

## Optimal separable partitioning in the plane

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### Abstract

Sets of points are called separable if their convex hulls are disjoint. We suggest a technique for optimally partitioning of a set  $N$  into two separable subsets,  $N_1, N_2$ . We assume that a monotone measure,  $\mu$ , is defined over the subsets of  $N$ , and the objective is to minimize  $\max\{\mu(N_1), \mu(N_2)\}$ .

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### 1. Introduction

Let  $V = \{v_1, v_2, \dots, v_n\}$  be a set of  $n$  points in the plane. The problem of partitioning  $V$  into clusters has a wide range of applications and exact and heuristic algorithms have been developed to solve such problems.

A partition  $(V', \bar{V}')$  of  $V$  is called *separable* if the convex hulls of  $V'$  and  $\bar{V}'$  are disjoint.

In this paper we suggest a new approach for computing optimal separable partitions with respect to a class of objective functions of the min–max type. The basic idea of our approach relies on the possibility, under certain conditions, to apply binary search in order to select the best separable partition from a sequence of *nested* separable partitions, that is, a sequence of separable partitions  $(V_j, \bar{V}_j)$ ,  $j = 1, \dots, l$ , such that  $V_1 \subset V_2 \subset \dots \subset V_l$ .

For this idea to be useful, there must be a relatively small number of nested sequences of separable partitions whose union contains all of those partitions that are candidates for optimality. In such a case, the binary search is applied to find the best partition in each sequence, and then the best (over all the sequences) partition is selected.

We continue this introduction with a formal definition of the class of objective functions that fit into our framework. We also mention important special cases and refer to the relevant literature. The next section describes a method for efficiently

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constructing the smallest set of nested sequences whose union is the set of candidates for optimality. This construction assumes a convenient representation of the partitions, and the subject of efficiently obtaining such a representation is discussed in Section 3. The last two sections discuss bounds and the expected value of a minimum size set of sequences required to solve the optimal partitioning problem.

Let  $\mu: 2^V \rightarrow \mathcal{R}_+$  be a given measure. Let  $f: \mathcal{R}_+^2 \rightarrow \mathcal{R}_+$ . Denote for  $V' \subset V$ ,  $\bar{V}' = V \setminus V'$ . The problem is to compute a separable partitioning  $(V', \bar{V}')$  of  $V$  such that  $f(\mu(V'), \mu(\bar{V}'))$  is minimum.

The function  $f$  is *unimodal* if for any nested sequence of partitions there exists  $k$ ,  $1 \leq k \leq l$ , such that

$$\begin{aligned} f(\mu(V_1), \mu(\bar{V}_1)) &\geq f(\mu(V_2), \mu(\bar{V}_2)) \geq \dots \geq f(\mu(V_k), \mu(\bar{V}_k)) \\ &\leq \dots \leq f(\mu(V_l), \mu(\bar{V}_l)). \end{aligned}$$

In particular, the unimodality property holds when  $f$  is the max operator and  $\mu$  is monotone (i.e.,  $\mu(W) \leq \mu(W')$  whenever  $W \subset W'$ ).

Let a *cover* be a set of nested sequences of separable partitions such that each separable partition appears in at least one sequence. Given that  $f$  is unimodal, binary search can be applied to compute the best partition in any nested sequence. Our task reduces then to computing a cover of small size. The algorithm that we present below requires  $O(n^2 \log n)$  time to compute a cover of minimum size in a given class of nested sequences of partitions. If the size of such a cover is  $c$  then the total number of partitions to be compared in order to compute an optimal partition is  $O(c \log n)$ . Since the number of separable partitions can be as high as  $\binom{n}{2}$ , our method is especially useful when  $c$  is much smaller than  $n^2/\log n$ , and when evaluating the cost of each partition is time consuming.

We now describe several important special cases where the monotonicity property holds. In some of them it has been shown that the optimal separable solution also solves the relaxed problem where any partitioning is allowed. We implicitly assume that distances are euclidean; however, in some cases the separability property also holds with respect to other metrics. [13, 18] contain a discussion of some other natural kinds of such measures used in partitioning problems.

(1) The *diameter* of a set is the largest distance between any two points of the set. The *radius* of a point set in the plane is the radius of the smallest enclosing circle. Both measures are clearly monotone. In [7] it is proved that the optimal partitioning under each of these measures is separable even when the points are partitioned into several subsets and  $f$  is any monotone increasing function.

