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A Cyclic Queueing Game

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by

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Contents

Abstract

We define a cyclic queueing game with a fixed number of players and analyze strategies and symmetric equilibria for three variations. Every unit of time a prize with a random value is offered to the players according to their order in the queue, and a player who accepts a prize moves to the end of the queue. Each player wishes to maximize his expected profit rate. The process of choosing which prizes to accept in each position is presented as a non-cooperative multiplayer game. Our goal is to find a symmetric Nash equilibrium strategy. It turns out that, generally, there exists a unique symmetric equilibrium. We also analyze the price of anarchy and its asymptotic behavior.

1 Introduction

Many real life services work in a cyclic queueing manner. The following illustrative example can be a motivation for the model and its applications: A company provides services and has a finite number of technicians. Every time a service is needed, the job is offered to the technicians according to their order in the queue. When offered a job, a technician can either accept it, or decline it. After service, the technician moves to the end of the queue. We assume that the time of service is shorter than the inter-arrival time, and therefore the technician will finish his job and rejoin the queue before another service is needed.

In this paper we analyze three variation of a cyclic queueing game. The game includes a fixed number n of players waiting in a queue. Every unit of time, which is interpreted as a **round**, a prize is offered to the player at the head of the queue. Once offered a prize, a player may accept the prize and move to the end of the queue, or decline the prize and keep his position in the queue. If a player declines a prize, the prize is immediately offered to the next player in the queue. After a prize is taken, a new round begins, i.e., a new prize is offered to the player at the head of the queue. The prize values are i.i.d positive random variable, and the players maximize their expected profit (the expected value of the prize they take) per round.

A pure **strategy** determines for every position which prizes to receive and which prizes to decline.

A player's expected average profit per round, or expected profit rate, E, of a strategy in this game, is calculated per **cycle**, meaning, from the moment he is the last in the queue (immediately after receiving a prize) until the next moment he is the last in the queue (immediately after receiving a prize). We will simply write "**profit**" to describe the expected profit rate. According to renewal theory, the profit can be calculated as the following ratio:

$$E = \frac{\text{Expected prize value}}{\text{Expected cycle length}}. \tag{1}$$

In the first variation analyzed in this paper, there are two possible prize values, $0 < b < a$. The prize value is a with probability p and b with probability $1-p$. The best-response strategies are defined by thresholds, i.e., the minimum positions in the queue from which a player accepts prize b.

For the first variation, there exists (in general) a unique pure symmetric equilibrium. Two ways of solution are presented, one by calculating the best response, and the other by calculating maximum-profit strategies.

In the second variation, there are two values, $0 < b < a$, and every new prize is a realization of the continuous uniform distribution $U(b, a)$. The best-response strategies are defined by thresholds per positions, $(c_1, c_2, \ldots, c_n)$ such that a player in position k accepts a prize x iff $x \geqslant c_k$.

For the second variation, there exists a unique symmetric equilibrium. The solution is calculated recursively, by computing the maximum profit threshold c_k in the k-th stage.

In the third variation, the players have different preferences, and prize values for different players are independent. For every player, a prize value is a with probability p and b with probability $1 - p$, for $0 < b < a$. The best-response strategies are defined by thresholds. There exist externalities in this variation, as a player may decide to accept a prize which is valued b for him and is valued a for a player who is behind him in the queue.

For the third variation, there exists a unique mixed symmetric equilibrium. Zones of the unique equilibrium strategies as a function of the parameter values $\frac{a}{b}$ and p are presented. For a given n, the maximum price of anarchy is a linear function of the equilibrium threshold strategy, and it is bounded by 2. For $n \to \infty$ the equilibrium strategy is not a function of p, and the price of anarchy approaches 1.

2 Literature review

Strategic behavior in queues has been extensively studied, dating back to Naor (1969). For surveys of the literature on strategic behavior in queues see Hassin and Haviv (2003) and Hassin (2016). The model discussed in this paper is new and has not been studied in this specific form. However, it draws similarities to previous models. In this section we mention related papers.

In some rational queueing models, players are offered a prize in order of their position in a queue, however, they do not return to the end of the queue after receiving a prize, but rather leave the queue.

Su and Zenios (2004, 2006) examine models for analyzing a Poisson process of kidney allocation in an overloaded kidney transplant waiting system. The first model is a queue with homogeneous patients. Different organs have different values, and patients have the option to refuse an organ offer if they expect future offers to be better. In the second model, there are several transplant queues corresponding to different candidate and kidney types. Candidates report their type strategically to join the queue that maximizes their utility.

Leshno (2017) studies the problem of a planner who allocates stochastically arriving items, such as public housing units, to agents in an overloaded waiting list. Agents arrive sequentially over time and have privately known preferences. An agent expecting a long wait for his preferred item will choose a mismatched item, and cause externalities. The planner will try to encourage agents to decline mismatched items by promoting in the queue agents who decline an item. The optimal policy requires the planner to have precise knowledge of the environment and requires agents to have correct beliefs. Therefore, a simpler policy is presented. This policy is more robust and performs almost as well as the optimal policy.

There are some strategic behavior models with cyclic perspective. Usually in models of queues with feedback, the customer returns to the end of the queue forcibly, as a result of an unsuccessful service. In our model, he begins a new cycle, after a successful service.

Lu, Van Mieghem, and Savaskan (2009) examine rework allocation schemes, in which rework is assigned back to the server who generates the defect or to another server dedicated to rework.

Brooms and Collins (2013) consider a FCFS queue with Bernoulli feedback. Each arriving customer joins the system, or balks, on the basis of the number of customers already present. The sojourn time of a customer already in the system depends on the joining decisions taken by future arrivals to the system.

Xia (2014) discusses closed Jakson networks from game-theoretical perspective. The payoff function consists of a holding cost and an operating cost, and servers optimize their service-rate strategy to maximize average payoff.

Wang and Honnappa (2017) dilate the interplay between feedback routing and strategic arrival behavior in single class queueing networks.

There are some models with the feature of cyclic queues, or customers returning to the end of the queue, without the element of games and strategic behavior.

Koenigsberg (1958) analyzes queues in which after a unit completes service, it rejoins the queue at the initial stage.

Finch (1959) and van den Berg and Boxma (1988) analyze cyclic queues with feedback. Chow (1980) analyzes cycle-time distribution of cyclic queues. Koenigsberg (1982) reviews cyclic queues development along the years. Kouvatsos and Almond (1988) analyze two-station cyclic queues. Liu and Buzacott (1992) extend the reversibility property of production lines to the case of cyclic queues and analyze the throughput of the system if the direction of the customer movement in a cyclic queue system is reversed.

Boxma and Cohen (1991) consider a queue with two types of customers: ordinary customers who arrive according to a Poisson process, and permanent customers, who immediately return to the end of the queue after having received a service. Peköz and Joglekar (2002) consider a queue with delayed customer feedback, where upon service completion customers may return to the end of the queue after an independent general feedback delay. Choi, Kim and Choi (2003) consider a queue with multiple types of feedback, and suggest the option of customers returning to the end of the queue as a result of an unsuccessful service.

Cyclic queueing games in the form described in this paper are yet to be analyzed.

3 First variation - basic game

In the basic game analyzed in this paper there are n players waiting in queue for prizes of two positive random values. Every round, a prize is offered to the player at the head of the queue. Once facing a prize, a player may choose to receive the prize and move to the end of the queue, or to decline the prize and keep his position in the queue. If a player declines a prize, the prize is immediately offered to the next player in the queue. Every new prize is valued a with probability p and b with probability $1-p$, $0 < b < a$.

