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OPTIMAL CONTESTS

AMIHAI GLAZER and REFAEL HASSIN*

Firms frequently use contests to compensate their employees: an employee's pay depends on the ranking of his output compared to that of others, rather than on the absolute level of his output. This paper analyzes the design of a contest which maximizes the contestants' expected aggregate output, and describes two settings which yield opposite results. In one, prizes should be equal except for that given to the contestant with the lowest output. In the other setting, only the contestant with the highest output should obtain a meaningful prize.

I. INTRODUCTION

Contests are widely used as a reward mechanism. Automobile dealers hold sales contests in which a salesman's remuneration is a function of his success compared to that of other salesmen. The reward offered young professors may consist of granting tenure to the best candidate, regardless of the absolute quality of his work. A manager will be promoted not if his work is excellent, but if it is better than that of his competitors. These examples illustrate an essential feature of contests: any one person's reward depends on the performances of all other contestants and not only on his own achievements. A contest thus differs from a conventional wage system in which a worker's reward is independent of that received by others.

Not all contests are designed solely for the purpose of providing contestants with appropriate incentives. Some contests, such as sports competitions, are held largely for thrill and excitement. Other contests, such as those for beauty queens, provide a forum for selecting a person out of a large group. Yet providing incentives is an integral feature of all these contests, and we would do well to focus on it.

The study of contests has attracted increasing attention in recent years. The earliest modelling efforts appear in studies of animal behavior. In particular, Maynard Smith [1973; 1974; 1976a; 1976b] and Dawkins [1976] study fights between animals in which the winner earns some benefit, but where both the winner and the loser may suffer injuries. It will come as no surprise that the emphasis in these studies is to characterize equilibrium strategies; fights are not viewed as incentive mechanisms. More specifically, Maynard Smith, Hirshleifer and Riley [1978], Riley [1979] and Eshel [1983] study evolutionary equilibrium strategies: such a strategy, if adopted by most members of the population, gives a higher expected gain than does any other strategy. The concept is identical to a Nash equilibrium only for infinite populations.

* Associate Professor, School of Social Sciences, University of California, Irvine, and Associate Professor, Department of Statistics, Tel Aviv University. We are grateful to an anonymous referee for helpful observations and suggestions.

The seminal work in the economic analysis of contests is by Lazear and Rosen [1981], who devote most of their attention to a contest between two contestants. The analysis is generalized by Green and Stokey [1983] and by Nalebuff and Stiglitz [1983]; Dye [1984] criticizes this approach by arguing that collusion by contestants may make contests a very imperfect system. The most important difference between their papers and ours concerns the source of a contestant's uncertainty. The aforementioned papers assume that contestants are identical in ability, but that measures of their output are subject to error. Under those assumptions each contestant will choose the same effort, though the reward he obtains is subject to random variation.

In our paper the rank order of output can be measured with no error, but no one contestant knows for sure what level of output others choose. This means that in equilibrium identical contestants will choose different levels of effort. (O'Keefe et al. [1984] consider this situation, but allow for only two prizes.) We also analyze contests in which contestants have different abilities. What matters then in the design of a contest is not only how to induce persons to work, but also how to get persons with high ability to work harder than persons with low ability.

In that sense, the present work is related to some recent papers on the optimal design of auctions (particularly Hirshleifer and Riley [1978]; Myerson [1981]; and Riley and Samuelson [1981]), and to other papers concerning the sale of differentiated goods to consumers who value quality differently (for example, Mussa and Rosen [1978]; Spence [1980]). The solutions obtained there are not, however, directly applicable here. In the literature just cited, persons differ in the value they place on the good. We assume all contestants have identical utility functions and therefore agree on the values of the prizes. The way in which they do differ is the ease with which they can produce something (output) of value to the employer. Though Maskin and Riley [1984] study a problem similar to ours, they assume that the employer can measure each worker's level of output where we assume that only *relative* rankings are observed.

