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Parking Strategy in an Infinite Line

A thesis submitted in the partial fulfilment of the requirements for the degree of
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by

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Contents

Abstract

We examine the problem of drivers arriving to a city center with Poisson rate, search for a parking spot, and park for an exponentially distributed duration. In this paper, we focus on a simple linear system, where there is a one-way road. We prove that when all drivers have the same destination, and the parking cost is the distance between the parking and the destination, there is a unique equilibrium strategy, where a strategy is threshold strategy, which means that a driver will park at the first vacant space after arriving to a certain distance to the destination. We find the dependency of the strategy and cost on the arrival rate and the time spent at the destination. Additionally, we compare the equilibrium and socially-optimal strategies in order to check whether an external intervention is needed. We further extend our model to allow for two-way traffic. We show that with one-way traffic, there is a significant difference between the equilibrium and socially-optimal costs, whereas will two-way traffic, the difference is minor. In the case of one-way traffic, the equilibrium strategy is to start looking for parking too close the destination, a possible solution is to increase the parking fee for parking space beyond the destination, and by that, motivate drivers to start the search earlier.

We also find an interesting phenomenon regarding the expected cost of a driver as function of the strategy; the overall shape of the graph is convex whereas each interval between two consecutive integer values is concave.

1 Introduction

Suppose you drive to a city center for shopping, will you park at the first vacant parking spot you see on the street of the shop? Will you park only from a certain distance from your destination? Or maybe drive to your destination and look for parking right after passing it? Many of our journeys are to a center, like a mall or a business complex for a meeting. In both examples, drivers arrive during the whole day and park for different periods of time.

In this paper we investigate the case where a common destination is located on an infinite one-way or two-way road, and park for a period of time, and the parking cost is the distance between the parking spot and the destination. We use simulation to find the equilibrium and socially-optimal strategies and costs. We find that the costs are approximately linearly dependent on the ratio of the arrival rate to the departure rate (the reciprocal of the expected parking time). We also find that in a one-way road, the socially-optimal costs are significantly lower than the equilibrium costs, whereas in a two-way road, the differences are minor.

2 Literature review

Parking models have been analysed from various angles, and we describe below the most relevant literature, divided into several topics

- How to choose where to park in a one-dimensional road, where the question is whether to park at the vacant spot or look for a closer one.
- Traffic congestion and time spent on cruising for parking.
- Ways of reducing traffic congestion and cruising costs using parking fee or by adding parking lots.
- Search Strategies on a network.

Closest to our work is the first topic and we briefly describe this literature. Similar to this paper, **Krapivsky and Redner (2019)** compare three types of strategies for minimizing costs for a driver in a linear system where the road is a half-infinite line which ends at the destination. The three strategies are

- park at the first vacant parking spot after a parked car,
- detect a gap and park at the closest spot to destination,
- go all the way to the destination, turn around, and park at the first vacant parking spot.

There are several differences between our paper and **Krapivsky and Redner (2019)**

- **Krapivsky and Redner (2019)** restrict driver's behaviour to the three strategies and **compares their costs**, whereas this paper examines the **equilibrium** and **socially-optimal** strategies.
- **Krapivsky and Redner (2019)** use a combination of driving costs and the cost of walking to the destination, whereas in this paper, the cost function is the distance between the parking spot and the target.
- **Krapivsky and Redner (2019)** use a constant arrival rate and assume drivers leave after a a constant parking time, whereas this paper uses a Poisson arrival process and exponentially distributed parking period.
- **Krapivsky and Redner (2019)** investigate an half-finite road, whereas this paper investigates an infinite road.

Krapivsky and Redner (2020) look at the same system like in **Krapivsky and Redner (2019)**, only now cars can see only the parking spot next to them, similar to this paper. The authors look for the optimal relation between the place of the first parked car new drivers meet and the place a new driver should start looking for parking spot, i.e., if the first car they see is at L, the authors compute a constant τ ($\tau < 1$) that maximizes the probability of parking at the closest available parking spot to the destination, if drivers park at the first vacant parking spot after passing $L \times \tau$ cars.
Arnott and Rowse (1998) suggest a model for parking search in a spatially homogeneous metropolis, where a driver choose where to start looking for parking spot, and parks at the first vacant spot. The probability of a parking spot to be vacant is fixed. The authors find the equilibrium and social optimum with and without parking fee. The cost is the expected time at the trip which includes travel time, visiting time and waiting for trip opportunity at home. Trip opportunities are offered with Poisson rate to the driver, and if he accept them and leaves immediately he will make a profit that will at least cover the trip costs.
Levy and Benenson (2015) derive an algorithm for evaluating the distribution of the distance between the driver's parking and destination under heterogeneous urban parking demand and supply. The algorithm uses supply and demand in small city zones, where a new driver gets to his target zone and looks for a parking spot,and if there is no vacant spot, he will go to the next closest zone etc.. The paper uses data of the city of Bat-Yam and analyzes the current and future state of overnight parking, and also investigates the consequences of parking regulation.

Other types of work assess the congestion caused by cruising for parking, and the time spent while searching. Here are some examples.
Shoup (2006) presents the average (of 16 studies) percentage of cars that are cruising for parking spot in congested cities, in any given time is 30%. The paper describes a model for choosing how much time to spend on cruising for on-street parking before parking off-street. The model's equilibrium (how much time to spend on searching) depends on the prices of on-street and off-street parking, parking duration, fuel cost, number of people in the car and the value of time spent on cruising.

Dowling, Ratliff and Zhang (2020) re-examines the 30% of cruising cars described in **Shoup (2006)**, and create a model based on finite-capacity Markovian queues. The whole system is built from a finite number of queues, and drivers arrive to the system with a known rate. If all servers are busy, the driver moves to a different queue. Using data of Seattle Department of Transportation that includes information of parking payment fees and traffic volume, the author concludes that up to 50% of the cars are cruising for parking during rush hours.
Arnott and Williams (2017) use simulation for the case of cruising for parking in a circle in a stochastic steady-state with Poisson entry and exponentially distributed parking time, in order to find the cruising time spent in search for parking. The results show that the binomial approximation, like in **Shoup (2006)**, underestimates the cruising time for parking.
Fulman and Benenson (2018) use simulation to find the occupancy rate and cruising time in both homogeneous and heterogeneous demand systems. They apply the model on the city of Bat-Yam and show that even in 61% average demand-to-supply ratio, the heterogeneity of the system causes long parking search time for a significant part of the drivers. The driver's search behaviour is based on **Benenson** *et* **al. (2015)**, where 35 participants played overall 200 game sessions in order to find the probability of a driver to make a turn closer or farther from the destination at a crossroad, as a function of the distance from the destination, while looking for parking.

