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Equilibrium in Queueing Systems with Complementary Products*

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Abstract. We study a model of a queueing system with two complementary products/services. In our model, there is one M/M/1 system and another facility that provides instantaneous service. The two services are complementary and the customer has no benefit from obtaining just one of them. We investigate the model under various price structures and ownership assumptions.

Keywords: game theory, queueing theory, complementary products, Nash equilibrium

AMS subject classification: 90B22, 91A10

1. Introduction

Competition among owners of service facilities has been the subject of several research papers, see Hassin and Haviv [3] for a survey. A typical model describes the situation as a game with two stages. For any set of prices set by the servers, customers select servers according to a (Nash) equilibrium strategy. Given this information, servers select their prices and again an equilibrium is attained.

The models of competition in queueing systems deal with almost no exception with servers who provide substitutable service. Thus, a customer may obtain the service at any one of the competing servers. However, it is very common that customers need complementary service provided by several entrepreneurs; a customer benefits from the service given by one server only if he also obtains service from the other entrepreneurs. Consider for example a medical facility, where a customer has to stand in line to be admitted, then for a blood check and then in order to consult a physician.

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In a common situation, which is the subject of our paper, one has to purchase a product in addition to a service in order to enjoy a benefit, and the seller of the good and the server are independent profit maximizing agents. We view the seller of the product as a facility with instantaneous service and therefore there is no queue in front of this server. The length of the other queue is unbounded. As the motivating example for our model we consider a parking service; a customer parks his car in order to spend the time at another service facility. Following this example, we call the server who provides instantaneous service the "parking provider" and the other server will be the "service provider". As another example, consider a customer who needs to buy a new pair of glasses; he also has to stand in line to be serviced by an optometrist.

We analyze two pricing models. In both models, the service provider charges a flat price. In the first, the parking provider charges a price that is proportional to the duration the facility is in use by the customer. The parking provider under this pricing structure may be viewed as any agent who rents a good and charges a price proportional to the time of use. In the second model, we assume that the parking provider charges a fixed sum. The parking provider under this price structure may be viewed as a seller of a product. Finally, we compare the two models, and conclude that the parking provider is better off if he charges a flat rate.

In this paper we investigate the equilibrium solutions of this model. We also compare the outcome with the socially optimal solution.

The basic model is formally defined in Section 2. In Section 3 we study this model, find the strategies of the suppliers and customers in equilibrium, show that the service-provider's profits are higher than the parking-provider's, and prove that the prices are non-increasing with the potential rate of arrival (which might be interpreted as customers' demand). We give graphical interpretation to the main model, to support intuition. In Section 4, we present the equilibria reached when a social-planner or a monopolist provides both services. We show that the prices set by a monopolist are also socially optimal. We show that the equilibrium rate of arrival under competition is lower than the socially optimal one, as the chosen prices are too high. We show how a social-planner will act if he controls one of the two services. In Section 5, we study the variation in which the parking-provider charges a fixed rate and show that he will prefer this option. We use the graphical interpretation to show convergence to equilibrium in Section 6. We conclude briefly in Section 7.

2. The model

We focus our discussion on profit-maximizing service-provider and parking-provider. When a customer arrives to receive service he needs to use a "parking facility" for the duration of his stay in the system.

For comparison, we also check the optimal pricing-policy of a monopolist (owning both the service and the parking facilities), and of a social planner (maximizing the collective social utility).

We analyze a specific model under the following assumptions (following Naor[5] except for the 8th assumption which follows Edelson and Hildebrand[2]):

1. A stationary Poisson stream of customers—with parameter Λ—arrives at a single-server station. Customers are identical except for their arrival times. Λ is the rate at which the need for the service arises, and not necessarily the actual rate of customers who join the queue to be served (which we refer to as λ).
2. Service times are independently, identically, and exponentially distributed with intensity parameter μ.
3. On successful completion of service, each customer receives a reward of R (expressible in monetary units).
4. The cost to a customer for staying in the system (either waiting or being served) is C monetary units per time unit.
5. Customers are risk neutral. That is, they maximize the expected value of a linear utility function.
6. From the public (social) point of view, utilities of individuals (customers and servers) are identical and additive.
7. For the model to make sense, it is assumed that any customer will choose to join the queue if nobody else joins (when he only incurs the costs of his own service-time), thus: $R \geq \frac{C}{\mu}$.
8. At the time a customer's need for service arises, he does not know the queue size. The decision to join or balk is irrevocable.

Thus we are dealing with a queueing system of the type M/M/1 with infinite population, infinite queueing positions and a First-Come-First-Served discipline.

