CONTROL OF ARRIVALS AND DEPARTURES IN A STATE-DEPENDENT INPUT–OUTPUT SYSTEM

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At each decision epoch, an offer for a unit to either enter or leave a system is received. These offers arrive according to a Poisson process. With each offer is associated a value revealed upon the arrival of the offer. The distribution of the value of the offer is given and is a function of what kind of offer is received (enter or leave). The decision to accept or reject an offer is allowed to depend on the current state and the current value received. The objective is to maximize the expected discounted difference between the sum of the accepted output offers and the sum of the accepted input offers. The key result of the paper is that under various conditions, the decision to accept or reject an offer depends on whether or not its value is above or below, respectively, a critical value that depends on the state of the system.

optimal control * birth and death processes * inventory and queues control

1. Introduction

We consider a system whose input and output can be controlled to maximize profit. At each decision epoch, an offer for a unit to either enter (input offer) or leave (output offer) the system is received. These offers arrive according to a Poisson process, with intensity parameters that may depend on the current number of units in the system (i.e., the system state). With each offer there is associated a value; a non-negative random variable whose distribution depends on the type of the offer (input or output). The decision to accept or reject an offer naturally depends on the system state and the offered value. The objective is to maximize the expected discounted profit, which is the difference between the sum of accepted output offers and the sum of the accepted input offers minus the inventory holding costs.

An example to a system of this type is a dealer facing streams of customers willing to sell or buy quantities of a commodity at various prices.

In this paper we seek for sufficient conditions under which the optimal acceptance policy is characterized by a decreasing function of the system state. In this case, an output (input) offer with a given value will be accepted if and only if the system state is above (below) a critical number.

The monotonicity of optimal policies is an issue in several birth and death processes described in the literature. For an example and references, see Albright and Winston [1]. Following the seminal work of Naor [5] monotone acceptance policies have been largely dealt in queueing control models. The paper of Stidham [6] contains a survey of such models. Recently, David and Yechiali [3] applied a similar model where the offers consist of live organs to be transplanted. The items in the system are the patients and the value of an offer is the expected lifetime of the patient.

An assumption, common to our model and to some of the works described above, is that the decision maker is not able, or does not want, to announce a unit price of the item. For related models where a unit price is announced, the reader is referred to Amihud and Mendelson [2].

Our model has several distinctive features: The arrival rates may depend on the system state, which in queueing models means the queue length. This extends the model to include situations where the storage capacity is finite (so that the input rate becomes zero when the storage is full), and where output offers emerge from the inventory (e.g., a multi-server repair shop). We allow offers to differ in their value, consider inventory holding costs, and include control of output offers. Finally, in a separate section, we consider batch arrivals of input offers. Queueing models with finite capacity can be obtained from our model by letting the items to be the vacant places of the waiting room.

In Section 2 we prove that if the arrival rates of the offers are non-increasing and the inventory holding costs are non-decreasing functions of the system state, then there exists an interval $0, 1, \ldots, M$ where the optimal acceptance policy is monotone. Moreover, once in this interval, the system will stay in it forever. In Section 3 we generalize this result, allowing input offers to arrive in batches. It is interesting to note that although we consider an infinite horizon problem, our proof is by backward induction.

2. The optimal policy

Let $m \geq 0$ denote the inventory level. Let μ_m and λ_m be the arrival rates of input and output offers, respectively, when the inventory level is m. We assume that $0 \leq \lambda_m \leq \bar{\lambda}$ and $0 \leq \mu_m \leq \bar{\mu}$ for $m = 0, 1, \ldots$. Let G and H be the probability distribution functions of the input and output offers, respectively. We assume that output offers have finite expected value. Let $\alpha > 0$ be the discount factor, and $h(m)$ be the holding costs associated with each unit time when the inventory level is m. We assume that $h(0) = 0$ and $h(m+1) \geq h(m)$ for $m = 0, 1, \ldots$.