The next section of the paper sets forth our assumptions. Sections III and IV concern contests in which all contestants have the same ability. We find that if contestants have linear cost functions, total expected output is maximized by giving equal prizes to all but the lowest producer. Section V considers a model in which contestants have different abilities; it describes the conditions under which persons with greater ability will exert more effort, and demonstrates that in general only a few of the prizes should be greater than the reservation wage.

II. ASSUMPTIONS

Let the number of potential contestants be exogenously fixed at N. Each potential contestant decides whether to participate in the contest; he who does produces a nonnegative output.

A maximum of N prizes, $m_1, m_2, \ldots, m_N$, are allocated; a nonparticipant obtains no prize. The prizes are ordered so that $m_1 \geq m_2 \geq \ldots \geq m_N \geq 0$. The prizes are distributed according to the ranks of the participants' outputs; the kth highest prize, m_k, is given to the contestant who produces the kth highest output. If several contestants produce the same output, the tie among them is broken in some fair way.

Each contestant is characterized by a parameter, α, that reflects his ability. His cost of producing output y is $c(\alpha, y)$, which function is twice differentiable. We assume that $c(\alpha, 0) = 0$, that $\partial c(\alpha, y)/\partial y > 0$, that $\partial c(\alpha, y)/\partial \alpha < 0$, and that $\partial^2 c(\alpha, y)/\partial \alpha \partial y < 0$. The last inequality means that a contestant can more easily increase his output the greater his ability. A potential contestant can find alternative employment which yields utility r.

A contestant's utility from obtaining a prize of size m is $u(m)$, a monotone increasing and concave function. For any value of output, each contestant is not sure what other contestants will do or what his prize will be, so that he views the value of m as a random variable. Let $F(y)$ denote, for $y \geq 0$, the probability that a randomly chosen contestant will choose an output of y or less, so that each contestant's problem can be viewed as maximizing a function $v[y, \alpha, F(y)]$, where α and $F(y)$ are given. The function $F(y)$ is called an equilibrium distribution if and only if $dv/dy = 0$ for some α whenever $dF(y)/dy > 0$; that is, any chosen level of output must be a possible solution to some contestant's maximization problem. Define $y_{\max}$ as $\sup[y \mid F(y) < 1]$.

Lemma 1. *$F(y)$ is continuous in the interval $(0, y_{\max})$.*
Proof. Suppose otherwise, that $F(y)$ is discontinuous at some value, y_0. This means that with a positive probability all contestants will choose output y_0, and that a contestant who increases his output to a level infinitesimally greater than y_0 increases the expected value of his prize and his expected utility by a noninfinitesimal amount. This violates the equilibrium condition that y_0 maximizes expected utility for some contestant.

Lemma 2. *$F(y)$ is strictly increasing in the interval $(0, y_{\max})$.*
Proof. Suppose otherwise, and let (a, b) be the subinterval of maximum length in $(0, y_{\max})$ on which F is constant. Then a contestant who chooses $y = b$ can increase his expected utility by producing $y = a$ instead; he reduces his costs without reducing his expected prize. This contradicts the assumption that $F(y)$ describes an equilibrium.

From lemmas 1 and 2 we conclude that though there may be a positive probability that a potential contestant decides not to participate, those who do participate choose different output levels. Thus, the probability that a contestant obtains the kth prize is the probability that $k - 1$ of the contestants choose an output greater than his, and that $N - k$ of them choose a lower level or decide not to participate. The expected utility from the prize accorded to a contestant who produces an output y is then

$$B(y) = \sum_{k=1}^{N} \binom{N-1}{k-1} u(m_k) F^{N-k}(y)[1 - F(y)]^{k-1}. \tag{1}$$

A contestant's cost of producing output y is $c(\alpha, y)$, so that his net benefit is

$$R(\alpha, y) \equiv B(y) - c(\alpha, y). \tag{2}$$

This value, $R(\alpha, y)$, can be called "contestant's surplus."

