{R\mu C^2} - \sqrt{C}$$

$$\left(\sqrt{R\mu} - \sqrt[6]{R\mu C^2}\right)^2 \geq \left(\sqrt[6]{R^2\mu^2 C} - \sqrt{C}\right)^2$$

And we're done.

2.4.1.2 Theorem 2.7

Here, we find the first derivatives of the parking provider's price and profit to C, and show that they are positive for certain low values of C (and then negative).

The derivative of P_N is: $\frac{\partial P_N}{\partial C} = \frac{2}{3} \cdot \sqrt[3]{\frac{\mu R}{C}} - 1$.

Comparing to zero, we can see that it is positive when $\boxed{C < \frac{8}{27} \cdot \mu R}$, and it is negative when $C > \frac{8}{27} \cdot \mu R$.

The derivative of π_N is: $\frac{\partial \pi_N}{\partial C} = 2\left(\sqrt[6]{CR^2\mu^2} - \sqrt{C}\right) \cdot \left(\frac{1}{6}\sqrt[6]{\frac{R^2\mu^2}{C^5}} - \frac{1}{2}\sqrt{\frac{1}{C}}\right)$.

Under assumption 7 ($\mu R \geq C$), we know that $\sqrt[6]{CR^2\mu^2} - \sqrt{C} > 0$, so we only have to compare $\frac{1}{6}\sqrt[6]{\frac{R^2\mu^2}{C^5}} - \frac{1}{2}\sqrt{\frac{1}{C}}$ to zero.

The comparison shows that the derivative is positive when $\boxed{C < \frac{1}{27} \cdot \mu R}$, and it is negative when $C > \frac{1}{27} \cdot \mu R$.

2.4.2 Bounded solution (Limiting Λ) - Competition

We now focus our attention on the case where Λ is actually smaller than the equilibrium-λ we found, meaning when it actually poses a limitation.

Clearly, adding a limitation of Λ smaller than the optimal λ will result in a situation where the actual rate of arrival will be Λ (every potential customer will wish to join the system as long as the optimal λ for them is at least Λ).

What about the two entrepreneurs ? Obviously, if they know that raising their prices won't lower the actual rate of arrival (which is Λ), they will raise them. They will raise the prices until the customers' optimal λ will be exactly Λ. They won't raise them above that level, because as we showed already, their derivatives to their own prices are negative beyond the optimal result, when there is no bound (unimodal profit functions).

Together with 2.8, we get the more general results for λ and w:

$$\boxed{\lambda = Min \quad \left\{\mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}, \Lambda\right\}} \qquad (2.12)$$

$$w = Min \left\{ \sqrt[3]{\frac{R}{C \cdot \mu^2}}, \frac{1}{\mu - \Lambda} \right\} \quad (2.13)$$

λ can be expressed as a function of the prices (from 2.1) as follows:

$$\lambda = Min \left\{ \mu - \frac{P_N + C}{R - P_S}, \Lambda \right\} \quad (2.14)$$

Since Λ is the maximal possible value for λ, we can extract from 2.14 the reaction functions for the extreme (bounded) case:

$$P_S^{\min} = R - \frac{P_N + C}{\mu - \Lambda} \quad (2.15)$$

or: $$P_N^{\min} = (\mu - \Lambda)(R - P_S) - C \quad (2.16)$$

We use the terms $P_S^{\min}, P_N^{\min}$ to express the lower-bound response-function strategies of the service- and parking- provider, respectively.

We note at this point that both lower-bound response-functions (2.15 and 2.16) are monotone decreasing functions of Λ. Since in the unbounded case, the response-functions are not influenced by Λ, this proves theorem 2.6.

For each of our two players we have a strategy which is a combination of the higher between our former strategy rule and our new rule of limitation.

That is: $P_S(P_N)$ = Max { (2.4a) , (2.15) }, and $P_N(P_S)$ = Max { (2.5a) , (2.16) }:

$$P_S(P_N) = Max\left\{ R - \sqrt{\frac{(P_N + C) \cdot R}{\mu}}, R - \frac{P_N + C}{\mu - \Lambda} \right\} \quad (2.17a)$$

$$P_N(P_S) = Max\left\{ \sqrt{C \cdot \mu \cdot (R - P_S)} - C, (\mu - \Lambda)(R - P_S) - C \right\} \quad (2.17b)$$

Whenever $\Lambda \geq \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}}$ (2.17c), holds, we have the non-limited results as before.

When 2.17c does not hold, we have a continuum of equilibria as follows:

From 2.17b we get the maximal value for P_S as the intersection point of the two values for P_N(the interpretation is that only the service-provider increases his price to meet the bounded demand, and the parking provider remains on his original response function – at the point where it crosses the lower-bound response function):

$$\sqrt{C\cdot\mu\cdot(R-P_S)}-C=(\mu-\Lambda)(R-P_S)-C$$

or: $$R-P_S=\frac{C\cdot\mu}{(\mu-\Lambda)^2}$$

which is associated with the minimal value for P_N (by 2.16): $P_N=\frac{C\cdot\mu}{\mu-\Lambda}-C$.

From 2.17a we similarly get the maximal value for P_N:

$$R-\sqrt{\frac{(P_N+C)\cdot R}{\mu}}=R-\frac{P_N+C}{\mu-\Lambda},$$

or: $$P_N=\frac{(\mu-\Lambda)^2\cdot R}{\mu}-C$$

so we have:

$$(P_N,P_S)=\begin{cases}\left(\sqrt[3]{\mu\cdot C^2\cdot R}-C,\,R-\sqrt[3]{\frac{C\cdot R^2}{\mu}}\right) & \Lambda\geq\mu-\sqrt[3]{\frac{C\cdot\mu^2}{R}}\\ \left\{\left(x,R-\frac{x+C}{\mu-\Lambda}\right)\middle|\,x\in\left[\frac{C\cdot\mu}{\mu-\Lambda}-C,\frac{(\mu-\Lambda)^2\cdot R}{\mu}-C\right]\right\} & 0<\Lambda<\mu-\sqrt[3]{\frac{C\cdot\mu^2}{R}}\end{cases} \quad (2.18)$$

We check for the solution of the equilibrium with the customers in the bounded case:

$$\lambda=\mu-\frac{P_N+C}{R-P_S}=\mu-\frac{x+C}{R-\left(R-\frac{x+C}{\mu-\Lambda}\right)}=\mu-(\mu-\Lambda)=\Lambda$$

Looking at the structure of $\pi_S(P_S)$ and $\pi_N(P_N)$ (see 2.2-2.5), we see that they are both unimodal and concave. It is trivial to conclude that when we impose an upper bound on the prices' range, which is lower than the optimal unbounded choice, the bound will constitute the optimal choice, so that each of the points defined in 2.18 will be a Nash-equilibrium.

In order to derive the results for the profits in the bounded case, we remember that in the bounded case:

$$\pi_S = \Lambda \cdot P_S = \Lambda R - \Lambda \cdot \frac{P_N + C}{\mu - \Lambda}$$

$$\pi_N = \Lambda \cdot P_N \cdot \frac{1}{\mu - \Lambda} \qquad \text{and equivalently:} \qquad P_N = \frac{\pi_N \cdot (\mu - \Lambda)}{\Lambda}$$

so that we have: $\boxed{\pi_S = \Lambda R - \frac{\Lambda C}{\mu - \Lambda} - \pi_N}$, and notice that this result emphasizes the fact that the customers' utility is zero, as the entrepreneurs' utilities complement each other to the collective social utility.

To find the range of results of this kind, we substitute P_N using our limitations of its range from 2.18 in π_N and get:

$$\pi_N^{\min} = \frac{C\Lambda^2}{(\mu - \Lambda)^2}$$

$$\pi_N^{\max} = \frac{(\mu - \Lambda) \cdot \Lambda \cdot R}{\mu} - \frac{C\Lambda}{\mu - \Lambda}$$

$$\boxed{(\pi_N, \pi_S) = \begin{cases} \left(\left(\sqrt[6]{C \cdot R^2 \cdot \mu^2} - \sqrt{C} \right)^2, \left(\sqrt{R \cdot \mu} - \sqrt[6]{C^2 \cdot R \cdot \mu} \right)^2 \right) & \Lambda \geq \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \\ \left\{ \left(x, \Lambda \cdot R - \frac{\Lambda \cdot C}{\mu - \Lambda} - x \right) \middle| x \in \left[\frac{C \cdot \Lambda^2}{(\mu - \Lambda)^2}, \frac{(\mu - \Lambda) \cdot \Lambda \cdot R}{\mu} - \frac{C \cdot \Lambda}{\mu - \Lambda} \right] \right\} & 0 < \Lambda < \mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \end{cases}}$$

(2.19)

This completes the proof of theorem 2.1

2.4.3 The monopolist and the social planner

The objective functions of the monopolist and of the social planner are quite straight forward, and as was defined in 2.2.2:

$\pi_M(P_S, P_N) = \lambda \cdot P_S + \lambda \cdot P_N \cdot w(\lambda)$

$\pi_W(P_S, P_N) = \lambda \cdot (R - C \cdot w(\lambda))$

Case I - unbounded

We begin by intuitively proving theorem 2.4 (an equilibrium reached by the monopolist is socially optimal).

To do this, we show that in this model, for the case of non-limiting Λ the objectives coincide (when Λ is limiting the solution this does not necessarily hold, and we will discuss this later).