Other works look for ways of reducing congestion and cruising costs by parking fees or by adding parking lots. **Larson and Sasanuma (2010)** create a Markovian queueing model estimating traffic congestion caused by cars that are looking for parking spots, and present a parking pricing model in order to reduce the congestion. **Pierce and Shoup (2013)** present the results of a curb-parking price reform made in San-Francisco, where the goal was to have empty spaces at every block. The authors use a smart parking meter that changes the prices according to the occupancy rate. Other papers, like **Arnott and Inci (2015)** consider a reduction in the traffic congestion by increasing the parking capacity in private sector's parking lots.

Finally, there are several papers presenting simulations for parking search on a network. For example, **Bar-Gera (2002)**, **Gentile (2009)** and **Boyles (2014)** focus on a two-dimensional traffic network model, where the drivers choose a route of search for a parking spot, and incur driving time costs, parking costs, walking and searching costs.

3 The Model

In this paper we examine a common destination at a city center, in which cars arrive, search for parking, and leave after a period of time. The cars move on an infinite line in one direction, with discrete parking spots $(-\infty, ..., -2, -1, 0, 1, 2, ..., \infty)$, and start their travel at a far point to a city center located at $i = 0$. Cars arrive according to a Poisson process with rate λ, search for parking, and stay for an exponentially-distributed time with parameter μ.

A *pure threshold search strategy* is defined by an integer l and means that the driver starts searching from location $-l$ and parks in the first vacant spot. For a real number $l+q$, where l is integer and, $q \in [0,1)$, the *mixed threshold strategy* $l+q$ means that the driver starts searching from location $-(l+1)$, if this spot is vacant the driver parks there with probability q, and otherwise the driver parks in the first vacant spot. In other words, strategy $l+q$ means adopting strategy $l+1$ with probability q and strategy l with probability $1-q$.

If the threshold $l+q$ is too large, most drivers will park far ahead from their destination, and if it is too small, they will probably not find a parking spot, pass the destination, and miss closer opportunities. We assume that the system (drivers) is in steady-state and restrict our attention to symmetric threshold strategies. A symmetric strategy is a strategy where all drivers choose the same strategy independently (an asymmetric strategy where different players follow different strategies requires some sort coordination among the drivers).

A *best-response strategy*, $l_{br}(l_{co})$, generates the lowest expected cost for a single driver, when all the other drivers follow the *common strategy* l_{co}. Generally, l_{br} is a pure strategy, although there might be cases where the driver is indifferent between two consecutive integer thresholds. In this case $l_{br}(l_{co})$ denotes the set of pure best-response strategies, and all mixed strategies that randomly select one of the best-response pure strategies.

A strategy l_{eq} is an *equilibrium strategy* if it is a best response to itself, i.e., if l_{eq} is the common strategy then no driver will gain from deviating from it. If a mixed strategy $l+q, 0<q<1$, defines an equilibrium, it means that both strategies l and $l+1$ are best response strategies, and a driver is indifferent between the two. As we show in the proof of Lemma 1 below, it is not possible that pure strategies corresponding to non-consecutive integers are best response strategies, and therefore it is sufficient to consider threshold strategies that mix between consecutive integers, as defined here.

The condition for the equilibrium strategy is that for l_{co}, there is no other strategy that a single driver can use and get a lower cost. i.e.

$$\arg\min_{l} cost(l, l_{co}) = l_{co}, \tag{1}$$

for a pure strategy, and,

$$\arg\min_{l} cost(l, l_{co}) = [\lfloor l_{co} \rfloor, \lceil l_{co} \rceil], \tag{2}$$

for a mixed strategy.

The *social cost* is the average rate of cost (distance from the destination) incurred by the drivers in the system. A strategy l_s is *socially optimal* if it is the common strategy that generates the lowest possible social cost.

Our goal is to compute the equilibrium and socially-optimal strategies for given arrival rate λ and expected parking time $1/\mu$. We note, however, that the only relevant parameter is $\rho = \lambda/\mu$.

3.1 State transition

Denote the system states as $x = (..., x_{-1}, x_0, x_1, ...)$, $x_i \in \{0, 1\}$, $i \in (-\infty, \infty)$, where $x_i = 1$ if a car is parked at i and $x_i = 0$ if parking i is vacant.

For a given state x, a parked driver may leave or a new driver may arrive at the next event. Denote by P_j, the probability that a newly arriving driver will park at j when all use strategy l_{co}, where $l_{co} = l + q$ with l an integer and $q \in [0, 1)$. [1] Then, for $j \in \{-l-1, -l, \ldots\}$,

$$\begin{aligned} P_j = (1-q) \cdot P(x_{-l} = 1, \ldots, x_{j-1} = 1, x_j = 0) + \\ q \cdot P(x_{-l-1} = 1, x_{-l} = 1, \ldots, x_{j-1} = 1, x_j = 0). \end{aligned} \tag{3}$$

For $j = -(l+1)$ we get

$$P_{-l-1} = q \cdot P(x_{-l-1} = 0). \tag{4}$$

Now we obtain the expected cost function for a driver when all drivers follow strategy l:

$$cost(l) = \sum_{j=-l-1}^{\infty} P_j(l) \cdot |j|. \tag{5}$$

Denote by $P_j(l, l_{co})$, the probability that a driver with a pure strategy l ($l \leq \lceil l_{co} \rceil$) who arrives at the system will park at j. Denote by $P(x)$ the probability that the system is at state x. Then,

$$P_j(l, l_{co}) = \sum_{x | x_{-l} \ldots x_{j-1} = 1, x_j = 0} P(x), \tag{6}$$

and the expected cost for this driver is

$$cost(l, l_{co}) = \sum_{j=-l}^{\infty} P_j(l, l_{co}) \cdot |j|. \tag{7}$$

Since solving the problem analytically is very complicated, we decided to create an agent-based simulation.

[1] When $q = 0$, j starts at $-l_{co}$.

3.2 Two-way traffic

In a two-way traffic, we have two one-way roads, competing for the same inventory of parking spots (no U-turns). Like in one-way traffic, all the drivers share the same destination.
The calculations of equilibrium and socially optimal strategies are similar to the ones of one-way traffic, Where for a given system (with a given rho), the probabilities of arriving/leaving cars is like in one-way traffic, but when car arrives, there is an equal probability for arriving from either side of the road.

4 Equilibrium uniqueness

In this section we prove that in the one-way traffic case, there exists an equilibrium threshold strategy in our model, and that it is unique.