Define $F_+(m, x)$ to be the expected discounted profit associated with an optimal policy when an input offer with value x is considered and the inventory level is m. Similarly, define $F_-(m, x)$ for an output offer. Standard application of the principal of optimality and basic results in the theory of Poisson processes give

$$\begin{aligned} &F_+(m, x) = \max\{-x + f(m+1), f(m)\}, && m = 0, 1, \ldots, \\ &F_-(m, x) = \max\{x + f(m-1), f(m)\}, && m = 1, 2, \ldots, \\ &F_-(0, x) = f(0), \end{aligned} \tag{1}$$

where

$$f(m) = a_m \int_0^\infty F_-(m, x) \mathrm{d}H(x) + b_m \int_0^\infty F_+(m, x) \mathrm{d}G(x) - c_m, \tag{2}$$

$$a_m = \lambda_m/(\lambda_m + \mu_m + \alpha), \quad b_m = \mu_m/(\lambda_m + \mu_m + \alpha) \quad \text{and} \quad c_m = h(m)/(\lambda_m + \mu_m + \alpha).$$

Thus $f(m)$ can be interpreted as the expected discounted profit associated with an optimal policy at some random point of time when the inventory level is m.

Let

$$l_m = f(m) - f(m-1), \qquad m = 1, 2, \ldots$$

(and for convenience $l_0 = \infty$). Then, by (1)

$$\begin{aligned} F_-(m, x) &= x + f(m-1), && x \geqslant l_m, \\ &= f(m), && x < l_m, \end{aligned} \tag{3a}$$

and

$$\begin{aligned} F_+(m, x) &= -x + f(m+1), && x \leqslant l_{m+1}, \\ &= f(m), && x > l_{m+1}, \end{aligned} \tag{3b}$$

for $m = 0, 1, \ldots$. We can substitute (3) in (2) to obtain

$$f(m) = a_m\left[f(m)G(l_m) + f(m-1)(1-G(l_m)) + \int_{l_m}^{\infty} x\mathrm{d}G(x)\right]$$

$$+ b_m\left[f(m+1)H(l_{m+1}) - \int_0^{l_{m+1}} x\mathrm{d}H(x) + (1-H(l_{m+1}))f(m)\right] - c_m$$

$$= a_m\left[l_m G(l_m) + f(m-1) + \int_{l_m}^{\infty} x\mathrm{d}G(x)\right]$$

$$+ b_m\left[l_{m+1}H(l_{m+1}) + f(m) - \int_0^{l_{m+1}} x\mathrm{d}H(x)\right] - c_m, \qquad m = 1, 2, \ldots,$$

$$f(0) = a_0 f(0) + b_0\left[l_1 H(l_1) + f(0) - \int_0^{l_1} x\mathrm{d}H(x)\right].$$

It can be written also as

$$f(m) = \frac{1}{\alpha}\left[\lambda_m\int_{l_m}^{\infty}(x-l_m)\mathrm{d}G(x) + \mu_m\int_0^{l_{m+1}}(l_{m+1}-x)\mathrm{d}H(x) - h(m)\right], \qquad m = 1, 2, \ldots,$$

$$f(0) = \frac{1}{\alpha}\mu_0\int_0^{l_1}(l_1 - x)\mathrm{d}H(x). \tag{4}$$

The integrals in (4) are the expected profit of input and output offers, respectively. The term inside the brackets is the expected rate of profit inflow if the system state is m. If this rate of profit is continued forever, then the total discounted profit is $f(m)$.

We turn now to prove that there exists an interval $0, 1, \ldots, M$ for some $M \in \{0, 1, \ldots, \infty\}$ such that the critical values $l_1, \ldots, l_M$ form a decreasing sequence, and if the system starts with an inventory level within this interval then it will never go beyond it. The monotonically of the critical values reflects, by definition, the decreasing marginal worth of units in the inventory. This result need not hold if, for example, large inventory atracts more offers. Thus we assume that the arrival rates are non-increasing in the system state. The marginal worth of units in inventory becomes negative for $m > M$ both because of the inventory holding costs and the decrease in the arrival rates. It can be shown that $M = \infty$ if $h(m) = 0$, $\mu_m = \mu$, and $\lambda_m = \lambda$ for $m = 0, 1, \ldots$ (except of course $\lambda_0 = 0$). On the other hand, if the holding costs are high, M may be zero so that all input offers will be rejected even when $m = 0$. Fast decrease in the arrival rates may also decrease M, but if $h(1) = 0$ then $M \geq 1$ (Hassin and Henig [4]).