[2] contains an  $O(n \log n)$  algorithm for the problem of partitioning  $V$  into two clusters that minimize the maximum diameter.

The radius is the relevant measure in location problems where centers are to be located in order to serve a set of customers, and each customer will be served by the center which is closer to him. In the *two-center problem*, we wish to find two closed discs whose union contains  $V$  so as to minimize the maximum radius.

[17] contains a linear time algorithm to compute  $\mu$  and [1] uses it in an  $O(n^2 \log^3 n)$  time algorithm for the problem. A slightly improved expected running time is achieved in [9].

It may be worth noting that when the center must be located in one of the cluster's points,  $\mu$  is no longer monotone and the problem does not fit into our framework.

(2) Let  $\mu$  be the average (over the points in the cluster) sum of distances from a point to all the others in the cluster, i.e.,  $\mu(V') = (\sum_{i,j \in V'} d_{ij})/|V'|$ . We claim that  $\mu$  is monotone, i.e., let  $V \subset N$  and  $k \in N \setminus V$ , then

$$\frac{\sum_{i,j \in V} d_{ij}}{|V|} < \frac{\sum_{i,j \in V \cup \{k\}} d_{ij}}{|V| + 1}.$$

This inequality holds if  $\sum_{i,j \in V} d_{ij} < 2|V| \sum_{j \in V} d_{kj} = \sum_{i,j \in V} (d_{ik} + d_{kj})$ , which holds by the triangle inequality. In [4] it is proved that there exists a separable partition minimizing the *sum* of values of this measure over the clusters (see [6]). A related measure is the sum of squares of all distances in the same cluster. This measure is clearly monotone. (In [5] it is proved that under this measure an optimal solution can be separated by a circle.)

(3) The Steiner–Weber problem (see [10]) is to find a point  $P$  in the plane minimizing the sum of Euclidean distances from any point in  $V$  to the point  $P$ . This measure is monotone. However, the partition of  $V$  that minimizes the maximum of this measure over the clusters is not necessarily separable. For example, consider four points located at  $(-1, 0)$ ,  $(1, 0)$ ,  $(0, 1.9)$ ,  $(0, -0.1)$ . The solution inserts the two first points into a cluster, and the two last points into another. The value of this solution is 2. It is clearly not separable. Similarly, the sum of distances over all of the pairs in a subset is unimodal but, as shown by the same example, it is not true that a separable optimal partition always exists.

(4) Two other related measures are the area and perimeter of the convex hull of the point set. Both measures are monotone. As pointed in [7], it is obvious that when the area is considered the solution need not be separable. On the other hand, the question of whether such a result holds for all monotone functions is stated there as open.

Finally, we note that [3] considers partitioning problems of the following type:  $k$ -dimensional vectors  $A^1, \dots, A^n$  are to be partitioned into  $m$  sets so as to maximize a (quasi) convex function of the sum of vectors in each set. The size of the sets is constrained by lower and upper bounds. They show that there exists a separable optimal partition. We do not know of an application of our approach to this class of problems.

## 2. Minimum chain covers

Let the set of separable partitions of  $V$  be denoted by  $\mathcal{P}$ . We say that a slope  $S \in [0, \pi)$  is *formed* by two points  $v, v'$  if it is the slope of the line connecting these points. Let the set of slopes of lines formed by pairs of points in  $V$  be denoted by  $\mathcal{S}$ .

The following proposition is quite straightforward.

**Proposition 2.1.**  $|\mathcal{S}| \leq |\mathcal{P}| \leq \binom{n}{2}$  and in particular, if  $|\mathcal{S}| = \binom{n}{2}$  then also  $|\mathcal{P}| = \binom{n}{2}$ .

A slope  $s$  induces a partition  $(V', \bar{V}')$  if there exists a line of slope  $s$  separating  $V'$  and  $\bar{V}'$  in the sense that each subset lies in a different (open) halfspace defined by that line. For  $P \in \mathcal{P}$  denote by  $S(P)$  the set of slopes that induce  $P$ .

Consider a given separable partition  $P = (V', \bar{V}')$ . A slope induces  $P$  if and only if there is a line with this slope separating the convex hulls of  $V'$  and  $\bar{V}'$ . From this it is clear that  $S(P)$  is an open interval (mod  $\pi$ ), bounded by two distinct members of  $\mathcal{S}$ .