We use the following notation:

$w = w(\lambda)$	Expected time spent in the system by a joining customer given the joining rate λ. $w = \frac{1}{\mu-\lambda}$.
P_s	Price charged by the service provider.
P_N	Price per time unit charged by the parking provider.
U	Expected utility of a joining customer. $U = R - P_s - P_N w - Cw$.
π_s	Expected profit per time unit for the service provider. $\pi_s = \lambda P_s$.
π_N	Expected profit per time unit for the parking provider. $\pi_N = \lambda P_N w$.
π_M	Expected profit per time unit of a monopoly that provides both parking and service. $\pi_M = \lambda[P_s + P_N w]$.
π_W	Expected social welfare per time unit in the system. $\pi_W = \lambda \cdot [R - C \cdot w]$.

Customers' equilibrium

Following Assumption 1, there is a potential stream Λ of customers in need of service. Each of these customers decides for himself whether it is worthwhile for him to join the system or not, considering the known statistical properties of the system.

We have two kinds of possible equilibrium: in the first, $\lambda = \Lambda$, so that all the customers in need of service join the system. In this case, the expected utility for the customers is clearly non-negative. However, if the expected utility is strictly positive then the suppliers have an incentive to increase the prices (without causing the stream of customers to lessen) until the expected utility for the customers is zero again.

In the second, customers follow a mixed strategy where they join the queue with some probability $0 < \bar{p} < 1$, such that an arriving stream $\lambda = \bar{p}\Lambda$ is realized, for which the expected utility of the customers is zero, and there is no incentive for any of the customers to change their behavior.

In the following, we divide our analysis into two cases: (1) when the natural unlimited equilibrium (call it λ^*) is such that $\lambda^* < \Lambda$, and (2) when the natural unlimited equilibrium is not reached because it is bigger than Λ.

3. Competition

Let $\lambda^* = \mu - \sqrt[3]{\frac{C\cdot\mu^2}{R}}$.

Theorem 1.

(1) Suppose that $\lambda^* \leq \Lambda$. Then there exists a unique equilibrium where $\lambda = \lambda^*$, $w = \sqrt[3]{\frac{R}{c\cdot\mu^2}}$, $P_N = \sqrt[3]{\mu C^2 R} - C$, $P_S = R - \sqrt[3]{\frac{CR^2}{\mu}}$, $\pi_N = (\sqrt[6]{CR^2\mu^2} - \sqrt{C})^2$ and $\pi_s = (\sqrt{R\mu} - \sqrt[6]{RC^2\mu})^2$.

(2) Suppose that $\lambda^* > \Lambda$. Then in equilibrium $\lambda = \Lambda$ and $w = \frac{1}{\mu-\Lambda}$. There exists a continuum of equilibria where for $x \in [\frac{C\mu}{\mu-\Lambda} - C, \frac{(\mu-\Lambda)^2 R}{\mu} - C]$ $P_N = x$, $P_S = R - \frac{x+C}{\mu-\Lambda}$, $\pi_N = \frac{\Lambda}{\mu-\Lambda}x$, and $\pi_S = \Lambda(R - \frac{x+C}{\mu-\Lambda})$.

Proof. We start our discussion assuming that Λ is very large. In the customers' equilibrium: $R = P_s + P_N w + Cw$, or $R = P_s + (P_N + C)\frac{1}{\mu-\lambda}$. Hence, given a set of prices, we get a joining rate:

$$\lambda = \mu - \frac{P_N + C}{R - P_S}. \tag{1}$$

Notice that in order to have $\lambda \geq 0$, the condition: $P_N + \mu P_s \leq \mu R - C$ has to be satisfied, and it is easy to check that for the equilibrium prices we find in (4) and (5), this condition is equivalent to assumption 7.

Using (1) and the equations for π_S and π_N,

$$\pi_s = P_S\mu - P_S\frac{P_N + C}{R - P_S}, \tag{2}$$

and

$$\pi_N = P_N\frac{\mu(R - P_S) - (P_N + C)}{P_N + C} \tag{3}$$

Maximizing π_s with respect to P_s and π_N with respect to P_N, yields the response-function strategies

$$P_S = R - \sqrt{\frac{(P_N + C)R}{\mu}} \tag{4}$$

and

$$P_N = \sqrt{C\mu(R - P_S)} - C, \tag{5}$$

which give

$$P_S = R - \sqrt[3]{\frac{CR^2}{\mu}} \tag{6}$$

and

$$P_N = \sqrt[3]{\mu C^2 R} - C. \tag{7}$$

It can be easily shown that π_s and π_N are concave in P_s and P_N respectively. Substituting (6) and (7) in equations (1)–(3) we obtain

$$\lambda = \mu - \sqrt[3]{\frac{C\mu^2}{R}} = \lambda^*, \tag{8}$$

$$w = \sqrt[3]{\frac{R}{C\mu^2}}, \tag{9}$$

$$\pi_S = (\sqrt{R\mu} - \sqrt[6]{RC^2\mu})^2, \tag{10}$$

$$\pi_N = (\sqrt[6]{CR^2\mu^2} - \sqrt{C})^2. \tag{11}$$

Consider next the case where Λ poses an effective limitation, namely $\lambda^* > \Lambda$. In this case, the providers may increase their prices relative to those in (6) and (7) without affecting the joining rate to the system, up to the point where