A strategy l_{co} is a *crossing point* if there is a neighborhood of l_{co} such that $l_{br}(l) > l$ for $l < l_{co}$ and $l_{br}(l) < l$ for $l > l_{co}$ in this neighborhood.

Theorem 1. *An equilibrium symmetric threshold strategy exists, and it is unique.*

We prove the theorem using multiple lemmas:

Lemma 1. *There exists an equilibrium.*

Proof. Clearly, $l_{br}(0) \in \{0, 1\}$. If $l_{br}(0) = 0$ then it is an equilibrium. Suppose $l_{br}(0) = 1$. Clearly, $\lim_{l_{co}\to\infty} l_{br}(l_{co}) = 0$, and therefore there must be a point where $l_{br}(l_{co})$ decreases from being above l_{co} to being below l_{co}. We claim that $l_{br}(l_{co})$ is a step function with unit-sized steps, and therefore the crossing point either satisfies $l_{br}(l_{co}) = l_{co}$ or drivers are indifferent between two consecutive integer strategies, $\lceil l_{co} \rceil$ and $\lfloor l_{co} \rfloor$, meaning that l_{co} is an equilibrium that mixes between these integers. We now prove that a step of size greater than 1 in the function $l_{br}(l_{co})$ is not possible. Otherwise, drivers are indifferent between starting their search from location $-L$ and $-(L + k)$ where $k > 1$. This means that the expected cost of the strategy is exactly $L + k$. This means that the driver prefers to park at $-(L + 1)$ and incur cost $L + 1$ rather than start the search at $-L$ and incur expected cost $L+k$, and this contradicts the assumption that L is a best response strategy. $\square$

Lemma 2. *Suppose there exists an integer l ,such that $l_{br}(l) = l$. Then $cost(l, l+1) < cost(l + 1, l + 1)$.*

Proof. From the assumption that $l_{br}(l) = l$ we have

$$cost(l, l) \leq l + 1. \tag{8}$$

Therefore, by increasing the probabilities of finding vacant parking spot near the destination, we get

$$cost(l, l + 1) < l + 1. \tag{9}$$

Denote $\pi_i(l_{co})$ the probability that parking spot i is vacant when all follow l_{co}. Then,

$$\begin{gathered} cost(l+1,l+1) = \\ \pi_{-(l+1)}(l+1)\cdot(l+1)+(1-\pi_{-(l+1)}(l+1))\cdot cost(l,l+1) > \\ \pi_{-(l+1)}(l+1)\cdot cost(l,l+1)+(1-\pi_{-(l+1)}(l+1))\cdot cost(l,l+1) = cost(l,l+1). \end{gathered} \quad (10)$$

□

Lemma 3. *Suppose there exists $l+q$ (l is an integer and $q \in (0,1)$) such that $\{l,l+1\} \in l_{br}(l+q)$, then $l_{br}(l+q') \leq l$, for $q < q' < 1$.*

Proof. $\{l,l+1\} \in l_{br}(l+q)$ means that a driver who arrives to a system where the common strategy is $l+q$ is indifferent between parking at $l+1$ (if vacant), or start looking at l. This means that

$$cost(l,l+q) = cost(l+1,l+q) = l+1. \quad (11)$$

When l_{co} changes to $l+q'$ ($q' > q$), the costs decreases for a driver who follows a strategy l by increasing the probabilities of finding vacant parking spot near the destination. Therefore,

$$cost(l,l+q') < l+1. \quad (12)$$

Therefore, a driver who arrives to a system and faces the common strategy $l+q'$, will not park at $l+1$ if vacant. Therefore,

$$l_{br}(l+q') < l+1. \quad (13)$$

□

Lemma 4. *Suppose $l_{br}(l_{co}) < l_{co}$, then $l_{br}(l'_{co}) < l'_{co}$ for $l'_{co} > l_{co}$.*

Proof. $l_{br}(l_{co}) < l_{co}$ means that for l_{co}, the best-response would be not to park at l_{co} or l'_{co} when are vacant. Therefore $cost(l_{br}(l_{co})) < l_{co}$. When increasing l_{co} to l'_{co}, the probability that a parking spot i, where $i > -l_{co}$, to be vacant increases. This means that a driver who starts looking for parking at $l_{br}(l_{co})$ encounters vacant parking spots with higher probabilities, and therefore can improve his costs, compared to what that was for l_{co}. Therefore, he would not park at $-\lceil l'_{co} \rceil$, meaning that $l_{br}(l'_{co}) < l'_{co}$. □

Proof of Theorem 1. We proved that there is at least one equilibrium point (Lemma 1). We also showed that when increasing l_{co} beyond the crossing point, whether it is pure or mixed, the best-response strategy will be closer to the destination (Lemma 2 + Lemma 3), and that once the best-response strategy is closer to the destination than the common strategy, it keeps that way for every greater common strategy (Lemma 4).
Therefore, the crossing point is unique and corresponds to a unique equilibrium. □

5 The Simulation

5.1 Introduction

In this paper we want to find a way to obtain the best-response to a given strategy l_{co}. We don't know to do it analytically. Instead, we use an algorithm to compute the best response for a given problem parameters. We describe the algorithm in this section.

5.2 A single destination, one-way traffic

Given the problem in which all drivers wish to arrive at a single destination located at $i = 0$ in a one-way traffic, the algorithm computes $cost(l, l_{co})$, then we find $l_{br}(l_{co})$, and by finding the crossing point we retrieve l_{eq}, the symmetric-equilibrium strategy. The algorithm's input contains basic parameters, such as ρ, l and l_{co} and calculative boundaries.

5.2.1 Calculative boundaries - State vector

The state vector is $x \in \{0,1\}^{n+N+1}$ for a given n ($n = \lceil l_{co} \rceil$), where $x_i = 1$ means that parking spot i is occupied. In order to use a finite-sized vector, we first calculate N such that the probability of arriving to the system when all parking spots up to N are taken is at most ϵ. We find N by Erlang-C formula for M/M/N.

5.2.2 Calculative boundaries - Examined strategies

We define bounds L_0 and L_1, and for every common strategy $l_{co} \in [L_0, L_1]$, we find the best-response strategy l_{br} by calculating the expected cost for an l driver in a l_{co} system. That is to say that $cost(l, l_{co})$ is the expected cost for a single player, using the strategy l, where everyone else uses the strategy l_{co}.