We showed that the customers' expected utility will be zero in equilibrium. We recall that the expected utility for each customer is:

$U=R-P_S-P_N\cdot w-C\cdot w$, and the total utility is: $\lambda\cdot(R-P_S-P_N\cdot w-C\cdot w)$.

This total utility is exactly equal to the difference between the objective functions of the social planner and of the monopolist: $\pi_W(P_S,P_N)-\pi_M(P_S,P_N)$. Since this value is zero, the results coincide, and they will reach the same solution.

This is not a formal proof, but consider that in order to reach this outcome, they both have to achieve a specific λ, which in turn is based on a specific set of prices.

To make this proof more formal, we substitute λ in both objective functions (the social planner's and monopolist's), using our available result 2.1. This shows that the objectives totally coincide in our model for the unbounded case:

$$\pi_M(P_S,P_N)=\pi_W(P_S,P_N)=\frac{(\mu\cdot R-\mu\cdot P_S-P_N-C)(C\cdot P_S+R\cdot P_N)}{(P_N+C)(R-P_S)}$$

The relation between the prices can be derived by equating the first derivatives of the above expression to zero. We use here a simpler way (based on theorem 2.4) which produces the same results:

We know that the social planner (and by theorem 2.4 this applies to the monopolist as well), wants to maximize:

$$\pi_W=\lambda\cdot R-\frac{\lambda\cdot C}{\mu-\lambda}$$

$$\frac{\partial\pi}{\partial\lambda} = R - \frac{C\cdot\mu}{(\mu-\lambda)^2} = 0$$

which leads us to the optimal result for the social planner and the monopolist:

$$\boxed{\lambda_M = \lambda_W = \mu - \sqrt{\frac{C\cdot\mu}{R}}}, \qquad (2.20)$$

where λ_M,λ_W represent the optimal solution for λ, in the eyes of the monopolist and social planner respectively.

Of course, from the customers' point of view, 2.1 still holds, so that:

$$\mu - \frac{P_N + C}{R - P_S} = \mu - \sqrt{\frac{C\cdot\mu}{R}}$$

and we have:

$$\boxed{P_S^M = R - \sqrt{\frac{R\cdot\left(P_N^M + C\right)^2}{\mu\cdot C}}} \qquad (2.21)$$

Basically, any result of this type will be true mathematically (and therefore, this is the result given in sec. 2.3), but a more reasonable result will be one with no negative prices. This easily translates to:

$$\boxed{0 \le P_N^M \le \sqrt{\mu\cdot R\cdot C} - C} \qquad (2.21a)$$

We can calculate the profit to be: $\boxed{\pi_M = \left(\sqrt{\mu\cdot R} - \sqrt{C}\right)^2}$ (2.20a)

Case II – Bounded solution ($\Lambda \ge \mu - \sqrt{\frac{C\cdot\mu}{R}}$)

Now consider the case where $\Lambda \ge \mu - \sqrt{\frac{C\cdot\mu}{R}}$.

Clearly (according to similar reasoning to that given in the case of competition), the optimal solution for both the social planner and the monopolist is $\lambda=\Lambda$.

However, in this case, there can be price policies, which are optimal for the social planner but not for the monopolist.

The monopolist will raise his prices as long as the rate of arrival does not drop below Λ, and make sure the utility for the customers is zero.

The social planner is indifferent about the distribution of the utility between the facility and the customers, so that the optimal prices for the monopolist serve as an upper bound for his optimal prices (this completes the proof of theorem 2.4).

Setting the customers' utilities to zero, the monopolist will make sure the customers' optimal λ is Λ:

$$\mu - \frac{P_N + C}{R - P_S} = \Lambda$$

which turns to:

$$\boxed{P_S^M = R - \frac{P_N^M + C}{\mu - \Lambda}} \qquad (2.22)$$

and: $\boxed{0 \le P_N^M \le R \cdot (\mu - \Lambda) - C}$ for positive prices (2.22a)

The results of the objective functions are easily found by using any set of the results in the objective functions. The explicit arguments are given in section 2.3 (theorem 2.3).

This completes the proof of theorem 2.3.

As mentioned above, 2.22 is the upper bound for the prices set by the social planner:

$$\boxed{P_S^W \le R - \frac{P_N^M + C}{\mu - \Lambda}} \qquad (2.23)$$

and with positive prices:

$$\begin{cases} 0 \le P_N^W \le R \cdot (\mu - \Lambda) - C \\ 0 \le P_S^W \le R - \dfrac{P_N^W + C}{\mu - \Lambda} \end{cases} \qquad (2.23a)$$

To complete this section we need to show that (2.20) ≥ (2.8) in order to prove Theorem 2.5:

We use assumption 7 - $\mu R/C \ge 1$, than (rearranging):

$$\sqrt[6]{\frac{C^3 \cdot \mu^3}{R^3}} \quad \ge \quad \sqrt[6]{\frac{C^2 \cdot \mu^4}{R^2}}$$

and: $$\mu - \sqrt[3]{\frac{C \cdot \mu^2}{R}} \quad \ge \quad \mu - \sqrt{\frac{C \cdot \mu}{R}}$$

which is the desired result. Now, since λ is a decreasing function of both prices, this means that the set of prices in competition is socially too high.

2.5 Graphical Interpretation, Convergence and Stability

We now try to show these results graphically.

The mathematical proofs given above should be considered sufficient. However, we feel that a graphical interpretation might help with the intuition about what is going on, especially in the case where the rate of arrival is limited by the potential rate of arrival Λ. Another important issue we address in this section is the convergence to and the stability of the Nash-equilibria we found.

The importance of the convergence and stability outcomes is emphasized in light of the question we raised about the adequacy of assumption 5. We expect the customers to know what is best for them – the stability/convergence outcome is important for the notion that this will be done through a learning process.

Stidham [16] relates to "demand" and "supply" curves in analyzing the dynamics in his model and the resulting equilibrium. Our model is different from that of Stidham, and so, we refer to curves of benefit and expenditure for the customers, rather than pure demand and supply.

The following is (at least at some parts) parallel to the mathematical proofs given in the previous section. We try to avoid giving the same proofs again, and instead refer whenever we can to the available results from section 2.4.

2.5.1 Customers

We begin with the market itself. Our curves are not those of demand and supply suggested by classic economics. We use the function $D(\lambda)$ (which may be considered the parallel to Stidham's demand) for the net **benefit** to the customers, subtracting their costs of time (which is basically what they will be willing to pay, and $S(\lambda)$ (which may be considered the parallel to the supply function) for the **expenditure** (which is what they will be asked to pay). These are not the standard demand\supply curves, because we relate here to a predetermined set of prices, and the changes in the "supply curves" here, are due to changes in the rate of arrival (demand) and not due to marginal costs. We draw the curves as functions of λ, and expect an equilibrium in the intersection of the curves, because as long the benefit exceeds the expenditure, each of the customers will benefit from increasing his chosen probability for joining the queue, thus increasing λ. Similarly, we will draw the same curves multiplied by λ: this will represent **total benefit** $DD(\lambda)$ vs. **total expenditure** $SS(\lambda)$ with the same intersection.

At this point, we regard the set of prices as externally given (we will consider how they are set shortly).

We know that with a rate of arrival - λ, each arriving customer will get

$\boxed{D(\lambda) = R - \frac{C}{\mu - \lambda}}$ from the service, (2.24a)

and pay for it: $\boxed{S(\lambda) = P_S + \frac{P_N}{\mu - \lambda}}$. (2.25a)

Alternatively, we have: $\boxed{DD(\lambda) = \lambda \cdot R - \frac{\lambda \cdot C}{\mu - \lambda}}$ (2.24b) and: $$SS(\lambda) = \Pi(\lambda) = \pi_S(\lambda) + \pi_N(\lambda) = \lambda \cdot P_S + \frac{\lambda \cdot P_N}{\mu - \lambda} \qquad (2.25b)$$

$SS(\lambda)$ represents the total revenues for both entrepreneurs together, while $DD(\lambda)$ equals the collective utility in our model.

Our case is different from that of a monopolist in the sense that we have two companies in competition that choose their price independently, forming the total price to be paid for the service. Moreover, the price-structure is different here, as can be easily observed from 2.25a (a predetermined price in regular models of supply and demand will be represented by a straight horizontal line – S=P).

We draw 2.24a and 2.25a in figure 2.1a. One can easily find $D(\lambda)$ to be strictly decreasing and concave with slope $-\frac{C}{(\mu-\lambda)^2}$, while $S(\lambda)$ is strictly increasing and convex with slope $\frac{P_N}{(\mu-\lambda)^2}$.

In order to have a solution we need S(0) not to exceed D(0). This means:

$$R - \frac{C}{\mu} \geq P_S + \frac{P_N}{\mu} \qquad (2.26)$$

We check this condition for the optimal prices we found earlier: using our results 2.6 and 2.7, we get:

$$R - \frac{C}{\mu} \geq R - \sqrt[3]{\frac{C \cdot R^2}{\mu}} + \frac{\sqrt[3]{\mu \cdot C^2 R} - C}{\mu}$$ which reduces to:

$\frac{R \cdot \mu}{C} \geq 1$, which is our basic requirement in assumption 7.

It is interesting to observe figure 2.1b. $DD(\lambda)$ here represents the total benefit of the system which correlates with the usual interpretation of social welfare (assumption 6).