III. CONTESTANTS WITH IDENTICAL ABILITIES

In this and the following section we study contests among participants who have identical abilities. For succinctness we write $R(y)$ and $c(y)$ instead of $R(\alpha, y)$ and $c(\alpha, y)$. Lemmas 1 and 2 imply that in such contests there is no Nash equilibrium in pure strategies. The reason for this result is best explained by an example. Suppose two contestants produce outputs y_1 and y_2 with the aim of obtaining the prize awarded to the one with the higher output. In equilibrium, y_1 cannot equal y_2. For were they equal, a contestant who raised his output by an infinitesimal amount would certainly win the highest prize. Suppose next that in equilibrium y_1 is greater than y_2. Clearly, contestant 2 will not win the highest prize and he would do well to let $y_2 = 0$. But then contestant 1 will find it profitable to choose a value of y_1 only infinitesimally greater than y_2. Thus, no deterministic Nash equilibrium exists.

We will accordingly examine a Nash equilibrium represented by a cumulative distribution function, $F(y)$, that gives the probability that a contestant chooses an output less than or equal to y. That is, we wish to find a function $F(y)$ such that if any one contestant believes that $F(y)$ describes the distribution of outputs for all other contestants, then each contestant believes his expected net benefit, $R(y)$, to be invariant with respect to all values of y he may choose. That is, $R(y)$ must be a constant R for all values of y that contestants may actually choose. In particular, by Lemma 1, $R = R(0) = B(0)$. Substitute this condition in (1) to obtain

$$R = \sum_{k=1}^{N} \binom{N-1}{k-1} u(m_k) F^{N-k}(0)[1 - F(0)]^{k-1}, \tag{3}$$

where $F(0)$ is the probability that a potential contestant chooses not to participate.

It is conceivable for the firm to set the prizes so low as to make the benefit of participation, R, negative; the firm would then have to attract contestants by giving them lump sum payments before a contest began. There could, however, be great difficulties in implementing such a system. A contestant, for example, might immediately spend any lump sum payment he obtains and then declare insolvency or bankruptcy when the results of the contest reveal that he owes a large sum of money to the employer. It is not unreasonable, therefore, to consider a contest which involves no lump sum payments, so that a contestant's remuneration consists solely of his prize. The condition of no advance grants to participants therefore requires that $R(y) \geq r$ in

$(0, y_{max})$, where r is a contestant's opportunity cost of participating. Two possibilities arise.

a) If the value of the smallest prize is so low that $u(m_N) < r$, then $F(0)$ must be positive. For were $F(0)$ equal to 0, equation (3) would yield $R = u(m_N) < r$, in violation of the condition that $R \geq r$. We conclude that if $u(m_N) < r$, with positive probability some potential contestants choose not to participate, and equilibrium requires that $R = r$. Substitute this in (3) to obtain the value of $F(0)$.

b) If $u(m_N) > r$, every participant is assured of obtaining at least a utility $u(m_N)$, even if he chooses $y = 0$. Therefore in this case $F(0) = 0$, and all potential contestants participate. Substitute this in (3) to obtain $R = u(m_N)$.

We can use equation (2) to find the value of y_{max}. By definition $F(y_{max}) = 1$, so that a contestant who chooses output y_{max} obtains the largest prize, m_1. From equation (2), $R(y_{max}) = u(m_1) - c(y_{max})$, which in equilibrium must equal R. Therefore

$$y_{max} = c^{-1}[u(m_1) - R]. \tag{4}$$

To summarize, the equilibrium value of $F(y)$ for each value of y must satisfy equations (1) through (4), so that in equilibrium

$$\begin{aligned} &F(y) = 0, \qquad \text{for } y < 0; \\ &\sum_{k=1}^{N} \binom{N-1}{k-1} u(m_k)F^{N-k}(y)[1 - F(y)]^{k-1} = R + c(y), \\ &\qquad\qquad \text{for } 0 \leq y \leq c^{-1}[u(m_1) - R]; \\ &F(y) = 1, \qquad \text{for } y \geq c^{-1}[u(m_1) - R], \end{aligned} \tag{5}$$

where $R = \max[r, u(m_N)]$.