$$R - P_s = \frac{P_N + C}{\mu - \Lambda} \tag{12}$$

(by substituting $\lambda = \Lambda$ in (1)). Any solution satisfying this relation and the bounds $P_S \geq R - \sqrt{\frac{(P_N+C)R}{\mu}}$ and $P_N \geq \sqrt{C\mu(R - P_S)} - C$ (following (4) and (5) respectively) will form an equilibrium. These solutions are therefore all those satisfying:

$$P_N = x,$$
$$P_s = R - \frac{x + C}{\mu - \Lambda}$$

for all values of $x \in [\frac{C\mu}{\mu-\Lambda} - C, \frac{(\mu-\Lambda)^2 R}{\mu} - C]$.

Using the definitions for w, π_s and π_N we obtain the desired results. □

Remark. Notice from the second case that $\pi_s + \pi_N = \Lambda(R - \frac{C}{\mu-\Lambda})$. This result emphasizes the fact that the customers' utility is zero, as the entrepreneurs' utilities add up to the collective social utility.

We stop here to show the graphical interpretation of the situation. This should help to increase the intuition about the proof, and of the kind of equilibrium involved.

Stidham [6] relates to "demand" and "supply" curves in analyzing the dynamics in his model and the resulting equilibrium. Our model is different from that of Stidham, and so, we refer to curves of benefit and expenditure for the customers, rather than pure demand and supply.

Customers

We use the function $D(\lambda)$ for the net benefit to the customers, and $S(\lambda)$ for the expenditure (which is what they will be asked to pay). We draw the curves as functions of λ, and expect an equilibrium in the intersection of the curves, because as long as the benefit exceeds the expenditure, λ will be increased. Similarly, we will draw the same curves multiplied by λ: this will represent total benefit $DD(\lambda)$ vs. total expenditure $SS(\lambda)$ with the same intersection. With a rate of arrival λ, each arriving customer will get from the service

$$D(\lambda) = R - \frac{C}{\mu - \lambda} \tag{13a}$$

and pay for it

$$S(\lambda) = P_s + \frac{P_N}{\mu - \lambda}. \tag{14a}$$

Alternatively, we have

$$DD(\lambda) = \lambda R - \frac{\lambda C}{\mu - \lambda} \tag{13b}$$

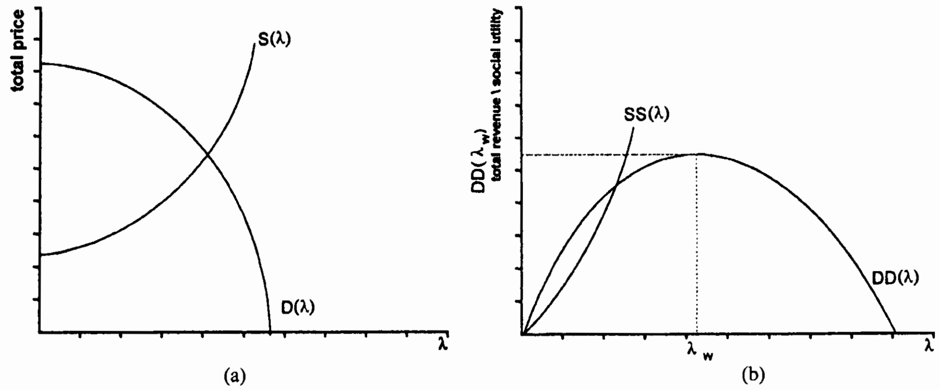

Figure 1.

and

$$SS(\lambda) = \lambda P_s + \frac{\lambda P_N}{\mu - \lambda}. \tag{14b}$$

$SS(\lambda)$ represents the total revenue for both entrepreneurs together, whereas $DD(\lambda)$ equals the collective utility in our model.

We draw (13a) and (14a) in figure 1(a). In order to have a positive solution (equilibrium point) we need $S(0)$ not to exceed $D(0)$. When the equilibrium prices are considered, this condition reduces to our basic requirement in Assumption 7.

It is interesting to observe figure 1(b). $DD(\lambda)$ represents the total benefit of the system which correlates with the usual interpretation of social welfare (Assumption 6). $DD(\lambda)$ has a peak when $R = \frac{C\mu}{(\mu-\lambda)^2}$, or $\lambda_w = \mu - \sqrt{\frac{C\mu}{R}}$, where λ_w is the socially optimal solution for λ. We show this result later in Theorem 6.

The value at this (socially optimal) peak equals $DD(\lambda_w) = (\sqrt{\mu \cdot R} - \sqrt{C})^2$ (as in 13a).