The slope of the demand curve is $DD'(\lambda) = R - \frac{C \cdot \mu}{(\mu-\lambda)^2}$. That means $DD(\lambda)$ is strictly concave ($0 \leq \lambda < \mu$), and it has a peak when:

$$R = \frac{C \cdot \mu}{(\mu - \lambda)^2}$$

$$\boxed{\lambda_W = \mu - \sqrt{\frac{C \cdot \mu}{R}}}$$

(where λ_W is the optimal solution for λ in the eyes of the social planner). This is the result we got in 2.20.

The value at this (socially optimal) peak equals: $DD(\lambda_W) = \left(\mu - \sqrt{\frac{C \cdot \mu}{R}}\right)\left(R - \frac{C}{\sqrt{C \cdot \mu / R}}\right)$, or

alternatively: $DD(\lambda_W) = \left(\sqrt{\mu \cdot R} - \sqrt{C}\right)^2$ (as in 2.20a)

Since we do not consider marginal costs here, it is obvious that a single monopolist would have chosen $\lambda_M = \lambda_W$ (λ_M is the optimal λ for the monopolist) , thus gaining a "social optimization" and collecting the maximum possible.

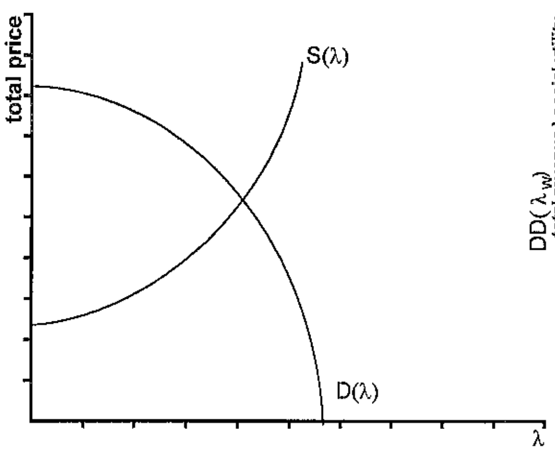

figure 2.1a

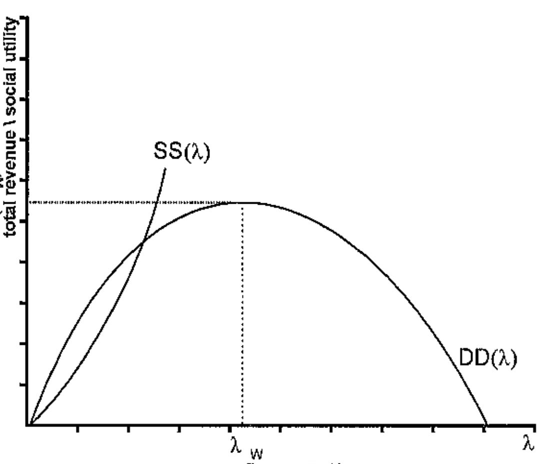

figure 2.1b

We have an intersection of DD(λ) with the x-axis (2.24b=0) either when λ=0 (trivial solution) or when $\lambda = \mu - \frac{C}{R}$.

The slope of the expenditure curve is: $SS'(\lambda) = P_S + \frac{P_N \cdot \mu}{(\mu - \lambda)^2}$. That means SS($\lambda$) is strictly increasing and convex in $[0,\mu)$.

Both curves begin from the origin, so we will have a solution if DD'(0)≥SS'(0) which is parallel to condition 2.26.

It is easy to see that the intersection occurs at 2.1, and for the competitive-equilibrium set of prices: 2.8

We have already showed in the previous section that the intersection point (2.8) is smaller than the peak of DD(λ) (2.20):

Since SS(λ) increases with the chosen prices, this result means that our set of prices is too high from a social point of view.

2.5.2 Entrepreneurs

Now, let's consider the decision of the service and parking suppliers about the set of prices. We observe their behavior in the prices-plane, where we have P_S on the vertical axis and P_N on the horizontal axis.

Each of the two "players" has an optimal price for each "state of nature"-composed of the customers' known preferences and the specific behavior of the other "player". These strategies can be drawn in this plane as the functions $P_S(P_N)$ for the service-supplier's strategy, and $P_N(P_S)$ for the parking-supplier's strategy (which we will define on P_N as $P_N^{-1}(P_N)$. See figure 2.2.

We already have these functions from the previous section:

For the service supplier:

(2.4a): $$\boxed{P_S(P_N) = R - \sqrt{\frac{(P_N + C)\cdot R}{\mu}}}$$

with a slope: $-\frac{1}{2}\sqrt{\frac{R}{\mu(P_N + C)}}$, so we have a strictly decreasing and convex function.

For P_N=0 we get $P_S = R - \sqrt{\frac{C \cdot R}{\mu}}$.

For P_S=0 we get $P_N = \mu \cdot R - C$.

For the parking supplier:

(2.5a): $\boxed{P_N(P_S)=\sqrt{C\cdot\mu\cdot(R-P_S)}-C}$

which turns to:

$$P_S=\boxed{P_N^{-1}(P_N)=R-\frac{(P_N+C)^2}{C\cdot\mu}} \qquad (2.28)$$

with a slope: $-2\frac{P_N+C}{C\cdot\mu}$, so we have a strictly decreasing and concave function.

For P_N=0 we get $P_S=R-\frac{1}{C\cdot\mu}$.

For P_S=0 we get $P_N=\sqrt{R\cdot C\cdot\mu}-C$.

Using assumption 7 again, we can compare the intersection points with the horizontal and vertical axis of both functions to find that $P_S(P_N)$ begins below $P_N^{-1}(P_N)$ and ends above, so we have one intersection point between them (and hence, the one solution we found).

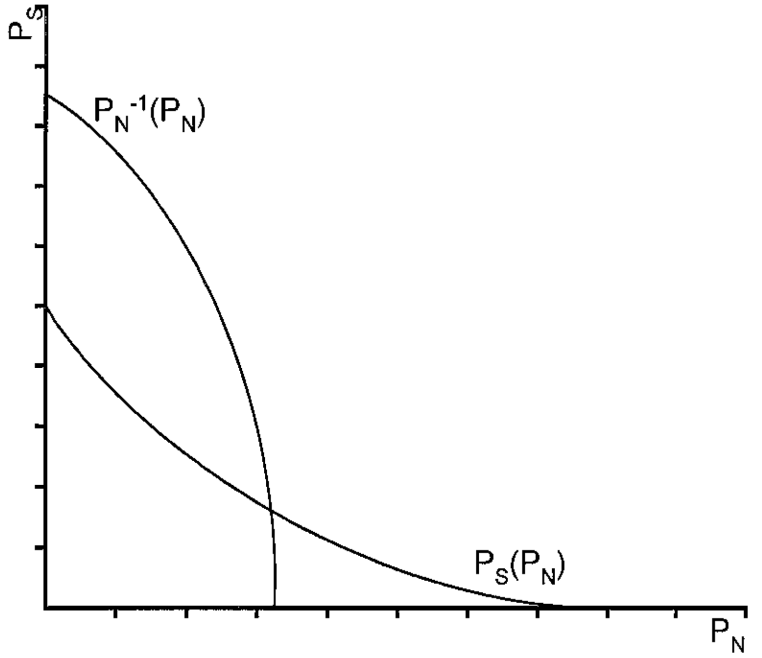

figure 2.2

2.5.3 Bounded case

We now focus our attention on the case where Λ is smaller than the equilibrium-λ we found, meaning when it actually poses a limitation.

First consider figures 2.1b and 2.3: adding a limitation of Λ to the left of our optimal λ will result in a situation where the actual rate of arrival will be Λ, and for the "optimal" set of prices, the expenditure curve will be lower than the benefit curve at the point. Since the benefit curve is given and unchangeable, we expect the entrepreneurs (either one, or both) to raise the prices, so that the expenditure curve will rise, until the curves intersect at $\lambda=\Lambda$.

In case of a limitation imposed by Λ, the prices in equilibrium will be higher than those achieved without a limitation.

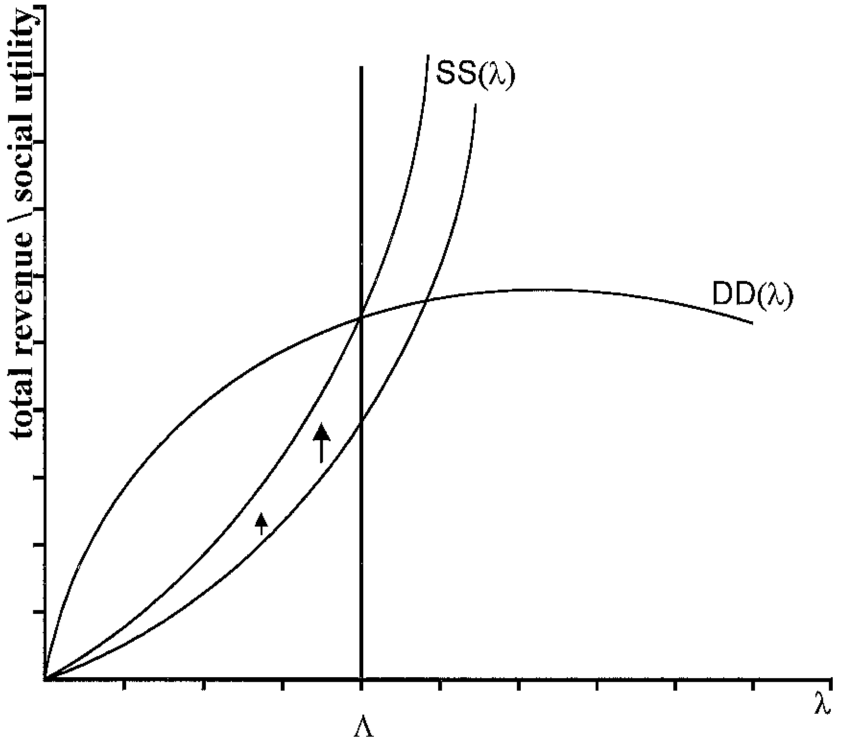

figure 2.3

With this intuition in mind we turn to figure 2.2.