A strategy is a subset A of $\{1, \dots n\}$, and a player in position k accepts a prize b iff $k \in A$. Players always accept a prizes.

For any integer $1 \leqslant k \leqslant n-1$, being in position k is strictly better than being in position $k+1$. Therefore, if it is optimal to accept b in k, then it is optimal to accept b in $k+1$. Therefore, all the best-response strategies are threshold strategies, and every symmetric equilibrium strategy is a threshold strategy.

Therefore, we are looking for a symmetric equilibrium strategy of the form k, where players in positions $\geqslant k$ accept any prize, and players in positions $< k$ only accept a.

We will prove in the rest of this section that in general, there exists a unique symmetric equilibrium strategy, $k = \left\lceil \frac{np(a-b)}{p(a-b)+b} \right\rceil$.

In subsection 3.1 we present a solution by calculating the best responses, and in subsection 3.2 we present an interpretation by calculating the maximum-profit strategies.

Generally in this section, our focus is on pure strategies. As we will prove, it is enough in order to find an equilibrium. A discussion about behavior strategies is presented in section 6. We prefer to avoid mixed strategies, as explained in the conclusion in section 7.

3.1 Conventional solution

For a symmetric equilibrium strategy, we assume everyone's strategy is k and my strategy is k'. For which values of k, is my best response $k' = k$?

For $k' = k$ (I accept b in the same positions as others accept b), according to Equation (1), my profit rate is:

$$\begin{aligned} E &= \frac{pa + (1-p)b}{(n-k) + 1 + \frac{k-1}{p} \cdot p + 0 \cdot (1-p)} \\ &= \frac{pa + (1-p)b}{n}. \end{aligned}$$

It takes $n - k$ rounds after I was at the end of the queue until I reach position k (since there are players who are ahead of me and will accept any prize). Then a new prize is offered. If the prize is a (with probability p), then the first player in the queue will take it, and I am now in position $k - 1$. Therefore, I will accept only a prizes from now on (by the definition of the solution) - and it will take $\frac{k-1}{p}$ rounds, in expectation, until an a prize is offered to me. If the prize is b (with probability $1-p$), then the first player to accept it is the player in position k (by the definition of the solution), i.e., me. This result is also intuitive, since the expected value of the prize is $pa + (1-p)b$, and the profit is divided by n, the number of the identical players in the system.

If $k' > k$ (others accept b in positions where I reject b), then, according to Equation (1):

$$\begin{aligned} E &= \frac{a}{(n-k) + \frac{k}{p}} \\ &= \frac{pa}{k + p(n-k)}. \end{aligned}$$

It takes $n - k$ rounds after I was at the end of the queue until I reach position k (since there are players who are ahead of me and will accept any prize). In this position and further on, I will accept only a prizes, and it will take $\frac{k}{p}$ rounds, in expectation, until an a prize is offered to me.

For equilibrium, we need $k' > k$ to be less profitable than $k' = k$. In this case, we say that $k' > k$ is **unprofitable**. $k' > k$ is unprofitable if $\frac{pa+(1-p)b}{n} \geqslant \frac{pa}{k+p(n-k)}$. Equivalently, $k \geqslant \frac{np(a-b)}{p(a-b)+b}$, or (since k is an integer), $k \geqslant \left\lceil \frac{np(a-b)}{p(a-b)+b} \right\rceil$.

In the range of $k' < k$ (I accept b in positions where others reject b), we check only the case of $k' = k-1$. This is sufficient because, as we now explain, if there is a profitable integer strategy $k_0 < k$, then $k' = k-1$ must be profitable as well (meaning, while all the others play k, my profit is higher if I play $k-1$ instead of k). This can be seen by looking at the following decision problems: If there is a profitable integer strategy $k' < k$, then there is a best response in this group of strategies, k'_0, and it is profitable. If the best response is $k'_0 < k-1$, then it is profitable to accept b in **position** k'_0 (otherwise, $k'_0 + 1$ is a better response then k'_0, and k'_0 is not the best response of the group of strategies $k' < k$).

But, if it is profitable to accept b in position $k'_0 < k-1$, then, all the more, it is profitable to accept b in position $k-1$ (since the expected profit of staying in a position is monotonically decreasing with the position), and therefore the **strategy** $k-1$ is profitable (while all the others play k). All in all, if the best response is $k'_0 < k-1$, then $k-1$ is profitable.

In this case ($k' = k-1$), according to Equation (1):

$$\begin{aligned} E &= \frac{ap^2 + b(1-p^2)}{(n-k) + 1 \cdot (1-p) + 2 \cdot p(1-p) + (2 + \frac{k-2}{p}) \cdot p^2} \\ &= \frac{ap^2 + b(1-p^2)}{n - (1-p)(k-1)}. \end{aligned}$$

It takes $n-k$ rounds after I was at the end of the queue until I reach position k (since there are players who are ahead of me and will accept any prize). If one of the next two prizes will be b, I will accept it. Otherwise, I will have to wait for an a prize.

For equilibrium, we demand $\frac{pa+(1-p)b}{n} \geqslant \frac{ap^2+b(1-p^2)}{n-(1-p)(k-1)}$. This is equivalent to $k-1 \leqslant \frac{np(a-b)}{p(a-b)+b}$, or (since k is an integer, and only if $\frac{np(a-b)}{p(a-b)+b}$ is not an integer), $k \leqslant \left\lceil \frac{np(a-b)}{p(a-b)+b} \right\rceil$.

In conclusion, we proved that the only symmetric equilibrium strategy is $k = \left\lceil \frac{np(a-b)}{p(a-b)+b} \right\rceil$, i.e., a player in position k agrees to receive prize b iff $k \geqslant \frac{np(a-b)}{p(a-b)+b}$.

The exception is when $\frac{np(a-b)}{p(a-b)+b}$ is integer, in which there are two symmetric equilibrium strategies, $k = \frac{np(a-b)}{p(a-b)+b}$ and $k = \frac{np(a-b)}{p(a-b)+b} + 1$.

Since $n > \frac{np(a-b)}{p(a-b)+b}$, we can see that a player in position n agrees to receive any prize that is offered to him, which is reasonable, as the prizes are positive and he can receive the prize **and** keep his position in the queue (as he is the last in the queue one way or another).

3.2 Intuitive interpretation

In subsection 3.1 we found the equilibrium solution by looking for the best response. In this subsection, we find it directly by looking for maximum profit.

When a prize b is offered to a player in position k, what are his options? He can accept the prize, or decline it and wait for a better prize to come. Notice that "time equals money", and a round you wait after declining a prize is a wasted round. Our first question here is, how much wasted? What is the cost of a wasted round? Since we consider an infinite cyclic game and symmetric strategies of homogeneous players, we can analyze wasted time as following: Every round, i.e., every time a prize appears, the prize's expected value is $pa + (1-p)b$. The game is infinite and cyclic and the strategies are symmetric, therefore, the expected profit of each one of the n homogeneous players from having another general playing round is $\frac{pa+(1-p)b}{n}$ - and this is the cost of a wasted round.

Equipped with the cost of a wasted round, we now find the most profitable decision, depending of the player's position. When a prize b is offered to a player in position k, by receiving it, he earns b. By declining it, he will finally receive a, but only after k rounds of an a prize (if it is profitable for a player in position k to decline b, all the more it is profitable for players in positions $< k$ to decline b), meaning, $\frac{k}{p}$ rounds, in expectation.