A graphic representation of the sets  $S(P)$  will be seen to be useful. Recall that the slopes are in  $[0, \pi)$ . Map  $[0, \pi)$  onto (and one to one) the points of a circle. Each  $S(P)$ ,  $P \in \mathcal{P}$ , becomes an interval called an *arc*. A cover corresponds now to a set of points on the circle intersecting all of the arcs. Note that each slope  $S \notin \mathcal{S}$  covers exactly  $n - 1$  partitions, and these partitions are nested.

Consider now a maximal interval  $I$  of the circle with the property that each  $S \in I$  intersects exactly the same set of  $n - 1$  (nested) partitions. Thus  $I = (S_1, S_2)$  where  $S_1, S_2 \in \mathcal{S}$  and no slope of  $\mathcal{S}$  corresponds to a point of  $I$ . We call the sequence of nested partitions covered by such an interval a *chain*. Let the set of chains be denoted by  $\mathcal{C}$ . Note that the cycle is partitioned into chains by the points of  $\mathcal{S}$ . Thus we obtain the following proposition.

**Proposition 2.2.**  $|\mathcal{C}| = |\mathcal{S}|$ .

Our goal is to find a minimum number of chains covering  $\mathcal{P}$ . Given the circular arc representation, the minimum chain cover problem is an instance of the following problem.

**Problem 2.3.** Let  $M$  be a circularly ordered set of points on a circle. Let  $\mathcal{A}$  be a set of open intervals of the circle, called arcs, such that  $(x, y) \in \mathcal{A}$  implies  $x, y \in M$ . Find a set of minimum size  $M'$  of points on the circle such that for each  $A \in \mathcal{A}$   $A \cap M' \neq \emptyset$ .

A minimum cover can then be computed in  $O(|\mathcal{A}| \log |\mathcal{A}|)$  time by applying the algorithm of [14] (see also, [11, 12, 16, 19]). In our case  $|\mathcal{A}| = |\mathcal{S}| = O(n^2)$  so that a minimum chain cover is computed in  $O(n^2 \log n)$  time. Our task remains now to present an efficient algorithm for constructing a circular-arc representation of the chain cover problem, and in particular, identifying the arcs  $S(P)$  for all  $P \in \mathcal{P}$ . In the next section we describe how to accomplish this in  $O(n^2)$  time.

### 3. Constructing a circular-arc representation

Let  $i, j \in V$ . Let  $L$  be the common line on which these points lie. (If  $i, j$  are contained in a larger set of colinear points then we assume that they are adjacent on  $L$ . We make this choice because if  $i_1, \dots, i_t$  is a colinear set and the points lie on the common line in

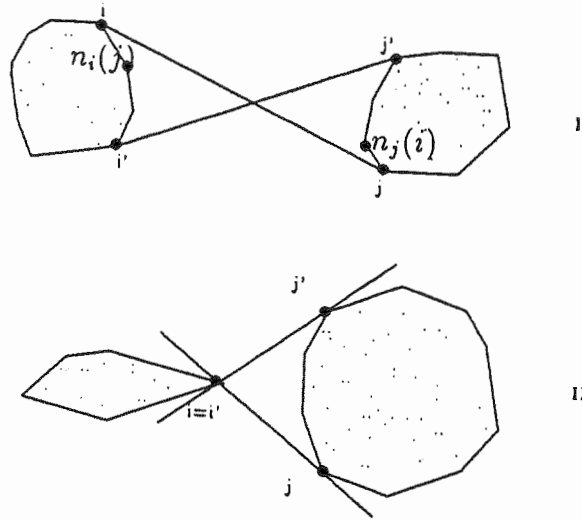


Fig. 1. The slope formed by  $i, j$  is the upper end of  $S(P)$  and the slope formed by  $i', j'$  is its lower end.

this order, then every separable partition will have for some  $1 \leq k \leq l$  that  $i_1, \dots, i_k$  are in one part while  $i_{k+1}, \dots, i_l$  are in the other part, so that the set is separated between an adjacent pair.)