Think of a given set of prices for which an equilibrium is reached in figure 2.3 at $\lambda=\Lambda$. As we have already seen, lowering one of the prices (without changing the other) will result in a bigger λ, which is "impossible" with our limitation. If we draw this set of prices as a point in figure 2.2, we will expect the equilibrium not to be below or to the left of that point. This is true for any set of prices for which the intersection in figure 2.3 will be at $\lambda=\Lambda$.

The set of points in the P_S-P_N plane satisfying this rule generate a straight line which is defined by the limitation rule on the result of figure 2.3 (for λ):

$$\boxed{Upper\ Bound(\lambda) = \mu - \frac{P_N + C}{R - P_S} = \Lambda}$$

Equivalently, we have the line:

$$\text{(2.15):} \qquad \boxed{Lower\ Bound(P_S) = R - \frac{P_N + C}{\mu - \Lambda}}$$

Notice that in case $\Lambda \geq \mu$, there will be no points of this kind in figure 2.2.

We can easily add this line to figure 2.2 (we do it in figure 2.4), which is strictly decreasing with slope: $-\frac{1}{\mu-\Lambda}$ beginning at (0, $R-\frac{C}{\mu-\Lambda}$) and ending at ($R(\mu-\lambda)-C$, 0).

Each of the two entrepreneurs will now choose a strategy, which is a combination of the higher between his former strategy (reaction curve) and the lower bound.

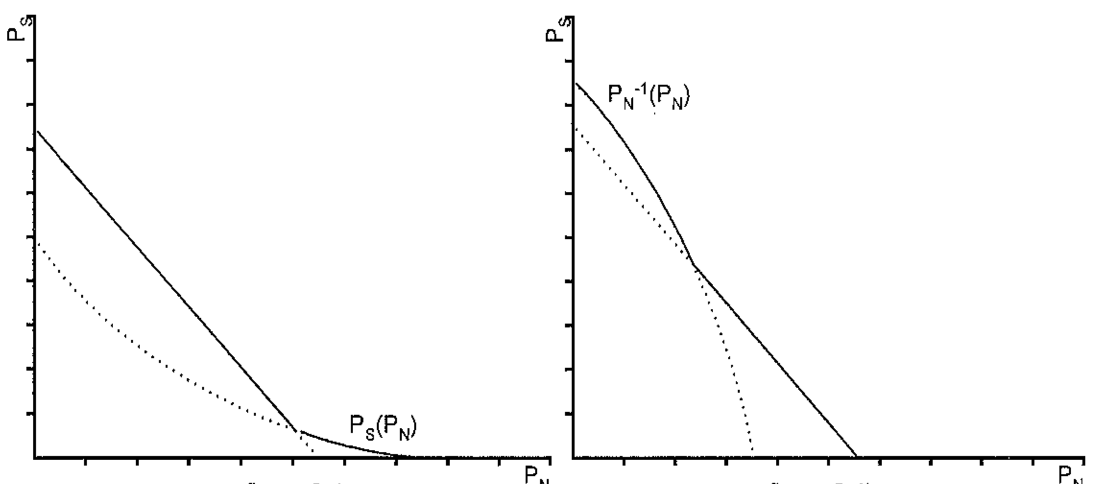

figure 2.4a figure 2.4b

Figure 2.4 is separated to the resulting reaction curves of the service-provider (figure 2.4a), and the parking-provider (figure 2.4b). In figure 2.4c, the combination of the two integrated curves is shown, to point out the range of possible Nash-equilibria.

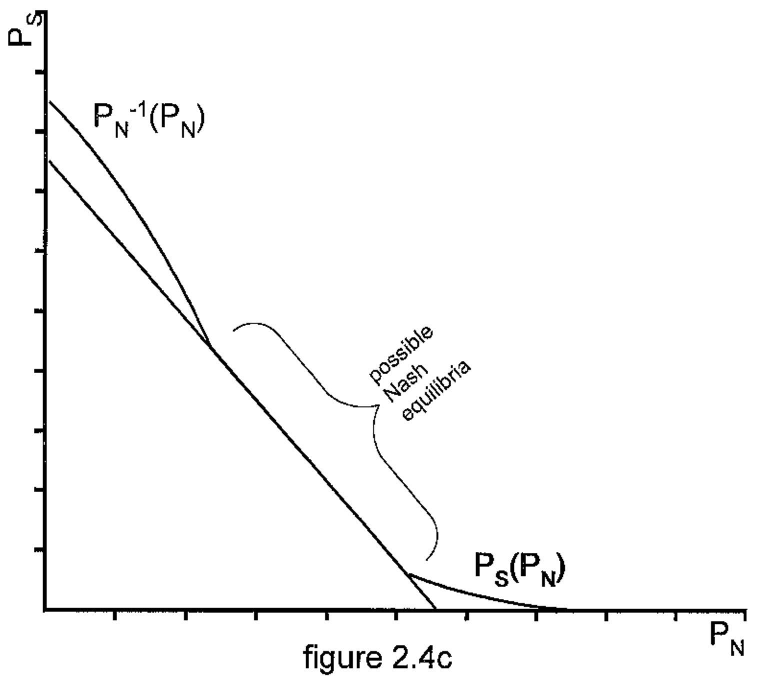

figure 2.4c

2.5.4 Convergence \ Stability

Now that we understand the graphical system we try to see how it reaches the designated equilibrium \ equilibria, and whether it stays in equilibrium.

First, consider figure 2.1b. Unfortunately, the benefit and expenditure functions, are both functions of λ, so there is no point engaging a method of "cobweb" diagrams here.

Still, one can see from the diagram, that whenever the market will be to the left of the convergence point, there will be an incentive for the customers to increase their rate of arrival, as they will encounter on average positive net utilities. Whenever the market will be to the right of the convergence point, there will be an incentive for the customers to decrease their rate of arrival, since they will encounter on average negative net utilities.

It is true that this fact was used to construct the customers' strategy. Hopefully, the graphical interpretation helps the intuitive justification of this strategy.

Now, consider the figure 2.2. Remember that the service provider's reaction function is a function on P_N, and the parking supplier's reaction function is a function on P_S.

In addition, based only on assumption 7 ($\mu R \geq C$), we found that that $P_S(P_N)$ begins below $P_N^{-1}(P_N)$ and ends above.

This allows us to easily follow where the system will converge to from any starting point. We draw a segmented line to show the dynamics of the reactions of the two entrepreneurs to each other from any starting state, until equilibrium is reached.

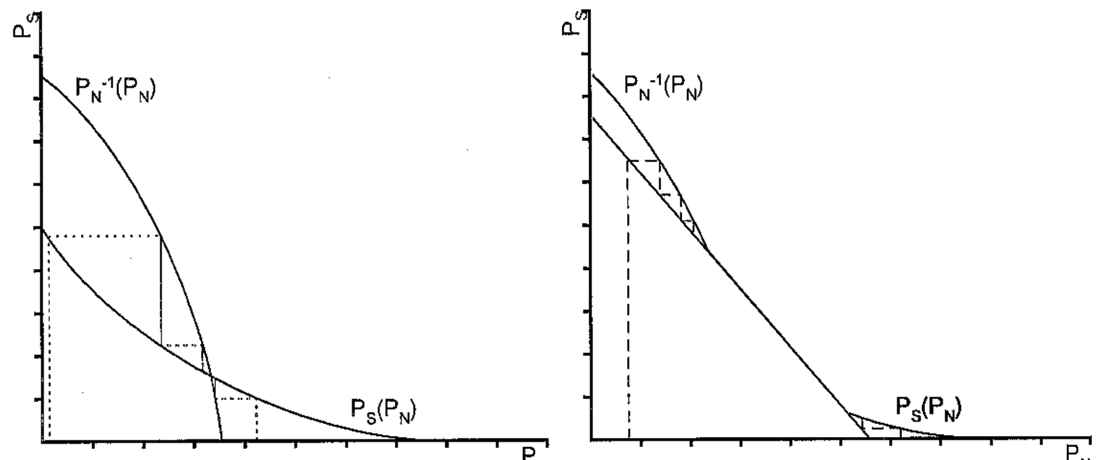

figure 2.5a figure 2.5b

It can be seen in figures 2.5a and 2.5b that the system will always converge to equilibrium, with no extra conditions needed. To show this we use here "cobweb" diagrams to follow the process of convergence (following stidham).

Part 3 – Variations to the Model

In this part, we present some variations on the basic model.

Each variation is aimed to clarify a certain point, to strengthen our general grasp of the model and sensitivity to different possibilities, and to raise further questions to be explored in the future. A part of the models produce results which could be reached intuitively – we use this chance to verify the intuition.