Therefore, receiving b in position k is profitable iff $b \geqslant a - \frac{k}{p} \cdot \frac{pa+(1-p)b}{n}$, or $k \geqslant \frac{np(a-b)}{p(a-b)+b}$, exactly as we proved in subsection 3.1. Notice that for $k = \frac{np(a-b)}{p(a-b)+b}$ the player is indifferent: His expected profit is identical whether he receives b or declines it. Therefore, if $\frac{np(a-b)}{p(a-b)+b}$ is an integer, then there are two profit-maximizing strategies with different thresholds and identical profit: $k = \frac{np(a-b)}{p(a-b)+b}$ and $k = \frac{np(a-b)}{p(a-b)+b} + 1$.

Hence, if there is a symmetric equilibrium, it is unique and the symmetric strategy is $k = \left\lceil \frac{np(a-b)}{p(a-b)+b} \right\rceil$.

Another interpretation is as follows: I am in position k and offered b. If I accept it I will return to the same position after $n-k$ rounds, so it is worth accepting b iff $b > (n-k) \cdot \frac{pa+(1-p)b}{n}$. This is equivalent to the condition $k > \frac{np(a-b)}{p(a-b)+b}$.

4 Second variation - uniformly distributed prizes

In the second variation, there are two values, $0 < b < a$, and the value of every new prize x has a uniform distribution, $x \sim U[b, a]$. A strategy is a sequence of sets $(C_1, \ldots, C_n)$, and a player in position k accepts a prize x iff $x \in C_k$.

We are looking for a symmetric equilibrium strategy. For any integer $1 \leqslant k \leqslant n-1$, being in position k is strictly better than being in position $k+1$. Therefore, if it is optimal to accept c in k, then it is optimal to accept c in $k+1$. Therefore, a best response strategy is a sequence of thresholds $(c_1, \ldots, c_n)$, such that $c_n \leqslant c_{n-1} \leqslant \ldots \leqslant c_2 \leqslant c_1$ and a player in position k accepts a prize x iff $x \geqslant c_k$.

We will prove that there exists a unique symmetric equilibrium strategy, with $c_k = a - \sqrt{\frac{k}{n}(a^2 - b^2)}$, by a similar method to the one used in subsection 3.2. The solution is calculated recursively, by computing the maximum profit threshold c_k in every stage.

The expected profit of a round in this variation is $\frac{b+a}{2}$, and therefore, the expected profit of each of the n homogeneous players from having another general playing round is $\frac{b+a}{2n}$. This is the cost of a wasted round.

We can calculate the value of every c_k in the equilibrium strategy, as the strategy with the maximal profit expectation for a player in position k, assuming $c_1, \ldots, c_{k-1}$ are given. First, we define f_k as the expected profit to the player in position k until the first time he returns to the end of the queue, and t_k as the expected number of rounds until the first time the player in position k returns to the end of the queue.

To calculate f_k and t_k, we examine the situation for a player in position k, assuming all the players play $c_1, \ldots, c_{k-1}$ in positions $1, \ldots, k-1$ respectively, as these are the strategies with the maximal profit expectation for these positions.

A new prize x is offered to the players. What can happen? If $x \in [c_{k-1}, a]$, it will be taken by a player in position prior to k, and the player in position k advances one position in the queue. If $x \in [c_k, c_{k-1})$, it will be taken by the player in position k, and its expected value is $\frac{c_{k-1}+c_k}{2}$ (since x is from a uniform distribution, then $\{x \in [b, a] \mid x \in [c_k, c_{k-1})\}$ is also from a uniform distribution). In this case, the player in position k moves to the end of the queue. If $x \in [b, c_k)$, the player in position k declines the prize, and keeps his position in the queue. Therefore, for $k = 2, \ldots, n$:

$$f_k = \frac{a - c_{k-1}}{a - b} \cdot f_{k-1} + \frac{c_{k-1} - c_k}{a - b} \cdot \frac{c_{k-1} + c_k}{2} + \frac{c_k - b}{a - b} \cdot f_k$$
$$(a - c_k) \cdot f_k = (a - c_{k-1}) \cdot f_{k-1} + \frac{c_{k-1}^2 - c_k^2}{2}.$$

The solution to the recursive equation is $f_k = \frac{a-c_1}{a-c_k} \cdot f_1 + \frac{c_1^2 - c_k^2}{2(a-c_k)}$. Clearly, $f_1 = \frac{a+c_1}{2}$ and therefore $f_k = \frac{a+c_k}{2}$. Moreover, this result can be proven directly by induction, as the distribution of the prize value received by the player in position k is $U[c_k, a]$: The distribution of the relevant prizes, $x \geqslant c_k$, is $U[c_k, a]$. With probability $\frac{a-c_{k-1}}{a-c_k}$ the first prize will be in $[c_{k-1}, a]$, therefore the player in position k will move to position $k-1$ and then by the induction hypothesis the distribution of his prize value will be $U[c_{k-1}, a]$. With probability $\frac{c_{k-1}-c_k}{a-c_k}$ the first prize will be in $[c_k, c_{k-1}]$, therefore the player in position k will receive it and then his prize value will be $U[c_k, c_{k-1}]$. A distribution $U[c_{k-1}, a]$ with probability $\frac{a-c_{k-1}}{a-c_k}$ and $U[c_k, c_{k-1}]$ with probability $\frac{c_{k-1}-c_k}{a-c_k}$ is exactly a distribution $U[c_k, a]$.

We proved $f_k = \frac{a+c_k}{2}$. Similarly, we can solve the recursive equation of t_k:

$$t_k = 1 + \frac{a - c_{k-1}}{a - b} \cdot t_{k-1} + \frac{c_k - b}{a - b} \cdot t_k$$
$$(a - c_k) \cdot t_k = (a - c_{k-1}) \cdot t_{k-1} + (a - b)$$
$$t_k = \frac{a - c_1}{a - c_k} \cdot t_1 + \frac{k(a - b)}{a - c_k}$$
$$t_1 = \frac{a - b}{a - c_1}$$
$$t_k = k \cdot \frac{a - b}{a - c_k}.$$

Now we can find c_k. When a prize x is offered to a player in position k, by receiving it, he earns x. By declining it, he will receive f_k in expectation, after t_k rounds in expectation. For $x = c_k$ the player is indifferent between these two options, and therefore $c_k = f_k - \frac{b+a}{2n} \cdot t_k$. Taking into consideration the value of a new prize is bounded by a, the unique solution is $c_k = a - \sqrt{\frac{k}{n}(a^2 - b^2)}$.

In particular, $c_n < b$. Meaning, a player in position n accepts any prize $x \in [b, a]$ that is offered to him, which is reasonable, as all the prizes are positive and he can receive the prize **and** keep his position in the queue (as he is the last in the queue one way or another).

Another interpretation is as follows: Taking into consideration the cost of wasted rounds, the strategy with the maximal profit expectation for a player in position k is $\arg\max_{c_k}[f_k - \frac{b+a}{2n} \cdot t_k] \equiv \arg\max_{c_k} F_k(c_k)$. We can calculate $F_k(c_k)$ and find its maximum by solving $F_k'(c_k) = 0$ and checking $F_k''(c_k) < 0$. Taking into consideration the value of a new prize is bounded by a, the unique solution is $c_k = a - \sqrt{\frac{k}{n}(a^2 - b^2)}$.

5 Third variation - unrelated prizes

In the third variation, the players have different preferences, and prize values for different players are independent. For every player, a prize value is a with probability p and b with probability $1-p$, for $0 < b < a$.