Now,  $S$ , the slope of  $L$ , is the upper end of an interval,  $S(P)$ , corresponding to some partition  $P$ . In particular, this is the partition separated by the line formed by a slight clockwise rotation of  $L$  around any point between  $i$  and  $j$ . Let  $S'$  be the lower end of  $S(P)$ .  $S'$  is the slope of some line  $L'$  formed by some pair  $i', j' \in V$ . (If this pair is part of a larger colinear subset of  $V$  then we choose as  $i', j'$  the unique adjacent pair on  $L'$  which is separated by  $P$ .) Note that  $i$  and  $i'$  are in one part of  $P$  and  $j$  and  $j'$  in the other part (see Fig. 1). For  $n > 2$ , either  $i \neq i'$  or  $j \neq j'$  and we assume without loss of generality that  $j \neq j'$ . We distinguish then between the cases  $i \neq i'$  and  $i = i'$ , as in Fig. 1(I) and 1(II) respectively.

Suppose that we have for each  $v \in V$  a circular list of  $V \setminus \{v\}$  sorted by the slopes of the directions from  $v$  to these points in clockwise order. Such lists can be obtained by the algorithm of [8] in  $O(n^2)$  total time. Denote by  $n_v(l)$  the next point in  $v$ 's sorted list after  $l$ . See Fig. 1 for  $n_j(i)$  and  $n_i(j)$ .

**Lemma 3.1.** *If either  $j' \neq n_j(i)$  or  $i' \neq n_i(j)$  then  $S(P') \subset S(P)$  for some  $P' \in \mathcal{P}$ .*

**Proof.** Suppose for example that  $j' \neq n_j(i)$ . Let  $P = (A, B)$  where  $i \in A$  and  $j \in B$  (see Fig. 2).

Let  $A$  be the line common to  $j$  and  $j'$ . Let  $\Sigma$  be its slope. Let  $C$  be the subset of  $B$  consisting of the points which lie on the same open halfspace defined by  $A$  as  $A$ . The partition  $P' = (A \cup C, B \setminus C)$  is in  $\mathcal{P}$  since it is induced by  $\Sigma$ . Since  $\Sigma$  also induces  $P$  it

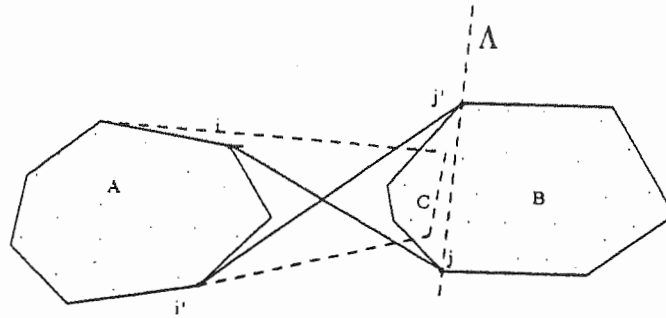


Fig. 2. The interval  $S(P)$  contains some other interval  $S(P')$ .

follows that  $S(P)$  and  $S(P')$  have a nonempty intersection. Since  $S$  and  $S'$  are the ends of  $S(P)$  while they are neither in  $S(P')$  nor its ends, it follows that  $S(P') \subset S(P)$ .  $\square$

We conclude that in the case described by the lemma the arc corresponding to  $P$  in the circular-arc representation of the problem can be deleted without affecting the chain cover problem. This is so since every point covering  $S(P')$  will also cover  $S(P)$ .

The algorithm we propose checks whether a perturbation of each of the following slopes induces  $P$ : The slope of the lines connecting  $n_j(i)$  and  $n_i(j)$ ;  $j$  and  $n_i(j)$ ;  $n_j(i)$  and  $i$ . If one does then it is the lower end of  $S(P)$ . Else, by the lemma,  $S(P)$  need not be represented.

Checking whether each of the three slopes induces  $P$  can be done in constant time. For example, for the slope of the line connecting  $n_j(i)$  and  $n_i(j)$  all we have to check is that  $n_i(n_i(j))$  is on the same closed halfspace as  $i$  and that  $n_j(n_j(i))$  is on the same closed halfspace as  $j$ .

Altogether, forming the arcs or concluding which of them can be deleted requires  $O(n^2)$  time, as the preprocessing stage of sorting the lists of directions.

#### 4. Bounds on the cover size

There is no lower bound on the size of a chain cover except for the trivial bound of 1. This bound is achieved when all the points are colinear. A nontrivial bound exists if we assume that no three points are colinear and no two slopes formed by points in  $V$  are identical.

**Proposition 4.1.** *Suppose that  $|\mathcal{S}| = \binom{n}{2}$ , then at least  $\lceil \frac{1}{2}n \rceil$  chains are necessary to form a cover. Furthermore, this bound is tight.*

**Proof.** By Proposition 2.1 there are  $\binom{n}{2}$  partitions. Since each chain covers  $n - 1$  partitions, then at least  $\lceil \binom{n}{2} / (n - 1) \rceil = \lceil \frac{1}{2}n \rceil$  chains are necessary to form a cover.