IV. OPTIMAL CONTESTS WITH IDENTICAL CONTESTANTS

We characterized the equilibrium distribution of output as a function of the fixed set of prizes, $m_1 \ldots m_N$; hence to each set of prizes there corresponds an expected value of total output. This relation is not, however, an intuitively obvious one. Suppose that the value of the first prize is increased, and that the second prize is decreased by the same amount. The increase should induce contestants to work harder. But the decrease in the second prize might either induce the contestants to work more in an attempt to obtain the first prize, or lead them to work less since the marginal benefit of ranking second rather than third has declined. Moreover, an analysis of such changes in the prizes must consider not only the incentives any one person faces, but also changes in the distribution of effort, $F(y)$, and how changes in this distribution affect each contestant's incentives. It is thus important to discover how to design a contest that maximizes the expected value of contestants' output. That problem is the topic of this section.

From equation (5) we know that $y = c^{-1}[B(y) - R]$, so that $Ey = \int_0^{y_{max}} y \; dF(y)$. Let $x = F(y)$ to obtain as the maximand

$$Ey = \int_{F(0)}^{1} c^{-1}\left[\sum_{k=1}^{N}\binom{N-1}{k-1}u(m_k)x^{N-k}(1-x)^{k-1} - R\right]dx. \tag{6}$$

We suppose the firm has a fixed amount, M, to be allocated as prizes, and wishes to maximize the expected value of total output produced by the contestants. The firm's choice variables are the values of the prizes $m_1, m_2, \ldots, m_N$; thus, the firm's problem is to maximize Ey subject to

$$\sum_{k=1}^{N} m_k = M,$$

$$m_1 \geq m_2 \geq \ldots \geq m_N \geq 0,$$

$$B(y) - c(y) = R \text{ for all } y \text{ in } (0, y_{max}),$$

$$R \geq r,$$

$$y_{max} = c^{-1}[u(m_1) - R].$$

We shall demonstrate that optimality requires that $u(m_N) \leq r$ and that $R = r$. Two cases need to be considered. If $u(m_N) < r$, then as we showed in the previous section, not all potential contestants will participate; the probability distribution of the number of participants will be such that $R = r$, or that contestants' surplus equals the opportunity cost of participating in the contest.

If $u(m_N) \geq r$, then all N potential contestants participate, $F(0) = 0$, and $R = u(m_N)$. Equation (6) then states that the coefficient of $u(m_N)$ is $-[1 - (1-x)^{N-1}]$, which is negative. Since Ey decreases in m_N, optimality requires that in this case $u(m_N) = r$, so that $R = r$.

In summary, optimality requires that $u(m_N) \leq r$, and that $R = r$. In general this problem has no analytic solution; we can however solve an important special case.

PROPOSITION 1. *Suppose that $r = u(0)$, that $u(m_N) \geq r$, and that $c(y) = y/\alpha$. Then maximizing output requires that $m_1 = m_2 = \ldots = m_{N-1} = M/(N-1)$ and that $m_N = 0$. That is, the smallest prize is zero, and all other prizes are equal.*

Proof. We showed above that optimality requires that $u(m_N) \leq r$ and that $R = r$. Together with the assumption that $u(m_N) \geq r$, this implies that $u(m_N) = r$. By assumption, $r = u(0)$ and therefore $m_N = 0$. Recall also that if $u(m_N) \geq r$, then in equilibrium $F(0) = 0$. Form the Lagrangean

$$L = Ey + \lambda\left(M - \sum_{k=1}^{N-1} m_k\right),$$

and differentiate with respect to $m_1, m_2, \ldots, m_{N-1}$ to obtain the first order conditions

$$0 = \partial L/\partial m_k = \alpha \int_0^1 \binom{N-1}{k-1} u'(m_k) x^{N-k}(1-x)^{k-1}\,dx - \lambda,$$