The intersection of $S(\lambda)$ and $D(\lambda)$ (and similarly for $SS(\lambda)$ and $DD(\lambda)$) occurs at λ satisfying (1), and for the competitive-equilibrium set of prices as in (8). We have already showed in the previous section that the intersection point (8) is smaller than the peak of $DD(\lambda)$ as in figure 1(b). Since $SS(\lambda)$ increases with the chosen prices, this means that the equilibrium set of prices under competition is too high from a social point of view. (See also Theorem 7)

Entrepreneurs

We consider next the price decisions of the service and parking providers about the set of prices. Each of the two "players" has an optimal price for each "state of nature", composed of the customers' known preferences and the specific behavior of the other "player". These strategies can be drawn in this plane as the functions $P_s(P_N)$ for the service-provider's strategy, and $P_N(P_s)$ for the parking-provider's strategy. See figure 2.

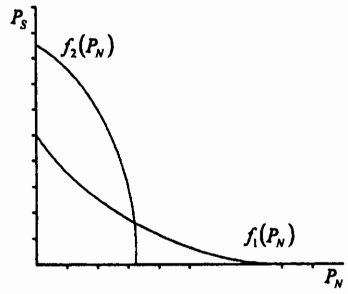

Figure 2.

We already have these functions in (4) and (5). For the service provider $P_s(P_N) = R - \sqrt{\frac{(P_N+C)\cdot R}{\mu}} \equiv f_1(P_N)$ and for the parking provider $P_N(P_s) = \sqrt{C \cdot \mu \cdot (R - P_S)} - C$, or $P_s = R - \frac{(P_N+C)^2}{c\cdot\mu} \equiv f_2(P_N)$.

Using Assumption 7 again, we can compare the intersection points with the horizontal and vertical axis of both functions to find that $f_1(P_N)$ begins below $f_2(P_N)$ and ends above. Since f_1 is convex and f_2 is concave, there is a single intersection in this (positive) range.

Bounded case

First consider figures 1(b) and 3: adding a limitation of Λ to the left of our optimal λ will result in a situation where the actual rate of arrival will be Λ, and for the "optimal" set of prices, the expenditure curve will be lower than the benefit curve at the point. Since the benefit curve is given and unchangeable, we expect the entrepreneurs (either one or both) to raise the prices, so that the expenditure curve will rise until the curves intersect at $\lambda = \Lambda$. See figure 3.

This limitation, seen in figure 3 (for λ), defines a straight line in the $P_s - P_N$ plane:

$$\mu - \frac{P_N + C}{R - P_S} \leq \Lambda, \text{ or}$$

$$P_S \geq R - \frac{P_N + C}{\mu - \Lambda} \qquad \text{(see (12)).}$$

Notice that if $\Lambda \geq \mu$, there will be no points of this kind in figure 2.

We can add this line to figure 2 (we do it in figure 4).

Each of the two entrepreneurs will now choose a strategy which is a combination of the higher between his former strategy (reaction curve) and the lower bound. Figures 4(a) and b show the resulting reaction curves of the service-provider (figure 4(a)), and the parking-provider (figure 4(b)). In figure 4(c), the combination of the two integrated curves is shown, to point out the range of possible Nash-equilibria.

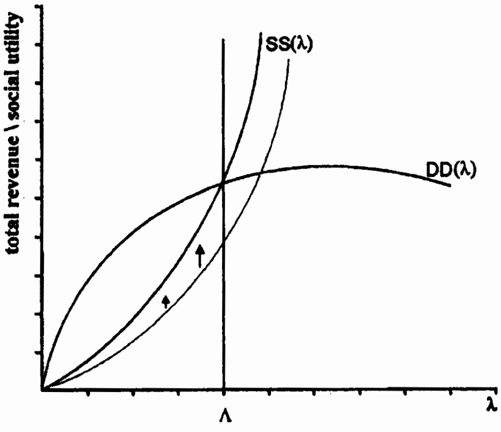

Figure 3.

Corollary 2. When: $\Lambda \geq \lambda^*$:

The parking provider's price is a unimodal function of C with a maximum at $C = \frac{8}{27}\mu R$.

The parking provider's profit is a unimodal function of C with a maximum at $C = \frac{1}{27}\mu R$.

Proof. The result for the parking provider's price is reached by setting its derivative: $\frac{\partial P_N}{\partial C} = \frac{2}{3} \cdot \sqrt[3]{\frac{\mu R}{C}} - 1$ to zero.

The derivative of π_N is: $\frac{\partial \pi_N}{\partial C} = 2(\sqrt[6]{CR^2\mu^2} - \sqrt{C}) \cdot (\frac{1}{6}\sqrt[6]{\frac{R^2\mu^2}{C^5}} - \frac{1}{2}\sqrt{\frac{1}{C}})$.