We now prove that this lower bound on cover size is achieved when the points in  $V$  are the vertices of a perfect polygon.

Let the points be ordered on the polygon. Recall that each partition is covered by exactly  $n - 1$  chains.

$\mathcal{S}$  can be partitioned into subsets  $\mathcal{S}_1, \dots, \mathcal{S}_{n-1}$  such that  $\mathcal{S}_k$  contains the slopes formed by pairs  $v_i, v_{i+k}, i = 1, \dots, n$  (by  $i + k$  we mean  $i + k \pmod n$ ). There are  $n$  slopes in each set. All the slopes in a set, when positively perturbed, cover the same number of partitions due to symmetry. On the other hand, between two consecutive slopes in the same set there is a fixed difference, namely  $2\pi/n$ . Between them there is exactly one slope from each set  $\mathcal{S}_j, j \neq k$ , and there are  $n - 1$  partitions covered by each slope. These partitions belong to consecutive members of  $\mathcal{C}$ . Thus, to find a cover for  $\mathcal{P}$  we may pick a slope at random, skip  $n - 1$  slopes of  $\mathcal{S}$ , take as the next one the chain induced by such slopes, etc., till we cover all the partitions. Altogether, we pick this way  $\lceil \frac{1}{2}n \rceil$  chains.  $\square$

We now discuss the other extreme where a large number of chains is needed to form a cover. This is done by presenting an example.

**Proposition 4.2.** *A minimum chain cover may contain  $\Theta(n^2)$  chains.*

**Proof.** We prove the proposition by constructing such an example. The construction is the following (see Fig. 3):

We form three sets of points, each one with  $\frac{1}{3}n$  points.

Set I contains  $\frac{1}{3}n$  points,  $p_1, \dots, p_{n/3}$ , on a tiny arc of a huge circle. They form an almost straight line but no three points are colinear.

Set II contains  $\frac{1}{3}n$  points  $q_1, \dots, q_{n/3}$ , constructed successively so that the ranges of slopes of the lines formed between any two successive points in Set II and all the points in Set I are successive but will not overlap.

Each point in Set III,  $r_1, \dots, r_{n/3}$ , is very close to a corresponding point in Set II so that the slopes of the lines formed between the two points  $r_i, q_i$  and all the points in Set I are alternating.

Positive perturbations of the slopes formed by one point in Set I and the second point in Set II or Set III will form the following sequence in  $\mathcal{S}$  (we denote the slopes by the points that form them):

- $S_{p_1, r_1}$
- $S_{p_1, q_1}$
- $S_{p_2, r_1}$
- $S_{p_2, q_1}$
- etc.
- $S_{p_1, r_2}$
- $S_{p_1, q_2}$
- $S_{p_2, r_2}$
- $S_{p_2, q_2}$
- etc.