We do not fully explore the variations in the way we did with the basic model, usually just checking the unbounded case (which is the case usually explored in related models). We try to avoid giving the explicit proofs whenever we feel they are unnecessary since they resemble the ones given for the basic model.

We generally just give the results, and discuss them later in section 4.

3.1 Marginal Costs

We add marginal costs to the basic model, to see how the basic results change, and to make sure we know how these costs influence the outcomes (for example, how the marginal cost of the parking supplier affects the service supplier's profits) so we don't have to rely on intuition alone.

We assume:

A marginal cost γ_S per served customer for the service-supplier.

A time-based marginal cost γ_N per customer per time unit for the parking-supplier.

We skip the proof, which exactly follows the proof of the basic model, and give the results:

$$\boxed{P_S = R - \sqrt[3]{\frac{(C+\gamma_N)(R-\gamma_S)^2}{\mu}}} \qquad (3.1)$$

$$\boxed{P_N = \sqrt[3]{\mu(C+\gamma_N)^2(R-\gamma_S)} - C} \qquad (3.2)$$

$$\lambda = \mu - \sqrt[3]{\frac{\mu^2 (C+\gamma_N)}{R-\gamma_S}} \quad (3.3)$$

$$\pi_S = \left(\sqrt{\mu (R-\gamma_S)} - \sqrt[6]{\mu (C+\gamma_N)^2 (R-\gamma_S)} \right)^2 \quad (3.4)$$

$$\pi_N = \left(\sqrt[6]{(C+\gamma_N)(R-\gamma_S)\mu^2} - \sqrt{(C+\gamma_N)} \right)^2 \quad (3.5)$$

Also, similar arguments to those given for the basic model show that theorem 2.4 holds here as well:

$\pi_M(P_S,P_N) = \lambda \cdot (P_S + P_N \cdot w(\lambda) - \gamma_S - \gamma_N \cdot w(\lambda))$

$\pi_W(P_S,P_N) = \lambda \cdot (R-C \cdot w(\lambda) - \gamma_S - \gamma_N \cdot w(\lambda))$

$\pi_W - \pi_M = \lambda \cdot (R - C \cdot w(\lambda) - P_S - P_N \cdot w(\lambda)) = \lambda \cdot U = 0$

So that the objectives of the monopolist and the social planner coincide.

3.2 Fixed Parking Price

We now change the model just slightly, by using a different price-structure, in which both the service-supplier and parking-supplier charge a fixed sum (each still choosing his price independently).

We want to find out whether the basic results hold under this price-structure as well, and we compare the result for the parking provider. Since the arguments are similar to those used in the proof of the basic model (but still, with some differences), we'll run through them.

The single customer's utility from receiving the service is now:

$U=R-P_S-P_N-C \cdot w$

Instead of (2.1) we have:

$$\lambda=\mu-\frac{C}{R-P_S-P_N} \quad (3.6)$$

The Service- and parking- suppliers now have the same structure for their revenues\profits:

$\pi_i = \lambda \cdot P_i$ where $i \in (S,N)$

Using our available result for λ, we get:

$$\pi_i = P_i \cdot \mu - \frac{C \cdot P_i}{R - P_i - P_j} \qquad i \neq j \;\; ; \;\; i,j \in (S,N) \tag{3.7}$$

To maximize the profits, we have:

$$\frac{\partial \pi_i}{\partial P_i} = \mu - \frac{C \cdot R - C \cdot P_j}{(R - P_i - P_j)^2} \tag{3.8}$$

And to check that we will get a Max-point:

$\frac{\partial^2 \pi_i}{\partial^2 P_i} = (C \cdot R - C \cdot P_j) \cdot 2(R - P_i - P_j)^{-3} \cdot (-1)$ which is obviously negative as long as $R > P_S + P_N$, which is a trivial demand.

From (3.8) we get (equating to zero): $$P_i = R - P_j - \sqrt{\frac{C \cdot (R - P_j)}{\mu}} \tag{3.9}$$

This symmetry leads us to conclude that in equilibrium, both prices are the same:

$$P_S = R - P_N - \sqrt{\frac{C \cdot (R - P_N)}{\mu}} \quad \Rightarrow \quad P_N = R - P_S - \sqrt{\frac{C \cdot (R - P_N)}{\mu}}$$

And we also have: $$P_N = R - P_S - \sqrt{\frac{C \cdot (R - P_S)}{\mu}}$$

So we get $$P = P_S = P_N \tag{3.10}$$

We can now try to solve for P:

$$P = R - P - \sqrt{\frac{C(R - P)}{\mu}}$$

$$(R-2P)^2 = \frac{C(R-P)}{\mu}$$

$$4\mu \cdot P^2 + (C-4\mu R)\cdot P + (\mu R^2 - CR) = 0$$

Solving, we get:

$$P = \frac{4\mu R - C \pm \sqrt{C^2 + 8\mu CR}}{8\mu}$$

On account that P should be smaller than R/2, we can ignore the (+) possibility immediately, so:

$$\boxed{P = \frac{4\mu R - C - \sqrt{C^2 + 8\mu CR}}{8\mu}} \qquad (3.11)$$

Now, let's consider a monopolist, collecting one fixed sum from every customer who joins the queue and receives the service. This fixed sum covers both the parking and the service. We'll denote it by P_M.

Given the price set by the monopolist, the customers will choose λ such that:

$$R = P_M + \frac{C}{\mu - \lambda} \qquad \Rightarrow \qquad \lambda = \mu - \frac{C}{R - P_M}$$

$$\pi_M = \lambda \cdot P_M = P_M \cdot \mu - \frac{P_M \cdot C}{R - P_M}$$

$$\frac{d\pi_M}{dP_M} = \mu - \frac{C(R-P_M) + P_M \cdot C}{(R-P_M)^2} = 0$$

$$\mu \cdot P_M^2 - 2\mu R \cdot P_M + \mu R^2 - CR = 0$$

Solving, we get:

$$P_M = R \pm \sqrt{\frac{CR}{\mu}}$$

Again, a result higher than R is not relevant, so we have:

$$\boxed{P_M = R - \sqrt{\frac{CR}{\mu}}} \qquad (3.12)$$

First we note, that the monopolist still takes all the customers' utility, and reaches as before, the socially optimal result.

We want to show that Theorem 2.5 still holds. We show that the price chosen by the monopolist and social planner is lower than the set of prices chosen by two separate entrepreneurs:

<u>Monopolist-Price (P_M)</u>		<u>Competition-Price (2P)</u>
$R-\sqrt{\frac{CR}{\mu}}$	$\leq$	$R-\frac{C}{4\mu}-\sqrt{\frac{C^2+8\mu RC}{16\mu^2}}$
$\frac{C}{4\mu}+\sqrt{\frac{C^2+8\mu RC}{16\mu^2}}$	$\leq$	$\sqrt{\frac{CR}{\mu}}$
$\frac{\sqrt{C}+\sqrt{C+8\mu R}}{4\mu}$	$\leq$	$\frac{4\sqrt{R}}{4\sqrt{\mu}}$
$\frac{\sqrt{C}+\sqrt{C+8\mu R}}{4\mu}$	$\leq$	$\frac{\sqrt{\mu R}+3\sqrt{\mu R}}{4\mu}$
$\frac{\sqrt{C}+\sqrt{C+8\mu R}}{4\mu}$	$\leq$	$\frac{\sqrt{\mu R}+\sqrt{\mu R+8\mu R}}{4\mu}$

Clearly, The competition set of prices is higher (as $\mu R \geq C$) than that of the monopolist and the social planner (and the rate of arrival will be smaller).

We also note that since the price structure is of no consequence for the social planner, the optimal λ and the socially optimal collective utility (which equals the monopolist's maximal revenue) are the same as before.

Using the result we got for the set of prices (3.11) in (3.6) and in our definition of each of the firms' profits (3.7), we get:

$$\boxed{\lambda = \mu - \frac{4\mu C}{C+\sqrt{C^2+8\mu R\cdot C}}} \qquad (3.13)$$

$$\pi_i = \frac{2\mu \cdot R + C}{4} - \frac{12\mu \cdot R \cdot C}{4C + 4\sqrt{C^2 + 8\mu \cdot R \cdot C}} \qquad i \in (S, N) \tag{3.14}$$

For the monopolist \ social planner, the parallel results are as in theorem 2.3.

Theorem 3.1: If the parking supplier can choose a pricing system (between a fixed price and a time-based price), he will prefer the fixed price system, which is also socially preferred (the social utility is higher).

Proof:

(Following, we will use superscript writing to distinguish the equilibrium results for the *modular* parking-price system and the *fixed* parking-price system. For example: λ^{modular})

We begin by proving that the value of λ under the fixed pricing system is bigger than its value under the modular (time based) pricing.