Under Assumption 7 ($\mu R \geq C$), we know that $\sqrt[6]{CR^2\mu^2} - \sqrt{C} > 0$, so we only have to set $\frac{1}{6}\sqrt[6]{\frac{R^2\mu^2}{C^5}} - \frac{1}{2}\sqrt{\frac{1}{C}}$ to zero, and find the above result for the profit. □

We next consider the relation between the profits of the two entrepreneurs.

Theorem 3. If $\Lambda \geq \lambda^*$, the service provider is better off than the parking provider: $\pi_S \geq \pi_N$.

Proof. From Assumption 7, $\mu R \geq C$. Therefore, by (10) and (11):

$$\begin{aligned}
\sqrt{\pi_S} &= \sqrt{R\mu} - \sqrt[6]{R^2\mu^2 C} \\
&= \sqrt[3]{R\mu}(\sqrt[6]{R\mu} - \sqrt[6]{C}) \\
&\geq \sqrt[3]{C}(\sqrt[6]{R\mu} - \sqrt[6]{C}) \\
&= \sqrt[6]{R\mu C^2} - \sqrt{C} = \sqrt{\pi_N}.
\end{aligned}$$

□

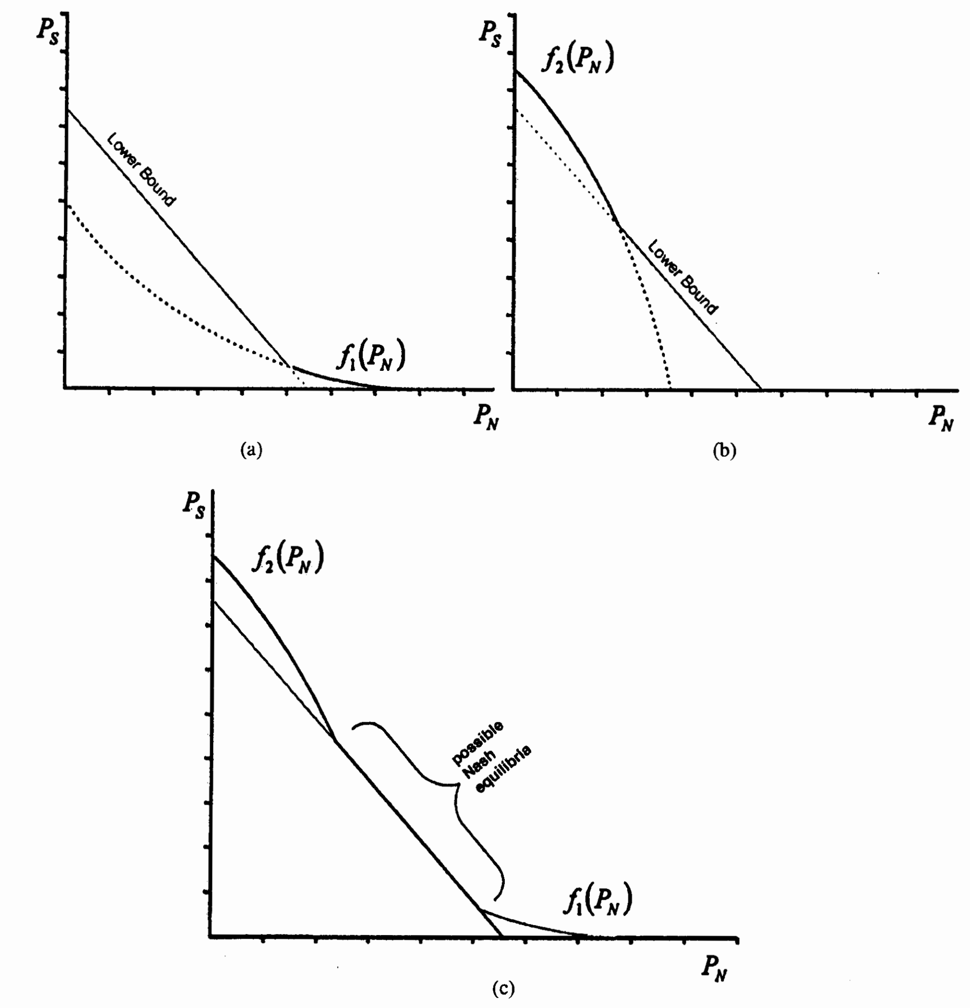

Figure 4.

When $\Lambda < \lambda^*$, the relation between the revenues of the service-provider and the parking-provider depends on the specific values of the parameters, and the parking-provider's profit may be greater, equal to, or less than that of the service-provider.

Note that the lower bound for the range of possible equilibria in the second part of Theorem 1, established by (12), is a monotone decreasing function of Λ. Since in the unbounded case, the response-functions are not influenced by Λ, we have the following:

Corollary 4. For a given price by one of the providers, the price of the other one is monotone non-increasing as a function of the potential rate of arrival (Λ).

4. The monopolist and the social planner

In this section we consider a profit maximizing monopolist who owns both service and parking facilities.