A pure threshold strategy in this game is an integer k, such that players in positions $\geqslant k$ accept any prize, and players in positions $< k$ only accept a. A mixed strategy which assigns a probability r to the pure threshold strategy $k+1$ and a probability $1-r$ to the pure threshold strategy k is indicated $s = k+r$. We refer to this strategy as "I decline b in position k with probability r". Then, for $r \in \{0, 1\}$, the mixed strategy $k+r$ is identical to the pure strategy $k+r$. Notice that the random decision whether to accept b in position k or not is determined only once, at the beginning of the game, and not every time the player is offered b in position k. A discussion about behavior strategies is presented in section 6.

Generally, a mixed strategy will assign a probability $0 \leqslant r_k \leqslant 1$ to accept b in position k for any $k \in \{1, \ldots, n\}$. For a given set of strategies for all the other players, there can be no more than one position in which I am indifferent whether to receive b and move to position n, or decline b and keep my position in the queue (since every position k is strictly more profitable to be in than the position $k+1$, while b and the profit of position n are fixed). Therefore, if k is the position in which I am indifferent (for position$< k$ I accept only a, for position k I am indifferent and for position$> k$ I accept any prize), then all my best response mixed strategies are $k+r$, for $0 < r < 1$. If there is no position in which I am indifferent, then there are no best-response mixed strategies.

We are looking for a mixed symmetric equilibrium strategy.

First, we calculate the profit of a pure equilibrium strategy, in subsection 5.1. Second, we check the conditions for a pure symmetric equilibrium, in subsection 5.2. Finding it complicated to explicitly present the symmetric equilibrium solution, we examine the special cases of $n = 2$ and $n = 3$, in subsections 5.3 and 5.4, respectively. Motivated by the results of these cases, we examine numerically the more general case of $n = 10$ in subsection 5.5, and the results support the conjecture the equilibrium is unique. Analysis for asymptotic cases is presented in subsection 5.6. Last, a discussion about the social optimum and the price of anarchy appears in subsection 5.7.

5.1 Profit of a pure symmetric strategy

The first interesting difference from the analysis of the basic game is that in this variation, the average profit of a player from a symmetric strategy is not fixed, but rather depends on the strategy: If the symmetric strategy is k, then, $n-k$ rounds after I was at the end of the queue, I reach position k (since there are players who are ahead of me and will accept any prize). Then, a new prize is offered to the players. If the prize will be b for all of the $k-1$ players who are ahead of me (with probability $(1-p)^{k-1}$), then they will decline it and I will accept the prize (regardless of its value for me). Otherwise, a player who is ahead of me will receive the prize. In this case, I will advance to position $k-1$, and will accept only a prizes from now on. What is the expected number of rounds until an a prize will be offered to me? To answer this, we will use a recursion.

We define N_i as the expected number of rounds until an a prize is offered to a player in position i for $i \leqslant k-1$. Our goal is to find N_{k-1}. The base case is $N_1 = \frac{1}{p}$ and the recursion step is $N_i = 1 + (1-p)^{i-1}p \cdot 0 + (1-p)^i \cdot N_i + [1-(1-p)^{i-1}] \cdot N_{i-1}$, representing the following options for the next round: The next prize will be b for all the players who are ahead of me and a for me (an a prize is offered to me), the next prize will be b for both the players who are ahead of me and myself (I decline it and keep my position in the queue), or the next prize will be a for a player who is ahead of me (I advance in the queue). This is equivalent to $f_i N_i = f_{i-1}N_{i-1}+1$ for $f_i = 1-(1-p)^i$. Therefore, $N_i = \frac{f_1 N_1 + i - 1}{f_i} = \frac{i}{1-(1-p)^i}$ for every $i \leqslant k-1$. In particular, $N_{k-1} = \frac{k-1}{1-(1-p)^{k-1}}$.

Having N_{k-1}, we can use Equation (1) for the profit of the symmetric strategy k:

$$\begin{aligned} E &= \frac{(1-p)^{k-1} \cdot [pa + (1-p)b] + [1-(1-p)^{k-1}] \cdot a}{(n-k+1) + (1-p)^{k-1} \cdot 0 + [1-(1-p)^{k-1}] \cdot N_{k-1}} \\ &= \frac{a \cdot [1-(1-p)^k] + b \cdot (1-p)^k}{n}. \end{aligned} \tag{2}$$

This result is also intuitive, since the expected value of the prize a player receives is $a \cdot [1-(1-p)^k] + b \cdot (1-p)^k$ (a player receives b iff there are k prizes b in a row, from the moment he reaches position k), and the profit is divided by n, the number of the identical players in the system.

5.2 Conditions for a pure symmetric equilibrium

For a pure symmetric equilibrium strategy, assume everyone's strategy is k. For which values of k, is my best response $k' = k$? According to Equation (2), if $k' = k$, then the profit of each player is $E = \frac{a \cdot [1-(1-p)^k] + b \cdot (1-p)^k}{n}$.

If $k' > k$ (others accept b in positions where I reject b), then, according to Equation (1):

$$\begin{aligned} E &= \frac{a}{(n-k) + N_k} \\ &= \frac{a}{n - k + \frac{k}{1-(1-p)^k}}. \end{aligned} \tag{3}$$

It takes $n-k$ rounds after I was at the end of the queue until I reach position k (since there are players who are ahead of me and will accept any prize). In this position and further on, I will accept only a prizes, and it will take N_k rounds until an a prize is offered to me.

In the range of $k' < k$ (I accept b in positions where others reject b), we check only the case of $k' = k-1$, by a similar justification to the one used in the basic game in subsection 3.1. In this case ($k' = k-1$), according to Equation (1), and with the notation $q := 1-p$:

$$\begin{aligned} E &= \frac{[q^{k-1} + (1-q^{k-1})q^{k-2}](pa+qb) + (1-q^{k-1})(1-q^{k-2})a}{(n-k) + q^{k-1}\cdot 1 + (1-q^{k-1})q^{k-2}\cdot 2 + (1-q^{k-1})(1-q^{k-2})\cdot(2+N_{k-2})} \\ &= \frac{a + (b-a)[(1-p)^{k-1} + (1-p)^k - (1-p)^{2k-2}]}{n + (1-p)^{k-1} - k(1-p)^{k-1}}. \end{aligned} \tag{4}$$

It takes $n-k$ rounds after I was at the end of the queue until I reach position k (since there are players who are ahead of me and will accept any prize). If one of the next two prizes will be b to all the players who are ahead of me (probability $(1-p)^{k-1}$ for the first of them, probability $[1-(1-p)^{k-1}](1-p)^{k-2}$ for the second), I will accept it (regardless of its value to me). Otherwise, I will reach position $k-2$ and will wait another N_{k-2} rounds for an a prize.

For a symmetric equilibrium, we demand both $k' > k$ and $k' < k$ to be less profitable, compared to $k' = k$. In the general case, solving these conditions analytically seem to be very complicated. Nevertheless, the existence of a symmetric equilibrium is guaranteed due to Nash's Existence Theorem, if we allow mixed strategies (the conditions of the theorem are satisfied, as the game is symmetric, the number of players, n, is finite, and the number of pure strategies each player can choose from, n, is finite as well).

In the next subsections we will examine in detail the special cases of $n = 2$ and $n = 3$, and explicitly present the unique symmetric equilibrium in each of these cases. Afterwards, we will calculate the equilibrium numerically for some arbitrarily selected parameter values, and check if it is indeed unique.

5.3 The game with $n = 2$

In this subsection we analyze the case $n = 2$ in three ways: intuitive, analytic and graphic.