5.2.3 Average costs

Denote $n(x) = \sum_{i=-\infty}^{\infty} x_i$, the number of parked cars in the system.
The average cost is calculated by having a large number of *iterations*. In each iteration a driver arrives or departs with $P_{arrived}(x) = \frac{\lambda}{\lambda + n(x) \cdot \mu} = \frac{\rho}{\rho + n(x)}$ and $P_{departure}(x) = \frac{n(x) \cdot \mu}{\lambda + n(x) \cdot \mu} = \frac{n(x)}{\rho + n(x)}$ with the same probability for every x_i such that $x_i = 1$, and the number of occupied parking places $n(x)$. We calculate the average cost for a virtual driver with strategy $l = 0, 1, \ldots, \lceil l_{co} \rceil + 1$ that arrive every time a real driver, with a strategy l_{co} does. The virtual drivers do not occupy a parking spot, the purpose of a virtual driver is to compute $cost(l, l_{co})$. An arriving driver parks at the first vacant place after reaching $l+1$ with probability q or l with probability $1-q$ for $l_{co} = l + q$ where l is integer and $q \in (0, 1)$.

By calculating $cost(l, l_{co})$ for $l \in [0, \lceil l_{co} \rceil + 1]$, we find the best-response strategy for l_{co}.

5.3 Equilibrium strategy

We look for the crossing point of $l_{br}(l_{co})$ and l_{co} for $l_{co} \in [L_0, L_1]$ with step-size of $\Delta_{l_{co}}$. If needed, we adjust L_0 a, L_1 and $\Delta_{l_{co}}$ to accurately capture the crossing point. The crossing point satisfy $l_{br} = l_{co} = l_{eq}$ for a pure strategy or $l_{br} = \{\lfloor l_{co} \rfloor, \lceil l_{co} \rceil\}$ for a mixed strategy.

5.4 Social optimality

We compute the optimal strategy by calculating $cost(l_{co}, l_{co})$ for $l_{co} \in [L_0, L_1]$ with step-size of $\Delta_{l_{co}}$, and find $\arg\min_{l_{co}} cost(l_{co}, l_{co})$.

5.5 Summary

To summarize, the algorithm's input and calculative parameters are:

- $\rho = \frac{\lambda}{\mu}$.
- ϵ - The probability that all parking spot are taken.
- *iterations* - Number of arrivals/departures for averaging.
- L_0, L_1 - Boundaries for calculating l_{br}.[2]
- $\Delta_{l_{co}}$ - Interval steps for $l_{co} \in [L_0, L_1]$ in which we derive $l_{br}(l_{co})$.

6 Simulation examples

In order to reduce the algorithm's running time, we took two steps, the first was to get a sense of where the equilibrium strategy is by a small number of iterations, and in the following step we applied a large number of iterations with a shorter interval $[L_0, L_1]$ and smaller $\Delta_{l_{co}}$.
Note that it is possible to calculate a free parking spot distribution for a given strategy, and then shift the graph along the x-axis. Specifically, the graph for strategy $l + q$ where l is integer and $0 < q < 1$, is the same as for $l + 1 + q$, etc., and this saves computations in the first phase of the search for the equilibrium or optimal strategies. However, this phase is not the main time consuming part of the algorithm, and in the second phase where we apply a small step size this idea is not applicable because the range of the search is smaller than one. This idea is also not applicable for two-way traffic - because of the "superposition" of the two roads. The selection of high number of iterations gain a small STD of the expected costs, for example, we got $STD = 0.0017$ for an average costs of 2.1 over 50 runs, for the case of one-way traffic, and $\rho = 5$ with 10^7 iterations.
This section presents one-way and two-way traffic examples. In each of them we find the equilibrium and socially-optimal strategies.

[2]Starting with wide boundaries and adjusted repeating the first step parameters.

6.1 One-way traffic example

In this example we used the following parameters:

- $\rho = 5$.
- $\epsilon = 0.001$.
- $iterations = 10^5$ and $\Delta_{l_{co}} = 0.2$ for the first step.
- $iterations = 10^7$ and $\Delta_{l_{co}} = 0.02$ for the second step.

As a result, in order to have at least one vacant parking spot with probability larger than 0.999, we get $N = 14$. [3]

6.1.1 Equilibrium

The results for the first step are shown in Figure 1, where the line represents a 45° slope.

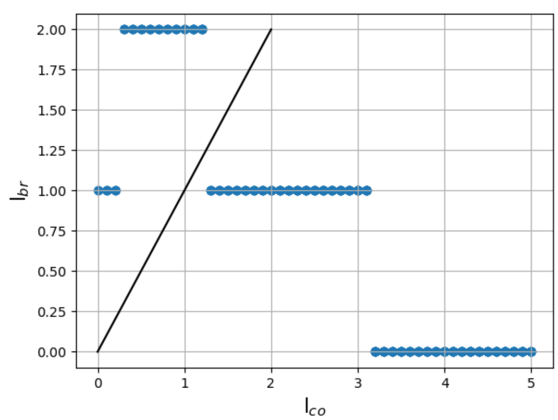

Figure 1: One-way traffic - **first** step
$\rho = 5$

We suspect the crossing point to be in the interval (1.1,1.4). The results for the second step are shown in Figure 2.

[3] $P_C(\rho = 5, N = 14) = \frac{\frac{\rho^N N}{N!(N-\rho)}}{\sum_{i=}^{N-1} \frac{\rho^i}{i!} + \frac{\rho^N N}{N!(N-\rho)}} = 0.00073$

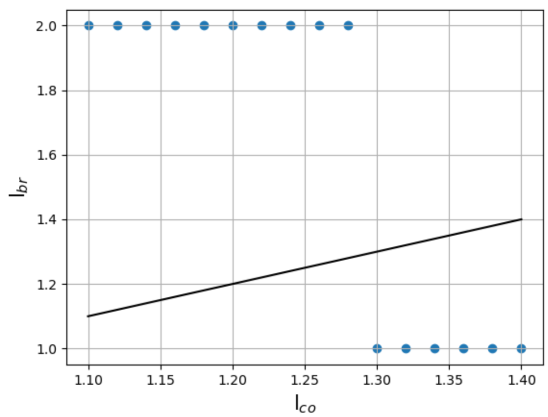

Figure 2: One-way traffic - **second** step
$\rho = 5$

We identify the crossing point at the interval (1.28,1.3) so we assume that $l_{eq} = 1.29$ and for this strategy we got that the average cost is 2.22. Figures 3 and 4 show the distribution of parked cars and the vacancy probability for each parking spot.