Theorem 5. The Nash-equilibrium in the case of a monopolist is socially optimal. Specifically, the possible sets of prices chosen by the monopolist are either the same as those chosen by a social planner or serve as an upper bound of those set by a social planner.

Proof. We first show that the objectives coincide. The objective functions of the monopolist and of the social planner are $\pi_M(P_S, P_N) = \lambda P_S + \lambda P_N w$ and $\pi_w(P_S, P_N) = \lambda(R - Cw)$, respectively.

We recall that the customers' total expected utility is: $\lambda U = \lambda(R - P_S - P_N w - Cw)$. When the arrival rate is not constrained, this expected utility is set equal to zero in the customers' equilibrium, which implies that

$$\pi_M(P_S, P_N) = \lambda P_S + \lambda P_N w = \lambda(R - Cw) = \pi_w(P_S, P_N),$$

so that the two objectives coincide.

When this optimal λ cannot be reached because the solution is bounded by Λ, the optimal solution for both the social planner and the monopolist is $\lambda = \Lambda$. In this case, there may be price policies which are optimal for the social planner but not for the monopolist. The monopolist will raise his prices as long as the rate of arrival does not drop below Λ, and make sure the utility for the customers is zero. The social planner is indifferent about the distribution of the utility between the facility and the customers, so that the optimal prices for the monopolist serve as an upper bound for his optimal prices. □

Let $\lambda_M = \mu - \sqrt{\frac{C\mu}{R}}$.

Theorem 6. In the case of a monopolist, an equilibrium always exists. Specifically:

(1) Suppose that $\lambda_M \leq \Lambda$. Then in equilibrium $\lambda = \lambda_M$, $w = \sqrt{\frac{R}{C\mu}}$, $\pi_M = (\sqrt{\mu R} - \sqrt{C})^2$. There exists a continuum of equilibria where for $x \in \Re$, $P_N = x$ and $P_S = R - \sqrt{\frac{R(X+C)^2}{\mu C}}$.

(2) Suppose that $\lambda_M > \Lambda$. Then in equilibrium $\lambda = \Lambda$, $w = \frac{1}{\mu - \Lambda}$, $\pi_M = \Lambda R - \frac{\Lambda C}{\mu - \Lambda}$. There exists a continuum of equilibria where for $x \in \Re$, $P_N = x$ and $P_s = R - \frac{x+C}{\mu - \Lambda}$.

Notice, that the profit here is the same as the case when there is no parking facility at all. Indeed, the result for the profit can be found in Edelson and Hildebrand [2] and Table 3.2 in Hassin and Haviv [3].

Proof.

Case I. $\lambda_M \leq \Lambda$

We know that the social planner (and by Theorem 5 this applies to the monopolist as well), wants to maximize $\pi_w = \lambda R - \frac{\lambda C}{\mu - \lambda}$. Differentiating and equating to 0 we obtain $\frac{\partial \pi_w}{\partial \lambda} = R - \frac{C\mu}{(\mu-\lambda)^2} = 0$.

This leads to the optimal result for the social planner and the monopolist:

$$\lambda = \mu - \sqrt{\frac{C\mu}{R}} = \lambda_M. \tag{15}$$

We can calculate the profit to be:

$$\pi_M = (\sqrt{\mu R} - \sqrt{C})^2. \tag{15a}$$

Of course, from the customers' point of view, (1) still holds, so that:

$$\mu - \frac{P_N + C}{R - P_S} = \mu - \sqrt{\frac{C\mu}{R}}$$

and we have:

$$P_S = R - \sqrt{\frac{R(P_N + C)^2}{\mu C}} \tag{16}$$

Case II. $\lambda_M > \Lambda$

Setting the customers' expected utility to zero, the monopolist will make sure the customers' equilibrium joining rate is Λ. Hence

$$\mu - \frac{P_N + C}{R - P_S} = \Lambda,$$

which gives:

$$P_S^M = R - \frac{P_N^M + C}{\mu - \Lambda}. \tag{17}$$

By Theorem 5, (17) gives the upper bound for the prices set by the social planner:

$$P_S^W \leq R - \frac{P_N^M + C}{\mu - \Lambda} \tag{18}$$

□

It is reasonable to require nonnegative prices. In this case (16)–(18) become

$$\begin{aligned}
&0 \le P_N^M \le \sqrt{\mu R C} - C,\\
&0 \le P_N^M \le R(\mu - \Lambda) - C,\\
&0 \le P_N^W \le R(\mu - \Lambda) - C, \quad \text{and}\\
&0 \le P_S^W \le R - \frac{P_N^W + C}{\mu - \Lambda}.
\end{aligned}$$

Theorem 7. In the case of a competition between revenue maximizing service and parking providers, the arrival rate is lower than the socially optimal one. The set of prices set in equilibrium is socially too high.

Proof. We need to show that the expression for λ in (8) is less than or equal to the expression for λ in (15).