5.3.1 Abstract explanation of ATC

In the two players game, the best responses of the players match the concept of *avoid the crowd* (ATC), as will be shown immediately. Therefore, for every set of parameters $\frac{a}{b}$ and p, one and only one symmetric equilibrium exists.

In this game (for a general n), ATC means that the individual's best response threshold is a nonincreasing function of the other players' threshold.

For $n = 2$, the strategy of a player is the probability he will accept a prize b in position 1, as everyone accepts an a prize everywhere, and everyone accepts a b prize in the last position (which is position 2, for the game with $n = 2$).

The question of ATC is how the probability p' the other player will accept a prize b in position 1 affects the probability p I will accept a prize b in position 1.

When offered a prize b in position 1, I can either accept it, or decline it and wait a fixed number of rounds, in expectation, until a prize a is offered to me. The number of rounds I wait is independent of the other player's strategy. Generally in the game, when I am in position 2 I prefer to get as many offers as possible (rather than "to waste" a round in order to move to position 1). In other words, the lower p' is, the more value I have from rounds in the game (the value of being first does not chance, and the value of being second increases). Meaning, if p' decreases, then it becomes less profitable for me (than before) to decline a prize b in position 1 and "waste rounds". Therefore, the game matches the concept of ATC.

5.3.2 The game in normal-form

Using the conditions from subsection 5.2 for the case of $n = 2$ provides a complete solution for the game with two homogeneous players and two pure strategies in normal-form. Every player has two possible pure strategies, 1 and 2, and we are looking for a symmetric equilibrium. Let $E_{i,j}$ denote the profit of a player if his strategy is i and the other player's strategy is j. According to Equations (2), (3) and (4), we have $E_{1,1} = \frac{ap+b(1-p)}{2}$, $E_{1,2} = \frac{ap+b(1-p)}{1+p}$, $E_{2,1} = \frac{ap}{1+p}$ and $E_{2,2} = \frac{ap+b(1-p)+p(1-p)(a-b)}{2}$. Then, $(1,1)$ is equilibrium iff $E_{1,1} \geqslant E_{2,1}$, or $\frac{a}{b} \leqslant 1 + \frac{1}{p}$. Likewise, $(2,2)$ is equilibrium iff $E_{2,2} \geqslant E_{1,2}$, or $\frac{a}{b} \geqslant 1 + \frac{1}{p^2}$. Hence, there cannot exist two pure symmetric equilibria at the same time (since $1 + \frac{1}{p} < 1 + \frac{1}{p^2}$ for $0 < p < 1$), and for $1 + \frac{1}{p} < \frac{a}{b} < 1 + \frac{1}{p^2}$ there are no pure symmetric equilibria at all. The complete collection of best responses appears in Table 1.

Table 1: Best responses for $n = 2$

Parameters' range	**Player two plays** 1	**Player two playes** 2	**Symmetric equilibrium**
$1 < \frac{a}{b} \leqslant 1 + \frac{1}{p}$	Player one plays 1	Player one plays 1	Pure $(1, 1)$
$1 + \frac{1}{p} < \frac{a}{b} < 1 + \frac{1}{p^2}$	Player one plays 2	Player one plays 1	Mixed $(1 + r, 1 + r)$
$1 + \frac{1}{p^2} \leqslant \frac{a}{b} < \infty$	Player one plays 2	Player one plays 2	Pure $(2, 2)$

We will now use Table 1 to provide a graphical representation of the dependence of the equilibrium value on the parameters' range, and presenting equilibrium zones. All different mixed equilibria are clustered in Figure 1 in the same zone, the zone of $k = 1$ and $0 < r < 1$.

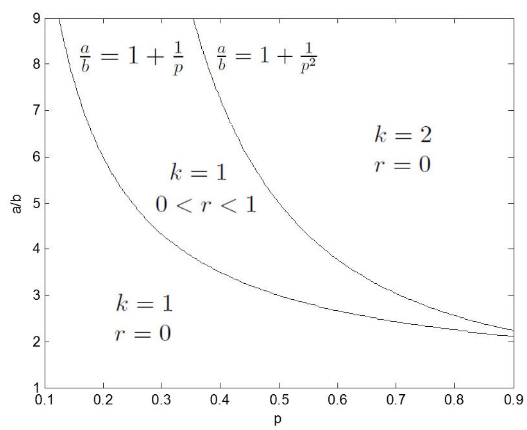


Figure 1: Equilibrium zones, $n = 2$

5.4 The game with $n = 3$

In this subsection we increase the number of players from two to three, and compute the profit of every pure strategy k' of player one, as a response for every pure strategy k of the other players, using Equations (2), (3) and (4). Having this information, we identify all the symmetric equilibria in the game, as presented in Table 2.

Table 2: Equilibriums for $n = 3$

Parameters' range	**Symmetric equilibrium**
$1 < \frac{a}{b} \leqslant 1 + \frac{1}{2p}$	Pure $(1, 1)$
$1 + \frac{1}{2p} < \frac{a}{b} < 1 + \frac{1}{p^2+p}$	Mixed $(1 + r, 1 + r)$
$1 + \frac{1}{p^2+p} \leqslant \frac{a}{b} \leqslant 1 + \frac{2}{2p-p^2}$	Pure $(2, 2)$
$1 + \frac{2}{2p-p^2} < \frac{a}{b} < 1 + \frac{2}{3p^2-2p^3}$	Mixed $(2 + r, 2 + r)$
$1 + \frac{2}{3p^2-2p^3} \leqslant \frac{a}{b} < \infty$	Pure $(3, 3)$

The first and most important conclusion from the results is the existence and uniqueness of the symmetric equilibrium, for every given parameters $\frac{a}{b}$ and p, as can be seen in the table. Moreover, the concept of ATC holds for three players as, for every given set of parameters, the best response k' when $k = 3$ is smaller or equal to the best response k' when $k = 2$, which is smaller or equal to the (maximal) best response k' when $k = 1$. Notice that for $k = 1$, there is no difference in practice whether $k' = 2$ or $k' = 3$, as the player may face a prize only when he is first in the queue (since $k = 1$), and therefore the only important aspect of his strategy is his decision in position 1.

The results and conclusions for two and three players support the conjecture the equilibrium is unique for any number of players.

5.5 Numerical results

In this subsection we describe the way to numerically analyze the game for a higher number of players, and present the results for $n = 10$.

For a fixed n, we are looking for symmetric equilibria as a function of the parameters $\frac{a}{b}$ and p. Having specific values of $\frac{a}{b}$ and p, we can check whether k or $k+r$ (for $0 < r < 1$) are equilibrium thresholds, as explained below. Then, we have the function $k = E(\frac{a}{b}, p)$, to return a unique value (*if not unique - print an assertion*) of equilibrium strategy for every set of parameters. The uniqueness is in the meaning that there are not two values, $k \neq k'$, such that both $k+r$ and $k' + r'$ are equilibrium strategies. In order to prove there cannot be two values, $r \neq r'$, such that both $k + r$ and $k + r'$ are equilibrium strategies, we have to prove ATC for the game with n players - which we have not done. Moreover, our condition for mixed equilibria (which is explained below) is sufficient, but we have not proved it is necessary, and therefore there may be mixed equilibria which we do not find. The uniqueness is up to our condition.

The next steps are standard, as we define a range and a step size for the parameters, and plot the zones of the different equilibrium values in the two-dimensional plane.