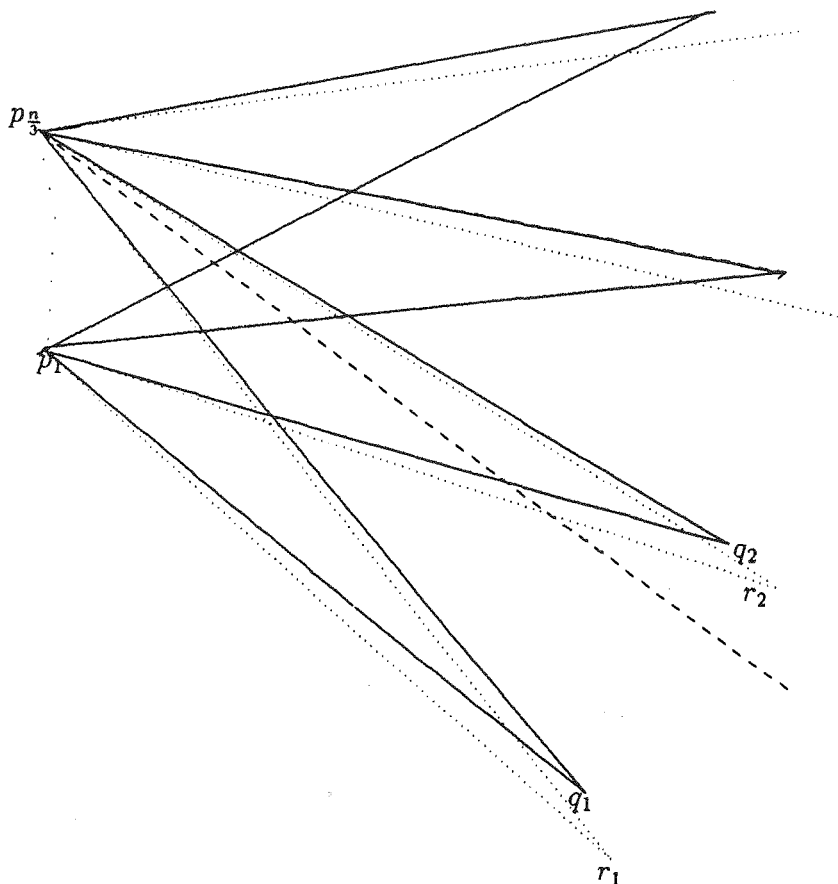


Fig. 3. Solid lines are of type  $L_{p_i, q_j}$ . Dotted lines are of type  $L_{p_i, r_j}$ . The dashed line is parallel to  $L_{p_1, q_1}$ .

Without loss of generality (overlooking different behavior by first and the last line of each sequence), any slope formed by  $p_i, q_j$  is the upper end of  $S(P)$  of some partition  $P$ , obviously  $n_{q_j}(p_i) = p_{i+1}$  and  $n_{p_i}(q_j) = r_j$  and  $|S(P)| = 3$ .

The magnitude of the set of such partitions is  $2n^2/9$ , hence the magnitude of its minimum chain cover is  $\Theta(n^2)$ .  $\square$

### 5. The average cover size

Processing random sets of points in the plane shows that in the average case the number of chains covering all partitions is of order  $n^{1.6}$ .

In the actual experiment ten sets of random points in the unit square were generated for each  $n, n = 10, 20, \dots, 90$ . The sizes of the minimum chain covers are given by Table 1.



Table 1

n	Random points									
10	7	8	7	9	8	8	7	9	8	8
20	22	26	22	25	24	22	25	24	26	22
30	44	45	43	50	46	46	54	47	46	49
40	72	71	74	73	71	80	75	75	76	80
50	105	107	101	100	106	114	105	109	100	103
60	144	142	144	141	143	139	142	147	149	138
70	183	174	184	178	175	178	174	180	176	173
80	221	212	223	210	211	220	220	218	226	214
90	271	266	274	270	261	245	258	279	269	251

Table 2

n	Average	$n^{1.6}/5$
10	7.900000	7.962143
20	23.799999	24.136705
30	47.000000	46.176819
40	74.699997	73.168808
50	105.000000	104.563957
60	142.899994	139.981934
70	177.500000	179.137650
80	217.500000	221.806351
90	264.399994	267.804596

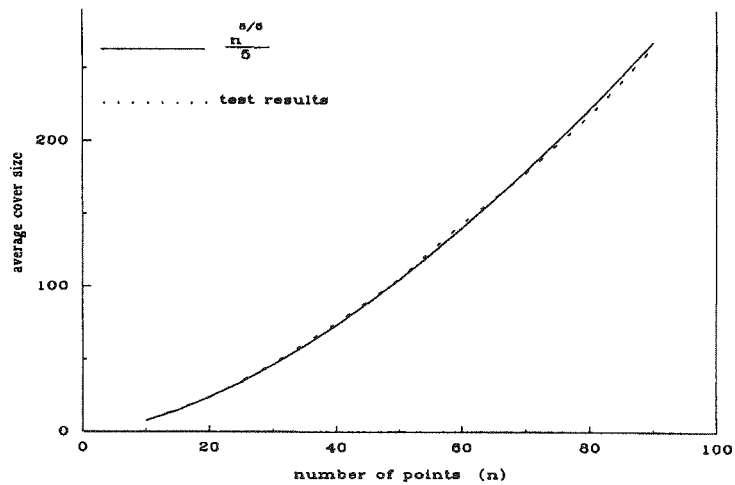


Fig. 4. The numerical results.

In Table 2 we compare the results with the function  $(n^{1.6}/5)$ .

In Fig. 4 we show the results of the test. The dotted line illustrates the numerical results and the solid line is the function  $\frac{1}{5}n^{1.6}$ .

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