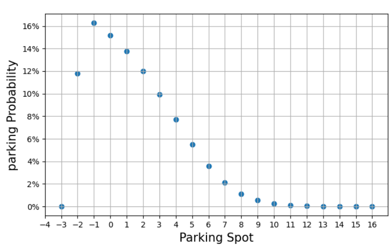

Figure 3: Parking probability distribution in equilibrium. $\rho = 5, \quad l_{co} = 1.29$

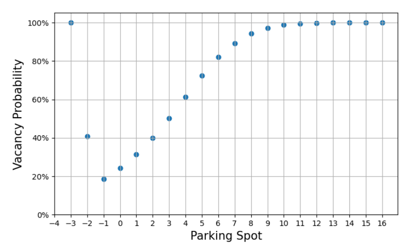

Figure 4: Vacancy probability distribution in equilibrium. $\rho = 5, \quad l_{co} = 1.29$

Note that while the values in each graph look like one minus the values of the other, this is not the case. One graph is the probability for a driver to park at this parking spot, whereas the second is the probability that the parking spot is vacant. It can be vacant and not be taken by the driver, and therefore the sum of the probabilities are less than 1.

Another example for the distribution of parked cars and the vacancy probability is shown in Figures 5 and 6 for $\rho = 10$.

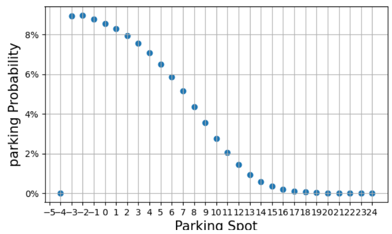

Figure 5: Parking probability distribution in equilibrium. $\rho = 10, \quad l_{co} = 2.86$

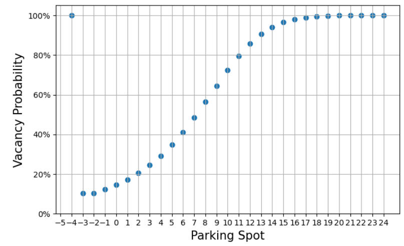

Figure 6: Vacancy probability distribution in equilibrium. $\rho = 10, \quad l_{co} = 2.86$

The physical meaning of ρ is the average served customers in a system, which, in our case, corresponds to the average number of parked cars. For this example of $\rho = 5$ we get 4.6 as the average of parked cars, which makes a decent approximation.

6.1.2 Social optimality

Now, we calculate the average costs for any symmetric strategy and find the socially-optimal strategy, see Figure 7.

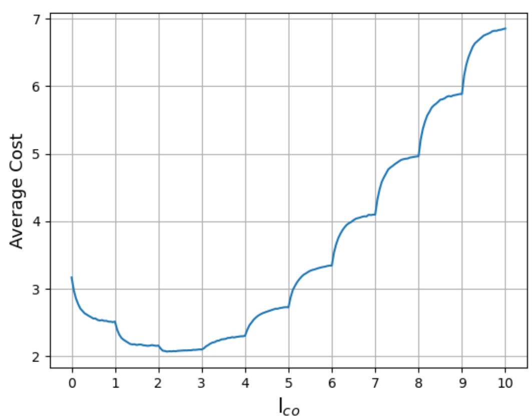

Figure 7: Average cost vs l_{co}
one-way traffic, $\rho = 5$

Figure 7 shows two properties. First, the graph between two integer values of l_{co} is concave. Second, the graph on the integer values is convex. A better way to see it is by using only integer values of l_{co}, like presented in Figure 8.

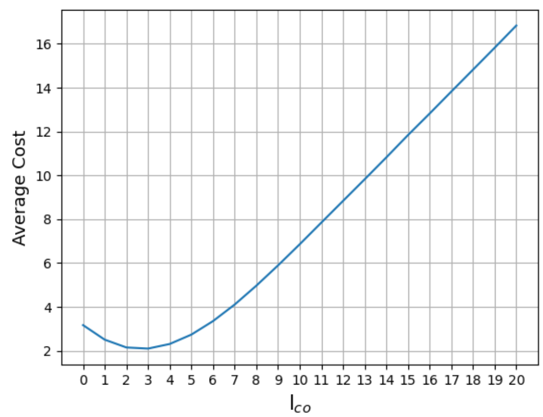

Figure 8: Average cost vs integer values of l_{co}
one-way traffic, $\rho = 5$

We suggest intuitive explanations for these phenomena:
Consider $l_{co} = l + q$ where l is fixed and $q \in [0, 1]$. For low values of q, the drivers who start searching from $l + 1$, will probably park at $l + 1$ since only few drivers search there even if the place is vacant. Therefore, for every small increase of q, the number of cars that will park there is approximately proportional to λq. For higher values of q, the probability of a driver who starts looking for a parking spot from $l + 1$ to park there, is decreasing. Therefore, increasing q for low values of q has more impact than for high q (decreasing marginal cost).
Consider integer values of l_{co}. When l is replaced by $l + 1$, the parking probability distribution looks the same, it only moves across the parking spots. We divide the parking spots into two groups, those between the starting spot and the destination ("before") and those "after". Every one step of increasing l_{co}, increases the average cost of those before the destination because they are getting farther from the destination, and decreases the average cost of those after, because they are getting closer to the destination. Since the parking probability distribution is monotonically decreasing as function of the distance from the search start spot, when increasing l_{co} (start looking farther from the destination), the probabilities of parking after the destination are getting smaller, therefore, their reduction of the average cost, is getting smaller, and every increase of l_{co} by one unit increases the average cost by 1. Then the slope of the graph is close to 1. The slope of the graph is close to 1 for large values of l_{co} can be clearly seen in Figure 8 from $l_{co} = 10$.
Clearly, the equilibrium strategy average cost (2.22 for $\rho = 5$) is greater than the social-cost minimizing strategy average cost (2.07 for $\rho = 5$), and as expected, the equilibrium strategy l_{eq} is smaller than the socially-optimal maximizing strategy l_s. By starting to search earlier, a driver leaves the spots that are closer to the destination to other drivers and thus imposes smaller negative externalities.
Choosing a high number of iterations makes an accurate results. We calculated the

standard deviation of the averaged cost and got $STD = 0.0017$ over 50 runs of $\rho = 5$ with 10^7 iterations.

6.2 Two-way traffic example

In this section we consider a single destination and two-way traffic with same arrival rates for both sides. The simulation is similar to one-way traffic, except that here we randomly choose whether the driver arrives from left or right, with 0.5 probability to each side.

6.2.1 Equilibrium

In the next example, we used the following parameters:

- $\rho = 10$
- $\epsilon = 0.001$
- $iterations = 10^5$ and $\Delta_{l_{co}} = 0.2$ for the first step.
- $iterations = 10^7$ and $\Delta_{l_{co}} = 0.02$ for the second step.