In order to check whether k or $k+r$ (for $0<r<1$) is an equilibrium for specific values of $\frac{a}{b}$ and p, we use a two-stage process: First, we calculate the best response $k' := B(k)$ of a player for every pure strategy k of all the other players, from the options $k' \in \{k-1, k, k+1\}$, using Equations (2), (3) and (4) for the profit comparisons. Then, for each of the n pure strategies and the $n-1$ mixed strategies ranges in between them, k is an equilibrium iff $B(k)=k$ and a mixed strategy $k+r$ (for $0<r<1$) is an equilibrium iff $B(k)=k+1$ and $B(k+1)=k$. Having this information, it is also straightforward to check the uniqueness of the equilibrium, for these specific parameter values.

The example below is for $n=10$, and the parameters' ranges are as follows: $\frac{a}{b} \in [1.1, 11.1]$ and $p \in [0.1, 0.9]$. The step sizes are chosen such that the number of realizations of each parameter is 1000. The plot is presented in Figure 2. Zones of pure equilibria are uncolored and are indicated by mentioning the equilibrium strategy, from $k=1$ (lower zone) to $k=10$ (upper zone). Zones of mixed equilibrium strategies are colored gray. For each k, all mixed equilibria $\{k+r \mid 0<r<1\}$ are clustered in the same zone.

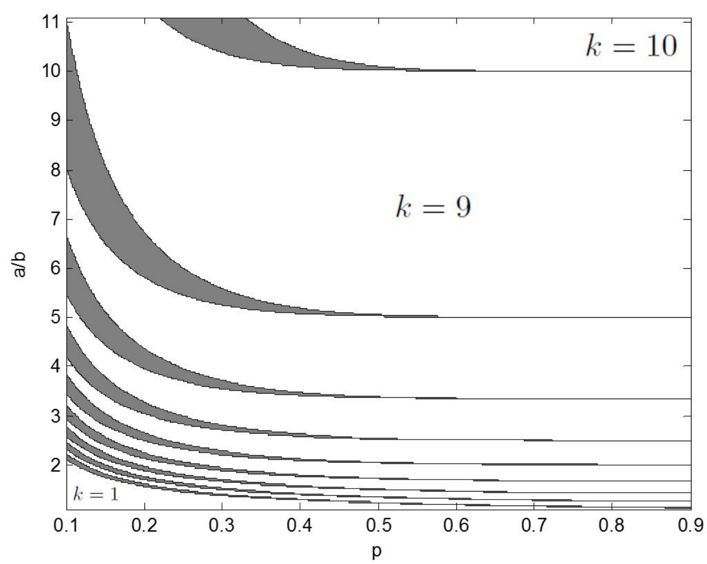


Figure 2: Equilibrium zones, $n=10$

Another result is the uniqueness of the equilibrium for the realizations. Moreover, the equilibrium strategy k is weakly increasing with each of the parameters $\frac{a}{b}$ and p, as one could expect. It is also interesting to see that every possible threshold strategy may be an equilibrium by appropriated parameter values. Moreover, the mixed zones do not disappear for high values of p (although they do become very narrow), and $\forall p \forall k \exists r \in (0,1)$ such that $k+r$ is an equilibrium. The exact behavior of the equilibrium strategy in the extreme values of the parameters will be discussed in the next subsection.

5.6 Asymptotic behavior

In Figure 2 we examined the behavior of the equilibrium strategy k for an arbitrary sample of parameters. In this subsection, we examine it for extreme values of the parameters $\frac{a}{b}$, p and n. We will prove the following properties:

1. $k=1$ iff $\frac{a}{b} \leqslant 1 + \frac{1}{(n-1)p}$ and $k=n$ iff $\frac{a}{b} \geqslant 1 + \frac{n-1}{1-(1-p)^{n-1}[(n-1)p+1]}$.
2. $\lim_{\frac{a}{b} \longrightarrow \infty} k = n$, $\lim_{\frac{a}{b} \longrightarrow 1} k = 1$ and $\lim_{p \longrightarrow 0} k = 1$.
3. $\lim_{n \longrightarrow \infty} \frac{k}{n} = 1 - \frac{b}{a}$ and $\lim_{p \longrightarrow 1} k = \lceil n(1-\frac{b}{a}) \rceil$.

Proof:

1. By comparing the profit in Equation (2) to the profit in Equations (3) and (4), it is possible to explicitly write conditions for $k=1$ and $k=n$ to be equilibria, respectively: $k=1$ is equilibrium iff $\frac{a \cdot [1-(1-p)^1] + b \cdot (1-p)^1}{n} \geqslant \frac{a}{n-1+\frac{1}{1-(1-p)^1}}$, or $\frac{a}{b} \leqslant 1 + \frac{1}{(n-1)p}$, and $k=n$ is equilibrium iff $\frac{a \cdot [1-(1-p)^n] + b \cdot (1-p)^n}{n} \geqslant \frac{a+(b-a)[(1-p)^{n-1}+(1-p)^n-(1-p)^{2n-2}]}{n+(1-p)^{n-1}-n(1-p)^{n-1}}$, or $\frac{a}{b} \geqslant 1 + \frac{n-1}{1-(1-p)^{n-1}[(n-1)p+1]}$.
2. The conditions from property 1 give us immediate asymptotic results for $\frac{a}{b}$: $\lim_{\frac{a}{b} \longrightarrow \infty} k = n$ and $\lim_{\frac{a}{b} \longrightarrow 1} k = 1$. It is also easy to see that $\lim_{p \longrightarrow 0} k = 1$, a behavior that cannot be noticed in Figure 2.
3. For $n \to \infty$, we first prove that $k \to \infty$, and then check what happens to $\frac{k}{n}$. For given $\frac{a}{b}$ and p, and a given strategy k for all the players but one, we will prove that $k' > k$ is more profitable than $k' = k$ for the one player for $n \to \infty$, and therefore $\lim_{n \longrightarrow \infty} k = \infty$. We define his profit from the strategy $k' = k$ as $E_{k'=k}$ and his profit from the strategy $k' > k$ as $E_{k'>k}$. Then, according to Equations (2) and (3), $E_{k'=k} = \frac{a \cdot [1-(1-p)^k] + b \cdot (1-p)^k}{n} = \frac{c_1}{n}$, and $E_{k'>k} = \frac{a}{n-k+\frac{k}{1-(1-p)^k}} = \frac{c_1+\Delta}{n+c_2}$ for some positive constants c_1, c_2 and Δ. Therefore, $k' > k$ is a better response than $k' = k$ if $E_{k'>k} > E_{k'=k}$, or $c_1 c_2 < n\Delta$, which indeed happens for $n \to \infty$.

   Interestingly, for $n \to \infty$ (and therefore $k \to \infty$), or as well for $p \to 1$, the conditions for equilibrium from subsection 5.2 are solvable analytically, and the asymptotic equilibrium is identical for these two limits. For a symmetric equilibrium at k, we demand $k' = k$ to be a better response than both $k' > k$ and $k' < k$.

For the first condition, $k' = k$ is a better response than $k' > k$ if, according to Equations (2) and (3), $\frac{a\cdot[1-(1-p)^k]+b\cdot(1-p)^k}{n} \geqslant \frac{a}{n-k+\frac{k}{1-(1-p)^k}}$, or $0 \leqslant k(a+(b-a)(1-p)^k)+n((b-a)+(a-b)(1-p)^k)$. For both extreme cases, this is equivalent to $0 \leqslant ka+(b-a)n$, or $k \geqslant n(1-\frac{b}{a})$.