$$\text{for } k = 1, \ldots, N-1. \tag{7}$$

We show that equation (7) is satisfied for $m_N = 0$ and $m_1 = m_2 = \ldots = m_{N-1} = M/(N-1)$. Make this substitution in (7) and use the assumption that $c(y) = y/\alpha$ to obtain the condition

$$\lambda = \alpha u'[M/(N-1)] \binom{N-1}{k-1} \int_0^1 x^{N-k}(1-x)^{k-1}\,dx,$$

$$\text{for } k = 1, \ldots, N-1. \tag{8}$$

The integral is the Beta function, equal to $(N-k)!(k-1)/N!$, so that equation (8) becomes $\lambda = \alpha u'[M/(N-1)]/N$ which is satisfied for $k = 1, 2, \ldots, N-1$. This completes the proof.

We note that the solution described in the proposition is also the limiting solution when N tends to infinity, even if the function $c(y)$ is strictly convex rather than linear. For if $m_1, m_2, \ldots, m_{N-1} = M/(N-1)$, the value on the right-hand side of (8) converges to a constant when N tends to infinity.[1]

It is clear that a contest is not the most efficient form of compensation. Suppose that $r = 0$. Then a contest will lead to a lower level of total output than the following compensation scheme: each of the N participants is awarded a prize of M/N if and only if he produces that value of output at which $c(y) = u(M/N)$. Nevertheless, use of a contest does have some advantages over other

1. Recent research (see Myerson [1981], Riley and Samuelson [1981]) shows that for auctions, which are similar to contests, the seller's revenue can be maximized by setting a minimum bid. The equivalent requirement here would be that no prize be given for an output less than some critical level, say y^*. In fact, however, such a scheme would reduce the total level of output.

Were there such a minimum output level, no contestant would choose an output in the range $(0, y^*)$. The equilibrium distribution of output would be given by the following conditions:

$$\begin{array}{ll} F(y) = 0 & \text{for } y < 0 \\ F(y) = J(y^*) & \text{for } 0 \le y \le y^* \\ F(y) = J(y) & \text{for } y^* \le y \le c^{-1}[u(m_1) - R] \\ F(y) = 1 & \text{for } y \ge c^{-1}[u(m_1) - R] \end{array}$$

where $R = \max[r, u(m_N) - c(y^*)]$, and $J(y)$ is defined by the condition that

$$\sum_{k=1}^{N} \binom{N-1}{k-1} u(m_k) J^{N-k}(y)[1 - J(y)]^{k-1} = R + c(y).$$

Optimal allocation of prizes requires that $u(m_N) - c(y^*) \le r$ so that $R = r$; for an output greater than y^* this distribution, $J(y)$, is identical to that for $F(y)$ given by equations (5). Output above y^* is therefore not increased, but output below y^* is lost. Letting y^* be greater than zero therefore reduces total output.

The difference between this solution and the auction results lies in our assumption that all persons place the same valuations on the prize.

systems. The most important is that with a contest the employer need measure output ordinally and not cardinally. This avoids all the difficulties of a system in which the parties must set the piece rate and later agree on exactly how much was produced. And for a large number of contestants, the inefficiency arising from contests becomes insignificantly small.

V. CONTESTANTS WITH DIFFERENT ABILITIES

This section considers contests among persons with different, instead of identical, abilities. Let at most N persons, randomly selected from the population, participate in the contest. The cumulative distribution of ability in the population is represented by a continuous function, $G(\alpha)$. Each contestant knows his own ability, but has only probabilistic estimates of the abilities of fellow potential contestants. Let a contestant with ability α choose output $y(\alpha)$.

As explained before, we assume that a contestant's remuneration consists solely of his prize, so that a person with ability α chooses to participate in the contest only if his surplus, $R[\alpha, y(\alpha)]$, is no less than the opportunity cost of participation.