As a result, $N = 22$.[4] The results for the first step are shown in Figure 9.

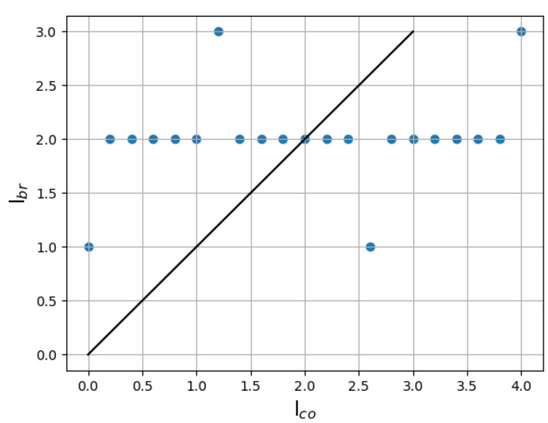

Figure 9: Two-way traffic - **first** step
$\rho = 10$

We suspect the crossing point to be at 2. The results for the second step are shown in Figure 10.

[4] $P_C(\rho = 10, N = 22) = \frac{\frac{\rho^N N}{N!(N-\rho)}}{\sum_{i=}^{N-1} \frac{\rho^i}{i!} + \frac{\rho^N N}{N!(N-\rho)}} = 0.00074$

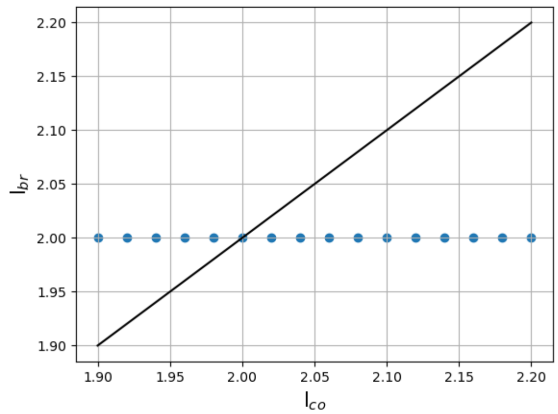

Figure 10: Two-way traffic - **second** step
$\rho = 10$

We identify the crossing point at the interval at $l_{co} = 2$ and the average cost is 3.42. The distribution of parked cars and the vacancy probability for each parking spot are shown in Figures 11 and 12.

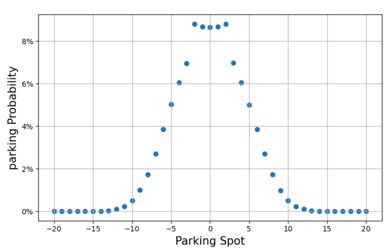

Figure 11: Parking spot distribution
$\rho = 10, \quad l_{co} = 2$

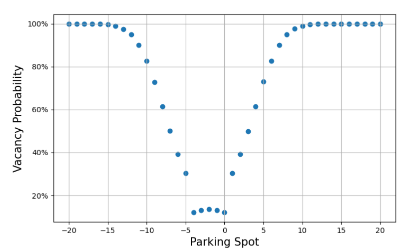

Figure 12: Parking spot vacancy probability, $\rho = 10, \quad l_{co} = 2$

We see in Figure 11 that although the system is symmetric, the maximum probability for parking is not at $i = 0$. We suggest an intuitive explanation for this phenomenon. In one-way traffic, the graph is monotonically decreasing (starting from $\lfloor l_{co} \rfloor$, and in two-way traffic, it is possible that the parking spot at $i = 0$ which is a combination from drivers from both sides, will not be parked at more frequently than the ones at $\lfloor l_{co} \rfloor$ or $-\lfloor l_{co} \rfloor$ which are a combination of those who start looking for parking spot from this spot plus the drivers from the other side. The paralleled phenomenon appear at Figure 12, and have the same explanation.

6.2.2 Social-optimality

We calculate the expected cost for each strategy, see Figures 13 and 14.

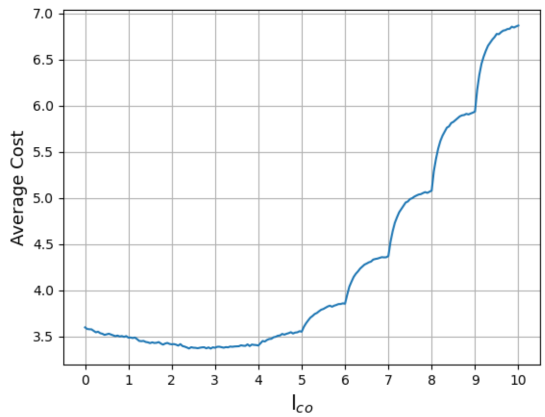

Figure 13: Average cost vs l_{co}
two-way traffic, $\rho = 10$

We see that there is not much of a difference between costs of strategies values for $l_{co} \in (2, 4)$ (minimum cost value 3.37 and maximum value 3.41). We also see that we get the same phenomenon as in one-way traffic where the overall shape of the graph is convex, while between two integer values, the graph is concave. Figure 14 shows the trend of the graph for integer values of l_{co}. Again, we see that slope increases until it approaches a slope of 1

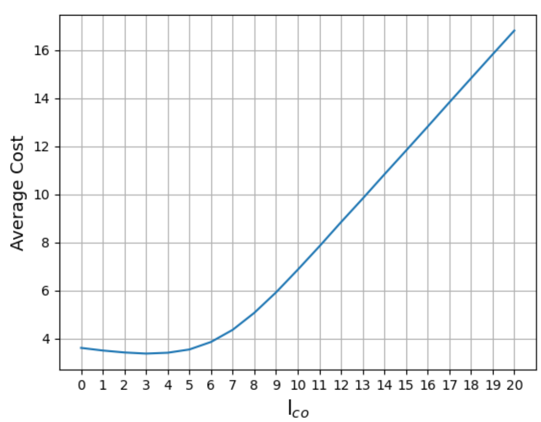

Figure 14: Average cost vs integer values of l_{co}
two-way traffic, $\rho = 10$

Again, we calculated the standard deviation of the averaged cost and got $STD = 0.0017$ over 50 runs of $\rho = 10$ with 10^7 iterations.

6.3 Conclusions

In two-way cases we see that there is not much of a difference in the average cost for a given value of ρ for $l \in (\lfloor l^* \rfloor, \lceil l^* \rceil]$ where l^* is the socially-optimal strategy. Therefore, in the next section of summarizing the simulation, we will look only at integer values of l_{eq}.