For the second condition, $k' = k$ is a better response than $k' < k$ if, according to Equations (2) and (4), $\frac{a\cdot[1-(1-p)^k]+b\cdot(1-p)^k}{n} \geqslant \frac{a+(b-a)[(1-p)^{k-1}+(1-p)^k-(1-p)^{2k-2}]}{n+(1-p)^{k-1}-k(1-p)^{k-1}}$, or $-a(k-1)+(a-b)(k-1)(1-p)^{k-1}(1-p) \geqslant -n(a-b)+n(a-b)(1-p)^{k-1}$. For both extreme cases, this is equivalent to $-a(k-1) \geqslant -n(a-b)$, or $k \leqslant n(1-\frac{b}{a})+1$.

In conclusion, for $n \to \infty$ or $p \to 1$, pure strategy k is equilibrium iff $n(1-\frac{b}{a}) \leqslant k \leqslant n(1-\frac{b}{a})+1$. The only pure strategy to fulfill these demands (for the likely case where $n(1-\frac{b}{a})$ is not an integer) is $k = \lceil n(1-\frac{b}{a})\rceil$. Notice the horizontal separating lines in Figure 2 are exactly the $\frac{a}{b}$ values in which $10\cdot(1-\frac{b}{a})$ is an integer.

5.7 Price of anarchy

In this subsection we will focus on pure strategies.

Unlike the other variations in this paper, for the game with unrelated prize values, the profit of each player from a symmetric strategy k is not fixed, but it rather depends on k. Moreover, the social profit during every round is not fixed, but depends on the players' strategies. The profit of a round is the value of the prize taken in the round, and therefore it is maximal when all the players take only the big prize whenever they can, i.e., when all the strategies are $k = n$. Therefore, for the pure symmetric equilibrium strategy k, the price of anarchy is PoA$=\frac{E_n}{E_k} = \frac{\frac{a}{b}(1-(1-p)^n)+(1-p)^n}{\frac{a}{b}(1-(1-p)^k)+(1-p)^k}$, where E_k is the profit of a player from the symmetric equilibrium strategy k, according to Equation (2). The PoA can be interpreted as the ratio of the expected values of the prize received in a round, under the socially optimal strategy and under the equilibrium strategy.

First, $\lim_{n\longrightarrow\infty} k = \infty$, and therefore $\lim_{n\longrightarrow\infty}PoA= \frac{\frac{a}{b}}{\frac{a}{b}} = 1$.

Moreover, PoA$\to \frac{1}{1} = 1$ for each of $\frac{a}{b} \to 1$ and $p \to 0$, and PoA$\to \frac{\frac{a}{b}}{\frac{a}{b}} = 1$ for $p \to 1$. For $\frac{a}{b} \to \infty$, $k = n$ and therefore PoA$\to 1$, as the socially optimal strategy and the equilibrium strategy are identical.

We conjecture that, for all the cases in which the equilibrium strategy is the integer value k and the number of players is n, then PoA$\leqslant 2-\frac{k}{n}$. Moreover, the bound is achievable asymptotically, with the maximal allowed $\frac{a}{b}$ (as a function of p, k and n, such that k is still the equilibrium strategy) and with $p \to 0$. We will prove it now for each of $n = 2$, $n = 3$, $k = 1$ and $k = n$, and check it numerically for $n = 10$.

The case $k = n$ is trivial, as the socially optimal strategy and the equilibrium strategy are identical, and therefore PoA$= 1 = 2 - \frac{n}{n}$.

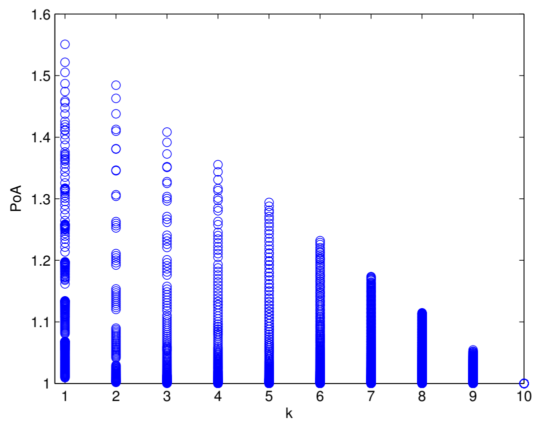
For $k < n$, PoA$= \frac{\frac{a}{b}(1-(1-p)^n)+(1-p)^n}{\frac{a}{b}(1-(1-p)^k)+(1-p)^k}$. The numerator is a weighted mean of $\frac{a}{b}$ and 1 and the denominator is a weighted mean of $\frac{a}{b}$ and 1 with a smaller weight for $\frac{a}{b}$, therefore the quotient is monotone increasing in $\frac{a}{b}$.

For $n = 2$, according to Table 1, we have $k = 1$ for $1 < \frac{a}{b} \leqslant 1 + \frac{1}{p}$ and $k = 2$ for $1 + \frac{1}{p^2} \leqslant \frac{a}{b} < \infty$. Then, for $\frac{a}{b} = 1 + \frac{1}{p}$, PoA$= \frac{(1+\frac{1}{p})(1-(1-p)^2)+1\cdot(1-p)^2}{(1+\frac{1}{p})(1-(1-p)^1)+1\cdot(1-p)^1} = \frac{3-p}{2}$, which is maximal when $p \to 0$ and equals $1\frac{1}{2} = 2 - \frac{1}{2}$.

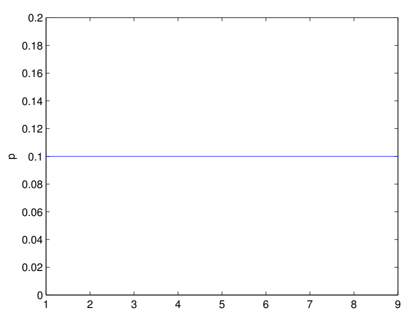
Similarly, for $n = 3$ we use Table 2, and check $k = 1$ and $k = 2$. For $k = 1$ and $\frac{a}{b} = 1 + \frac{1}{2p}$, PoA$= \frac{5-3p+p^2}{3}$, which is maximal when $p \to 0$ and equals $1\frac{2}{3}$. For $k = 2$ and $\frac{a}{b} = 1 + \frac{2}{2p-p^2}$, PoA$= \frac{2p^2-7p+8}{3(2-p)}$, which is maximal when $p \to 0$ and equals $1\frac{1}{3}$.

For general n, we already mentioned that the equilibrium strategy is $k = 1$ iff $\frac{a}{b} \leqslant 1 + \frac{1}{(n-1)p}$. For $k = 1$, PoA$= \frac{\frac{a}{b}(1-(1-p)^n)+(1-p)^n}{\frac{a}{b}\cdot p+(1-p)}$. The PoA is maximal with $\frac{a}{b} = 1 + \frac{1}{(n-1)p}$ and therefore,
PoA$\leqslant \frac{(1+\frac{1}{(n-1)p})(1-(1-p)^n)+(1-p)^n}{(1+\frac{1}{(n-1)p})p+(1-p)} = 1 + \frac{1-p-(1-p)^n}{np}$. By induction on n, $\frac{1-p-(1-p)^n}{np} \leqslant \frac{n-1}{n}$, and therefore PoA$\leqslant 1 + \frac{n-1}{n} = 2 - \frac{1}{n}$. Since $\lim_{p\longrightarrow 0} \frac{1-p-(1-p)^n}{np} = \frac{n-1}{n}$, then the maximal value of the PoA is achieved when $p \to 0$.