Consider the function: $f(x) = x^2 - 2x + 1$

It is easy to check that this function is always greater or equal (only when x=1) to zero: $x^2 \geq 2x - 1$

We now replace x with $\sqrt[3]{\frac{\mu R}{C}}$ and get:

$$\sqrt[3]{\left(\frac{\mu R}{C}\right)^2} \geq 2 \cdot \sqrt[3]{\frac{\mu R}{C}} - 1 \qquad \cdot 8 \cdot \sqrt[3]{\frac{\mu R}{C}} + 1$$

$$\frac{8\mu R}{C} + 1 \geq 16 \cdot \sqrt[3]{\left(\frac{\mu R}{C}\right)^2} - 8 \cdot \sqrt[3]{\frac{\mu R}{C}} + 1$$

$$\frac{8\mu R \cdot C + C^2}{C^2} \geq \left(4 \cdot \sqrt[3]{\frac{\mu R}{C}} - 1\right)^2$$

$$\frac{\sqrt{8\mu R \cdot C + C^2}}{C} \geq 4 \cdot \sqrt[3]{\frac{\mu R}{C}} - 1$$

$$\sqrt{8\mu R\cdot C+C^2}\geq 4\mu C\cdot\sqrt[3]{\frac{R}{\mu^2 C}}-C$$

$$-4\mu C\cdot\sqrt[3]{\frac{R}{\mu^2 C}}\geq-\left(\sqrt{8\mu R\cdot C+C^2}+C\right)$$

$$-\frac{4\mu C}{C+\sqrt{C^2+8\mu R\cdot C}}\geq-\sqrt[3]{\frac{\mu^2 C}{R}}$$

$$\mu-\frac{4\mu C}{C+\sqrt{C^2+8\mu R\cdot C}}\geq\mu-\sqrt[3]{\frac{\mu^2 C}{R}}$$

so that: $\lambda^{\text{fixed}}\geq\lambda^{\text{modular}}$.

Note that the equality only holds when $R= C/\mu$.

We've already seen that the social utility function is concave, and that under competition (as in this model) the result for λ is under the socially desired λ. This is true both when the parking pricing is fixed and when it is modular.

This means that between the two options, the one with the higher result for λ, is socially preferred, or in other words: the social utility gained under fixed pricing is higher than the social utility gained under modular time-based pricing.

Now consider the profits of the two entrepreneurs. Since They divide between themselves all of the social utility (the customers get zero utility), we just proved that the total profit ($\pi_S+\pi_N$) is bigger under fixed parking pricing.

We also know that under the fixed pricing system we got a symmetrical equilibrium ($\pi_S=\pi_N$), while under the modular pricing system, the service supplier was better off than the parking supplier ($\pi_S\geq\pi_N$).

Obviously: $\pi_N^{Fixed}\geq\pi_N^{Modular}$, which completes our proof.

3.3 Another Service provider

An alternative service provider decides to join the market. The parking service for his customers will be provided by the same parking-supplier, which cannot distinguish between the customers of the two facilities (which seems a reasonable assumption).

In equilibrium, the customers will choose λ_1 – the Poisson arrival rate of customers to be served by S_1: these customers pay the parking provider $P_N \cdot w_1(\lambda_1)$ and pay the first service provider P_S^1.

Similarly, they will choose λ_2 - the Poisson arrival rate of customers to be served by S_2: these customers pay the parking provider $P_N \cdot w_2(\lambda_2)$ and pay the second service provider P_S^2.

Service times at both facilities are exponentially distributed with parameter μ.

We can assume that the utility for the customers served at both facilities is zero.

Solving this kind of system is very similar to what was done in the basic model. The results are the same as well. For the parking provider, the result is as in (2.7), and for each of the service providers, the result is as in (2.6).

This means that for the 2-servers case there is actually no difference, each acts with the parking provider as a monopolist, taking all the utility from the customers (together with the parking provider).

Each of the separate service providers sets the same price and gets the same profit as the single service provider does in the basic model. In fact, the two service providers could actually offer different services (and not be in competition between themselves at all) – as long as the customers of both use the same parking facility, and the potential demand is unbounded, these results hold.

The parking provider doesn't change his price, and doubles his profits.

Obviously, even if the characteristics of the two facilities differ, each of them can be expected to behave in the same way, given the price for the parking service (same reaction functions).

However, if the service facilities differ, the parking provider may have to change his strategy (price).

3.3.1 Non-similar servers

We now change just one feature - the two facilities differ in the distribution of their service times (different intensity parameters) - and check how the results change.

Service provider i serves with intensity parameter μ_i, and charges P_S^i for the service.

$$\boxed{\lambda_i = \mu_i - \frac{P_N + C}{R - P_S^i} \qquad i = 1,2} \tag{3.15}$$

$$\boxed{\pi_S^i = \lambda_i \cdot P_S^i = P_S^i \cdot \mu_i - P_S^i \frac{P_N + C}{R - P_S^i} \qquad i = 1,2} \tag{3.16}$$

$$\pi_N = \left(\frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} \right) \cdot P_N$$

$$\boxed{\pi_N = \frac{P_N \mu_1 (R - P_S^1) + P_N \mu_2 (R - P_S^2)}{P_N + C} - 2P_N} \tag{3.17}$$

Again, equating the first derivatives to zero, we get:

$$\boxed{P_N = \sqrt[3]{C^2 R \left(\sqrt{\mu_1} + \sqrt{\mu_2}\right)^2 / 4} - C} \tag{3.18}$$

$$\boxed{P_S^i = R - \sqrt[6]{\frac{C^2 R^4 \left(\sqrt{\mu_1} + \sqrt{\mu_2}\right)^2}{4\mu_i^3}} \qquad i = 1,2} \tag{3.19}$$

$$\boxed{\lambda_i = \mu_i - \sqrt[6]{\frac{\mu_i^3 C^2 \left(\sqrt{\mu_1} + \sqrt{\mu_2}\right)^2}{4R^2}} \qquad i = 1,2} \tag{3.20}$$

$$\boxed{w_i = \sqrt[6]{\frac{4R^2}{\mu_i^3 C^2 \left(\sqrt{\mu_1} + \sqrt{\mu_2}\right)^2}} \qquad i = 1,2} \tag{3.21}$$

And note that $\boxed{\frac{w_1}{w_2} = \sqrt{\frac{\mu_2}{\mu_1}}}$ (3.22), the ratio between the times spent in the two facilities equals the root of the ratio between their service times.

$$\pi_S^i = \left(\sqrt{R\mu_i} - \sqrt[6]{C^2 R \left(\sqrt{\mu_1} + \sqrt{\mu_2} \right)^2 / 4} \right)^2 \qquad i = 1,2 \tag{3.23}$$

$$\pi_N = \left(\sqrt[6]{R^2 C \left(\sqrt{\mu_1} + \sqrt{\mu_2} \right)^4 / 2} - \sqrt{2C} \right)^2 \tag{3.24}$$

We can see that the parking-provider will use a kind of rooted average method, when relating to the servers' service-rate, and it can be easily seen that when $\mu_1 = \mu_2$, this will result in the original outcomes (while the parking provider doubles his revenue).

3.4 Partial Intervention of the Social Planner

Consider the case when one of the two separate entities (the service provider and the parking provider) is public (maximizes social utility) while the other one is private (maximizes revenues).

We denote:

$\pi^S = \lambda \cdot R - \dfrac{\lambda \cdot C}{\mu - \lambda}$ the social (governmental) objective function,

$\pi_S^P = \lambda \cdot P_S$ a private service provider's objective function,

$\pi_N^P = \dfrac{\lambda \cdot P_N}{\mu - \lambda}$ a private parking provider's objective function.

We use our result (2.1) for λ, to get:

$$\pi^S = \mu R - \frac{P_N R + C \cdot R}{R - P_S} - \frac{\mu C \cdot R - \mu C \cdot P_S}{P_N + C} - C$$

We already have the formulas for the private entrepreneurs' objective functions and their derivatives from the basic model (2.2-2.5).

We find the derivatives of π^S to P_S and to P_N, so we can solve both cases later:

$$\frac{\partial \pi^S}{\partial P_S} = -\frac{P_N R + CR}{\left(R - P_S \right)^2} + \frac{\mu C}{P_N + C} \tag{3.25}$$

$$\frac{\partial^2 \pi^S}{\partial^2 P_S} = -2\frac{P_N R + CR}{(R-P_S)^3} < 0 \quad \Rightarrow \quad Max$$

$$\boxed{\frac{\partial \pi^S}{\partial P_N} = -\frac{R}{(R-P_S)} + \frac{\mu C(R-P_S)}{(P_N+C)^2}} \tag{3.26}$$

$$\frac{\partial^2 \pi^S}{\partial^2 P_N} = -2\frac{\mu C(R-P_S)}{(P_N+C)^3} < 0 \quad (R > P_S) \Rightarrow Max$$

Case I – Governmental service provider Vs. private parking provider:

We use (3.29)=0 and (2.5)=0 to get:

$$\boxed{P_N = \sqrt{R\cdot C\cdot \mu} - C} \tag{3.27}$$

$$\boxed{P_S = 0} \tag{3.28}$$

$$\boxed{\lambda = \mu - \sqrt{\frac{C\mu}{R}}}$$

Case II – Governmental parking provider Vs. private service provider:

We use (3.30)=0 and (2.4)=0 to get:

$$\boxed{P_N = 0} \tag{3.29}$$

$$\boxed{P_S = R - \sqrt{\frac{CR}{\mu}}} \tag{3.30}$$

$$\boxed{\lambda = \mu - \sqrt{\frac{C\mu}{R}}}$$

In both cases, we find that the government will choose to "give way" by setting zero-prices, and allow the private entrepreneur to act as a monopoly, achieving the socially optimal result (in the basic model we showed that a monopoly will achieve the socially desired result for λ (2.20), which is the same result we got here).