Theorem 1. *Suppose that λ_m and μ_m are non-increasing, then there exists $M \in \{0, 1, \ldots, \infty\}$ such that $\infty = l_0 > l_1 > l_2 > \ldots > l_M > 0 \geq l_{M+1}$.*

Proof. Suppose that for some $m \geq 1$ $l_{m+1} > l_{m+2}$, and $l_{m+1} > 0$, so that

$$\alpha[f(m+1) - f(m)] = \alpha l_{m+1} > 0. \tag{5}$$

By (4), the left-hand side of (5) is the sum of the following two expressions:

$$\lambda_{m+1}\int_{l_{m+1}}^{\infty}(x - l_{m+1})\mathrm{d}G(x) - \lambda_m\int_{l_m}^{\infty}(x-l_m)\mathrm{d}G(x), \tag{6}$$

$$\mu_{m+1}\int_0^{l_{m+2}}(l_{m+2} - x)\mathrm{d}H(x) - \mu_m\int_0^{l_{m+1}}(l_{m+1} - x)\mathrm{d}H(x) - [h(m+1) - h(m)]. \tag{7}$$

Since $h(m+1) \geq h(m)$, and $\mu_{m+1} \leq \mu_m$, then (7) is non-positive. To obtain (5) the expression in (6) must be positive, and since $\lambda_m \geq \lambda_{m+1}$ it follows that $l_m > l_{m+1}$. Therefore, if there exists some M such that $l_M > 0$ and $l_{M+1} < l_M$ then the above induction proves that l_m is monotone decreasing over $m = 0$,

$1, \ldots, M$. Indeed, let $M = \sup\{m \mid l_M > 0,\ M = 0, 1, \ldots\}$. If $M < \infty$ then the theorem is proved by the above induction, since $l_M > 0 \geq l_{M+1}$. Suppose that $M = \infty$, then l_m converges to zero since $\sum_{i=1}^{m} l_i = f(m) - f(0)$ is bounded. Therefore, for every $m \geq 1$ there exists $\overline{m} \geq m + 1$ such that $l_{\overline{m}} > l_{\overline{m}+1} > 0$, and by the induction $l_m > l_{m+1}$. Hence l_m is monotone decreasing over all positive integers. □

3. Batch arrivals of input offers

In Section 2 we assumed that each offer consists of one unit. Here we let each input offer to consist of several units of equal value x, and show that Theorem 1 still holds. We note that the theorem will not necessarily hold if output offers arrive in batches. Suppose, for example, that each output offer consists of two units, and that $h(m) = h$ for $m = 1, 2, \ldots$, so that there is a fixed cost for holding positive inventory, independent of its size. The marginal worth of the first unit of inventory is reduced because of this cost, but not the second unit, since both units will leave simultaneously. Therefore, $l_2 > l_1$ is possible. Theorem 1 may hold in this case under more restrictive assumptions, but a different proof (such as Successive Approximations) will be needed since expression (7) will contain terms with l_{m-i} for positive values of i, for which the inductive hypothesis does not apply.

Let $p(i+)$ denote the probability that an input offer consists of a batch of size i or bigger. Suppose that $l_{m+1} > l_{m+2} > \ldots > l_M > 0 \geq l_{M+1}$, then an optimal acceptance policy for an input offer of size k and value x is as follows: reject the offer completely if $x \geq l_{m+1}$, accept all of it if $x < l_{m+k}$, accept i units of it if $l_{m+i+1} \leq x < l_{m+i}$. Note that since $l_{M+1} \leq 0$, no more than M-m units will be accepted.

Considerations similar to those presented in Section 2 yield that for the present case, eq. (4) is replaced by (8),

$$f(m) = \frac{1}{\alpha}\left[\lambda_m \int_{l_m}^{\infty} (x - l_m) \mathrm{d}G(x) + \mu_m \sum_{i=1}^{\infty} p(i+) \int_0^{l_{m+i}} (l_{m+i} - x) \mathrm{d}H(x) - h(m)\right], \tag{8}$$

where l_{m+i} is taken as zero for $m + i > M$.

The summation reflects the acceptance policy described above, where the ith unit will be accepted only if $x < l_{m+i}$, and then the expected value of x in this range is subtracted while $f(m+i) - f(m+i-1) = l_{m+i}$ is added.

The summation term is easily seen to decrease with m if we use the inductive hypothesis that $l_{m+1} > l_{m+2} > \ldots > l_M > 0$ and $l_{M+i} \equiv 0$ for $i \geq 1$. Therefore, as in the proof of Theorem 1, expression (6) must be positive and the rest of the proof follows identically.

References

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