Since for any $\alpha_1 > \alpha_2$ and for $y > 0$, we have $R[\alpha_1, y(\alpha_1)] \geq R[\alpha_1, y(\alpha_2)] > R[\alpha_2, y(\alpha_2)]$, we find that $R[\alpha, y(\alpha)]$ increases with α.

Let α_0 satisfy $R[\alpha_0, y(\alpha_0)] = r$; then only persons with ability $\alpha \geq \alpha_0$ will participate in the contest. Therefore a potential contestant with ability α_0 is certain to obtain the lowest prize among the participants (although he may not know beforehand which prize it will be since the number of participants is not known), and will produce zero output.

A necessary and sufficient condition that guarantees the participation of all N potential contestants is that the smallest prize be sufficiently attractive, or that $u(m_N) \geq r$. If $u(m_N) < r$, the least able potential contestants will not participate.

A contestant with ability $\alpha > \alpha_0$ chooses output y if and only if $\partial R(\alpha, y)/\partial y = 0$ and $\partial^2 R(\alpha, y)/\partial y^2 < 0$, that is if and only if

$$dB(y)/dy = \partial c(\alpha, y)/\partial y \tag{9}$$

and

$$d^2B(y)/dy^2 - \partial^2 c(\alpha, y)/\partial y^2 < 0. \tag{10}$$

But we know from Lemma 2 that each value of y in the range $(0, y_{max})$ may in fact be chosen by some contestant, so that conditions (9) and (10) must hold in equilibrium for all values of y in $(0, y_{max})$.

Surely a contest is an attractive reward system only if $y(\alpha)$ is an increasing function of α for all $\alpha > \alpha_0$: the most productive should work the most. Under our assumptions this is indeed the case. Differentiate (9) with respect to α, and solve for $dy(\alpha)/d\alpha$ to find

$$dy(\alpha)/d\alpha = [\partial^2 c(\alpha, y)/\partial y \partial \alpha]/[d^2B(y)/dy^2 - \partial^2 c(\alpha, y)/\partial y^2] \tag{11}$$

which is positive by our assumptions on $c(\cdot)$ and by condition (10).

These assumptions allow contests to be used for sorting; not only can a contest reveal which contestant has the greatest ability, but it also allows the estimation of a person's ability on the basis of his output. Moreover, *any* set of prizes can be used to elicit information on the contestants' abilities, and such information can be obtained on all contestants, not only on those who win a meaningful prize.

Recall that a contestant chooses a higher level of output the greater is his ability, so the probability that a contestant with ability α obtains the kth highest prize is simply the probability that he has the kth highest ability among the N potential contestants. Thus the expected utility of the prize won by a contestant with ability α is

$$B(\alpha) \equiv B[y(\alpha)] = \sum_{k=1}^{N} \binom{N-1}{k-1} u(m_k) G^{N-k}(\alpha)[1 - G(\alpha)]^{k-1}. \quad (12)$$

Use (9) to obtain

$$dB(\alpha)/d\alpha = \{dB[y(\alpha)]/d\alpha\}\ dy/d\alpha = \{\partial c[\alpha, y(\alpha)]/\partial y\}\ dy/d\alpha. \quad (13)$$

Substitute $dB(\alpha)/d\alpha$ from (12) in (13) to obtain a differential equation that can be solved for $y(\alpha)$.

The firm's objective is to allocate the prizes to maximize the expected value of total output. Unfortunately we are unable to find a general solution to this problem, and must therefore make some restrictive assumptions. We first suppose that all potential contestants, including the one with the lowest possible ability, actually participate. Since $y(\alpha_0) = 0$, and since $dy(\alpha)/d\alpha > 0$, this implies that the contestant with the lowest ability in the population is sure to obtain the smallest prize, with value m_N. This person will participate if and only if $u(m_N) \geq r$.

Thus, the firm's objective is to maximize $Ey = \int_{\alpha_0}^{\infty} y(\alpha)\ dG\ (\alpha)$ subject to equations (12) and (13), and subject to the conditions that $m_1 \geq m_2 \geq \ldots \geq m_N \geq r$, and that $\sum_{k=1}^{N} m_k = M$.