This can be concluded from Assumption 7 ($\mu R \ge C$).

Since λ is a decreasing function of both prices, this means that the set of prices in competition is socially too high. □

Partial Intervention of the Social Planner

Consider the case when one of the two separate entities (the service provider and the parking provider) is public (maximizes social utility) whereas the other one is private (maximizes revenues).

We denote:

$\pi^S = \lambda \cdot R - \dfrac{\lambda \cdot C}{\mu - \lambda}$ the social (governmental) objective function,

$\pi_S^P = \lambda \cdot P_S$ a private service provider's objective function,

$\pi_N^P = \dfrac{\lambda \cdot P_N}{\mu - \lambda}$ a private parking provider's objective function.

We use our result (1) for λ, to get:

$$\pi^S = \mu R - \frac{P_N R + C \cdot R}{R - P_S} - \frac{\mu C \cdot R - \mu C \cdot P_S}{P_N + C} - C.$$

We already have the formulas (2) and (3) for the private entrepreneurs' objective functions from the basic model.

Case I. Governmental service provider vs. private parking provider:

We set $\frac{\partial \pi^S}{\partial P_S} = 0$ and $\frac{\partial \pi_N^P}{\partial P_N} = 0$ to get:

$$P_N = \sqrt{R \cdot C \cdot \mu} - C$$
$$P_S = 0$$
$$\lambda = \mu - \sqrt{\frac{C\mu}{R}}.$$

Case II. Governmental parking provider vs. private service provider:

We set $\frac{\partial \pi^S}{\partial P_N} = 0$ and $\frac{\partial \pi_S^P}{\partial P_S} = 0$ to get:

$$P_N = 0$$
$$P_S = R - \sqrt{\frac{CR}{\mu}}$$
$$\lambda = \mu - \sqrt{\frac{C\mu}{R}}.$$

In both cases, we find that the government will choose to "give way" by setting zero-prices, and allow the private entrepreneur to act as a monopoly, achieving the socially optimal result (in the basic model we showed that a monopoly will achieve the socially desired result for λ (15), which is the same result we got here).

5. Fixed parking price

We now assume that both the service-provider and parking-provider charge a fixed sum. We focus on the differences between this case and the basic model with time based parking price and consider only the unlimited demand case. We avoid giving complete proofs whenever they resemble the ones given for the basic model. The resulting equilibrium prices are as follows:

$$P_S = P_N = P = \frac{R}{2} - \frac{C + \sqrt{C^2 + 8\mu C R}}{8\mu}. \tag{19}$$

This price is nonnegative by the assumption $\mu R \geq C$.

For the monopolist:

$$P_M = R - \sqrt{\frac{CR}{\mu}}. \tag{20}$$

We also have

$$\lambda = \mu - \frac{4\mu C}{C + \sqrt{C^2 + 8\mu RC}}, \tag{21}$$

$$\pi_i = \frac{2\mu R + C}{4} - \frac{12\mu RC}{4C + 4\sqrt{C^2 + 8\mu RC}} \quad i \in (S, N) \tag{22}$$

The monopolist still takes all the customers' utility, and reaches as before the socially optimal solution.

We note that Assumption 7, i.e. $\mu R \geq C$, implies that

$$4\sqrt{\mu R} \geq \sqrt{C} + \sqrt{C + 8\mu R}.$$

Therefore,

$$\begin{aligned} P_M &= R - \frac{\sqrt{C}}{\mu}\sqrt{\mu R} \\ &\leq R - \frac{\sqrt{C}}{4\mu}(\sqrt{C} + \sqrt{C + 8\mu R}) \\ &= 2P = P_S + P_N. \end{aligned}$$

We conclude that the competition set of prices is higher than that of the monopolist and the social planner (and the rate of arrival will be smaller). Thus, Theorem 7 holds also in this case.

Since the price structure is of no consequence for the social planner, the socially optimal λ and the socially optimal collective utility (which equals the monopolist's maximal revenue) are the same as before.

For the monopolist/social planner, the results in Theorems 5 and 6 hold for λ, π.

Theorem 8. If the parking provider can choose a pricing system (between a fixed price and a time-based price), he will prefer the fixed price system. This is also socially preferred.

Proof. We begin by proving that the value λ^{fixed} of λ under competition in the fixed price system is bigger than its value λ^{var} in the variable (time based) price system.

Denote $\alpha = R\mu/C$, then by (8) and (21)

$$\lambda^{\text{fixed}} = \mu\left(1 - \frac{4}{1 + \sqrt{1 + 8\alpha}}\right) \quad \text{and} \quad \lambda^{\text{var}} = \mu\left(1 - \sqrt[3]{\frac{1}{\alpha}}\right).$$

One can verify in a straightforward way that for every $\alpha \geq 1$, $\lambda^{\text{fixed}} \geq \lambda^{\text{var}}$.