3.5 Two Part Tariff (Socially Optimizing Monopolist with Balking)

To complete our work, we present a model where the customers are able to balk (give up service) after observing the queue. Naor [15] (see 1.1.1) found that in such a model, the monopolist's price is socially too high (with a socially too small rate of arrival). Edelson and Hildebrand [5] (see 1.1.2) changed the model by allowing the owner of the facility to sell in advance rights for the service (with a predetermined price for the service if the customers later decide to be served) to the all population. They found that in their model, the monopolist chooses to socially optimize the system (by his set of prices).

We describe here a model, which is basically similar to the one of Edelson and Hildebrand, using our interpretation for the separation of the parking-facility from the service-facility.

There are no explicit analytic results for this model, and we find no incentive in engaging in numerical research here, since such results are available from Edelson and Hildebrand. Rather, we show that **the result they reach (the monopolist reaching socially optimal Nash-equilibrium) can be expected from analyzing the model**, in the same way we did with the previous models..

3.5.1 Model's Structure

There is a potential Poisson arrival rate of customers Λ per time-unit who need service.

Those who decide to arrive at the system have to pay a lump sum P_N to the parking-supplier.

Having paid the Parking-supplier, they can go to the service-facility to observe the queue. Observing the current size of the queue at the time of his arrival, a customer can decide whether he wants to join the queue, or balk.

If a customer decides to join the queue, he has to pay another lump sum: P_S to the service-supplier, and wait his turn. When he is served, he is endowed with a reward of R. There is a fixed cost per time-unit spent in the system for each of the customers: C.

Say a customer arrives at the service-facility and finds i customers in the facility.

His expected utility, whether he decides to join the queue, is:

$E[U]=R-P_N-P_S-C\cdot E[W(i+1)]$

Where $E[W(i+1)]=(i+1)/\mu$ is the expected time it takes (i+1) customers to finish service under a exponential distribution of service-times.

Any customer (of the original stream of Λ) has 3 options (which divide to two separate decisions):

1. He may decide not to join the system at all, thus gaining zero.
2. Having decided to arrive at the system and observing the size of the queue, he may decide to balk, thus gaining a negative utility: $U= -P_N$.
3. Having decided to arrive at the system and observing the size of the queue, he may decide to join the queue, thus gaining an expected utility of:

   $R-P_N-P_S-C\cdot (i+1)/\mu$.

3.5.2 How the Result is Reached

The second decision is an "on-line" decision where each customer, upon is arrival at the system decides whether to join or balk, according to the specific queue-length he encounters.

This decision is based on whether the utility of the third "option" is expected to be higher (where he will join) or lower (where he will balk) than the utility of the second "option".

Before arriving at the service facility itself (after paying for the parking), the customers have no idea whether they will actually decide to join the queue and be served or not. Rather they evaluate the expected utility of arriving at the facility in order to decide what to do. Obviously, if this expected utility is positive, all the customers will decide to arrive at the facility. If it is negative, no customer will arrive at the facility. This means that we can expect the customers to adopt a mixed strategy, similar to the one from our basic model.

An actual rate of arrival λ will be set collectively, which will result in a zero expected utility. Of course, the calculation of λ is totally different than the one in the basic model.

This conclusion may be considered sufficient to understand that a monopolist will reach the socially optimal result, but we analyze the system a bit further to show more clearly why this is so.

We analyze the 2nd decision, so we can find out what will be the expected utility of arriving at the system, given λ.

We know that an arriving customer will decide to join the queue if he finds the expected utility of the 3rd option to be higher than that of the 2nd. This is easily translated to: $R - P_S - C(i+1)/\mu \geq 0$.

Or: $$\boxed{i+1 \leq \frac{(R-P_S)\cdot\mu}{C}} \qquad (3.36)$$

(3.36) is the decision-rule for the 2nd decision.

Thus, we can define an integer K, which is the maximum number of customers that we can find in the service-facility (either waiting in queue or being serviced).

This integer, K, is the highest possible (i+1) that will satisfy (3.36):

$$\boxed{K = \left\lfloor \frac{(R-P_S)\cdot\mu}{C} \right\rfloor} \qquad \text{(the highest integer from below)} \qquad (3.36)$$

This is obviously a blocked queueing system, of the type: M/M/1/K, with

state-probabilities: $\Pr_i = \rho^i \cdot \Pr_0$; $\Pr_0 = \left[\sum_{i=0}^{K} \rho^i\right]^{-1}$ where $\rho = (\lambda/\mu)$, as can be found in any textbook of queueing theory.

On finding K customers already in the service-facility, a customer will balk, otherwise, he will join the queue, and according to the specific number of customers he found in the facility, will gain an expected utility in accordance with option 3.

$$E[U] = \sum_{i=0}^{K-1}\left[\Pr_i \cdot \left(R - P_S - \frac{C(i+1)}{\mu}\right)\right] - P_N$$

3.5.3 Social Planner - Monopolist

We consider a solution to be a set (λ, K, P_S, P_N) which is the set of decision variables in the system. Notice that K is a function of P_S, while λ is derived from E[U]=0, which means it is a function of P_S and P_N. That means, solutions are reached through the setting of prices (P_S, P_N).

Since the queue is of the type M/M/1/K, the expected time E[w] in the system (before watching the queue), is a function of both λ and K.

We now define the objective functions involved:

$$E[U] = (1-\Pr(K))\cdot R - \lambda\cdot P_N - \lambda\cdot(1-\Pr(K))\cdot P_S - \lambda\cdot(1-\Pr(K))\cdot C\cdot E[w(\lambda,K)] = 0$$

$$\pi_S = \lambda\cdot P_S\cdot(1-\Pr(K))$$

$$\pi_N = \lambda\cdot P_N$$

$$\pi_M = \pi_S + \pi_N$$

$$\pi_W = \lambda\cdot R\cdot(1-\Pr(K)) - \lambda\cdot C\cdot E[w(\lambda,K)]\cdot(1-\Pr(K))$$

One can easily see that $\pi_W - \pi_M = \lambda\cdot E[U] = 0$

So we see that the objectives of the social planner and of the monopolist coincide, as we have seen before.

Part 4 – Discussion of Results

In this section, we try not to repeat the results we gave in parts 2 & 3. Rather, we try to understand how they help us in view of our goals (see 1.2).

The model analyzed is a model of a service facility where impatient customers have to line up in a queue to get the service. The customers are being served by a service-facility, but they have to use a parking-facility as well, for the duration of their stay in the system. The service-provider and the parking-provider, separately charge the customers for the service each of them provides. The service-provider charges a bulk sum, and the parking-provider charges a modular sum, according to the time the customer spends in the system.

4.1 Analytic Results

First, the reader is directed to section 2.3, where a summary of our main analytic results is given. These results speak for themselves: in any real-life application which resembles this model, they can be readily used in the process of decision-making. In part 3, there are a few extensions that show how these results change under different variations of the model. Explicit analytic results are available both for the case of two separate entrepreneurs, and for a monopolist who owns both the facilities (and follows the same strategy as that of a social planner).

The sensitivity of the results to the different parameters could be easily observed (sometimes, using assumption 7: $\mu R \geq C$), and are usually quite intuitive (the equilibrium arrival rate is a monotone increasing function of R and μ, etc.).

We find it unnecessary to bother the reader with a full list of all the derivatives of the results.

However, two specific issues should be noted:

4.1.1 Pricing sensitivity to Demand

Our intuition suggests that when demand for the service increases, so will the prices.

Our results (theorem 2.6) seem to suggest otherwise – when Λ increases, the entrepreneurs (be it a monopolist or two separate competing entities) respond by raising the prices. This could be observed most easily in our graphical interpretation of the price-plane (figure 2.4): we have seen that when Λ increases, the limitation-line lowers toward the origin. Since the limitation-line presents a lower bound for the set of prices in equilibrium, this results in the lowering of prices.

A similar result was reached by Chen and Frank [1], for the case of the monopolist.

Why does this happen ? Does this result contradicts existing results and our basic intuition ?

When considering this result more closely, we can see that this result is not really unique for this model or for congestion systems. We have to remember that Λ is not a demand-function, but rather an upper bound on the demand. Actually, observing the analytic result for the prices, we can see that they are raised when the relevant parameters (R, μ,C) change to increase the demand.

Think of the demand-supply market with a monopolist (figure 2.1) – regularly, whenever the demand curve rises, the monopolist will increase his price. Now, add an upper bound on the demand, which cuts the demand curve before it reaches the equilibrium point (lower demand), the monopolist will react by increasing his prices. When this upper bound increases, the monopolist will lower his price in order to capture all the demand again.

4.1.2 Sensitivity to Customers' Impatience

Considering the customers' behavior, we expect that when the customers' impatience increases (as is expressed through an increase in C), their demand will decrease, resulting in a lower rate of arrival, lower prices, and lower profits for the entrepreneurs.

Again, we find that our results (theorem 2.7) are in partial disagreement with this intuition.

Indeed, when the customers' evaluation of time increases, this always results in a lower rate of arrival, and lower price and profit for the service-provider.

However, the results for the parking-supplier suggest that for certain low values of C, the opposite is true – when the value of C is very low, the parking supplier will increase prices and profits when C increases.

The issue is sharpened when observing that without the assumption that the customers are impatient (C=0), the result is that the parking provider sets his price to zero and gets nothing.