7 Simulation results

This section presents the results of the simulation and makes comparisons of strategy and expected cost as function of ρ:

- Equilibrium vs. socially-optimal strategies.
- One-way vs. two-way traffic.

In the previous sections, we gave examples for a fixed ρ. In this section we calculate the equilibrium and optimal strategy and its cost as a function of ρ.

7.1 Equilibrium vs. socially-optimal strategies

Figures 15 and 16 show the simulation results for both equilibrium and socially-optimal strategies in one-way and two-way traffics.

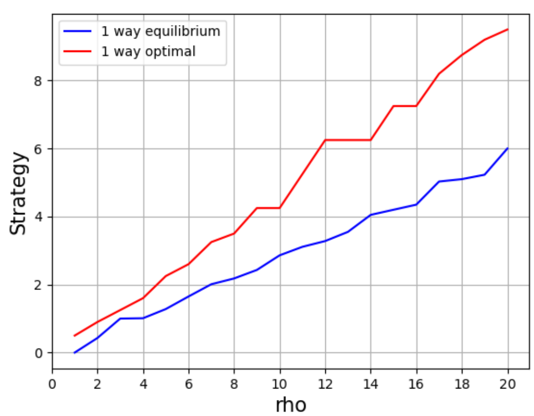

Figure 15: One-way traffic - equilibrium vs. socially-optimal strategy - the lower graph corresponds to the equilibrium strategy, and the upper graph to the socially-optimal strategy

We see that the two graphs are immediately separated and that the equilibrium strategy is always smaller than the socially-optimal strategy. The intuition behind this is that you need to convince the drivers to start looking for parking spot farther than what they would for their best interest. This is because a driver who starts to search for parking spot earlier, increases the probability for other drivers to find parking spot closer to the destination. But selfish drivers ignore the negative externalities of starting to search too late.

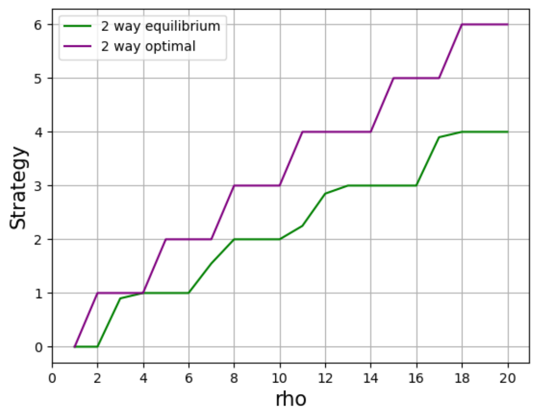

Figure 16: Two-way traffic - equilibrium vs. socially-optimal strategy - the lower graph corresponds to equilibrium strategy, and the upper graph to socially-optimal strategy

In Figure 16, the separation between the two graphs happens later than in Figure 15, and increase for larger values of ρ slower than in Figure 15.
Figures 17 and 18 show the results for the average cost as a function of ρ.

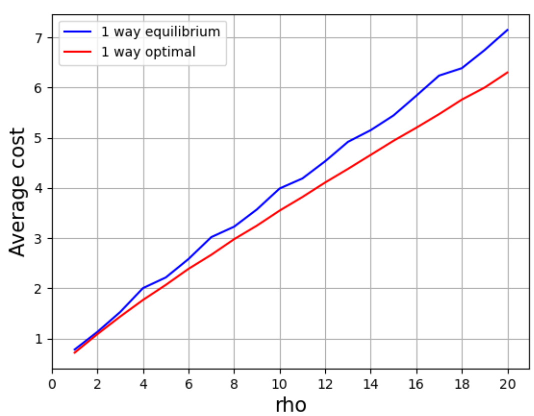

Figure 17: One-way traffic - equilibrium vs. socially-optimal average cost - the lower graph corresponds to socially-optimal costs, and the upper graph to equilibrium costs

Figure 17 shows that the ratio between the equilibrium and optimal costs is almost fixed, since both of the graphs are almost linear.

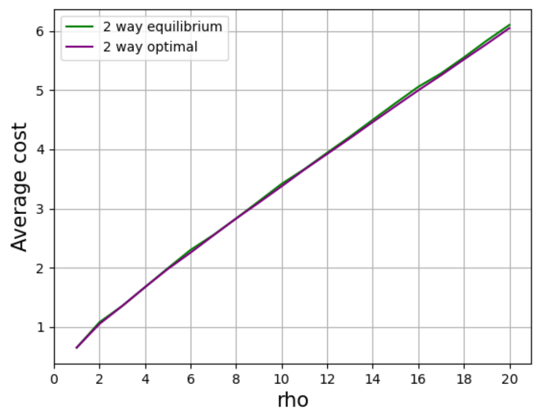

Figure 18: Two-way traffic - equilibrium vs. socially-optimal average cost

Figure 18 shows that the costs are very close in the case of two-way traffic. The reason for this is the flat curve near the minimum point of the graph in Figure 13. The results show that for one-way traffic, the equilibrium and optimal strategies graphs are smoother than in two-way traffic and has a higher effect on the average cost. The strategy graphs slopes are 0.29 and 0.49 for equilibrium and optimal strategies respectively. On the other hand, for two-way traffic, both the equilibrium and socially-optimal strategies are not linear, but as shown in Figure 18, the equilibrium and socially-optimal costs are with the same slope of 0.279.
Figures 19 and 20 show the strategy and average cost as functions of demand for higher values of demand for one-way traffic.

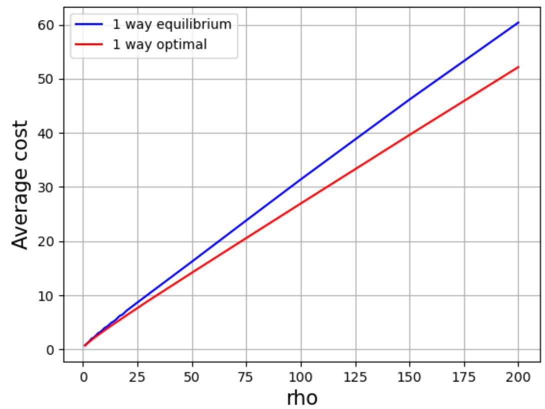

Figure 19: One-way traffic - average cost vs. ρ - the lower graph corresponds to socially optimal costs, and the upper graph to equilibrium costs

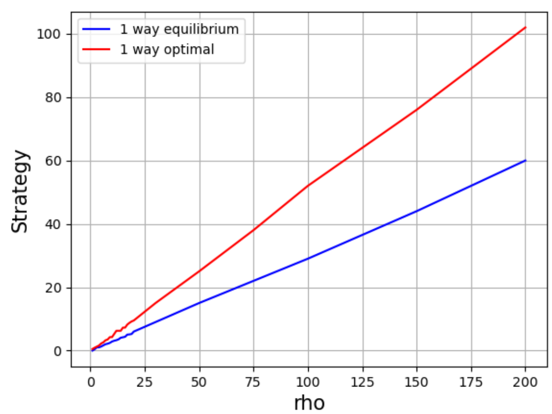

Figure 20: One-way traffic - strategy vs. ρ - the lower graph corresponds to equilibrium strategies, and the upper graph to socially-optimal strategies

In Figure 19, the graph slopes are 0.3 and 0.258 for equilibrium and optimal strategies respectively. Therefore, the ratio between the two costs, which is often called "the price of anarchy", is 1.163.