An increase in m_N, the smallest prize, above the value which satisfies $u(m_N) = r$ does not increase the number of participants, and from equation (1) we find that

$$\partial^2 B(y)/\partial y \partial m_N = -(N - 1)u(m_N)[1 - F(y)]^{N-2}\ dF(y)\ dy,$$

which is negative. Any increase in m_N above $u^{-1}(r)$ thus lowers each contestant's incentive to increase his output, and lowers the total value of expected output. Optimality thus requires that $u(m_N) = r$.

PROPOSITION 2. *Suppose that $c(\alpha, y) = y/\alpha$, that ability, α, is uniformly distributed on* (0, 1) *and that the opportunity cost of participation, r, equals* $u(0)$. *Then in equilibrium expected aggregate output, NEy, is equal to*

$\sum_{k=1}^{N} u(m_k)(N - 2k + 1)/(N + 1).$

Proof. Rewrite (13) in the form $dB(\alpha)/d\alpha = (1/\alpha)(dy/d\alpha)$, and integrate to obtain

$$y(\alpha) = \alpha B(\alpha) - \int_0^{\alpha} B(x)\, dx.$$

A contestant's expected output is

$$Ey = \int_0^1 y(\alpha)\, d\alpha = \int_0^1 \alpha B(\alpha)\, d\alpha - \int_0^1 \int_0^{\alpha} B(x)\, dx\, d\alpha$$
$$= \int_0^1 \alpha B(\alpha)\, d\alpha - \int_0^1 (1 - \alpha) B(\alpha)\, d\alpha.$$

Substitute (12) with $G(\alpha) = \alpha$ to obtain

$$Ey = \sum_{k=1}^{N} u(m_k) \binom{N-1}{k-1} \left[\int_0^1 \alpha^{N-k+1}(1 - \alpha)^{k-1}\, d\alpha - \int_0^1 \alpha^{N-k}(1 - \alpha)^{k}\, d\alpha \right]. \qquad (14)$$

Note that $\int_0^1 \alpha^a (1 - \alpha)^b\, d\alpha$ is the Beta function, and is equal to $(a!b!)/(a + b + 1)!$ so that (14) simplifies to the desired result.

COROLLARY 1. *Suppose that* $u'(\infty) = 0$. *Let* k *be the smallest integer greater than or equal to* $(N + 1)/2$. *Then expected output is maximized by setting the smallest prizes,* $m_k, \ldots, m_N$ *to zero and setting the largest prizes,* $m_1, \ldots, m_{k-1}$, *to positive values that sum to* M, *and that satisfy* $u'(m_k)/u'(m_1) = (N - 1)/(N - 2k + 1)$.

COROLLARY 2. *If* $u(m)$ *is linear, then expected output is maximized by setting* $m_1 = M$ *and* $m_2 = \ldots = m_N = 0$.

VI. CONCLUSION

We often think that contests are held primarily for the purpose of selecting the best person for some job. In many contests, such as the electoral race for the presidency or the rat race for executive promotion, the winner receives a large prize and the losers show no gain for their efforts. This is consistent with the optimal design of contests. We have also shown that in equilibrium output is an increasing function of ability, so that a contest can be used as a sorting mechanism. Moreover, *any* set of prizes that awards higher prizes to higher outputs will serve for such sorting. But clearly contests serve as incentive mechanisms as well; employees are promoted not only to fill a position, but also to provide them with incentives to perform well in their current jobs. For these purposes, as we have shown, we cannot be indifferent between one contest design and another.

Perhaps of greatest interest is the finding that an optimal contest may require that meaningful prizes be given not only to the top-ranked worker, but to others as well, and that these prizes be unequal. Indeed, although under a wage system we would expect identical workers to have identical earnings, such a result is simply incompatible with the use of a contest as an incentive mechanism; the further study of contests may be able to shed new light on the problems of income distribution.

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