Remember also that we found that the parking provider will generally prefer a fixed-price system than the one suggested in our model (theorem 3.1).

It seems that the suggested price construction is only rational when C is bigger than a minimum value, and even then, the parking provider will prefer to charge a bulk sum. We have also seen that a fixed price charged for the parking-service is socially preferred.

So how come we can observe modular prices for parking in real-life situations ?

This takes us to the part of future research, which will be discussed later. A lot of situations involve a mix of alternative assumption (we checked for a few separately, but a mix of them might produce different results) – for example, theorem 3.1 might not hold in the model of a few service providers with different service times (our guess is that it will not change), or a model could be thought of, where the service-providers are of different services with different values (R).

4.2 Rules-of-Thumb

In addition to the specific analytic results we found, there are several, more general, rules that we found, that may be applied to other similar models on more intuitive basis.

4.2.1 Socially optimizing Monopolist

We defined a solution to the model to be a set (P_N,P_S,λ); or in models that permit balking: (P_N,P_S,λ,K), where K represents the threshold in the customers' rule of decision.

From the monopolist's point of view, it seems that all of the solution's components influence the outcome.

However, from the social planner's point of view, the only components that influence his objectives are those representing the customers' behavior, namely, (λ,K). This is true because of assumptions 5 and 6 (all the individuals' utilities are identical, risk-neutral, and additive) – the social planner considers the payments as utility transfers, which do not influence the total utility.

So what makes the monopolist act to maximize the social utility in our models ? What is the underlying difference between the models constructed here and in the paper of Edelson and Hildebrand [5], and the model presented by Naor [15] ?

Hopefully, the answer to these questions should be clear at this point.

If in a certain model, a revenue-maximizing monopolist is able impose tolls on the customers in such a way that he collects all their expected gained utility (no consumer surplus) under any specific set of tolls, he will always choose to impose tolls in such a way that the solution reached, will be the socially optimal one. Otherwise, he might divert from this solution.

This is quite straightforward. No customer will agree to pay more than he is expected to gain (unless the service is imposed upon him, which is not the case in the discussed models). This means, that for any solution, there cannot be any way to collect more than all the customers' gains. In case this can be done for every solution, certainly the monopolist's interests coincide with those of society, and therefore we can expect him to reach (if possible) the socially optimal solution, along with the socially optimal toll.

(Note however, that since K is chosen stepwise (in models with balking), there may be a range of socially optimal tolls resulting in the same optimal behavior by the customers. The revenue-maximizer's optimal toll however, may be a specific point in that range, and not the all range).

The way to easily check this point is given in the basic model, and is repeated for the two-part-tariff model (sec. 3.5). The basic idea is that, under our assumption concerning the utilities' structure (5&6), the social utility is composed of that of the monopolist, together with that of all the customers. When we know that for any solution, the customers' utilities are set to zero (the expected utilities, of course), this means that the objectives of the monopolist coincide with the social objectives.

4.2.2 Competition between Service and Parking Suppliers

We found that when the service and parking suppliers are separated into two independent entities that engage in a price-competition, rather than a single monopolist, the set of prices reached in equilibrium is higher than that reached by the monopolist (and the resulting rate of arriving customers is lower).

This might sound a bit strange and opposed to the intuition usually related to the word "competition".

However, we have to remember that the service and parking facilities represent complementary products, as any customer who wants to be served has to pay both entrepreneurs.

Consider the equilibrium reached by a monopolist. At the equilibrium's set of prices, the monopolist knows that the extra revenues that will be gained by raising his prices will be contradicted by the decrease in demand. However, when this set of prices is separated for the two entrepreneurs, each of them will find it worthwhile to raise his price at this point, because the relative influence of the contradicting decrease in demand will be divided between himself and his counterpart.

This is certainly no new concept to economics: as presented in Economides and Salop [4], a model of two separate monopolists that produce complementary goods was considered by Cournot [3] back in 1838. He showed that indeed, a joint ownership by a single integrated monopolist reduces the sum of the two prices. The explanation given is similar in nature: the two independent firms ignore the effect of their individual markups on each other, while the integrated monopolist internalizes this externality.

4.2.3 Other rules

In the third section we checked a few alternatives to see how certain changes in the model might change the results.

We saw that when the parking provider sets a modular price that depends on the time the customer spends in the system, the service provider is better off than the parking provider (bigger revenue). When the price for parking is set as a bulk sum, the solution is symmetrical (for the two suppliers), and this is preferred both by the parking provider and socially.

We found that the basic results do not change dramatically when we add marginal costs, and gave the specific analytic expressions. Both costs have negative influence on the resulting rate of arrival by customers, and on both the entrepreneurs' profits.

When we increase the marginal cost of the service (parking) supplier, the service (parking) supplier will increase his price, while the parking (service) supplier will decrease his.

We saw that when there are two separate identical service providers (in the case of unbounded rate of arrival), they will both set the same price and get the same rate of arrival as the single service provider set in the basic model. The single parking provider sets the same price as before, and serves twice as many customers (doubling his profit). The social utility doubles as well.

In case the two servers differ in their rate of service, we saw that the parking provider will base his pricing-policy on a rooted-average rate of service, $\bar{\mu}$, as follows: $\bar{\mu} = \left(\sqrt{\mu_1} + \sqrt{\mu_2}\right)^2 / 4$. Also, the resulting ratio between the expected times spent in the two facilities equals the root of the ratio between their service times.

Last, we found that when he social planner gets hold of any one of the two facilities in our basic model (service \ parking), He will set the price at that facility to zero, so that his private counterpart will act as a monopolist and reach social optimality.

4.3 Dependence on Assumptions

In this section we want to specifically mention the strong dependence of the results given here on assumption 5 and 6. These assumptions concerning the structure of the individual and social utilities are very common in related research, and more generally- in economic modeling, as a tool to simplify the research and reach manageable results.

We found that a wide range of models will reach the very appealing result that a monopolist will reach a social optimality. But consider these solutions, the underlying rule for this to happen, is that the monopolist gains all the social utility and the customers are left with nothing. Does our society really considers this situation to be optimal ?

Actually, this problem may be attributed to the assumption concerning the individual utilities. Even if we consider the individual utilities to be similar and additive in the social view, if we considered concave individual utility functions, than society as a whole prefers a more even spread of incomes than the one we reach.

To justify the result as holding under different assumptions, one might claim that arguments concerning the spread of incomes are irrelevant since this could always be fixed through an efficient taxing system.

Another issue concerning the structure of the individual utility functions is that of the customers' behavior in equilibrium. In the models presented here, the customers choose what to do based on expected utilities under different alternatives. For non-linear utilities, this does not hold and we have to engage in much more complex calculations.

Chen and Frank [1] explored the option of non-linear preferences by the customers, and found that in models in which a monopolist will reach a socially optimal equilibrium, when preferences are not linear – the equilibrium (monopolist) will not normally be socially efficient.

This point might be excused by arguments of approximate local linearity of the customers' utilities, meaning that for the sums involved, the customers' preferences are more or less linear. This is quite reasonable, if we consider sums that are small relative to the customers' wealth.

Part 5 – Summary and Future Research

In this thesis, we investigated a model of a service facility, where impatient customers have to line up in queue to get the service.

We introduced complementary competition by dividing the service facility into two separate entrepreneurs: a service-supplier and a parking-supplier.

Explicit analytic results where found, and are given in sec. 2.3.

We found that under such a competition, the prices set in equilibrium are higher than those set by a single monopolist that supplies the parking and the service. This result is similar to that reached for regular (without time-cost considerations) models of supply and demand by Cournot [3].

We also found that in our model, a monopolist will reach a socially optimal result, and gave the conditions under which, such an outcome should be expected.

A more detailed discussion of the results is given in part 4.

Future Research

Some thought should be given to future research.

First, one might consider the long-run case, where the service-provider is able to choose the rate of service, and pays for it. This point was investigated by Chen and Frank [1], who found that a monopolist in a similar model will either choose not to produce at all ($\mu=0$) or set the rate of arrival as to accommodate all the population (and if the potential demand is infinite, will choose an infinite rate of service). However, the rational behind the result of Chen and Frank does not apply in the case of complementary products (basically, when the service-provider pays to increase the rate of service, the parking-provider will exploit it for his own good and raise his price). The task of analytically finding the long-run rate of service is not a simple one when considering our model, and it may be the case that one will have to be satisfied with numerical results.

Our results in part 3 of the thesis are true for the case of unbounded potential demand. However, whenever the potential demand poses a realistic bound on the results, other considerations should be taken into account. Specifically, in the case of two service-suppliers, in case of a realistic bound, we can assume that this bound will be the actual rate of arrival, but the specific prices that will be chosen, and the division of the total utility between the entrepreneurs and the customers is not obvious (existing works of competition between suppliers with impatient customers show why the prices will not be set to zero, see for example Levhari and Luski [8]).

Basically, by small changes in the assumptions, a lot of variations to this model can be explored, while finding which results are indeed robust. Basic examples may be heterogeneity between customers in relation to their appreciation of time-costs or appreciation of the value of service.

We found that in our model, the parking-provider will prefer to charge a bulk sum rather than a modular sum based on the time spent in the system. In real life situations, we observe cases in which parking-providers choose to charge modular sums. Different explanations can be offered as to the difference between our model and these situations, but indeed, further research is in order to arrive at a suitable model with this result.

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