7.2 One-way vs. Two-way traffic

Figures 21 and 22 show the simulation results for $l(\rho)$ for both one-way traffic and two-way traffic for equilibrium and socially-optimal strategies.

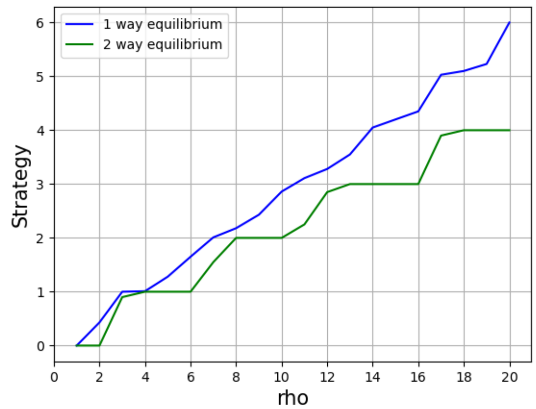

Figure 21: Equilibrium strategy - one-way traffic vs. two-way traffic - the lower graph corresponds to two-way traffic costs, and the upper graph to one-way traffic costs

Figure 21 shows that for low values of ρ, the strategies are very close.

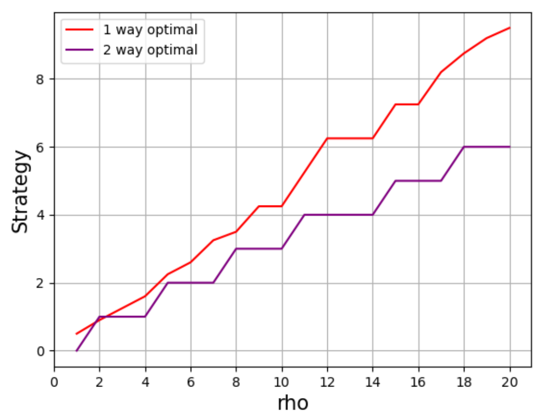

Figure 22: Socially-optimal strategy - one-way traffic vs. two-way traffic - the lower graph corresponds to two-way traffic costs, and the upper graph to one-way traffic costs

In Figure 22, the differentiation is more distinguish, but we get a crossing point at $\rho = 2$ due to integer values constraint for two-way traffic, when looking for mixed strategies we got that the strategy is lower than in one-way traffic. As expected, in two-way traffic, the drivers will look for parking spot closer to the destination. Figures 23 and 24 show the simulation result for $cost(\rho)$ for both one-way traffic and two-way traffic for equilibrium and socially-optimal strategies.

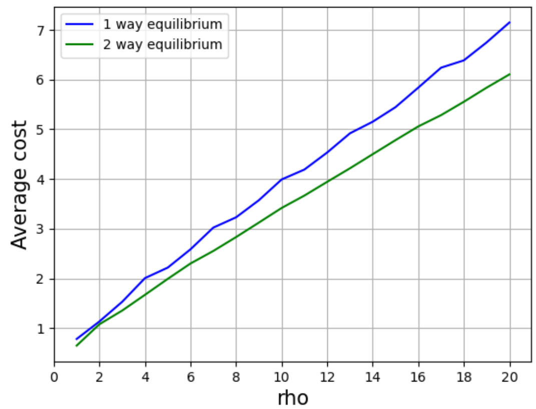

Figure 23: Equilibrium average cost - one-way traffic vs. two-way traffic

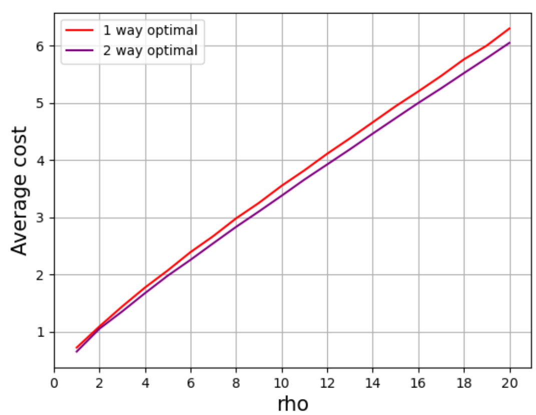

Figure 24: Socially-optimal average cost - one-way traffic vs. two-way traffic

Figure 24 shows that when using wasteful approach (equilibrium strategy), the ratio between one-way traffic and two-way traffic is higher than when using more economical approach. This time we get that the socially-optimal average costs are very close in one-way and two-way traffics (slopes of 0.288 and 0.278 respectively (ratio of 1.036), and R-square of 0.998 for both of them).
The ratios between the equilibrium costs are more significant with slopes of 0.328 and 0.281 (ratio of 1.167), and R-square of 0.998 for both of them.

8 Conclusions

This paper examined the equilibrium and socially-optimal strategies in an infinite one-dimensional road, and found that the equilibrium strategy is smaller than the socially-optimal.
We find that the cost dependency on the demand is almost linear.
We also find that for two-way traffic, there is little difference between the average cost of equilibrium and socially-optimal, therefore, there is no reason to control the drivers strategy.
On the other hand, in a one-way traffic the difference increases as the demand grows, therefore, the city planner might want to intervene at some points during the day, for example, by having a higher parking fee at parking spots that are beyond the destination, and by that, motivate the drivers to start looking for parking earlier.

8.1 Suggestions for future research

In this research we examined a case of one destination with one and two-way traffics. Here are our suggestions for future research:

- Adding and optimizing parking fees.
- Multiple destinations (with same/different arriving rate). The purpose is to examine how a destination with high demand effect a neighbour destination.
- Two-way traffic with non-even arriving rates for more realistic scenario.
- Temporally varying arrival rate.
- N-way traffic ($N > 2$ - cross section).
- Drivers can inspect further parking spots.
- Drivers can communicate information regarding parking availability.

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