We have already seen that the social utility function is unimodal concave in λ, and that under competition the joining rate is less than or equal to the socially desired value

(See figure 1(b) in Section 5.1). This is true both when the parking price is fixed and when it is variable. This means that between the two options, the one with the higher result for λ is socially preferred (the socially optimal λ is the same in both cases). In other words, the social utility gained under fixed pricing is higher than the social utility gained under variable time-based pricing.

Now consider the profits of the two entrepreneurs. Since they share all of the social utility (the customers get zero utility), we just proved that the total profit ($\pi_S + \pi_N$) is bigger under a fixed parking price.

We also know that under the fixed price system we get a symmetrical equilibrium ($\pi_S = \pi_N$), whereas under the variable pricing system the service provider is better off than the parking provider ($\pi_S \geq \pi_N$).

Obviously: $\pi_N^{\text{fixed}} \geq \pi_N^{\text{var}}$. □

6. Convergence and stability

We follow Stidham [6] in analyzing the dynamics of the model using cobweb diagrams. Convergence analysis can be also found in Yao [7].

Consider figure 1(b). One can see that whenever the market is to the left of the intersection point, there is an incentive for the customers to increase their rate of arrival, as they will encounter on average positive net utilities. Whenever the market is to the right of the intersection point, there is an incentive for the customers to decrease their rate of arrival, since they will encounter on average negative net utilities.

Consider figure 5. Broken lines denote the dynamics of the reactions of the two entrepreneurs to each other from a starting state, until equilibrium is reached. It can be seen in figures 5(a) and (b) that the system will always converge to equilibrium in both the limited and unlimited cases.

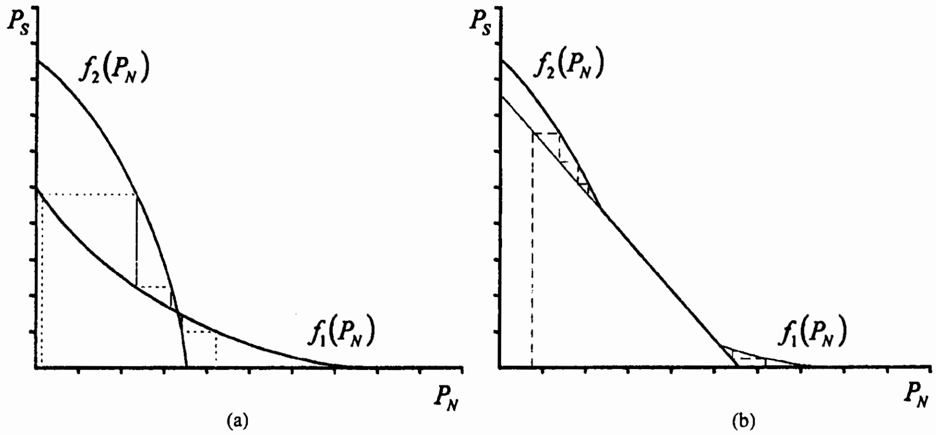

Figure 5.

7. Concluding remarks

In this paper we studied a model of queue-oriented competition between providers of totally complementary products. A customer has to pay both suppliers, and stand in line (once) to enjoy the benefits of the service. We considered the possibility that one of the providers asks for a price that depends on the time the customer spends in the system.

Some of the results might seem counter-intuitive at first, and are related to the fact that we are dealing with complementary products. First, consider the pricing sensitivity to the demand - Λ. Intuition suggests that when demand increases, so will the prices. However, in our model, when Λ increases the entrepreneurs (be it a monopolist or two separate competing entities) respond by lowering the prices. A similar result was reached by Chen and Frank [1]. The explanation is in the fact that an increase in the joining rate increases the expected waiting time, and this can be viewed as deterioration in the quality of service.

Second, as opposed to common competitive-oriented intuition, we found that two separate entities that are engaged in price competition reach a higher set of prices than that reached by a monopolist. Again, the reason for this lies in the fact that this is a competition between suppliers of completely complementary products.

We showed that a monopolist will set prices to reach the social optimum. This is a result that was dealt with in previous work (see Edelson and Hildebrand [2]). The reason for this phenomenon lies in the fact that it is possible for the monopolist to gain all the utility (zero customer surplus) under the structure of the model.

We found that if the "parking provider" had a choice, he would prefer to charge a fixed price like his counterpart. It turns out that within the specific structure of this model, there would probably not be such a pricing system.

One might consider the long-run case, where the service-provider is able to choose the rate of service. This point was investigated by Chen and Frank [1], who found that a monopolist in a similar model will either choose not to produce at all ($\mu = 0$) or set the rate of arrival as to accommodate all the population (and if the potential demand is infinite, may choose an infinite rate of service). However, the rationale behind the result of Chen and Frank does not apply in the case of complementary products (basically, when the service-provider pays to increase the rate of service, the parking-provider will exploit it for his own good and raise his price).

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