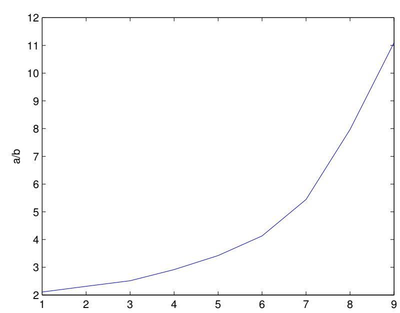
In order to support our conjecture that for a given n, the maximal value of the PoA is a linear function of the equilibrium strategy, we checked it for an arbitrary sample of parameters. We chose $n = 10$, checked some values of parameters for $\frac{a}{b}$ and p, and calculated the equilibrium strategy k and the PoA. All the pairs of (k,PoA) with a pure equilibrium strategy were presented together on a two-dimensional coordinate system in Figure 3(a). It is easy to see the maximal values of the PoA for the sample, and the equilibrium strategy in which it was obtained. Furthermore, for each k, the values of p and $\frac{a}{b}$ of the observation with the maximal PoA were kept, and then presented in separated graphs, in Figures 3(b) and 3(c). The parameters' ranges are as follows: $n = 10$, $\frac{a}{b} \in [1.1, 11.1]$ and $p \in [0.1, 0.9]$. The step sizes of $\frac{a}{b}$ and p are chosen such that the number of realizations of each parameter is 100. Every circle in Figure 3(a) represents an observation. It can be seen in the sample, that the maximal values of the PoA are monotonically decreasing with k, the chosen p values are the minimal possible in the simulation ($p = 0.1$), and the chosen $\frac{a}{b}$ values are monotonically increasing with k (as the maximal allowed $\frac{a}{b}$ is monotonically increasing with k, for fixed n and p).



(a) Strategies and PoA



(b) p of max PoA



(c) $\frac{a}{b}$ of max PoA

Figure 3: Arbitrary Sample

6 Discussion about behavior strategies

Threshold mixed strategies in the basic game and in the game with unrelated prizes are in the form $s = k + r$, $k \in \{1, \ldots, n-1\}$ and $0 \leqslant r \leqslant 1$, such that the player accepts b in positions $> k$, declines b in positions $< k$, and accepts b in position k with probability r. The random decision whether to accept b in position k is determined only once, at the beginning of the game. Another form of strategies can be similar to the mixed strategies, but with one change: The random decision is determined anew every time the player is offered b in position k. This is a behavior strategy. However, the game in which the behavior strategy takes place needs clarification.

While a mixed strategy assigns a probability distribution over pure strategies, a behavior strategy assigns at each information set a probability distribution over the set of possible actions. Yet, in the way we described the game, there is only one information set - at the beginning of the game. Then, the set of possible actions is identical to the set of pure strategies - choosing in which positions the player **always** accepts prize b, and in which positions the player **never** accepts prize b. Strategies such as "accept b in the first time it is offered to you in position k_0, and afterward decline b every time it is offered to you in position k_0" are not acceptable. As a result of this assumption, the number of pure strategies is finite, and the game can be easily presented in normal-form. A disadvantage of this assumption is losing some strategies, and in particular losing behavior strategies in the form described in the first paragraph of this section, which may be very natural to the game.

We can expand the space of pure strategies and allow players to have different decisions for different rounds, cycles, histories, or even have a new decision every time a prize b is offered to them in a position in which they were already offered a prize b before. Then, behavior strategies in the form described in the first paragraph of this section are possible (side by side with non-threshold behavior strategies). In this case, there is a continuum of pure strategies, as the game can be presented in extensive-form with infinite tree depth. There is an equilibrium with symmetric stationary strategies, and the stationary strategies are thresholds (as best response strategies are thresholds). These stationary strategies are exactly the strategies described in the original game (the game before the expansion), and the equilibrium in the original game remains an equilibrium in the extended game.

The extended game has perfect recall, and therefore Kuhn's theorem holds, as we can see in Aumann (1964). Therefore, for every behavior strategy there is a mixed strategy (in the extended game) that has an equivalent payoff, i.e., the strategies are equivalent. The behavior strategy $k + r$ is equivalent to a mixed strategy which gives positive probability to an infinite number of pure strategies. Every pure strategy has a threshold k or $k + 1$ for the i-th time a prize b is offered to the player, such that the expected value of the threshold for the i-th time is $k + r$, $\forall i \in \mathbb{N}$.

7 Conclusion

We defined a cyclic queueing game with a fixed number of players and analyzed strategies and symmetric equilibria for three variations: the basic game, the game with uniformly distributed prizes and the game with unrelated prizes. It turns out that, generally, there exists a unique symmetric equilibrium. We also analyzed the price of anarchy in the variation with externalities, and found out the price of anarchy is bounded by 2 and approaches 1 for $n \to \infty$.

A main difficulty was to calculate the profit of mixed strategies, both in the basic game and in the game with unrelated prizes. In order to calculate the profit of the strategy $k+r$, we need to calculate the profit when some of the players play k and others play $k+1$. This is very complicated, as it requires to compute the stationary probability distribution of the position of k players and $k+1$ players, and the stationary probabilities of a k player and a $k+1$ player to be in a given position are different. Therefore, being in position $k+1$, we do not know what is the probability to be offered a b prize, and as a result, we do not know what is the expected profit. In the basic game we were satisfied with proving the existence and uniqueness of a pure symmetric equilibrium, as pure strategies fit the motivation for the game better than mixed strategies anyway. In the game with unrelated prizes we proved there may not be a symmetric equilibrium with pure strategies, and therefore we had to work with mixed strategies. We claimed that, if my best response against all the others play strategy k is the strategy $k+1$, and if my best response against all the others play strategy $k+1$ is the strategy k, then there is a mixed equilibrium strategy $k+r$, due to the "continuous interpretation" of the step function. In order to prove there cannot be two values, $r \neq r'$, such that both $k+r$ and $k+r'$ are equilibrium strategies, we have to use a stronger claim (proving ATC, for example) - which we have not done. There may be more mixed equilibria which this condition does not find, and therefore the uniqueness is up to this condition.

Future research may include more variations. For example, a game with unrelated uniformly distributed prizes. Another interesting variation is a game with a random "traveling time", as a function of the prize, from the moment a player accepts a prize until he is back to the end of the queue. This can resemble the real-life situation of taxis waiting for passengers in the airport. It can also be a better fit for the example with the technicians from the beginning of section 1, if the time of service can be longer than the inter-arrival time.

An interesting point to take in consideration is the method of the proofs in subsection 3.2 and in section 4, using the cost of wasted rounds. The method is intuitive and its solution is identical to the solution of the conventional method of finding equilibrium in the basic game. Yet, a proof of its correctness in general cases is required.

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תקציר

אנו מגדירים משחק תור מחזורי עם מספר שחקנים קבוע ומנתחים אסטרטגיות ושיוויי משקל סימטריים לשלוש וריאציות. כל יחידת זמן פרס בעל ערך אקראי מוצע לשחקנים על פי הסדר שלהם בתור, ושחקן שמקבל פרס עובר לסוף התור. כל שחקן שואף למקסם את התוחלת של קצב הרווח שלו. תהליך הבחירה אילו פרסים לקבל בכל עמדה מוצג כמשחק רב שחקנים לא-שיתופי. מטרתנו למצוא אסטרטגיית שיווי משקל נאש סימטרית. מתקבל כי, באופן כללי, קיים שיווי משקל סימטרי יחיד. אנו מנתחים גם את מחיר האנרכיה ואת התנהגותו האסימפטוטית.

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על ידי

אדם נתנאל

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פרופסור רפאל חסין

אוקטובר 2017